

On a comprehensive class of linear control problems

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We discuss a class of linear control problems in a Hilbert space setting. This class encompasses such diverse systems as port-Hamiltonian systems, Maxwell’s equations with boundary control or the acoustic equations with boundary control and boundary observation. The boundary control and observation acts on abstract boundary data spaces such that the only geometric constraint on the underlying domain stems from requiring a closed range constraint for the spatial operator part, a requirement which for the wave equation amounts to the validity of a Poincaré–Wirtinger-type inequality. We also address the issue of conservativity of the control problems under consideration.

Keywords: linear control systems; well-posedness; conservativity; evolutionary equations.

1. Introduction

Finite-dimensional linear control problems are commonly discussed in the form of a differential-algebraic system. The first system equation links the *state* x taking values in \mathbb{R}^n to the *control* or *input* u , which takes values in \mathbb{R}^m via matrices A, B, μ_0 of appropriate size in the way

$$\mu_0 \dot{x}(t) = Ax(t) + Bu(t), \quad t \in]0, \infty[.$$

If μ_0 is boundedly invertible, the latter equation is also known as *state differential equation*, in general we could have here a *state differential-algebraic equation*. This equation is completed by some initial condition for the part of the state variable that gets differentiated, i.e. $(\mu_0 x)(0+) = \mu_0 x_0$. In control theory, one is mainly interested in the *observation* or *output* y , which is an \mathbb{R}^l -valued function given by the *observation equation*

$$y(t) = Cx(t) + Du(t), \quad t \in]0, \infty[,$$

for suitable matrices C and D .

Thus, denoting the time derivative by ∂_0 and using the whole real line \mathbb{R} instead of $]0, \infty[$, which transforms the initial condition into a Dirac- δ -source term on the right-hand side, we arrive at the following system:

$$\begin{pmatrix} \partial_0 \mu_0 - A & 0 \\ -C & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} B \\ D \end{pmatrix} u + \begin{pmatrix} \delta \otimes \mu_0 x_0 \\ 0 \end{pmatrix}. \quad (1)$$

Here for time-continuous states Φ , we have $(\delta \otimes x_0)\Phi := x_0^* \Phi(0)$.

In essence, with the added observation equation we are just considering a larger differential-algebraic equation with an implied specific block structure.

Making x, u the unknowns and treating y as a term on the right-hand side, we arrive at the alternative formulation

$$\begin{aligned} \begin{pmatrix} A - \partial_0 \mu_0 & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} &= \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} - \partial_0 \begin{pmatrix} \mu_0 & 0 \\ 0 & 0 \end{pmatrix} \right) \begin{pmatrix} x \\ u \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} y - \begin{pmatrix} \delta \otimes \mu_0 x_0 \\ 0 \end{pmatrix}. \end{aligned} \quad (2)$$

Whereas well-posedness issues are discussed in connection with respect to (1) (given control u , unknown output y) the—in a sense—inverse problem (2) (given output y , unknown control u) is the usual starting point of discussion of control system leading in the commonly discussed case $\mu_0 = 1$ to the analysis of 2×2 block matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

Systems of such general block structure have been generalized to the infinite-dimensional case. In this case, A, B, C and D are linear operators in suitable Hilbert spaces. A solution theory for this problem is rather straightforward, if one assumes that $\mu_0 = 1$ and A is a generator of a strongly continuous semi-group and the operators B, C and D are bounded linear operators.

If one studies systems with boundary control, the assumption on B and C to be bounded has to be lifted. Hence, more sophisticated techniques need to be used to establish well-posedness of such systems even if $\mu_0 = 1$ is assumed, see Salamon (1987); Salamon (1989), Curtain and Weiss (1989), Weiss (1989a), Engel (1998), Lasiecka & Triggiani (2000a,b), Weiss & Tucsnak (2003) and Jacob and Partington (2004). In the light of the rather sophisticated considerations required to deal with such a situation, the question arises if a different perspective may shed some new insight on this problem class. Taking our guidance from the discussion in a book by Lasiecka & Triggiani (2000a) and two seminal papers by Weiss & Tucsnak (2003) and Weiss *et al.* (2001), where a class of systems is specified by $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with A being a semi-group generator and B, C operators, which are not bounded operators between state and control space is considered, it has been found, Picard *et al.*, that by introducing an additional state variable we get an equivalent system with a different 2×2 -block structure $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where now A is even skew-selfadjoint¹ and B, C, D are all bounded linear operators. However, since μ_0 is not invertible the (semi-)group for A is of little help to obtain well-posedness. Fortunately, there is a whole machinery to attack differential-algebraic systems directly without resorting to one-parameter semi-group techniques. The solution strategy relies solely on the fact that—in a suitable Hilbert space setting—the whole differential-algebraic system operator together with its adjoint is strictly positive definite. Since—by elementary Hilbert space functional analysis—strict positive definiteness of a closed operator T and of its adjoint T^* implies that 0 is an element in the resolvent sets of both operators, it would probably be difficult to find a more basic well-posedness class than this one. Surprisingly, however, this class is spacious enough to cover all classical linear evolution problems of mathematical physics and allows for convenient generalization to more complex ‘material relations’. The solution concept does not require the existence of a fundamental solution. Therefore, questions naturally arising in the semi-group context such as whether an operator is admissible or not (Salamon, 1987; Weiss, 1989a; Engel, 1998; Lasiecka & Triggiani, 2000a,b; Jacob and Partington, 2004) can be by-passed and replaced by a mere regularization requirement rather than the well-posedness of the respective equations.

¹ For two operators A, B defined on a Hilbert space, we say that A is the adjoint of B if $A = B^*$, and we say that A is selfadjoint if $A = A^*$. In order to consistently extend this terminology to the case, when $A = -B^*$, we choose to say that A is the skew-adjoint of B . Therefore, if $A = -A^*$, we say that A is skew-selfadjoint.

In this note, we shall present a unified way of looking at control problems of this type as differential-algebraic systems, which may make the solution theory more easily accessible. More precisely, we will provide evidence that linear control problems can readily be understood as evolutionary equations, a particular class of differential-algebraic equations, which have been studied and used for many applications to other fields, see [Picard & McGhee \(2011\)](#). We will show that a large class of linear (boundary) control systems fits into this class. We exemplify these observations with linear boundary control problems studied by [Weiss \(1989b\)](#), [Weiss & Tucsnak \(2003\)](#), [Lasićka & Triggiani \(2000a,b\)](#) and [Weck \(2000\)](#). It should be noted, however, that the class presented here is much larger, since we are not limited to cases, where one-parameter semi-group strategies can successfully be utilized. This having been said, it also has to be admitted that the results of this paper are merely addressing the foundation of control problems. Actual control issues such as controllability, reachability, stability, etc. are beyond the scope of this paper and may constitute future research.

In the process of developing our framework for boundary control systems, we shall also make a particular effort at developing a theoretical setting for dealing with arbitrary boundaries of underlying domains, which is of importance in more realistic applications, where boundary smoothness is not reasonable to assume. This way we are saved from using boundary trace results, which are hard to come by or unavailable, for example, for domains with cuts, cusps, line segments or fractal boundaries. However, the general well-posedness results are independent of this theoretical setting, which in any case may also be substituted by more classical boundary trace ideas, if requiring sufficient smoothness of the boundary is not an issue.

A particular subclass of port-Hamiltonian systems ([Le Gorrec *et al.*, 2006](#); [Zwart *et al.*, 2010](#); [Jacob and Zwart, 2012](#)) can be discussed within this theory. As a by-product, we give a possible generalization of boundary control systems similar to port-Hamiltonian systems to the case of more than one spatial dimension, which appeared to be, at least to the best of the authors' knowledge, an open problem.

We will also address the issue of conservativity. In fact, we show a certain type of impedance conservativity ([Staffans, 2001, 2002](#); [Weiss *et al.*, 2001](#); [Ball and Staffans, 2006](#); [Malinen & Staffans, 2006, 2007](#); [Weiss & Tucsnak, 2003](#)). Thereby, we show that the hypotheses on the structure of the material law in [Picard *et al.*](#) can be weakened. We obtain a certain general energy-balance equality, imposing assumptions on the structure of the equation that are easily verified in applications.

In Section 2, we give the functional-analytic preliminaries needed to discuss evolutionary equations in the sense of [Picard \(2009\)](#). This includes the time derivative realized as a normal, continuously invertible operator and the notion of Sobolev chains.

Section 3 states the notion of abstract linear control systems defined as a subclass of particular evolutionary systems. We show well-posedness of the respective systems under easily verifiable conditions on the structure of the operators involved. In essence, this section recalls the well-posedness theorem of [Picard \(2009\)](#) including the notion of causality defined in [Kalauch *et al.* \(2014\)](#).

Section 4 discusses the qualitative property of conservativity for abstract linear control systems. In order to show conservativity of abstract linear control systems, a particular structure of the operators involved and a regularizing property of the solution operator associated to the system is needed. The regularizing property is slightly stronger than the one in [Picard *et al.*](#) As a trade-off, the structural requirements on the operators involved are less restrictive.

The subsequent section, Section 5, provides a way to embed linear boundary control systems into abstract linear control systems. For an account on boundary control systems dealt with in the literature, we refer the reader to [Arov *et al.* \(2011\)](#), [Malinen & Staffans \(2007\)](#), [Malinen & Staffans \(2006\)](#), [Salamon \(1987\)](#), [Weck \(2000\)](#), [Weiss \(1989a\)](#), [Weiss & Tucsnak \(2003\)](#) and [Zwart *et al.* \(2010\)](#), where also strategies from the theory of selfadjoint extensions of symmetric operators come into play, see

Behrndt *et al.* (2009), Behrndt & Kreusler (2007), Derkach *et al.* (2009), Waurick & Kaliske (2012) and Schubert *et al.*. As a first illustrative example of boundary control systems, we discuss in Section 5.1 the notion of port-Hamiltonian systems as introduced in Jacob and Zwart (2012), also see Jacob and Zwart. In order to give higher-dimensional analogues for a particular subclass of port-Hamiltonian systems, we define abstract boundary data spaces (Section 5.2). The latter can and will be introduced in a purely operator-theoretic framework. Consequently, in applications these spaces may be defined without any regularity assumptions on the underlying domain. The main idea is to replace the classical trace spaces, which may not be defined in the general situation of irregular boundaries, with an abstract analogue of ‘1-harmonic functions’. Section 5.3 provides the solution theory of a class of abstract linear control systems with boundary control and boundary observation.

The last section, Section 6, is devoted to illustrate our previous findings. We give an alternative way to show the well-posedness of Maxwell’s equation with boundary control similar to the one discussed in Weck (2000) (Section 6.2) and the well-posedness of a wave equation with boundary control and observation generalizing the one discussed in Weiss & Tucsnak (2003) (Section 6.1).

2. Functional-analytic framework

In this section, we introduce the framework for evolutionary equations, which will be defined in the next section. The relevant statements of the results can be found in more detail in Picard & McGhee (2011). First, following Kalauch *et al.* (2014), we define the time derivative as a normal, boundedly invertible operator in a suitable L^2 -type space.

DEFINITION 2.1 For $\nu \in]0, \infty[$, we denote by $H_{\nu,0}(\mathbb{R})$ the space of all square-integrable functions² with respect to the exponentially weighted Lebesgue-measure $\exp(-2\nu t) dt$, equipped with the inner product given by

$$\langle f | g \rangle_{H_{\nu,0}(\mathbb{R})} := \int_{\mathbb{R}} f(t)^* g(t) \exp(-2\nu t) dt \quad (f, g \in H_{\nu,0}(\mathbb{R})).$$

REMARK 2.2 From the definition of $H_{\nu,0}(\mathbb{R})$, we see that the operator $\exp(-\nu m) : H_{\nu,0}(\mathbb{R}) \rightarrow L^2(\mathbb{R})$, defined by $(\exp(-\nu m)f)(t) = \exp(-\nu t)f(t)$, $t \in \mathbb{R}$, is unitary. Furthermore, it is clear that the space $\mathring{C}_{\infty}(\mathbb{R})$, the space of indefinitely differentiable functions with compact support on \mathbb{R} , is dense in $H_{\nu,0}(\mathbb{R})$.

DEFINITION 2.3 Let $\nu > 0$. We denote by³ $\partial : H^1(\mathbb{R}) \subseteq L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ the usual weak derivative on $L^2(\mathbb{R})$, which is known to be skew-selfadjoint, i.e. $\partial^* = -\partial$. We set

$$\partial_{0,\nu} := \exp(-\nu m)^{-1}(\partial + \nu) \exp(-\nu m),$$

as the derivative operator on $H_{\nu,0}(\mathbb{R})$. For convenience, we will write ∂_0 instead of $\partial_{0,\nu}$ if the particular choice of $\nu > 0$ is clear from the context.

REMARK 2.4 The operator $\partial_{0,\nu}$ is normal with $\Re \partial_{0,\nu} = \nu$. Moreover, since the operator $\exp(-\nu m)^{-1} \partial \exp(-\nu m)$ is skew-selfadjoint, we get that $0 \in \rho(\partial_{0,\nu})$ and $\|\partial_{0,\nu}^{-1}\| \leq 1/\nu$. To justify our

² Throughout, we identify the equivalence classes induced by the equality almost everywhere with their representatives.

³ For the space of L^2 -functions defined on an open subset $\Omega \subseteq \mathbb{R}^n$ with distributional gradient lying in $L^2(\Omega)^n$, we use the notation $H^1(\Omega)$. If the gradient is only locally square-integrable, we write $H_{\text{loc}}^1(\Omega)$.

choice of $\partial_{0,v}$ as the derivative, we compute $\partial_{0,v}\phi$ for $\phi \in \overset{\circ}{C}_\infty(\mathbb{R})$:

$$\begin{aligned} (\partial_{0,v}\phi)(t) &= \exp(vt)((\partial + v) \exp(-vm)\phi)(t) \\ &= \exp(vt)(-v \exp(-vm)\phi + \exp(-vm)\phi' + v \exp(-vm)\phi)(t) \\ &= \phi'(t), \end{aligned}$$

for all $t \in \mathbb{R}$.

Next we need the (standard) concept of so-called Sobolev chains or rigged Hilbert spaces. The proofs of the following assertions can be found, for instance, in [Picard & McGhee \(2011, Chapter 2\)](#).

