Stability of Highly Nonlinear Hybrid Stochastic Integro-differential Delay Equations

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Abstract

For the past few decades, the stability criteria for the stochastic differential delay equations (SDDEs) have been studied intensively. Most of these criteria can only be applied to delay equations where their coefficients are either linear or nonlinear but bounded by linear functions. Recently, the stability criterion for highly nonlinear hybrid stochastic differential equations is investigated in [Fei, Hu, Mao and Shen, Automatica, 2017]. In this paper, we investigate a class of highly nonlinear hybrid stochastic integro-differential delay equations (SIDDEs). First, we establish the stability and boundedness of hybrid stochastic integro-differential delay equations. Then the delay-dependent criteria of the stability and boundedness of solutions to SIDDEs are studied. Finally, an illustrative example is provided.

Keywords: stochastic integro-differential delay equation (SIDDE); nonlinear growth condition; asymptotic stability; Markovian switching; Lyapunov functional

1. Introduction

In many real-world systems, such as science, industry, economics and finance etc., we will encounter a time delay. The differential delay equations (DDEs) including the functional differential equations have been used to model such time-delay systems. Since the time-delay often causes the instability of systems, stability of DDEs has been researched intensively for more than 50 years. Generally, the stability criteria are classified into the delay-dependent and delay-independent stability criteria. When the size of delays is incorporated into the delay-dependent stability criteria, the delay-dependent systems are generally less conservative than the delay-independent ones which work for any size of delays. There exists a very rich literature in this area (see, e.g., [10, 11, 12, 16, 20]).

In 1980's, the stochastic differential delay equations were developed in order to model the real-world systems which are subject to external noises (see, e.g., [30]). Since then, in the study of SDDEs the stability has been one of the most important topics (see, e.g., [9, 19, 23, 24, 37, 38]).

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Since 1990's, the hybrid SDDEs (called also SDDEs with Markovian switching) were developed to model the real-world systems where they may experience abrupt changes in their parameters and structures in addition to uncertainties and time lags. One of the important issues in the study of hybrid SDDEs is the analysis of stability of control systems. Moreover, the delay-dependent stability criteria have been created by many authors (see, e.g., [26, 27, 28, 29, 39, 40, 41, 2]). To our best knowledge, the existing delay-dependent stability criteria are mainly provided for the hybrid SDDEs where their coefficients are either linear or nonlinear but bounded by linear functions (or, satisfy the linear growth condition). Recently, [13, 14] initiate the investigation on the stability of the hybrid highly nonlinear stochastic delay differential equations. Based on the highly nonlinear hybrid SDDEs (see, e.g., [13, 14]), the stability of highly nonlinear systems is further explored in [6, 7, 8, 34, 35]. However, the current states of many real systems depend on several history states of some time interval. Thus multiple time delay systems are also discussed (see, e.g., [21, 31]).

On the other hand, a real system depends on not only discrete delays (single or multiple ones) but also a whole history of states of the system with some lag interval. [4] considers a nonhomogeneous Volterra integro-differential equation with the solution being a non-Markovian process. Moreover, the convergence and stability of the linear stochastic integro-differential delay equations were discussed in [1, 5, 15, 25, 17, 18, 32, 33, 36, 3]. With the energy of a hybrid system accumulated, a stable system might get unstable as it is disturbed by a white noise. In general, if a stable system has too long lag, then it might get unstable. Our problem is as follows: In how long lag time, the system can remain stable? To this end, we discuss an example as follows. Now we consider the stability of the following hybrid highly nonlinear SIDDE

$$dX(t) = \begin{cases} (-10X^3(t) - 2\hbar(X_t))dt + (\hbar(X_t))^2 dB(t), & \text{if } i = 1, \\ (\hbar(X_t) - 5X^3(t))dt + (\hbar(X_t))^2 dB(t), & \text{if } i = 2. \end{cases} \tag{1.1}$$

Here for $\tau \geq 0$ $X_t := \{X(t+u) : -\tau \leq u \leq 0\}, \hbar(X_t) := \frac{1}{\tau}\int_{-\tau}^{0} X(t+u)du$ with $\hbar(X_t) := X(t)$ for $\tau = 0$, $X(t) \in \mathbb{R}$ is the state of the highly non-linear hybrid system, $B(t)$ is a scalar Brownian motion, $r(t)$ is a Markovian chain with the state space $\mathbb{S} = \{1, 2\}$ and its generator Γ given by

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 8 & -8 \end{pmatrix}. \tag{1.2}$$