DEFINITION 2.5 Let H be a Hilbert space and $C : D(C) \subseteq H \rightarrow H$ be a densely defined, closed linear operator with $0 \in \rho(C)$. For $k \in \mathbb{Z}$, we set $H_k(C)$ as the completion of the domain $D(C^k)$ with respect to the norm $|C^k \cdot|_H$. Then $(H_k(C))_{k \in \mathbb{Z}}$ becomes a sequence of Hilbert spaces such that $H_k(C)$ is continuously and densely embedded into $H_{k-1}(C)$ for each $k \in \mathbb{Z}$. We call $(H_k(C))_{k \in \mathbb{Z}}$ the *Sobolev-chain of C* . We define

$$\begin{aligned} H_\infty(C) &:= \bigcap_{k \in \mathbb{Z}} H_k(C), \\ H_{-\infty}(C) &:= \bigcup_{k \in \mathbb{Z}} H_k(C). \end{aligned}$$

REMARK 2.6 For $k \in \mathbb{N} \setminus \{0\}$, the operator

$$\begin{aligned} C : H_k(C) &\rightarrow H_{k-1}(C), \\ x &\mapsto Cx \end{aligned}$$

is unitary. For $-k \in \mathbb{N}$, consider the operator

$$\begin{aligned} C : H_\infty(C) \subseteq H_k(C) &\rightarrow H_{k-1}(C), \\ x &\mapsto Cx. \end{aligned}$$

This operator turns out to be densely defined, isometric with dense range, hence it can be extended to a unitary operator (again denoted by C) $C : H_k(C) \rightarrow H_{k-1}(C)$.

REMARK 2.7 (a) The Hilbert space $H_k(C)$ for $k \in \mathbb{Z}$ can be identified with the dual space $H_{-k}(C^*)^*$ using the following unitary mapping:

$$\begin{aligned} U : H_k(C) &\rightarrow H_{-k}(C^*)^*, \\ x &\mapsto (y \mapsto \langle C^k x | (C^*)^{-k} y \rangle_H). \end{aligned}$$

This allows an extension of the inner product $\langle \cdot | \cdot \rangle$ in H to a continuous sesqui-linearform

$$\langle \cdot | \cdot \rangle : H_k(C) \times H_{-k}(C^*) \rightarrow \mathbb{C},$$

in the sense of the dual pairing $(H_k(C), H_{-k}(C^*))$. We will not distinguish between the inner product given on H and its extension to such pairings.

- (b) Let U be a Hilbert space and $A : H_1(C) \rightarrow U$ be a linear bounded operator. Then the dual operator $A' : U^* \rightarrow H_1(C)^*$ can be identified with the operator $A^\diamond : U \rightarrow H_{-1}(C^*)$, by identifying the dual space U^* with U and the space $H_1(C)^*$ with $H_{-1}(C^*)$ according to the aforementioned unitary mapping.

EXAMPLE 2.8 Choosing $H = H_{\nu,0}(\mathbb{R})$ for some $\nu > 0$ and $C = \partial_0$, we can construct the Sobolev-chain associated to ∂_0 . We will use the notation $H_{\nu,k}(\mathbb{R}) := H_k(\partial_0)$ for $k \in \mathbb{Z}$. The Dirac-distribution δ is an element of $H_{\nu,-1}(\mathbb{R})$ and $\partial_0^{-1}\delta = \chi_{]0,\infty[}$.

REMARK 2.9 For a densely defined closed linear operator $A : D(A) \subseteq H_0 \rightarrow H_1$, where H_0 and H_1 are two Hilbert spaces, we can construct the Sobolev-chain to $|A| + i$ and $|A^*| + i$, respectively. Then A and A^* can be established as bounded linear operators

$$A : H_k(|A| + i) \rightarrow H_{k-1}(|A^*| + i)$$

and

$$A^* : H_k(|A^*| + i) \rightarrow H_{k-1}(|A| + i),$$

for all $k \in \mathbb{Z}$.

Not only the concept of Sobolev chains is of use in the later sections but also the one of Sobolev lattices. A possible way to define them is with the help of tensor product constructions. For the theory of tensor products, see, e.g. [Weidmann \(1980\)](#) and for the concept of Sobolev lattices we refer the reader to [Picard & McGhee \(2011, Chapter 2\)](#).

REMARK 2.10 Let $\nu > 0$ and H be a Hilbert space. For a densely defined closed linear operator $C : D(C) \subseteq H \rightarrow H$ with $0 \in \rho(C)$, we consider the canonical extension $1_{H_{\nu,0}(\mathbb{R})} \otimes C$ of C to the space $H_{\nu,0}(\mathbb{R}) \otimes H$, where $1_{H_{\nu,0}(\mathbb{R})}$ denotes the identity on $H_{\nu,0}(\mathbb{R})$. Analogously, we extend ∂_0 to the space $H_{\nu,0}(\mathbb{R}) \otimes H$ by taking the tensor product $\partial_0 \otimes 1_H$ with the identity 1_H on H . We re-use the notation C and ∂_0 for their respective extensions to the space $H_{\nu,0}(\mathbb{R}) \otimes H$. Then the operators ∂_0 and C can be established as operators on $H_{\nu,-\infty}(\mathbb{R}) \otimes H_{-\infty}(C) := \bigcup_{k,j \in \mathbb{Z}} H_{\nu,k}(\mathbb{R}) \otimes H_j(C)$. More precisely,

$$\partial_0 : H_{\nu,k}(\mathbb{R}) \otimes H_j(C) \rightarrow H_{\nu,k-1}(\mathbb{R}) \otimes H_j(C)$$

and

$$C : H_{\nu,k}(\mathbb{R}) \otimes H_j(C) \rightarrow H_{\nu,k}(\mathbb{R}) \otimes H_{j-1}(C)$$

are unitary operators for each $k, j \in \mathbb{Z}$. As a matter of convenience, we will also write $H_{\nu,k}(\mathbb{R}, H)$ for all $k \in \mathbb{Z} \cup \{-\infty, \infty\}$ for $H_{\nu,k}(\mathbb{R}) \otimes H$ (or $\bigcup_{l \in \mathbb{Z}} H_{\nu,l}(\mathbb{R}) \otimes H$ or $\bigcap_{l \in \mathbb{Z}} H_{\nu,l}(\mathbb{R}) \otimes H$) to stress the unitary equivalence of the tensor products of these Hilbert spaces with the respective space of (generalized) Hilbert-space-valued functions.

3. Control systems as special evolutionary problems

In Section 5, we shall show that many linear control systems fit into the following particular class.

DEFINITION 3.1 Let H, V be Hilbert spaces, $M_0, M_1 \in L(H)$, $J \in L(V, H)$ and $A : D(A) \subseteq H \rightarrow H$ skew-selfadjoint. For $\nu \in]0, \infty[$, we define the set

$$\mathcal{E}_{M_0, M_1, A, J}^\nu := \{(x, f) \in H_{\nu, -\infty}(\mathbb{R}, H \oplus V) \mid (\partial_0 M_0 + M_1 + A)x = Jf\}.$$

The set $\mathcal{E}_{M_0, M_1, A, J} := \bigcup_{\nu > 0} \mathcal{E}_{M_0, M_1, A, J}^\nu$ is called *evolutionary system*. The system $\mathcal{E}_{M_0, M_1, A, J}$ is called *well-posed* if there exists $\nu_0 \in]0, \infty[$ such that for all $\nu \in [\nu_0, \infty[$ the relation

$$\mathcal{S}_{M_0, M_1, A, J}^\nu := \{(f, x) \mid (x, f) \in \mathcal{E}_{M_0, M_1, A, J}^\nu \cap H_{\nu, 0}(\mathbb{R}, H \oplus V)\} \subseteq H_{\nu, 0}(\mathbb{R}, V) \oplus H_{\nu, 0}(\mathbb{R}, H)$$

defines a densely defined, continuous linear mapping from $H_{\nu, 0}(\mathbb{R}, V)$ to $H_{\nu, 0}(\mathbb{R}, H)$. We call $\mathcal{S}_{M_0, M_1, A, J}^\nu$ *solution operator (for ν)*.

THEOREM 3.2 (Picard *et al.*; Picard, 2009) Let $\mathcal{E}_{M_0, M_1, A, J}$ be an evolutionary system. Assume that $M_0 = M_0^*$ and that there exists $c \in]0, \infty[$ such that

$$\nu M_0 + \Re M_1 \geq c > 0,$$

for all sufficiently large $\nu \in]0, \infty[$. Then $\mathcal{E}_{M_0, M_1, A, J}$ is well-posed and the corresponding solution operator $\mathcal{S}_{M_0, M_1, A, J}^\nu$ is *causal*, i.e. for all $a \in \mathbb{R}$ we have

$$\chi_{] - \infty, a[}(m_0) \mathcal{S}_{M_0, M_1, A, J}^\nu \chi_{] - \infty, a[}(m_0) = \chi_{] - \infty, a[}(m_0) \mathcal{S}_{M_0, M_1, A, J}^\nu,$$

where $\chi_{] - \infty, a[}(m_0)$ denotes the operator of multiplying with the cut-off function $\chi_{] - \infty, a[}$.

The following proposition can be found in Picard *et al.*. The basic fact, which is used in the proof is that $\partial_{0, \nu}^{-1}$ commutes with $\mathcal{S}_{M_0, M_1, A, J}^\nu$ for a well-posed evolutionary system $\mathcal{E}_{M_0, M_1, A, J}$, for all sufficiently large $\nu \in]0, \infty[$.

PROPOSITION 3.3 Let $\mathcal{E}_{M_0, M_1, A, J}$ be a well-posed evolutionary system. Then, for all sufficiently large $\nu \in]0, \infty[$, we have that $\mathcal{S}_{M_0, M_1, A, J}^\nu$ uniquely extends to a continuous linear operator from $H_{\nu, k}(\mathbb{R}, V)$ to $H_{\nu, k}(\mathbb{R}, H)$ for all $k \in \mathbb{Z}$.

REMARK 3.4 This proposition provides a way to model initial value problems, since initial conditions can be represented as a Dirac- δ -source term, which turns out to be an element of the space $H_{\nu, -1}(\mathbb{R}, H)$.

We can now describe abstract linear control systems as particular evolutionary systems.

DEFINITION 3.5 An evolutionary system $\mathcal{E}_{M_0, M_1, A, J}$ is called *abstract linear control system* if there exist Hilbert spaces H_0, H_1, Y, U_1 , a densely defined, closed linear operator $F : D(F) \subseteq H_0 \rightarrow H_1$, $B \in L(U_1, H)$ such that $H = H_0 \oplus H_1 \oplus Y$, $A = \begin{pmatrix} 0 & -F^* & 0 \\ F & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $V = H \oplus U_1$ and $J = (1 \ B)$. The Hilbert spaces $H_0 \oplus H_1$, U_1 and Y are called *state*, *control* and *observation space*, respectively. We also write $\mathcal{C}_{M_0, M_1, F, B}$ to denote an abstract linear control system.

COROLLARY 3.6 Let $\mathcal{C}_{M_0, M_1, F, B}$ be an abstract linear control system. Assume that M_0 is selfadjoint and that

$$\nu M_0 + \Re M_1 \geq c > 0$$

holds for all sufficiently large $\nu \in]0, \infty[$. Then $\mathcal{C}_{M_0, M_1, F, B}$ is well-posed and the corresponding solution operators are causal. The solution operators uniquely extend to continuous linear operators from $H_{\nu, k}(\mathbb{R}, H \oplus U_1)$ to $H_{\nu, k}(\mathbb{R}, H)$ for all $k \in \mathbb{Z}$ and $\nu \in]0, \infty[$ sufficiently large.

Proof. Observing that $\begin{pmatrix} 0 & -F^* & 0 \\ F & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is a skew-selfadjoint operator, we are in the situation of Theorem 3.2 and Proposition 3.3. □

4. Conservative systems

In this section, we consider a qualitative property of solutions to particular linear evolutionary equations, namely that of conservativity. For this, a suitable regularizing property has to be additionally imposed. As a slightly modified version to the definition given in Picard *et al.*, we define (locally) regularizing systems as follows.

DEFINITION 4.1 Let $\mathcal{E}_{M_0, M_1, A, J}$ be a well-posed evolutionary system. We say that $\mathcal{E}_{M_0, M_1, A, J}$ is (locally) regularizing if the following conditions are satisfied:

- (a) There exists $U \subseteq D(A)$ dense in H such that for all $T \in \mathbb{R}$ and $\nu \in]0, \infty[$ sufficiently large

$$\chi_{] - \infty, T[}(m_0) P_0 ((\partial_0 M_0 + M_1 + A)^{-1} \delta \otimes M_0 - \chi_{]0, \infty[} \otimes P_0)[U] \subseteq \chi_{] - \infty, T[}(m_0)[H_{\nu, 1}(\mathbb{R}, H)],$$

where $P_0 : H \rightarrow H$ denotes the orthogonal projector onto $M_0[H]$, the range of M_0 .

- (b) There exists $C \in]0, \infty[$ such that for all $\Phi \in H$ we have for all $T \in \mathbb{R}$ and $\nu \in]0, \infty[$ sufficiently large

$$\chi_{] - \infty, T[}(m_0) (\partial_{0, \nu} M_0 + M_1 + A)^{-1} (\delta \otimes M_0 \Phi) \in H_{\nu, 0}(\mathbb{R}, H)$$

and

$$|\chi_{] - \infty, T[}(m_0) ((\partial_0 M_0 + M_1 + A)^{-1} \delta \otimes M_0 \Phi)|_{H_{\nu, 0}(\mathbb{R}; H)} \leq C |\Phi|_H.$$

REMARK 4.2 As we shall see in our discussion of regularizing evolutionary systems, it often suffices to study the following weaker norm on the left-hand side of the estimate in (b): $|f|_{\epsilon, \nu, -1, 1} := \sup_{\phi \in H_{\nu, 1}(-\epsilon, \epsilon; H), |\phi| \leq 1} |(\phi, f)_{H_{\nu, 0}(\mathbb{R}, H)}| + |\chi_{] \epsilon, \infty[}(m) f|_{H_{\nu, 0}(\mathbb{R}, H)}$. Then the modified inequality to impose is: for all $T \in \mathbb{R}$ and $\nu \in]0, \infty[$ sufficiently large and all $\epsilon \in]0, \infty[$ there exists $C \in]0, \infty[$ such that

$$|\chi_{] - \infty, T[}(m_0) ((\partial_0 M_0 + M_1 + A)^{-1} \delta \otimes M_0 \Phi)|_{\epsilon, \nu, -1, 1} \leq C |\Phi|_H.$$

We first will consider a conservation property for evolutionary systems. In the light of Weiss & Tucsnak (2003), this can be interpreted as a energy-balance equality. In fact, we will see later on that this balance equality may be interpreted as impedance conservativity, see, e.g. Ball and Staffans (2006) and also Malinen & Staffans (2006, 2007), Staffans (2001, 2002) and Weiss *et al.* (2001).

THEOREM 4.3 Let $\mathcal{E}_{M_0, M_1, A, J}$ be a regularizing well-posed evolutionary system. Let $u_0 \in H$ and consider the solution $x \in H_{\nu, -1}(\mathbb{R}, H)$ of the equation

$$(\partial_0 M_0 + M_1 + A)x = \delta \otimes M_0 u_0.$$

Then the following *conservation equation* holds:⁴

$$\int_{[a,b]} \langle x | \Re M_1 x \rangle_H = \frac{1}{2} \langle x | M_0 x \rangle_H(a) - \frac{1}{2} \langle x | M_0 x \rangle_H(b),$$

for almost every $a, b \in]0, \infty[$ with $b > a$.

Proof. Let $v_0 \in U$. Since $\mathcal{E}_{M_0, M_1, A, J}$ is well-posed, there is a solution $y \in H_{v,-1}(\mathbb{R}, H)$ of

$$(\partial_0 M_0 + M_1 + A)y = \delta \otimes M_0 v_0.$$

This can be re-written as

$$\begin{aligned} & \partial_0 M_0(y - \chi_{]0, \infty[} \otimes v_0) + M_1(y - \chi_{]0, \infty[} \otimes v_0) + A(y - \chi_{]0, \infty[} \otimes v_0) \\ & = -\chi_{]0, \infty[} \otimes M_1 v_0 - \chi_{]0, \infty[} \otimes A v_0, \end{aligned} \tag{3}$$

from which we read off that $y - \chi_{]0, \infty[} \otimes v_0 \in H_{v,0}(\mathbb{R}, H)$ and hence $y \in H_{v,0}(\mathbb{R}, H)$. Let $\phi \in \overset{\circ}{C}_\infty(]0, \infty[)$ and set $T := \text{supp } \phi$. By assumption, we have that $\chi_{]1-\infty, T[}(m_0)P_0(y - \chi_{]0, \infty[} \otimes v_0) \in \chi_{]1-\infty, T[}(m_0)[H_{v,1}(\mathbb{R}, H)]$ and hence we get from (3) that

$$y - \chi_{]0, \infty[} \otimes v_0 \in \chi_{]1-\infty, T[}(m_0)[H_{v,0}(\mathbb{R}, H_1(A + 1))].$$

Since $v_0 \in U \subseteq D(A)$, we obtain that $y \in \chi_{]1-\infty, T[}(m_0)[H_{v,0}(\mathbb{R}, H_1(A + 1))]$. We apply $\Re \langle \phi y | \cdot \rangle_{H_{v,0}(\mathbb{R}, H)}$ to (3) and obtain

$$\Re \langle \phi y | \partial_0 M_0(y - \chi_{]0, \infty[} \otimes v_0) \rangle_{H_{v,0}(\mathbb{R}, H)} + \Re \langle \phi y | M_1 y \rangle_{H_{v,0}(\mathbb{R}, H)} + \Re \langle \phi y | A y \rangle_{H_{v,0}(\mathbb{R}, H)} = 0.$$

Since y takes values in the domain of A and since A is skew-selfadjoint, we get

$$\Re \langle \phi y | \partial_0 M_0(y - \chi_{]0, \infty[} \otimes v_0) \rangle_{H_{v,0}(\mathbb{R}, H)} + \Re \langle \phi y | M_1 y \rangle_{H_{v,0}(\mathbb{R}, H)} = 0. \tag{4}$$

Since this holds for every $\phi \in \overset{\circ}{C}_\infty(]0, \infty[)$, it follows that

$$\Re \langle y | \partial_0 M_0(y - \chi_{]0, \infty[} \otimes v_0) \rangle_H = -\Re \langle y | M_1 y \rangle_H \quad \text{a.e. on }]0, \infty[. \tag{5}$$

Let $a, b \in]0, \infty[$ with $a < b$. From $\chi_{]1-\infty, b[}(m_0)P_0(y - \chi_{]0, \infty[} \otimes v_0) \in \chi_{]1-\infty, b[}(m_0)[H_{v,1}(\mathbb{R}, H)]$, we get that $(P_0 y)' \in L^2(]a, b[, H)$ with

$$(P_0 y)' = \partial_0 P_0(y - \chi_{]0, \infty[} \otimes v_0) \quad \text{on }]a, b[,$$

and thus, integrating equation (5) over $[a, b]$ gives

$$\frac{1}{2} \langle y | M_0 y \rangle_H(a) = \int_{[a,b]} \langle y | \Re M_1 y \rangle_H + \frac{1}{2} \langle y | M_0 y \rangle_H(b).$$

⁴ Note that $\chi_{]1-\infty, T[}(m_0)x \in H_{v,0}(\mathbb{R}, H)$ for each $T \in \mathbb{R}$ according to the second assumption for regularizing systems.

Let now $u_0 \in H$ and $(v_n)_n$ be a sequence in U converging to u_0 in H . For $n \in \mathbb{N}$, let $y_n := (\partial_0 M_0 + M_1 + A)^{-1} \delta \otimes M_0 v_n$ and $x := (\partial_0 M_0 + M_1 + A)^{-1} \delta \otimes M_0 u_0$. Then for every $T \in \mathbb{R}$, we can estimate:

$$\begin{aligned} |\chi_{]-\infty, T[}(x - y_n)|_{H_{v,0}(\mathbb{R}, H)} &= |\chi_{]-\infty, T[}(\partial_0 M_0 + M_1 + A)^{-1}(\delta \otimes M_0 u_0 - \delta \otimes M_0 v_n)|_{H_{v,0}(\mathbb{R}, H)} \\ &\leq C|u_0 - v_n|_H, \end{aligned}$$

where C is chosen according to assumption (b) for regularizing systems. As $n \rightarrow \infty$, we may assume $y_n \rightarrow x$ almost everywhere on $] - \infty, b[$ by re-using the notation for a suitable subsequence of $(y_n)_{n \in \mathbb{N}}$ and consequently $\int_{[a,b]} \langle y_n | \Re M_1 y_n \rangle_H \rightarrow \int_{[a,b]} \langle x | \Re M_1 x \rangle_H$ for all $a, b \in \mathbb{R}$. Thus, the conservation equation for x holds almost everywhere. \square

4.1 On the structure of conservative control systems

For the particular case of an abstract linear control systems, we shall derive now a different conservation property based on our observation concerning evolutionary systems. Following the block structure of the operator matrix A for the operators M_0 and M_1 , we shall denote the corresponding entries of M_0 and M_1 as $M_{0,ij}$ and $M_{1,ij}$, respectively, for $i, j \in \{0, 1, 2\}$. Analogously, we may write the operator $B \in L(U_1, H) = L(U_1, H_0 \oplus H_1 \oplus Y)$ as a row vector $(B_0 \ B_1 \ B_2)$, where $B_i \in L(U_1, H_i)$ for $i \in \{0, 1\}$ and $B_2 \in L(U_1, Y)$.

THEOREM 4.4 Let $\mathcal{C}_{M_0, M_1, F, B}$ be an abstract linear control system. Assume that M_0 is selfadjoint and that there exists $c > 0$ such that for all $v > 0$ large enough, we have $vM_0 + \Re M_1 \geq c$. Moreover, assume that $\mathcal{C}_{M_0, M_1, F, B}$ is a locally regularizing evolutionary system and that $M_{0,20} = 0, M_{0,21} = 0, M_{0,22} = 0$. Assume the compatibility conditions⁵

$$(M_{1,22}^{-1} M_{1,20})^* B_2 = B_0 \quad \text{and} \quad (M_{1,22}^{-1} M_{1,21})^* B_2 = B_1.$$

Then for $(v, w, y) \in H_{v,-1}(\mathbb{R}, H_0 \oplus H_1 \oplus Y)$ and $u \in H_{v,0}(\mathbb{R}, U)$ satisfying

$$\left(\partial_0 M_0 + M_1 + \begin{pmatrix} 0 & -F^* & 0 \\ F & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \begin{pmatrix} v \\ w \\ y \end{pmatrix} = \delta \otimes M_0 \begin{pmatrix} v_0 \\ w_0 \\ y_0 \end{pmatrix} + Bu,$$

for some $(v_0, w_0, y_0) \in H_0 \oplus H_1 \oplus Y$ the control conservation equation holds:

$$\begin{aligned} &\frac{1}{2} \left\langle \begin{pmatrix} v \\ w \\ y \end{pmatrix} \middle| M_0 \begin{pmatrix} v \\ w \\ y \end{pmatrix} \right\rangle_H \quad (a) - \frac{1}{2} \left\langle \begin{pmatrix} v \\ w \\ y \end{pmatrix} \middle| M_0 \begin{pmatrix} v \\ w \\ y \end{pmatrix} \right\rangle_H \quad (b) \\ &= \int_{[a,b]} \left(\left\langle \begin{pmatrix} v \\ w \\ y \end{pmatrix} \middle| \Re M_1 \begin{pmatrix} v \\ w \\ y \end{pmatrix} \right\rangle_H - \langle B_2 u | \Re M_{1,22}^{-1} B_2 u \rangle_Y \right), \end{aligned}$$

for a.e. $a, b \in]0, \infty[$ with $a < b$.

Before we come to the proof of Theorem 4.4, we will discuss an easy example. More precisely, we discuss a connection to the so-called impedance conservativity in the sense of Ball and Staffans (2006), where the focus is on realization theory.

⁵ Note that the condition $vM_0 + \Re M_1 \geq c$ together with $M_{0,20} = 0, M_{0,21} = 0, M_{0,22} = 0$ implies that $M_{1,22}$ is continuously invertible.

EXAMPLE 4.5 If we let

$$-\tilde{A} = \begin{pmatrix} 0 & -F^* \\ F & 0 \end{pmatrix}, \quad M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -C_0 & -C_1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} B_0 \\ B_1 \\ D \end{pmatrix},$$

for suitable (bounded) operators B_0, B_1, C_0, C_1, D . Abbreviating $x = \begin{pmatrix} v \\ w \end{pmatrix}$, $C = (C_0 \ C_1)$ and $\tilde{B} = \begin{pmatrix} B_0 \\ B_1 \end{pmatrix}$, we may rewrite the equation⁶

$$\left(\partial_0 M_0 + M_1 + \begin{pmatrix} -\tilde{A} & 0 \\ 0 & 0 \end{pmatrix} \right) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \tilde{B} \\ D \end{pmatrix} u,$$

as

$$\begin{pmatrix} \partial_0 x \\ y \end{pmatrix} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ C & D \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}.$$

Note that in this particular situation the block structure of A corresponds to the one of M_0 , which we did not assume in Theorem 4.4. However, in this particular case, we may compare the asserted conservativity in Theorem 4.4 with the conservative realizations of transfer functions in Ball and Staffans (2006). Assume that the operators $\tilde{A}, \tilde{B}, C, D$ formally satisfy the equations in Ball and Staffans (2006, Formula (1.7)), i.e.

$$\tilde{A} + \tilde{A}^* = -\tilde{B}\tilde{B}^*, \quad C = \tilde{B}^*, \quad D = 1.$$

Then by the skew-selfadjointness of \tilde{A} , we deduce that $0 = \tilde{B} = C^*$. With the notation from Theorem 4.4, we get that

$$(M_{1,22}^{-1}M_{1,20})^*B_2 = (1 \cdot (-C_0))^*D = 0 = B_0$$

and

$$(M_{1,22}^{-1}M_{1,21})^*B_2 = (1 \cdot (-C_1))^*D = 0 = B_1,$$

thus the operator equations of the above theorem are satisfied. The corresponding control conservation equation reads

$$\frac{1}{2} \langle x|x \rangle(a) - \frac{1}{2} \langle x|x \rangle(b) = \int_{[a,b]} (\langle y|y \rangle - \langle u|u \rangle),$$

for a.e. $a, b \in]0, \infty[$ with $a < b$. A more sophisticated example will be discussed after the proof of Theorem 4.4.

Proof of Theorem 4.4. Similarly to the proof of the conservation equation for evolutionary systems, we show the conservation equation stated here for initial data $(v_0, w_0, y_0) \in U$, where U is chosen according to the definition of regularizing systems. Hence, analogously to the proof of Theorem 4.3 we get that (v, w, y) takes values in the domain of $\begin{pmatrix} 0 & -F^* & 0 \\ F & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and that $M_0 \begin{pmatrix} v \\ w \\ y \end{pmatrix}$ is locally differentiable

⁶ For simplicity, we assume zero initial conditions.

in $L^2_{\text{loc}}(]0, \infty[, H)$. Let $\phi \in \mathring{C}_\infty(]0, \infty[)$. Then, we obtain, similarly to (4), the equation

$$\begin{aligned} & \Re \left\langle \phi \begin{pmatrix} v \\ w \\ y \end{pmatrix} \middle| \partial_0 M_0 \left(\begin{pmatrix} v \\ w \\ y \end{pmatrix} - \chi_{]0, \infty[} \otimes \begin{pmatrix} v_0 \\ w_0 \\ y_0 \end{pmatrix} \right) \right\rangle_{H_{v,0}(\mathbb{R}, H)} \\ & + \left\langle \phi \begin{pmatrix} v \\ w \\ y \end{pmatrix} \middle| \Re M_1 \begin{pmatrix} v \\ w \\ y \end{pmatrix} \right\rangle = \Re \left\langle \phi \begin{pmatrix} v \\ w \end{pmatrix} \middle| \begin{pmatrix} B_0 \\ B_1 \end{pmatrix} u \right\rangle_{H_{v,0}(\mathbb{R}, H_0 \oplus H_1)} \\ & + \Re \langle \phi y | B_2 u \rangle_{H_{v,0}(\mathbb{R}, Y)}, \end{aligned}$$

and hence

$$\begin{aligned} & \Re \left\langle \begin{pmatrix} v \\ w \\ y \end{pmatrix} \middle| \partial_0 M_0 \left(\begin{pmatrix} v \\ w \\ y \end{pmatrix} - \chi_{]0, \infty[} \otimes \begin{pmatrix} v_0 \\ w_0 \\ y_0 \end{pmatrix} \right) \right\rangle_H + \left\langle \begin{pmatrix} v \\ w \\ y \end{pmatrix} \middle| \Re M_1 \begin{pmatrix} v \\ w \\ y \end{pmatrix} \right\rangle_H \tag{6} \\ & = \Re \left\langle \begin{pmatrix} v \\ w \end{pmatrix} \middle| \begin{pmatrix} B_0 \\ B_1 \end{pmatrix} u \right\rangle_{H_0 \oplus H_1} + \Re \langle y | B_2 u \rangle_Y, \end{aligned}$$

almost everywhere on $]0, \infty[$. We aim to substitute y in the mixed term on the right-hand side. For this, consider the last row equation of the general system

$$M_{1,20}v + M_{1,21}w + M_{1,22}y = B_2u.$$

Using that $M_{1,22}$ is continuously invertible due to the positive definiteness constraint on $\nu M_0 + \Re M_1$, we therefore get that

$$y = -M_{1,22}^{-1}M_{1,20}v - M_{1,22}^{-1}M_{1,21}w + M_{1,22}^{-1}B_2u.$$

Thus, we have

$$\begin{aligned} \Re \langle B_2 u | y \rangle_Y &= \Re \langle B_2 u | -M_{1,22}^{-1}M_{1,20}v - M_{1,22}^{-1}M_{1,21}w + M_{1,22}^{-1}B_2u \rangle_Y \\ &= \Re \langle B_2 u | M_{1,22}^{-1}B_2u \rangle_Y - \Re \langle B_2 u | M_{1,22}^{-1}M_{1,20}v + M_{1,22}^{-1}M_{1,21}w \rangle_Y. \end{aligned}$$