The above system (1.1) will switch from one mode to the other according to the probability law of the Markovian chain. If the time delay $\tau = 0.01$, the computer simulation shows it is asymptotically stable (see Figure 4.1). If the time-delay is large, say $\tau = 3$, the computer simulation shows that the hybrid SIDDE (1.1) is unstable (see Figure 4.2). In other words, whether the hybrid SIDDE is stable or not depends on how small or large the time-delay is. On the other hand, both drift and diffusion coefficients of the hybrid SIDDE affect the stability of the systems due to highly nonlinear. However, there is no delay dependent criterion which can be applied to the SIDDE to derive a sufficient bound on the time-delay τ such that the SIDDE is stable, although the stability criteria of the highly nonlinear hybrid SDDE have been discussed in [6] on the single delay. The aim of this paper is to establish the delay dependent criteria for the highly nonlinear hybrid SIDDEs.

Our main contributions are as follows:

(a). The hybrid highly nonlinear stochastic integro-differential delay equations first are investigated, where the coefficients are highly nonlinear on both the current state $X(t)$ and the history $\hbar(X_t)$ with lag time $\tau \geq 0$.

(b). We established the theorem of the stability and boundedness of the solutions to the hybrid highly nonlinear SIDDEs similar to [13] where they only investigate the highly nonlinear hybrid SDDE (see Theorem 2.4 in Section 2 below).

(c). The delay-dependent criteria are established first for the solutions to the hybrid highly nonlinear SIDDEs in Section 3.

(d). New mathematical techniques are well applied to solve our stability criteria, such as by constructing an appropriate Lyapunov functional.

2. Notation and Assumptions

Throughout this paper, unless otherwise specified, we use the following notation. If A is a vector or matrix, its transpose is denoted by $A^\top$. If $x \in \mathbb{R}^d$, then $|x|$ is its Euclidean norm. For a matrix A, we let $|A| = \sqrt{\text{trace}(A^\top A)}$ be its trace norm and $\|A\| = \max\{|Ax| : |x| = 1\}$ be the operator norm. Let $\mathbb{R}_+ = [0, \infty)$. Denote by $C([-\tau, 0]; \mathbb{R}^d)$ the family of continuous functions η from $[-\tau, 0] \to \mathbb{R}^d$ with the norm $\|\eta\| = \sup_{-\tau\leq u\leq 0} |\eta(u)|$. If A is a subset of Ω, denote by I_A its indicator function. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $B(t) = (B_1(t), \cdots, B_m(t))^\top$ be an m-dimensional Brownian motion defined on the probability space. Let $r(t)$, $t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $\mathbb{S} = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t + \Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$. Let

$$f(\cdot,\cdot,\cdot) : \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^d, \qquad g(\cdot,\cdot,\cdot) : \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^{d\times m}$$
$$F(\hbar(\cdot),\cdot,\cdot) : C([-\tau, 0]; \mathbb{R}^d) \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^d, \qquad G(\hbar(\cdot),\cdot,\cdot) : C([-\tau, 0]; \mathbb{R}^d) \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^{d\times m}$$

be Borel measurable functions, where $\hbar(\phi) := \frac{1}{\tau}\int_{-\tau}^0 \phi(u)du, \phi \in C([-\tau, 0]; \mathbb{R}^d)$ with $\hbar(\phi) := \phi(0)$ for $\tau = 0$. Let the functional $X_t := \{X(t + u) : -\tau \leq u \leq 0\}$. Consider a d-dimensional hybrid highly nonlinear SIDDE with Assumption 2.1 below

$$\begin{aligned} dX(t) =& [f(X(t), r(t), t) + F(\hbar(X_t), r(t), t)]dt \\ &+ [g(X(t), r(t), t) + G(\hbar(X_t), r(t), t)]dB(t) \end{aligned} \tag{2.1}$$

on $t \geq 0$ with initial data

$$\{\eta(t) : -\tau \leq t \leq 0\} = \eta \in C([-\tau, 0]; \mathbb{R}^d), \ r(0) = i_0 \in \mathbb{S}. \tag{2.2}$$

The classical conditions for the existence and uniqueness of the global solution are the local Lipschitz condition and the linear growth condition (see, e.g., [24, 29]). In this paper, we need only the local Lipschitz condition. However, we will consider highly nonlinear hybrid SIDDEs which, in general, do not satisfy the linear growth condition in this paper. Therefore, we impose the polynomial growth condition, instead of the linear growth condition. Let us state these conditions as an assumption for our aim.