The first term on the right-hand side of (6) may—using the compatibility condition—be computed as follows:

$$\begin{aligned} \Re \left\langle \begin{pmatrix} v \\ w \end{pmatrix} \middle| \begin{pmatrix} B_0 \\ B_1 \end{pmatrix} u \right\rangle_{H_0 \oplus H_1} &= \Re \langle v | B_0 u \rangle_{H_0} + \Re \langle w | B_1 u \rangle_{H_1} \\ &= \Re \langle M_{1,22}^{-1}M_{1,20}v | B_2 u \rangle_Y + \Re \langle M_{1,22}^{-1}M_{1,21}w | B_2 u \rangle_Y. \end{aligned}$$

Hence,

$$\Re \left\langle \begin{pmatrix} v \\ w \end{pmatrix} \middle| \begin{pmatrix} B_0 \\ B_1 \end{pmatrix} u \right\rangle_{H_0 \oplus H_1} + \Re \langle y | B_2 u \rangle_Y = \Re \langle B_2 u | M_{1,22}^{-1}B_2 u \rangle_Y.$$

Now, integrating equation (6) over $[a, b]$ yields

$$\begin{aligned} & \frac{1}{2} \Re \left\langle \begin{pmatrix} v \\ w \\ y \end{pmatrix} \middle| M_0 \begin{pmatrix} v \\ w \\ y \end{pmatrix} \right\rangle_H (a) - \frac{1}{2} \Re \left\langle \begin{pmatrix} v \\ w \\ y \end{pmatrix} \middle| M_0 \begin{pmatrix} v \\ w \\ y \end{pmatrix} \right\rangle_H (b) \\ &= \int_{[a,b]} \left(\left\langle \begin{pmatrix} v \\ w \\ y \end{pmatrix} \middle| \Re M_1 \begin{pmatrix} v \\ w \\ y \end{pmatrix} \right\rangle_H - \langle B_2 u | \Re M_{1,22}^{-1} B_2 u \rangle_Y \right), \end{aligned}$$

for all a, b positive with $a < b$. Using an approximation argument as in the proof of Theorem 4.3, we get the desired assertion. \square

EXAMPLE 4.6 In Picard *et al.*, we studied the conservation property of the following particular system, which is possible to deduce from the (abstract) system treated in Weiss & Tucsnak (2003) (take $z =: v$ and $\dot{z} =: \zeta$):

$$\begin{aligned} & \left(\partial_0 \begin{pmatrix} 1 & \begin{pmatrix} 0 & 0 \end{pmatrix} & 0 \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right) + \begin{pmatrix} 0 & \begin{pmatrix} 0 & 0 \end{pmatrix} & 0 \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ & + \begin{pmatrix} 0 & \text{DIV} & 0 \\ \text{GRAD} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} v \\ \zeta \\ w \\ y \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ -\sqrt{2} \\ -1 \end{pmatrix} u + \delta \otimes \begin{pmatrix} z^{(1)} \\ z^{(0)} \\ 0 \\ 0 \end{pmatrix}, \end{aligned}$$

where GRAD and DIV are suitable operators such that $\text{DIV}^* = -\text{GRAD}$. We remark here that the notation GRAD and DIV serve as a reminder of the fact that the former is the negative adjoint of the latter. In Picard *et al.*, these operators are similarly constructed as the operator F and $-F^*$ in Section 5.3. We also refer to Section 6.1 equation (14) for a more specific example. It was shown that this system is well-posed and locally regularizing. Furthermore, the compatibility conditions of Theorem 4.4 are satisfied with

$$\begin{aligned} M_{1,22} &= 1, & M_{1,20} &= 0, & M_{1,21} &= \begin{pmatrix} 0 & \sqrt{2} \end{pmatrix}, \\ B_0 &= 0, & B_1 &= \begin{pmatrix} 0 \\ -\sqrt{2} \end{pmatrix}, & B_2 &= -1. \end{aligned}$$

Thus, we end up with the conservation equation

$$\frac{1}{2} (|v(a)|^2 + |\zeta(a)|^2) - \frac{1}{2} (|v(b)|^2 + |\zeta(b)|^2) = \int_a^b (|w(t)|^2 + \sqrt{2} \Re \langle w(t) | y(t) \rangle + |y(t)|^2 - |u(t)|^2) dt.$$

From the last row, we read off the equation $\sqrt{2}w + y = -u$ and thus $w = -(1/\sqrt{2})(y + u)$. If we plug in this representation of w , we get

$$\frac{1}{2}(|v(a)|^2 + |\zeta(a)|^2) - \frac{1}{2}(|v(b)|^2 + |\zeta(b)|^2) = \int_a^b \frac{1}{2}|y(t)|^2 - \frac{1}{2}|u(t)|^2 dt,$$

which is the conservation equality in [Weiss & Tucsnak \(2003, Corollary 1.5\)](#).

5. Boundary control

We shall now consider particular types of control equations involving so-called boundary control. One may find the notion of boundary control systems in the literature, see, e.g. [Arov et al. \(2011\)](#) and [Malinen & Staffans \(2006, 2007\)](#). These are equations of the form

$$u = Gx, \quad \dot{x} = Lx, \quad y = Kx,$$

subject to certain initial conditions for suitable linear operators G, L, K on suitable Hilbert spaces. The operators G and K are thought of as trace mappings, where the first one is onto, and L is assumed to be a generator of a C_0 -semi-group if restricted to the kernel of G . The precise (abstract) definition of the latter operators is done with the help of so-called boundary triples. We infer that these kind of boundary control systems are, if we focus on well-posedness issues only, a mere non-homogeneous (abstract) Cauchy problem. Indeed, using that G is onto, we get w such that $Gw = u$. Introducing the new variable $\tilde{x} := x - w \in N(G)$, we arrive at the equation

$$\dot{\tilde{x}} = L\tilde{x} - \dot{w} + Lw,$$

which may be solved by the variation of constants formula. The output y can then be computed as follows $y = K(\tilde{x} + w)$. For a more specific account of this strategy, we refer the reader to Section 6.2.

We will mainly focus on a class of boundary control systems where both the equations on the boundary have terms of the input and output. These are, for example, special types of port-Hamiltonian systems or the control system discussed in [Weiss & Tucsnak \(2003\)](#). Moreover, in the later study, we will develop a framework that gives a possible generalization of (a subclass of) port-Hamiltonian systems to more than one spatial dimension.

As a first introductory example, we consider these types of port-Hamiltonian systems (cf., e.g. [Zwart et al., 2010](#); [Jacob and Zwart, 2012](#)).

5.1 Port-Hamiltonian systems

The notion of port-Hamiltonian systems with boundary control and observation as discussed in [Jacob and Zwart, Section 11.2](#) can be described as follows: Let $n \in \mathbb{N}$, $a, b \in \mathbb{R}$, $a < b$, $P_0, P_1 \in \mathbb{K}^{n \times n}$, $\mathcal{H} \in L^\infty([a, b], \mathbb{K}^{n \times n})$, $W_B, W_C \in \mathbb{K}^{n \times 2n}$. We assume the following:

- P_1 is invertible and selfadjoint,
- for a.e. $\zeta \in [a, b]$, we have $\mathcal{H}(\zeta)$ is selfadjoint and there exist $m, M \in]0, \infty[$ such that for a.e. $\zeta \in [a, b]$ we have $m \leq \mathcal{H}(\zeta) \leq M$,
- W_B and W_C have full rank and $\begin{pmatrix} W_B \\ W_C \end{pmatrix}$ is invertible.

Jacob and Zwart considered the problem of finding (x, y) such that for given $x^{(0)} \in L^2([a, b[, \mathbb{K}^n)$ and $u:]0, \infty[\rightarrow \mathbb{K}^n$ twice continuously differentiable the following equations hold:

$$\begin{aligned} \dot{x}(t) &= P_1 \partial_1 \mathcal{H}x(t) + P_0 \mathcal{H}x(t), \\ u(t) &= W_B \frac{1}{\sqrt{2}} \begin{pmatrix} P_1 & -P_1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (\mathcal{H}x(t))(b) \\ (\mathcal{H}x(t))(a) \end{pmatrix}, \\ y(t) &= W_C \frac{1}{\sqrt{2}} \begin{pmatrix} P_1 & -P_1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (\mathcal{H}x(t))(b) \\ (\mathcal{H}x(t))(a) \end{pmatrix}, \\ x(0) &= x^{(0)}, \end{aligned}$$

where ∂_1 is the distributional derivative with respect to the spatial variable. Under particular assumptions on the matrices involved a well-posedness result can be obtained by using C_0 -semi-group theory, see, for instance, Jacob and Zwart, Theorem 13.3.2. Our perspective to boundary control systems considers a particular subclass of port-Hamiltonian (boundary control) systems. This subclass shows the advantage that it can be generalized to an analogue of port-Hamiltonian systems in more than one spatial dimension. The key assumption is that P_1 is unitarily equivalent to a matrix of the form $\begin{pmatrix} 0 & N^* \\ N & 0 \end{pmatrix}$, where $N \in \mathbb{K}^{\ell \times \ell}$ with $2\ell = n$. Consequently, $P_1 \partial_1$ is replaced by $\begin{pmatrix} 0 & \partial_1 N^* \\ N \partial_1 & 0 \end{pmatrix}$ with suitable domain. The unknown x decomposes into (x_0, x_1) . Furthermore, we assume that we only control the boundary values of x_1 and that the output is given in terms of the boundary values⁷ of x_0 . We are led to study the following problem, which corresponds as we will see to port-Hamiltonian systems with boundary control and observation as considered in Jacob and Zwart in a pure Hilbert space setting provided our key assumptions are satisfied:

Let $\ell \in \mathbb{N}$, $N \in \mathbb{K}^{\ell \times \ell}$ invertible, $n := 2\ell$, $M_0 \in L(L^2([a, b[, \mathbb{K}^n))$ selfadjoint and strictly positive definite. Let $M_1 \in L(L^2([a, b[, \mathbb{K}^n) \oplus \mathbb{K}^{4\ell})$ with the restriction of $\Re M_1$ to a linear mapping in $\mathbb{K}^{4\ell}$ assumed to be strictly positive definite, and $B_0, B_1 \in \mathbb{K}^{n \times n}$. We define the operators

$$\begin{aligned} N\partial_1 : H^1([a, b[, \mathbb{K}^\ell) \subseteq L^2([a, b[, \mathbb{K}^\ell) &\rightarrow L^2([a, b[, \mathbb{K}^\ell), \\ f &\mapsto Nf', \\ \partial_1 N^* : H^1([a, b[, \mathbb{K}^\ell) \subseteq L^2([a, b[, \mathbb{K}^\ell) &\rightarrow H_{-1}(|\partial_1| + i) \\ f &\mapsto (N^*f)' - (N^*f)(b) \cdot \delta_b + (N^*f)(a) \cdot \delta_a. \end{aligned}$$

The expression $N^*f(b)$ is well-defined by the one-dimensional Sobolev embedding theorem and

$$N^*f(b) \cdot \delta_b : H^1([a, b[, \mathbb{K}^\ell) \rightarrow \mathbb{K}, \quad g \mapsto \langle N^*f(b) | g(b) \rangle.$$

We define the operator $C : H_1(|\partial_1| + i) \rightarrow \mathbb{K}^n, f \mapsto (-Nf(b), Nf(a))$, in other words $C = (-N\delta_b) \oplus N\delta_a$. Identifying $\mathbb{K}^n = \mathbb{K}^\ell \oplus \mathbb{K}^\ell$ with its dual, we get $C^\circ : \mathbb{K}^\ell \oplus \mathbb{K}^\ell \rightarrow H_{-1}(|\partial_1| + i), (x, y) \mapsto -N^*x \cdot \delta_b + N^*y \cdot \delta_a$.

⁷ This assumptions can be guaranteed for instance for the Timoshenko beam equation, the vibrating string equation or the one-dimensional heat equation with boundary control, see Jacob and Zwart. It does, however, not capture the one-dimensional transport equation.

We consider the following problem: Find $(x_0, x_1, w, y) \in H_{v,-1}(\mathbb{R}; L^2([a, b[, \mathbb{K}^n) \oplus \mathbb{K}^{2n})$ such that for given $u \in H_{v,0}(\mathbb{R}, \mathbb{K}^n)$ and $\xi_0, \xi_1 \in L^2([a, b[, \mathbb{K}^\ell)$ we have

$$\begin{aligned} & \left(\partial_0 \begin{pmatrix} M_{0,00} & \begin{pmatrix} M_{0,01} & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} M_{0,10} \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} M_{0,11} & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right) + M_1 + \begin{pmatrix} 0 & \begin{pmatrix} -\partial_1 N^* & C^\circ \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -N\partial_1 \\ -C \\ 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ w \\ y \end{pmatrix} \\ & = \delta \otimes \begin{pmatrix} \xi_0 \\ \xi_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ B_1 u \\ B_2 u \end{pmatrix}. \end{aligned} \tag{7}$$

In Section 5.3, we shall see that this type of problem is well-posed in $H_{v,-1}(\mathbb{R}, L^2([a, b[, \mathbb{K}^n) \oplus \mathbb{K}^{2n})$. For convenience⁸, assume that $(x_0, x_1, w, y) \in H_{v,0}(\mathbb{R}, L^2([a, b[, \mathbb{K}^n) \oplus \mathbb{K}^{2n})$ is a solution of the above system. Then, it follows that

$$\begin{pmatrix} 0 & \begin{pmatrix} -\partial_1 N^* & C^\circ \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -N\partial_1 \\ -C \\ 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ w \\ y \end{pmatrix}$$

is an element of $H_{v,-1}(\mathbb{R}, L^2([a, b[, \mathbb{K}^n) \oplus \mathbb{K}^{2n})$. Consequently, we get that

$$\begin{pmatrix} -\partial_1 N^* & C^\circ \end{pmatrix} \begin{pmatrix} x_1 \\ w \end{pmatrix} \in H_{v,-1}(\mathbb{R}, L^2([a, b[, \mathbb{K}^\ell)).$$

Thus, with $w = (w_1, w_2)$

$$-N^*x'_1 + (N^*x_1)(b) \cdot \delta_b - (N^*x_1)(a) \cdot \delta_a - N^*w_1\delta_b + N^*w_2\delta_a \in H_{v,-1}(\mathbb{R}, L^2([a, b[, \mathbb{K}^\ell)).$$

The latter, however, can only happen if $x_1(b) = w_1$ and $x_1(a) = w_2$. Hence, the first two equations read as

$$\begin{aligned} & \left(\partial_0 \begin{pmatrix} M_{0,00} & M_{0,01} & 0 & 0 \\ M_{0,10} & M_{0,11} & 0 & 0 \end{pmatrix} + \begin{pmatrix} M_{1,00} & M_{1,01} & M_{1,02} & M_{1,03} \\ M_{1,10} & M_{1,11} & M_{1,12} & M_{1,13} \end{pmatrix} \right. \\ & \left. + \begin{pmatrix} 0 & -\partial_1 N^* & C^\circ & 0 \\ -N\partial_1 & 0 & 0 & 0 \end{pmatrix} \right) \begin{pmatrix} x_0 \\ x_1 \\ w \\ y \end{pmatrix} = \delta \otimes \begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix}, \end{aligned}$$

or

$$\partial_0 \begin{pmatrix} M_{0,00} & M_{0,01} \\ M_{0,10} & M_{0,11} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} + \begin{pmatrix} M_{1,00} & M_{1,01} & M_{1,02} & M_{1,03} \\ M_{1,10} & M_{1,11} & M_{1,12} & M_{1,13} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ w \\ y \end{pmatrix} + \begin{pmatrix} -N^*x'_1 \\ -Nx'_0 \end{pmatrix} = \delta \otimes \begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix}.$$

⁸ This holds true if we assume the initial data ξ_0, ξ_1 and the control u to be smooth enough.

Thus, we arrive at the following system:

$$\begin{aligned} \partial_0 \begin{pmatrix} M_{0,00} & M_{0,01} \\ M_{0,10} & M_{0,11} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} + \begin{pmatrix} M_{1,00} & M_{1,01} & M_{1,02} & M_{1,03} \\ M_{1,10} & M_{1,11} & M_{1,12} & M_{1,13} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ w \\ y \end{pmatrix} - \begin{pmatrix} 0 & N^* \\ N & 0 \end{pmatrix} \partial_1 \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \\ = \delta \otimes \begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix}. \end{aligned}$$

In order to reproduce the formal structure of port-Hamiltonian systems, we are led to assume that $\begin{pmatrix} M_{0,00} & M_{0,01} \\ M_{0,10} & M_{0,11} \end{pmatrix} = \mathcal{H}^{-1}$ and $\begin{pmatrix} M_{1,00} & M_{1,01} \\ M_{1,10} & M_{1,11} \end{pmatrix} = -P_0$. Moreover, $\begin{pmatrix} M_{1,02} & M_{1,03} \\ M_{1,12} & M_{1,13} \end{pmatrix}$ must be assumed to be 0. To simplify matters further, we consider the second two rows of M_1 to be of the form

$$\begin{pmatrix} 0 & 0 & M_{1,22} & M_{1,23} \\ 0 & 0 & M_{1,32} & M_{1,33} \end{pmatrix}.$$

Then the second two rows of system (7) are

$$\begin{aligned} M_{1,22}w + M_{1,23}y - Cx_0 &= B_1u, \\ M_{1,32}w + M_{1,33}y &= B_2u. \end{aligned}$$

Using the above condition that $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} x_1(b) \\ x_1(a) \end{pmatrix}$, we get that

$$\begin{aligned} M_{1,22} \begin{pmatrix} x_1(b) \\ x_1(a) \end{pmatrix} + M_{1,23}y + \begin{pmatrix} Nx_0(b) \\ -Nx_0(a) \end{pmatrix} &= B_1u, \\ M_{1,32} \begin{pmatrix} x_1(b) \\ x_1(a) \end{pmatrix} + M_{1,33}y &= B_2u. \end{aligned}$$

In the spirit of boundary control and boundary observation, we have that the boundary values of x_0 are expressed as a linear combination of the output y . Thus, there is a linear operator $W \in L(H_{v,0}(\mathbb{R}, \mathbb{K}^n))$ such that $Wy = \chi_{]0,\infty[}(m_0) \begin{pmatrix} Nx_0(b) \\ -Nx_0(a) \end{pmatrix}$. Moreover, assuming suitable invertibility properties on the operators $B_1, B_2, M_{1,33}$ and $M_{1,23}$, we may express the above two equations as a system of two equations of the form:

$$\begin{aligned} u &= (B_1 - (M_{1,23} + W)M_{1,33}^{-1}B_2)^{-1} (M_{1,22} - (M_{1,23} + W)M_{1,33}^{-1}M_{1,32}) \begin{pmatrix} x_1(b) \\ x_1(a) \end{pmatrix}, \\ y &= (B_1B_2^{-1}M_{1,33} - (M_{1,23} + W))^{-1} (M_{1,22} - B_1B_2^{-1}M_{1,32}) \begin{pmatrix} x_1(b) \\ x_1(a) \end{pmatrix}. \end{aligned}$$

These equations are the control and the observation equations and they are of the same form as considered in [Jacob and Zwart](#). A similar reasoning is applied in Remark 5.6, where a more general situation is considered.

The discussion of boundary control within the context of port-Hamiltonian systems becomes accessible due to the Sobolev embedding theorem yielding a continuous boundary trace operator and a finite-dimensional boundary trace space. In higher-dimensional situations, the Sobolev embedding theorem

depends on the geometry of the underlying domain. A continuous boundary trace operator can only be defined for domains satisfying some regularity assumptions at the boundary, e.g. assuming a Lipschitz-continuous boundary. We shall approach boundary control systems from a more general perspective without assuming undue regularity of the boundary. In order to have the functional-analytic notions at hand to replace the boundary trace space by an appropriate alias that captures the boundary data, we implement the necessary concepts in the next section.

5.2 Boundary data spaces

Throughout this section, let H_0 and H_1 be Hilbert spaces and let⁹ $\overset{\circ}{G} \subseteq H_0 \oplus H_1$, $\overset{\circ}{D} \subseteq H_1 \oplus H_0$ be two densely defined, closed linear operators, which are assumed to be formally skew-adjoint linear operators, i.e.

$$\begin{aligned}\overset{\circ}{D} \subseteq D &:= - \left(\overset{\circ}{G} \right)^*, \\ \overset{\circ}{G} \subseteq G &:= - \left(\overset{\circ}{D} \right)^*.\end{aligned}$$

LEMMA 5.1 We have the orthogonal decompositions

$$H_1(|G| + i) = H_1(|\overset{\circ}{G}| + i) \oplus N(1 - DG), \quad (8)$$

$$H_1(|D| + i) = H_1(|\overset{\circ}{D}| + i) \oplus N(1 - GD). \quad (9)$$

Proof. Let $\phi \in H_1(|\overset{\circ}{G}| + i)^\perp$. Then for all $\psi \in H_1(|\overset{\circ}{G}| + i)$

$$\begin{aligned}0 &= \langle \psi | \phi \rangle_{H_1(|G|+i)} \\ &= \langle \psi | \phi \rangle_{H_0} + \langle |G|\psi | |G|\phi \rangle_{H_0} \\ &= \langle \psi | \phi \rangle_{H_0} + \langle G\psi | G\phi \rangle_{H_1} \\ &= \langle \psi | \phi \rangle_{H_0} + \langle \overset{\circ}{G}\psi | G\phi \rangle_{H_1}.\end{aligned}$$

We read off that $G\phi \in D((\overset{\circ}{G})^*) = D(D)$ and

$$DG\phi = \phi.$$

The remaining case follows analogously. □

⁹ The notation $\overset{\circ}{G}$, $\overset{\circ}{D}$ is chosen as a reminder of the basic situation taking these as the closure of the classical operations grad and dive defined on C_∞ -functions with compact support in an open set Ω of \mathbb{R}^n , $n \in \mathbb{N}$. In other practical cases, these operators can change role or can be totally different operators such as curl.

We define¹⁰

$$\begin{aligned} BD(G) &:= N(1 - DG), \\ BD(D) &:= N(1 - GD), \end{aligned}$$

and obtain

$$\begin{aligned} G[BD(G)] &\subseteq BD(D), \\ D[BD(D)] &\subseteq BD(G). \end{aligned}$$

For later purposes, we also introduce the canonical projectors $\pi_{BD(G)} : H_1(|G| + i) \rightarrow BD(G)$ and $\pi_{BD(D)} : H_1(|D| + i) \rightarrow BD(D)$ onto the component spaces $BD(G)$, $BD(D)$ according to the direct sum decompositions (8), (9), respectively. The orthogonal projectors $P_{BD(G)} : H_1(|G| + i) \rightarrow H_1(|G| + i)$, $P_{BD(D)} : H_1(|G| + i) \rightarrow H_1(|G| + i)$ associated with (8) and (9) can now be expressed as

$$P_{BD(G)} = \pi_{BD(G)}^* \pi_{BD(G)}, \quad P_{BD(D)} = \pi_{BD(D)}^* \pi_{BD(D)}.$$

Note that $\pi_{BD(G)}^*$, $\pi_{BD(D)}^*$ are the canonical embeddings of $BD(G)$ in $H_1(|G| + i)$ and of $BD(D)$ in $H_1(|D| + i)$, respectively.

Thus, on $BD(D)$ we may define the operator $\overset{\bullet}{D}$ by¹¹

$$\begin{aligned} \overset{\bullet}{D} : BD(D) &\rightarrow BD(G), \\ \phi &\mapsto D\phi, \end{aligned}$$

¹⁰ The notation $BD(\cdot)$ is supposed to be a reminder that in applications these spaces will serve as the spaces of boundary data.

¹¹ These operators are an abstract version of the Dirichlet-to-Neumann operator since the ‘boundary data’ space for G is transformed into the ‘boundary data’ space for D . Indeed, if u is a solution of the inhomogeneous ‘Dirichlet boundary value problem’

$$\begin{aligned} (1 - DG)u &= 0, \\ u - g &\in D(\overset{\circ}{G}), \end{aligned}$$

for given data $g \in BD(G)$ then also

$$(1 - D\overset{\circ}{G})(u - g) = 0,$$

implying

$$u = g.$$

This implies

$$\overset{\bullet}{G}u = \overset{\bullet}{G}g,$$

and u is therefore also the solution of the inhomogeneous ‘Neumann boundary value problem’

$$\begin{aligned} (1 - DG)u &= 0, \\ Gu - \overset{\bullet}{G}g &\in D(\overset{\circ}{D}), \end{aligned}$$

and vice versa.

and the operator $\overset{\circ}{G}$ by

$$\begin{aligned} \overset{\circ}{G} : BD(G) &\rightarrow BD(D), \\ \phi &\mapsto G\phi. \end{aligned}$$

The operators $\overset{\circ}{D}$ and $\overset{\circ}{G}$ enjoy the following surprising property.

THEOREM 5.2 We have that¹²

$$(\overset{\circ}{G})^* = \overset{\circ}{D} = (\overset{\circ}{G})^{-1}.$$

In particular, $\overset{\circ}{G}$ and $\overset{\circ}{D}$ are unitary.

Proof. Obviously is $\overset{\circ}{D}\overset{\circ}{G}$ the identity on $BD(G)$ and $\overset{\circ}{G}\overset{\circ}{D}$ the identity on $BD(D)$. Consequently,

$$\overset{\circ}{D} = (\overset{\circ}{G})^{-1}.$$

Moreover, for $\phi \in BD(G)$ and $\psi \in BD(D)$

$$\begin{aligned} \langle \overset{\circ}{G}\phi | \psi \rangle_{BD(D)} &:= \langle \overset{\circ}{G}\phi | \psi \rangle_{H_1(|D|+i)} = \langle \overset{\circ}{G}\phi | \psi \rangle_{H_0(|D|+i)} + \langle \overset{\circ}{D}\overset{\circ}{G}\phi | \overset{\circ}{D}\psi \rangle_{H_0(|G|+i)} \\ &= \langle \overset{\circ}{G}\phi | \overset{\circ}{G}\overset{\circ}{D}\psi \rangle_{H_0(|D|+i)} + \langle \phi | \overset{\circ}{D}\psi \rangle_{H_0(|G|+i)} \\ &= \langle \phi | \overset{\circ}{D}\psi \rangle_{H_1(|G|+i)} =: \langle \phi | \overset{\circ}{D}\psi \rangle_{BD(G)}, \end{aligned}$$

leading to

$$(\overset{\circ}{G})^* = \overset{\circ}{D},$$

in $BD(D) \oplus BD(G)$. □

EXAMPLE 5.3 As an application, let us calculate the dual mapping $\pi_{BD(G)}^\circ$ of

$$\pi_{BD(G)} : H_1(|G| + i) \rightarrow BD(G),$$

¹² Note, however, that in contrast we have

$$(G)^* = -\overset{\circ}{D},$$

in $H_0(|\overset{\circ}{D}| + i) \oplus H_0(|G| + i)$.

according to the Gelfand-triplet $H_1(|G| + i) \subseteq H_0(|G| + i) \subseteq H_{-1}(|G| + i)$ ¹³, which would be a mapping from $BD(G)$ (identified with $BD(G)^*$) into $H_{-1}(|G| + i)$. We find

$$\begin{aligned} (\pi_{BD(G)})^\diamond &= R_{H_1(|G|+i)}^* \pi_{BD(G)}^* \\ &= (|G|^2 + 1) \pi_{BD(G)}^* \\ &= \pi_{BD(G)}^* - \overset{\circ}{D}G \pi_{BD(G)}^* \\ &= \pi_{BD(G)}^* - \overset{\circ}{D} \pi_{BD(D)}^* \overset{\bullet}{G}. \end{aligned}$$

REMARK 5.4 In the literature, in order to discuss boundary control systems in an operator-theoretic framework, the concept of boundary triples is used, see, e.g. Malinen & Staffans (2006), Behrndt *et al.* (2009), Behrndt & Kreuzler (2007) and Derkach *et al.* (2009), we also refer to Schubert *et al.* and Waurick & Kaliske (2012), where in Schubert *et al.* a unified perspective is given. A boundary triple is a symmetric operator S defined in a Hilbert space H and two continuous linear operators $\Gamma_0, \Gamma_1 : H_1(|S^*| + i) \rightarrow K$, mapping onto a Hilbert space K . Moreover, for all $x, y \in D(S^*)$ the following equality should be satisfied:

$$\langle S^*x|y \rangle_H - \langle x|S^*y \rangle_H = \langle \Gamma_0x|\Gamma_1y \rangle_K - \langle \Gamma_1x|\Gamma_0y \rangle_K.$$