Assumption 2.1. *Assume that for any $h > 0$, there exists a positive constant K_h such that*

$$\begin{aligned}&|f(x,i,t) - f(\bar{x},i,t)| \vee |g(x,i,t) - g(\bar{x},i,t)|\\ &\vee |F(\hbar(\phi),i,t) - F(\hbar(\bar{\phi}),i,t)| \vee |G(\hbar(\phi),i,t) - G(\hbar(\bar{\phi}),i,t)| \le K_h(|x-\bar{x}| + \|\phi - \bar{\phi}\|)\end{aligned}$$

for all $x, \bar{x}, \in \mathbb{R}^d$ and $\phi, \bar{\phi} \in C([-\tau,0];\mathbb{R}^d)$ with $|x| \vee |\bar{x}| \vee \|\phi\| \vee \|\bar{\phi}\| \le h$ and all $(i,t) \in \mathbb{S} \times \mathbb{R}_+$. Assume moreover that there exist three constants $K > 0$, $q_1 \ge 1$ and $q_2 \ge 1$ such that

$$\begin{aligned}&|f(x,i,t) + F(\hbar(\phi),i,t)| \le K(1 + |x|^{q_1} + |\hbar(\phi)|^{q_1}),\\ &|g(x,i,t) + G(\hbar(\phi),i,t)| \le K(1 + |x|^{q_2} + |\hbar(\phi)|^{q_2}) \qquad (2.3)\end{aligned}$$

for all $x \in \mathbb{R}^d, (i,t) \in \mathbb{S} \times \mathbb{R}_+, \phi \in C([-\tau,0];\mathbb{R}^d)$.

We emphasize that we are here interested in highly nonlinear SIDDEs which have either $q_1 > 1$ or $q_2 > 1$. We will refer condition (2.3) as the polynomial growth condition. It is known that Assumption 2.1 only guarantees that the SIDDE (2.1) with the initial data (2.2) has a unique maximal solution, which may explode to infinity at a finite time (see [24]). To avoid such a possible explosion, we need to impose an additional condition in terms of Lyapunov functions.

For our aim, we now give the following useful lemma.

Lemma 2.2. *For nonnegative integers $m \ge k$, let the non-negative coefficient quasi-polynomial $U(x) = a_m|x|^m + \cdots + a_k|x|^k, x \in \mathbb{R}^d$, where $a_i \ge 0, i = k+1, \cdots, m-1$, and $a_k, a_m > 0$. Assume that $x(t) : [-\tau, \infty) \to \mathbb{R}^d$ is a continuous function, where $x(t) = \xi(t), t \in [-\tau, 0]$ and $\tau \ge 0$. Then, for any $\varepsilon \ge 0$, we have*

$$\begin{aligned}&\int_0^T e^{\varepsilon t} U\Big(\int_{-\tau}^0 x(t+u)du\Big)dt\\ &\le e^{\varepsilon\tau}\Big(\int_{-\tau}^0 e^{\varepsilon s} U(\xi(s))ds + \int_0^T e^{\varepsilon s} U(x(s))ds\Big), \quad \forall T > 0. \qquad (2.4)\end{aligned}$$

Proof. We first consider the case of $\tau > 0$. Let $U(x) = |x|^p, p \ge 1$. For $p > 1$, by the Hölder inequality with $q = \frac{p}{p-1}$, we obtain

$$U\Big(\tfrac{1}{\tau}\int_{-\tau}^0 x(t+u)du\Big) = \Big|\tfrac{1}{\tau}\int_{-\tau}^0 x(t+u)du\Big|^p \le \tfrac{1}{\tau}\int_{-\tau}^0 |x(t+u)|^p du.$$

For $p = 1$, the inequality above holds still. Therefore, we have

$$\begin{aligned}&\int_0^T e^{\varepsilon t}\Big(\tfrac{1}{\tau}\int_{-\tau}^0 |x(t+u)|^p du\Big)dt = \tfrac{1}{\tau}\int_{-\tau}^0\int_0^T e^{\varepsilon t}|x(t+u)|^p dtdu\\ &= \tfrac{1}{\tau}\int_{-\tau}^0\int_u^{T+u} e^{\varepsilon(s-u)}|x(s)|^p dsdu \le \tfrac{1}{\tau}\int_{-\tau}^0\int_{-\tau}^T e^{\varepsilon s}e^{\varepsilon\tau}|x(s)|^p dsdu\\ &= e^{\varepsilon\tau}\int_{-\tau}^T e^{\varepsilon s}|x(s)|^p ds = e^{\varepsilon\tau}\Big(\int_{-\tau}^0 e^{\varepsilon s}|\xi(s)|^p ds + \int_0^T e^{\varepsilon s}|x(s)|^p ds\Big).\end{aligned}$$