In the literature, one finds the notation (K, Γ_0, Γ_1) , which explains the name. In the situation of this section, we also have a boundary triple: Setting

$$S = -i \begin{pmatrix} 0 & \overset{\circ}{D} \\ \overset{\circ}{G} & 0 \end{pmatrix}, \quad K = BD(G), \quad \Gamma_0 = (\pi_{BD(G)} \quad 0), \quad \Gamma_1 = (0 \quad i \overset{\bullet}{D} \pi_{BD(D)}),$$

we get a boundary triple. Indeed, let $(u, v), (x, y) \in H_1(|S^*| + i) = H_1(|G| + i) \oplus H_1(|D| + i)$. Denoting $P_{\overset{\circ}{D}} := 1 - P_{BD(D)}$ and $P_{\overset{\circ}{G}} := 1 - P_{BD(G)}$, we compute

$$\begin{aligned} & - \left\langle S^* \begin{pmatrix} u \\ v \end{pmatrix} \middle| \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle_{H_0(|S^*|+i)} + \left\langle \begin{pmatrix} u \\ v \end{pmatrix} \middle| S^* \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle_{H_0(|S^*|+i)} \\ &= i(\langle Dv|x \rangle_{H_0(|G|+i)} + \langle Gu|y \rangle_{H_0(|D|+i)} + \langle u|Dy \rangle_{H_0(|G|+i)} + \langle v|Gx \rangle_{H_0(|D|+i)}) \\ &= i(\langle DP_{\overset{\circ}{D}}v + DP_{BD(D)}v|x \rangle_{H_0(|G|+i)} + \langle GP_{\overset{\circ}{G}}u + GP_{BD(G)}u|y \rangle_{H_0(|D|+i)} \\ &\quad + \langle P_{\overset{\circ}{G}}u + P_{BD(G)}u|Dy \rangle_{H_0(|G|+i)} + \langle P_{\overset{\circ}{D}}v + P_{BD(D)}v|Gx \rangle_{H_0(|D|+i)}) \\ &= i(\langle DP_{BD(D)}v|x \rangle_{H_0(|G|+i)} + \langle GP_{BD(G)}u|y \rangle_{H_0(|D|+i)} \\ &\quad + \langle P_{BD(G)}u|Dy \rangle_{H_0(|G|+i)} + \langle P_{BD(D)}v|Gx \rangle_{H_0(|D|+i)}) \end{aligned}$$

¹³ Note that the Riesz-mapping $R_{H_1(|G|+i)} : H_{-1}(|G| + i) \rightarrow H_1(|G| + i)$ is given by $R_{H_1(|G|+i)}\phi = (1 + |G|^2)^{-1}\phi = (1 + G^*G)^{-1}\phi = (1 - \overset{\circ}{D}G)^{-1}\phi$.

$$\begin{aligned}
 &= i(\langle DP_{BD(D)}v|x\rangle_{H_0(|G|+i)} + \langle GP_{BD(G)}u|y\rangle_{H_0(|D|+i)} \\
 &\quad + \langle DGP_{BD(G)}u|Dy\rangle_{H_0(|G|+i)} + \langle GDP_{BD(D)}v|Gx\rangle_{H_0(|D|+i)}) \\
 &= i(\langle DP_{BD(D)}v|x\rangle_{H_1(|G|+i)} + \langle GP_{BD(G)}u|y\rangle_{H_1(|D|+i)}) \\
 &= i(\langle \dot{D}\pi_{BD(D)}v|\pi_{BD(G)}x\rangle_{BD(G)} + \langle \dot{G}\pi_{BD(G)}u|\pi_{BD(D)}y\rangle_{BD(D)}) \\
 &= i(\langle \dot{D}\pi_{BD(D)}v|\pi_{BD(G)}x\rangle_{BD(G)} + \langle \pi_{BD(G)}u|\dot{D}\pi_{BD(D)}y\rangle_{BD(G)}) \\
 &= -\langle i\dot{D}\pi_{BD(D)}v|\pi_{BD(G)}x\rangle_{BD(G)} + \langle \pi_{BD(G)}u|i\dot{D}\pi_{BD(D)}y\rangle_{BD(G)}.
 \end{aligned}$$

5.3 Control systems with boundary control and boundary observation

We apply our previous findings in this section to model problems with boundary control and boundary observation in more complex situations. For this purpose, we consider abstract linear control systems $C_{M_0, M_1, F, B}$ where the operator F is given in the following form:

$$F := \begin{pmatrix} -G \\ C \end{pmatrix} : H_1(|G| + i) \subseteq H_0(|G| + i) \rightarrow H_0(|\dot{D}| + i) \oplus V, \tag{10}$$

with $C \in L(H_1(|G| + i), V)$ for some Hilbert space V and G, D are as in Section 5.1. As a variant of Picard *et al.*, Lemma 5.1, we compute the adjoint of F explicitly under the additional constraint that G is boundedly invertible.

THEOREM 5.5 Let F be given as above and let G be boundedly invertible. Then

$$\begin{aligned}
 F^* : D(F^*) \subseteq H_0(|\dot{D}| + i) \oplus V &\rightarrow H_0(|G| + i), \\
 (\zeta, w) &\mapsto \dot{D}\zeta + C^\circ w,
 \end{aligned}$$

where C° is the dual operator of C with respect to the Gelfand-triplet $H_1(|G| + i) \subseteq H_0(|G| + i) \subseteq H_{-1}(|G| + i)$ and

$$D(F^*) = \{(\zeta, w) \in H_0(|\dot{D}| + i) \oplus V \mid \dot{D}\zeta + C^\circ w \in H_0(|G| + i)\}.$$

Proof. We define

$$\begin{aligned}
 K : D(K) \subseteq H_0(|\dot{D}| + i) \oplus V &\rightarrow H_0(|G| + i), \\
 (\zeta, w) &\mapsto \dot{D}\zeta + C^\circ w,
 \end{aligned}$$

with $D(K) := \{(\zeta, w) \in H_0(|\dot{D}| + i) \oplus V \mid \dot{D}\zeta + C^\circ w \in H_0(|G| + i)\}$. From

$$((\dot{D} \quad C^*) : H_1(|\dot{D}| + i) \oplus V \subseteq H_0(|\dot{D}| + i) \oplus V \rightarrow H_0(|G| + i)) \subseteq K,$$

we get that K is densely defined. Furthermore, K is closed. Thus, it suffices to prove $K^* = F$. Let $v \in D(K^*)$. Then there exists $\begin{pmatrix} f \\ g \end{pmatrix} \in H_0(|\mathring{D}| + i) \oplus V$ such that for all $\begin{pmatrix} \zeta \\ w \end{pmatrix} \in D(K)$ we have

$$\left\langle K \begin{pmatrix} \zeta \\ w \end{pmatrix} \middle| v \right\rangle_{H_0(|G|+i)} = \left\langle \begin{pmatrix} \zeta \\ w \end{pmatrix} \middle| \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle_{H_0(|\mathring{D}|+i) \oplus V}.$$

Choosing $w = 0$ and $\zeta \in H_1(|\mathring{D}| + i)$, we get

$$\langle \mathring{D}\zeta | v \rangle_{H_0(|G|+i)} = \langle \zeta | f \rangle_{H_0(|\mathring{D}|+i)},$$

yielding $v \in H_1(|G| + i)$ and $f = -Gv$. Let now $w \in V$ be arbitrarily chosen. Like in [Trostorff & Waurick \(2012, Theorem 2.1.4\)](#), we find an element $\zeta \in H_0(|\mathring{D}| + i)$ such that $\mathring{D}\zeta = -C^\diamond w$. For this choice of ζ , we get $(\zeta, w) \in D(K)$ with $K \begin{pmatrix} \zeta \\ w \end{pmatrix} = 0$ and thus we compute

$$\begin{aligned} 0 &= \left\langle \begin{pmatrix} \zeta \\ w \end{pmatrix} \middle| \begin{pmatrix} -Gv \\ g \end{pmatrix} \right\rangle_{H_0(|\mathring{D}|+i) \oplus V} \\ &= \langle \zeta | -Gv \rangle_{H_0(|\mathring{D}|+i)} + \langle w | g \rangle_V \\ &= \langle \mathring{D}\zeta | v \rangle_{H_0(|G|+i)} + \langle w | g \rangle_V \\ &= \langle -C^\diamond w | v \rangle_{H_0(|G|+i)} + \langle w | g \rangle_V \\ &= \langle w | -Cv + g \rangle_V. \end{aligned}$$

This shows $g = Cv$ and hence $K^* \subseteq F$. Let now $v \in D(F)$ and $\begin{pmatrix} \zeta \\ w \end{pmatrix} \in D(K)$. Then

$$\begin{aligned} \left\langle K \begin{pmatrix} \zeta \\ w \end{pmatrix} \middle| v \right\rangle_{H_0(|G|+i)} &= \langle \mathring{D}\zeta + C^\diamond w | v \rangle_{H_0(|G|+i)} \\ &= \langle \mathring{D}\zeta | v \rangle_{H_0(|G|+i)} + \langle C^\diamond w | v \rangle_{H_0(|G|+i)} \\ &= \langle \zeta | -Gv \rangle_{H_0(|\mathring{D}|+i)} + \langle w | Cv \rangle_V, \end{aligned}$$

which shows $F \subseteq K^*$. □

REMARK 5.6 With this choice of F , we can model systems with boundary observation and boundary control in the following way: Let M_0 and M_1 be of the following form:

$$M_0 = \begin{pmatrix} M_{0,00} & M_{0,01} & 0 & 0 \\ M_{0,10} & M_{0,11} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} M_{1,00} & M_{1,01} & M_{1,02} & M_{1,03} \\ M_{1,10} & M_{1,11} & M_{1,12} & M_{1,13} \\ M_{1,20} & M_{1,21} & M_{1,22} & M_{1,23} \\ M_{1,30} & M_{1,31} & M_{1,32} & M_{1,33} \end{pmatrix},$$

for suitable bounded linear operator $M_{i,jk}$ such that M_0 is selfadjoint and $\nu M_0 + \Re M_1$ is uniformly strictly positive definite for all $\nu \in]0, \infty[$ sufficiently large. Consider the abstract linear control system

$$\left(\partial_0 M_0 + M_1 + \begin{pmatrix} 0 & -F^* & 0 \\ F & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \begin{pmatrix} v \\ \zeta \\ w \\ y \end{pmatrix} = \delta \otimes M_0 \begin{pmatrix} v_0 \\ \zeta_0 \\ w_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ B_1 \\ B_2 \end{pmatrix} u, \quad (11)$$

where F is chosen as in (10) and $B_1 \in L(U, V)$, $B_2 \in L(U, Y)$. We characterize the domain of F^* . By Theorem 5.5, a pair (ζ, w) belongs to $D(F^*)$ if and only if $\overset{\circ}{D}\zeta + C^\circ w \in H_0(|G| + i)$. Using the invertibility of $\overset{\circ}{D}$ on the related Sobolev chains, this is equivalent to

$$\zeta + \overset{\circ}{D}^{-1} C^\circ w \in H_1(|\overset{\circ}{D}| + i).$$

Hence, using the results on boundary data spaces this reads as

$$\pi_{BD(D)}(\zeta + \overset{\circ}{D}^{-1} C^\circ w) = 0. \quad (12)$$

This means that w prescribes the boundary data of ζ . We read off the last two lines of equation (11) and get

$$\begin{aligned} M_{1,20}v + M_{1,21}\zeta + M_{1,22}w + M_{1,23}y + Cv &= B_1u, \\ M_{1,30}v + M_{1,31}\zeta + M_{1,32}w + M_{1,33}y &= B_2u. \end{aligned}$$

Since the operator matrix $\begin{pmatrix} M_{1,22} & M_{1,23} \\ M_{1,32} & M_{1,33} \end{pmatrix} \in L(V \oplus Y, V \oplus Y)$ is boundedly invertible by the assumption, we get that

$$\begin{pmatrix} w \\ y \end{pmatrix} = \begin{pmatrix} M_{1,22} & M_{1,23} \\ M_{1,32} & M_{1,33} \end{pmatrix}^{-1} \begin{pmatrix} B_1u - (M_{1,20} + C)v - M_{1,21}\zeta \\ B_2u - M_{1,30}v - M_{1,31}\zeta \end{pmatrix}.$$

Thus, w can be expressed by v, u and ζ . If we plug this expression for w into equality (12), we obtain a boundary control equation. Likewise, we may assume that the operator matrix $\begin{pmatrix} -M_{1,22} & B_1 \\ -M_{1,32} & B_2 \end{pmatrix} \in L(V \oplus U, V \oplus Y)$ is boundedly invertible and hence we get that

$$\begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} -M_{1,22} & B_1 \\ -M_{1,32} & B_2 \end{pmatrix}^{-1} \begin{pmatrix} M_{1,23}y + (M_{1,20} + C)v + M_{1,21}\zeta \\ M_{1,33}y + M_{1,30}v + M_{1,31}\zeta \end{pmatrix}.$$

This yields an expression of w in terms of y, v and ζ and hence (12) becomes a boundary observation equation.

EXAMPLE 5.7 We discuss a possible choice for the observation space, which will come in handy when we consider the wave equation with boundary control and observation in the next section. This particular choice for the control and observation space can be interpreted as abstract implementation of $L^2(\Gamma)$ of the boundary Γ of the underlying region. To this end, assume that we are given a continuous linear

operator $N : BD(G) \rightarrow BD(D)$ satisfying

$$\langle (\dot{D}N + N^* \dot{G})\phi | \phi \rangle_{BD(G)} > 0 \quad (\phi \in BD(G) \setminus \{0\}).$$

Consider the following sesqui-linear form on $BD(G)$:

$$\langle \cdot | \cdot \rangle_U : BD(G) \times BD(G) \ni (f, g) \mapsto \frac{1}{2} \langle Nf | \dot{G}g \rangle_{BD(D)} + \frac{1}{2} \langle \dot{G}f | Ng \rangle_{BD(D)}.$$

For $f \in BD(G) \setminus \{0\}$, we get

$$\frac{1}{2} \langle Nf | \dot{G}f \rangle_{BD(G)} + \frac{1}{2} \langle \dot{G}f | Nf \rangle_{BD(G)} = \frac{1}{2} \langle (\dot{D}N + N^* \dot{G})f | f \rangle_{BD(G)} > 0.$$

Hence, $\langle \cdot | \cdot \rangle_U$ is an inner product on $BD(G)$. We denote by U the completion of $BD(G)$ with respect to the norm induced by $\langle \cdot | \cdot \rangle_U$. Then U is a Hilbert space and

$$\begin{aligned} j : BD(G) &\rightarrow U, \\ f &\mapsto f \end{aligned}$$

is a dense and continuous embedding. We compute j^* . Let $f \in BD(G)$ and $g \in BD(G) \subseteq U$. Then

$$\begin{aligned} \langle j^*g | f \rangle_{BD(G)} &= \langle g | jf \rangle_U \\ &= \frac{1}{2} \langle Ng | \dot{G}f \rangle_{BD(D)} + \frac{1}{2} \langle \dot{G}g | Nf \rangle_{BD(D)} \\ &= \frac{1}{2} \langle Ng | \dot{G}f \rangle_{BD(D)} + \frac{1}{2} \langle N^* \dot{G}g | f \rangle_{BD(G)} \\ &= \frac{1}{2} \langle Ng | \dot{G}f \rangle_{H_0(|D|+i)} + \frac{1}{2} \langle \dot{D}Ng | f \rangle_{H_0(|G|+i)} \\ &\quad + \frac{1}{2} \langle N^* \dot{G}g | f \rangle_{H_0(|G|+i)} + \frac{1}{2} \langle \dot{G}N^* \dot{G}g | \dot{G}f \rangle_{H_0(|D|+i)} \\ &= \frac{1}{2} \langle (D - D)\pi_{BD(D)}^* Ng + (1 - DG)\pi_{BD(G)}^* N^* \dot{G}g | f \rangle_{H_0(|G|+i)} \\ &= \frac{1}{2} \langle (D - DGD)\pi_{BD(D)}^* Ng + (1 - DG)\pi_{BD(G)}^* N^* \dot{G}g | f \rangle_{H_0(|G|+i)} \\ &= \frac{1}{2} \langle (1 - DG)\pi_{BD(G)}^* (\dot{D}N + N^* \dot{G})g | f \rangle_{H_0(|G|+i)} \\ &= \frac{1}{2} \langle (\dot{D}N + N^* \dot{G})g | f \rangle_{H_0(|G|+i)} + \frac{1}{2} \langle \dot{G}(\dot{D}N + N^* \dot{G})g | \dot{G}f \rangle_{H_0(|D|+i)} \\ &= \frac{1}{2} \langle (\dot{D}N + N^* \dot{G})g | f \rangle_{BD(G)}, \end{aligned}$$

which gives

$$j^*g = \frac{1}{2}(\dot{D}N + N^* \dot{G})g$$

or

$$\dot{G}j^*g = (\frac{1}{2}N + \frac{1}{2}\dot{G}N^*\dot{G})g.$$

This yields

$$(D - D)\pi_{BD(D)}^* \dot{G}j^*g = \frac{1}{2}(D - D)\pi_{BD(D)}^* (N + \dot{G}N^*\dot{G})g. \tag{13}$$

Let us try to interpret this equation in order to underscore that this can indeed be considered as an equation between classical boundary traces if the boundary is sufficiently smooth. So, let $\Omega \subseteq \mathbb{R}^n$ be open and let grad be the weak gradient in $L^2(\Omega)$ as introduced in Section 6.1 and let dive be the weak divergence from $L^2(\Omega)^n$ to $L^2(\Omega)$. We denote the boundary of Ω by Γ . Assume that $\Gamma \neq \emptyset$ and that any function $f \in D(\text{grad})$ admits a trace $f|_\Gamma \in L^2(\Gamma)$ with continuous trace operator. Moreover, assume that there exists a well-defined unit outward normal $n : \Gamma \rightarrow \mathbb{R}^n$ being such that there exists an extension to Ω in a way that this extension (denoted by the same name) satisfies $n \in L^\infty(\Omega)^n$ with distributional divergence lying in $L^\infty(\Omega)$. Then the operator $\tilde{N} : H_1(|\text{grad}| + i) \rightarrow H_1(|\text{dive}| + i), f \mapsto nf$ is well-defined and continuous. For the choices $D = \text{dive}$, $G = \text{grad}$ and $N = \pi_{BD}(\text{dive})\tilde{N}\pi_{BD}^*(\text{grad})$ in (13) we can interpret (13) as the equality of the Neumann trace of $\dot{G}j^*g$ and the trace of g . Indeed, for $f, g \in BD(\text{grad})$ we compute formally with the help of the divergence theorem

$$\begin{aligned} \int_\Gamma \text{grad}j^*g \cdot nf &= \frac{1}{2} \int_\Gamma Ng \cdot nf + \frac{1}{2} \langle N^* \text{grad}g|f \rangle_{H_0(|\text{grad}|+i)} + \frac{1}{2} \langle \text{grad}N^* \text{grad}g|\text{grad}f \rangle_{H_0(|\text{dive}|+i)} \\ &= \frac{1}{2} \int_\Gamma Ng \cdot nf + \frac{1}{2} \langle N^* \text{grad}g|f \rangle_{H_1(|\text{grad}|+i)} \\ &= \frac{1}{2} \int_\Gamma Ng \cdot nf + \frac{1}{2} \langle g|\text{dive}(Nf) \rangle_{H_1(|\text{grad}|+i)} \\ &= \frac{1}{2} \int_\Gamma Ng \cdot nf + \frac{1}{2} \int_\Omega \text{dive}(gNf) \\ &= \frac{1}{2} \int_\Gamma gf + \frac{1}{2} \int_\Gamma g(Nf) \cdot n \\ &= \int_\Gamma gf. \end{aligned}$$

6. Some further applications

6.1 Boundary control and observation for acoustic waves

We introduce the operator

$$\text{grad} : D(\text{grad}) \subseteq L^2(\Omega) \rightarrow L^2(\Omega)^n,$$

as the usual weak gradient in $L^2(\Omega)$ for a suitable domain $\Omega \subseteq \mathbb{R}^n$. We require that the geometric properties of Ω are such that grad is injective and that the range $\text{grad}[L^2(\Omega)]$ is closed¹⁴ in $L^2(\Omega)^n$. We choose to use this assumption to avoid technicalities. If grad is not injective, one has to proceed similarly to the way presented in the next section. However, the assumption on $\text{grad}[L^2(\Omega)] \subseteq L^2(\Omega)^n$ to be closed is essential. See also the discussion in [Trostorff & Waurick \(2012, Remark](#)

¹⁴ This holds if a Poincaré–Wirtinger-type inequality holds, which is, for example, the case, if Ω is connected, bounded in one direction, satisfies the segment property and possesses infinite Lebesgue-measure.

3.1(a). We denote by $\pi_{\text{grad}} : L^2(\Omega)^n \rightarrow \text{grad}[L^2(\Omega)]$ the canonical projector induced by the orthogonal decomposition of $L^2(\Omega)^n$ with respect to the closed subspace $\text{grad}[L^2(\Omega)]$ and consider the operator $\pi_{\text{grad}} \text{grad} : D(\text{grad}) \subseteq L^2(\Omega) \rightarrow \text{grad}[L^2(\Omega)]$. The negative adjoint of this operator is given by $\text{div} \pi_{\text{grad}}^* : D(\text{div}) \cap \text{grad}[L^2(\Omega)] \subseteq \text{grad}[L^2(\Omega)] \rightarrow L^2(\Omega)$, where div is defined as the closure of the divergence defined on the space of test functions $\mathring{C}_\infty(\Omega)^n$. In Weiss & Tucsnak (2003, Section 7), a control system for the wave equation has been discussed, which has its first-order representation in the system:

$$\begin{aligned} & \left(\partial_0 \begin{pmatrix} 1 & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & 0 \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 0 & \begin{pmatrix} 0 & 0 \end{pmatrix} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & 0 \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & \sqrt{2} \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 0 & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & 1 \end{pmatrix} \right) \\ & + \begin{pmatrix} 0 & (-\text{div}|_{\text{grad}[L^2(\Omega)]} - C^\diamond) & 0 \\ \begin{pmatrix} -\pi_{\text{grad}} \text{grad} \\ C \\ 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} v \\ \zeta \\ w \\ y \end{pmatrix} \\ & = \delta \otimes \begin{pmatrix} z^{(1)} \\ z^{(0)} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\sqrt{2} \\ -1 \end{pmatrix} u. \end{aligned} \tag{14}$$

Using the Hilbert space U from Example 5.7, we define the operator C by¹⁵

$$\begin{aligned} C : H_1(|\text{grad}| + i) &\rightarrow U, \\ u &\mapsto -bj\pi_{BD(\text{grad})}u, \end{aligned}$$

where $b \in L(U)$. Then we are in the situation of Theorem 5.5 and hence Corollary 3.6 is applicable. The state space of equation (14) is given by $H = L^2(\Omega) \oplus \text{grad}[L^2(\Omega)] \oplus U \oplus U$. We compute C^\diamond with respect to the Gelfand-triplet $H_1(|\text{grad}| + i) \subseteq H_0(|\text{grad}| + i) \subseteq H_{-1}(|\text{grad}| + i)$. For $u \in H_1(|\text{grad}| + i)$, $v \in BD(\text{grad}) \subseteq U$, using Example 5.3, we get that

$$\begin{aligned} \langle -C^\diamond v | u \rangle_{H_0(|\text{grad}| + i)} &= -\langle v | Cu \rangle_U \\ &= \langle v | bj\pi_{BD(\text{grad})}u \rangle_U \\ &= \langle j^* b^* v | \pi_{BD(\text{grad})}u \rangle_{BD(\text{grad})} \\ &= \langle \pi_{BD(\text{grad})}^\diamond j^* b^* v | u \rangle_{H_0(|\text{grad}| + i)} \\ &= \langle (\pi_{BD(\text{grad})}^* - \text{div} \pi_{BD(\text{div})}^* \text{grad}) j^* b^* v | u \rangle_{H_0(|\text{grad}| + i)} \end{aligned}$$

¹⁵ Note that $|\pi_{\text{grad}} \text{grad}| = |\text{grad}|$.

$$\begin{aligned} &= \langle (\pi_{BD(\text{grad})}^* \overset{\circ}{\text{div}} \overset{\bullet}{\text{grad}} - \overset{\circ}{\text{div}} \pi_{BD(\text{dive})}^* \overset{\bullet}{\text{grad}}) j^* b^* v | u \rangle_{H_0(|\text{grad}|+i)} \\ &= \langle (\overset{\circ}{\text{div}} - \overset{\circ}{\text{div}}) \pi_{BD(\text{dive})}^* \overset{\bullet}{\text{grad}} j^* b^* v | u \rangle_{H_0(|\text{grad}|+i)}, \end{aligned}$$

and we read off that $C^\circ v = -(\overset{\circ}{\text{div}} - \overset{\circ}{\text{div}}) \pi_{BD(\text{dive})}^* \overset{\bullet}{\text{grad}} j^* b^* v \in H_{-1}(|\text{grad}| + i)$ for all $v \in BD(\text{grad}) \subseteq U$. Hence, using (12), we write the boundary equation as

$$\pi_{BD(\text{dive})}(\zeta - \overset{\circ}{\text{div}}|_{\text{grad}[L^2(\Omega)]}^{-1}((\overset{\circ}{\text{div}} - \overset{\circ}{\text{div}}) \pi_{BD(\text{dive})}^* \overset{\bullet}{\text{grad}} j^* b^* w)) = 0.$$

Since

$$\overset{\circ}{\text{div}}|_{\text{grad}[L^2(\Omega)]}^{-1}((\overset{\circ}{\text{div}} - \overset{\circ}{\text{div}}) \pi_{BD(\text{dive})}^* \overset{\bullet}{\text{grad}} j^* b^* w) \in H_1(|\text{dive}| + i),$$

we get that

$$\begin{aligned} \pi_{BD(\text{dive})} \zeta &= \pi_{BD(\text{dive})} \overset{\circ}{\text{div}}|_{\text{grad}[L^2(\Omega)]}^{-1}((\overset{\circ}{\text{div}} - \overset{\circ}{\text{div}}) \pi_{BD(\text{dive})}^* \overset{\bullet}{\text{grad}} j^* b^* w) \\ &= - \overset{\bullet}{\text{grad}} j^* b^* w. \end{aligned}$$

To invoke the boundary control and observation equation, we compute

$$\begin{pmatrix} w \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \sqrt{2} & 1 \end{pmatrix}^{-1} \begin{pmatrix} -\sqrt{2}u - Cv \\ -u \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} -\sqrt{2}u - Cv \\ -u \end{pmatrix}$$

and

$$\begin{pmatrix} w \\ u \end{pmatrix} = - \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}^{-1} \begin{pmatrix} Cv \\ y \end{pmatrix} = \begin{pmatrix} 1 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} Cv \\ y \end{pmatrix}.$$

Thus, we get $w = -\sqrt{2}u - Cv$ and $w = Cv - \sqrt{2}y$. This yields

$$\pi_{BD(\text{dive})} \zeta = \sqrt{2} \overset{\bullet}{\text{grad}} j^* b^* u - \overset{\bullet}{\text{grad}} j^* b^* b j \pi_{BD(\text{grad})} v$$

and

$$\pi_{BD(\text{dive})} \zeta = \overset{\bullet}{\text{grad}} j^* b^* b j \pi_{BD(\text{grad})} v + \sqrt{2} \overset{\bullet}{\text{grad}} j^* b^* y.$$

REMARK 6.1 Let us assume that there exists a outward unit normal n on $\Gamma := \bar{\Omega} \setminus \overset{\circ}{\Omega}$ such that there exists a bounded, measurable extension to Ω with bounded, measurable distributional divergence. Using the interpretation from Example 5.7, the assumption $b^*u, b^*Cv, b^*y \in BD(\text{grad})$ ¹⁶ and imposing suitable

¹⁶ In Weiss & Tucsnak (2003), these assumptions are formulated with the help of a certain quotient space Z_0 .

additional requirements on the underlying domain, we can interpret the latter equations as

$$\begin{aligned} n \cdot \zeta &= -b^*bv + \sqrt{2}b^*u, \\ n \cdot \zeta &= b^*bv + \sqrt{2}b^*y, \end{aligned}$$

on Γ as boundary control and boundary observation equation, respectively. These correspond to the boundary equations originally considered in Weiss & Tucsnak (2003, Section 7).

REMARK 6.2 (a) It is also possible to consider a model, where the type of the partial differential equations changes over the space, i.e. there are regions, where the equation is parabolic others where the equation is hyperbolic and regions where the equation is described best by elliptic. More precisely, assume the open set Ω under consideration can be decomposed into three pairwise disjoint measurable parts $\Omega_e, \Omega_p, \Omega_h$ such that the evolutionary equation may be written as

$$\begin{aligned} &\left(\partial_0 \begin{pmatrix} \chi_{\Omega_h} + \chi_{\Omega_p} & (0 \ 0) & 0 \\ (0) & (\chi_{\Omega_h} \ 0) & (0) \\ (0) & (0 \ 0) & (0) \\ 0 & (0 \ 0) & 0 \end{pmatrix} + \begin{pmatrix} \chi_{\Omega_e} & (0 \ 0) & 0 \\ (0) & (\chi_{\Omega_e} + \chi_{\Omega_p} \ 0) & (0) \\ (0) & 0 & 1 \\ 0 & (0 \ \sqrt{2}) & 1 \end{pmatrix} \right) \\ &+ \begin{pmatrix} 0 & (-\overset{\circ}{\text{div}}|_{\text{grad}[L^2(\Omega)]} - C^\diamond) & 0 \\ (-\pi_{\text{grad}}\text{grad}) & \begin{pmatrix} (0 \ 0) \\ (0 \ 0) \\ (0 \ 0) \end{pmatrix} & \begin{pmatrix} (0) \\ (0) \\ 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} v \\ \zeta \\ w \\ y \end{pmatrix} \\ &= \delta \otimes \begin{pmatrix} (\chi_{\Omega_h} + \chi_{\Omega_p})z^{(1)} \\ (\chi_{\Omega_h}z^{(0)}) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ (0) \\ (-\sqrt{2}) \\ -1 \end{pmatrix} u. \end{aligned}$$

Obviously, the well-posedness condition in Corollary 3.6 is still satisfied. As it can be verified immediately from the equations in Remark 5.6, the control and observation equations remain the same. However, we find different types of equations describing the main physical phenomenon. In particular, on Ω_e we have

$$\begin{aligned} &\left(\begin{pmatrix} 1 & (0 \ 0) & 0 \\ (0) & (1 \ 0) & (0) \\ (0) & (0 \ 1) & (0) \\ 0 & (0 \ \sqrt{2}) & 1 \end{pmatrix} + \begin{pmatrix} 0 & (-\overset{\circ}{\text{div}}|_{\text{grad}[L^2(\Omega)]} - C^\diamond) & 0 \\ (-\pi_{\text{grad}}\text{grad}) & \begin{pmatrix} (0 \ 0) \\ (0 \ 0) \\ (0 \ 0) \end{pmatrix} & \begin{pmatrix} (0) \\ (0) \\ 0 \end{pmatrix} \end{pmatrix} \right) \begin{pmatrix} v \\ \zeta \\ w \\ y \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ (0) \\ (-\sqrt{2}) \\ -1 \end{pmatrix} u. \end{aligned}$$

Thus,

$$\begin{aligned} v - \operatorname{div}_{|\operatorname{grad}[L^2(\Omega)]} \zeta - C^\circ w &= 0, \\ \zeta - \pi_{\operatorname{grad}} \operatorname{grad} v &= 0, \\ w + Cv &= -\sqrt{2}u, \\ \sqrt{2}w + v &= -u, \end{aligned}$$

which gives

$$v - \operatorname{div} \operatorname{grad} v = 0,$$

with the (formal) boundary conditions

$$\begin{aligned} n \cdot \operatorname{grad} v &= -b^*bv + \sqrt{2}b^*u, \\ n \cdot \operatorname{grad} v &= b^*bv + \sqrt{2}b^*y. \end{aligned}$$

On Ω_p we get, by similar computations,

$$\partial_0 v - \operatorname{div} \operatorname{grad} v = \delta \otimes z^{(1)},$$

with the (formal) boundary conditions

$$\begin{aligned} n \cdot \operatorname{grad} v &= -b^*bv + \sqrt{2}b^*u, \\ n \cdot \operatorname{grad} v &= b^*bv + \sqrt{2}b^*y, \end{aligned}$$

and on Ω_h we get correspondingly

$$\partial_0^2 v - \operatorname{div} \operatorname{grad} v = \partial_0 \delta \otimes z^{(1)} + \delta \otimes z^{(0)},$$

with the same equations on the boundary.

(b) The last example treats local operators with respect to the spatial variables. Unless the well-posedness condition in Corollary 3.6 is not violated, we can also treat integral operators as coefficients. Indeed, the equation

$$\left(\partial_0 \begin{pmatrix} M_{0,00} & (M_{0,01} \ 0) & 0 \\ (M_{0,10}) & (M_{0,11} \ 0) & (0) \\ 0 & (0 \ 0) & 0 \end{pmatrix} + \begin{pmatrix} M_{1,00} & (M_{1,01} \ 0) & 0 \\ (M_{1,10}) & (M_{1,11} \ 0) & (0) \\ 0 & (0 \ \sqrt{2}) & 1 \end{pmatrix} \right)$$

$$\begin{aligned}
 & + \left(\begin{array}{ccc} 0 & (-\operatorname{div}|_{\operatorname{grad}[L^2(\Omega)]} - C^\diamond) & 0 \\ (-\pi_{\operatorname{grad}} \operatorname{grad}) & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ C & & \\ 0 & \begin{pmatrix} 0 & 0 \end{pmatrix} & 0 \end{array} \right) \begin{pmatrix} v \\ \xi \\ w \\ y \end{pmatrix} \\
 & = \delta \otimes \left(\begin{array}{ccc} M_{0,00} & (M_{0,01} \ 0) & 0 \\ (M_{0,10}) & (M_{0,11} \ 0) & \begin{pmatrix} z^{(1)} \\ z^{(0)} \\ 0 \end{pmatrix} \\ 0 & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & 0 \end{array} \right) + \begin{pmatrix} 0 \\ 0 \\ -\sqrt{2} \\ -1 \end{pmatrix} u
 \end{aligned}$$

leads to the same observation and control equation as in (a), but the operators $M_{i,jk}$ for $i, j, k \in \{0, 1\}$ can be matrices with variable coefficients or integral operators such as negative roots of the negative Laplacian.

6.2 Boundary control for electromagnetic waves

As a second example, we consider a boundary control problem for Maxwell’s system. We shall first introduce the operators involved. Throughout, let $\Omega \subseteq \mathbb{R}^3$ be an open domain.

DEFINITION 6.3 We define the operator curl as the closure of the operator

$$\begin{aligned}
 \overset{\circ}{C}_\infty(\Omega)^3 & \subseteq L^2(\Omega)^3 \rightarrow L^2(\Omega)^3, \\
 (\phi_1, \phi_2, \phi_3)^\top & \mapsto \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix},
 \end{aligned}$$

where ∂_i denotes the partial derivative with respect to the i th coordinate. The operator $\overset{\circ}{\operatorname{curl}}$ turns out to be symmetric and we set $\operatorname{curl} := (\operatorname{curl})^*$ and obtain the relation

$$\overset{\circ}{\operatorname{curl}} \subseteq \operatorname{curl}.$$

In Weck (2000), the exact controllability of the following problem was considered:

$$\begin{aligned}
 \partial_0 \varepsilon E + \operatorname{curl} H & = \delta \otimes E^{(0)}, \\
 \partial_0 \mu H - \operatorname{curl} E & = \delta \otimes H^{(0)},
 \end{aligned}$$

where the control $u \in BD(\operatorname{curl})$ prescribes the boundary behaviour of the tangential component of H , i.e. $\pi_{BD(\operatorname{curl})} H = u$. This problem can be dealt with in the following way: We introduce the function $\tilde{H} := H - \pi_{BD(\operatorname{curl})}^* u$ and formulate Maxwell’s equations for the pair (E, \tilde{H}) as follows:

$$\begin{aligned}
 \partial_0 \varepsilon E + \operatorname{curl} \tilde{H} & = \delta \otimes E^{(0)} - \operatorname{curl} \pi_{BD(\operatorname{curl})}^* u, \\
 \partial_0 \mu \tilde{H} - \operatorname{curl} E & = \delta \otimes H^{(0)} - \partial_0 \mu \pi_{BD(\operatorname{curl})}^* u,
 \end{aligned}$$

or in matrix-form

$$\left(\partial_0 \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} + \begin{pmatrix} 0 & \text{curl} \\ -\text{curl} & 0 \end{pmatrix} \right) \begin{pmatrix} E \\ \tilde{H} \end{pmatrix} = \delta \otimes \begin{pmatrix} E^{(0)} \\ H^{(0)} \end{pmatrix} - \begin{pmatrix} \text{curl} \pi_{BD(\text{curl})}^* \\ \partial_0 \mu \pi_{BD(\text{curl})}^* \end{pmatrix} u.$$

By our general solution theory (Theorem 3.2 and Proposition 3.3), this system is well-posed and we obtain a unique solution $(E, \tilde{H}) \in H_{v,-1}(\mathbb{R}; L^2(\Omega)^3 \oplus L^2(\Omega)^3)$. Since the time derivative of u occurs as a source term, we obtain a regularity loss of the solution (E, \tilde{H}) , although the system is locally regularizing. In order to detour this regularity loss, we may follow the strategy of Section 5.3 and point out, which type of boundary control equations can be treated in this way.

In the framework of Section 5.3, we want curl to play the role¹⁷ of $\overset{\circ}{D}$ and $-\text{curl}$ that of G . In view of Theorem 5.5, we have to guarantee that curl is boundedly invertible. For this purpose, we consider the restriction of the operator curl given by

$$\widetilde{\text{curl}} : D(\text{curl}) \cap N(\text{curl})^\perp \subseteq N(\text{curl})^\perp \rightarrow \overline{\text{curl}[L^2(\Omega)^3]}.$$

We require that Ω has suitable geometric properties such that $\text{curl}[L^2(\Omega)^3]$ is closed in order to obtain a boundedly invertible operator.¹⁸ An easy computation shows that $(\widetilde{\text{curl}})^* = \text{curl}|_{\text{curl}[L^2(\Omega)^3]}$. We decompose the Hilbert space $L^2(\Omega)^3$ into the following orthogonal subspaces:

$$\begin{aligned} L^2(\Omega)^3 &= N(\text{curl}) \oplus N(\text{curl})^\perp, \\ L^2(\Omega)^3 &= N(\overset{\circ}{\text{curl}}) \oplus N(\overset{\circ}{\text{curl}})^\perp, \end{aligned}$$

and denote by $\pi_0 : L^2(\Omega)^3 \rightarrow N(\text{curl})$, $\pi_1 : L^2(\Omega)^3 \rightarrow N(\text{curl})^\perp$, $\overset{\circ}{\pi}_0 : L^2(\Omega)^3 \rightarrow N(\overset{\circ}{\text{curl}})$ and $\overset{\circ}{\pi}_1 : L^2(\Omega)^3 \rightarrow N(\overset{\circ}{\text{curl}})^\perp$ the respective orthogonal projections. Since $\pi_{BD(\text{curl})} H = \pi_{BD(\text{curl})} \overset{\circ}{\pi}_1 H$ for each $H \in D(\text{curl})$ we may write the boundary control problem in the following way:

$$\left(\partial_0 \begin{pmatrix} \pi_1 \varepsilon \pi_1^* & \begin{pmatrix} 0 & 0 \end{pmatrix} & \pi_1 \varepsilon \pi_0^* & 0 \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \overset{\circ}{\pi}_1 \mu \overset{\circ}{\pi}_1^* & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \overset{\circ}{\pi}_1 \mu \overset{\circ}{\pi}_0^* \\ 0 \end{pmatrix} \\ \pi_0 \varepsilon \pi_1^* & \begin{pmatrix} 0 & 0 \end{pmatrix} & \pi_0 \varepsilon \pi_0^* & 0 \\ 0 & \begin{pmatrix} \overset{\circ}{\pi}_0 \mu \overset{\circ}{\pi}_1^* & 0 \end{pmatrix} & 0 & \overset{\circ}{\pi}_0 \mu \overset{\circ}{\pi}_0^* \end{pmatrix} \right)$$

¹⁷ This implies $\overset{\circ}{G} = -\overset{\circ}{\text{curl}}$ and $D = \overset{\circ}{\text{curl}}$.

¹⁸ For example, domains $\Omega \subseteq \mathbb{R}^3$ with conical points, wedges and cups with a cross section satisfying the segment property. In Picard et al. (2001), a large class of such domains is characterized for which the compactness of the embedding $D(\text{curl}) \cap D(\text{div}) \hookrightarrow L^2(\Omega)^3$ holds. This compact embedding result implies the desired properties for $\widetilde{\text{curl}}$.

$$\begin{aligned}
 & + \begin{pmatrix} 0 & \begin{pmatrix} 0 & 0 \end{pmatrix} & 0 & 0 \\ \begin{pmatrix} 0 \\ M_{1,31} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ M_{1,32} & M_{1,33} \end{pmatrix} & \begin{pmatrix} 0 \\ M_{1,34} \end{pmatrix} & \begin{pmatrix} 0 \\ M_{1,35} \end{pmatrix} \\ 0 & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & 0 & 0 \\ 0 & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & 0 & 0 \end{pmatrix} \\
 & + \begin{pmatrix} 0 & ((\widetilde{\text{curl}})^* & C^\diamond) & 0 & 0 \\ \begin{pmatrix} -\widetilde{\text{curl}} \\ -C \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 0 & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & 0 & 0 \\ 0 & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & 0 & 0 \end{pmatrix} \begin{pmatrix} \pi_1 E \\ \overset{\circ}{\pi}_1 H \\ w \\ \pi_0 E \\ \overset{\circ}{\pi}_0 H \end{pmatrix} \\
 & = \delta \otimes \begin{pmatrix} \pi_1 E^{(0)} \\ \overset{\circ}{\pi}_1 H^{(0)} \\ 0 \\ \pi_0 E^{(0)} \\ \overset{\circ}{\pi}_0 H^{(0)} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ Bu \\ 0 \\ 0 \end{pmatrix},
 \end{aligned}$$

where

$$C : H_1(|\widetilde{\text{curl}}| + i) \rightarrow U$$

is a bounded linear operator, U an arbitrary Hilbert space and $B \in L(U)$. The linear operators $M_{1,3i}$ for $i \in \{1, \dots, 5\}$ are bounded in the respective Hilbert spaces and $\Re M_{1,33}$ is assumed to be strictly positive definite. Since $\widetilde{\text{curl}}$ is boundedly invertible, Theorem 5.5 applies and Corollary 3.6 yields the well-posedness of the control problem. The domain of $((\widetilde{\text{curl}})^* \ C^\diamond)$ reads as

$$\begin{aligned}
 \begin{pmatrix} \overset{\circ}{\pi}_1 H \\ w \end{pmatrix} \in D(((\widetilde{\text{curl}})^* \ C^\diamond)) & \Leftrightarrow (\widetilde{\text{curl}})^* \overset{\circ}{\pi}_1 H + C^\diamond w \in H_0(|\widetilde{\text{curl}}| + i) \\
 & \Leftrightarrow (\widetilde{\text{curl}})^* (\overset{\circ}{\pi}_1 H + ((\widetilde{\text{curl}})^*)^{-1} C^\diamond w) \in H_0(|\widetilde{\text{curl}}| + i) \\
 & \Leftrightarrow \overset{\circ}{\pi}_1 H + ((\widetilde{\text{curl}})^*)^{-1} C^\diamond w \in H_1(|(\widetilde{\text{curl}})^*| + i) \\
 & \Leftrightarrow \pi_{BD(\text{curl})}(\overset{\circ}{\pi}_1 H + ((\widetilde{\text{curl}})^*)^{-1} C^\diamond w) = 0, \tag{15}
 \end{aligned}$$

for each $w \in U, H \in L^2(\Omega)^3$. By the third equation of the above boundary control problem, we get that

$$w = -M_{1,33}^{-1}((M_{1,31} - C)\pi_1 E + M_{1,32}\overset{\circ}{\pi}_1 H + M_{1,34}\pi_0 E + M_{1,35}\overset{\circ}{\pi}_0 H - Bu),$$

and hence (15) yields

$$\begin{aligned}
 \pi_{BD(\text{curl})}(\overset{\circ}{\pi}_1 H - ((\widetilde{\text{curl}})^*)^{-1} C^\diamond M_{1,33}^{-1}((M_{1,31} - C)\pi_1 E + M_{1,32}\overset{\circ}{\pi}_1 H \\
 + M_{1,34}\pi_0 E + M_{1,35}\overset{\circ}{\pi}_0 H - Bu)) = 0.
 \end{aligned}$$

Although this equation covers a number of possible control equations, it appears that in this setting the term $(M_{1,31} - C)\pi_1 E$ cannot be made to vanish, since we have to assume that $M_{1,31}$ is bounded on $H_0(|\text{curl}| + i)$ whereas in general C is not. This shows that in this setting only boundary control equations containing terms in $\pi_1 H$ and $\pi_1 E$ can be treated without more intricate adjustments.

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