Thus, we know, for $p = 0, 1, \cdots, m$,

$$a_p\int_0^T e^{\varepsilon t}\Big|\frac{1}{\tau}\int_{-\tau}^0 x(t+u)du\Big|^p dt \le e^{\varepsilon\tau}\Big(\int_{-\tau}^0 e^{\varepsilon s}a_p|\xi(s)|^p ds + \int_0^T e^{\varepsilon s}a_p|x(s)|^p ds\Big),$$

from which (2.4) holds.

Next, for the case of $\tau = 0$, the claim easily are obtained. Thus the proof is complete. □

Let $C^{2,1}(\mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+; \mathbb{R}_+)$ denote the family of non-negative functions $U(x, i, t)$ defined on $(x, i, t) \in \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+$ which are continuously twice differentiable in x and once in t. For such a function $U(x, i, t)$, let $U_t = \frac{\partial U}{\partial t}$, $U_x = \left(\frac{\partial U}{\partial x_1}, \cdots, \frac{\partial U}{\partial x_d}\right)$, and $U_{xx} = \left(\frac{\partial^2 U}{\partial x_k \partial x_l}\right)_{d\times d}$. Let $C(\mathbb{R}^d \times [-\tau, \infty); \mathbb{R}_+)$ denote the family of all continuous functions from $\mathbb{R}^d \times [-\tau, \infty)$ to $\mathbb{R}_+$. We can now state another assumption.

Assumption 2.3. *Let $H(x), x \in \mathbb{R}^d$ be the nonnegative coefficient quasi-polynomial $H(\cdot)$ (see, Lemma 2.2). Assume that there exist the function $\bar{U} \in C^{2,1}(\mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+; \mathbb{R}_+)$, and nonnegative constants c_0, c_1, c_2 and $q \geq 2(q_1 \vee q_2)$ (where q_1 and q_2 are the same as in Assumption (2.1)) such that*

$$c_2 < c_1, \quad |x|^q \leq \bar{U}(x, i, t) \leq H(x) \tag{2.5}$$

for $\forall (x, i, t) \in \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+$, and

$$\begin{aligned} &\mathbb{L}\bar{U}(x, \phi, i, t) := \bar{U}_t(x, i, t) + \bar{U}_x(x, i, t)(f(x, i, t) + F(\hbar(\phi), i, t)) \\ &+ \frac{1}{2}\text{trace}[(g(x, i, t) + G(\hbar(\phi), i, t))^\top \bar{U}_{xx}(x, i, t)(g(x, i, t) + G(\hbar(\phi), i, t))^\top] + \sum_{j=1}^{N} \gamma_{ij} \bar{U}(x, j, t) \\ &\leq c_0 - c_1 H(x) + c_2 H(\hbar(\phi)) \end{aligned} \tag{2.6}$$

for all $x \in \mathbb{R}^d, (i, t) \in \mathbb{S} \times \mathbb{R}_+, \phi \in C([-\tau, 0]; \mathbb{R}^d)$.

In what follows, similar to the discussion in [13], we have the following theorem which shows the existence and uniqueness, the stability and boundedness of the global solution to highly nonlinear hybrid SIDDEs. In order to complete the proof of the theorem, we provide the existence and uniqueness of the maximal solution to SIDDE (2.1) delegated to Appendix A.

Theorem 2.4. *Under Assumptions 2.1 and 2.3, the SIDDE (2.1) with initial data (2.2) has the following assertions:*

(i) *There is a unique global solution $X(t)$ to the SIDDE (2.1) on $t \in [-\tau, \infty)$.*

(ii) *The solution X(t) obeys*

$$\limsup_{t\to\infty} \mathbb{E}|X(t)|^q \leq \frac{c_0}{\varepsilon} \tag{2.7}$$

and

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{E}H(X(s))ds \leq \frac{c_0}{c_1 - c_2}, \tag{2.8}$$

where $\varepsilon > 0$ is the unique root to the equation

$$c_1 = \varepsilon + c_2 e^{\varepsilon\tau}. \tag{2.9}$$

(iii) *If, in addition, $c_0 = 0$, then the solution has the moment properties that*

$$\limsup_{t\to\infty} \frac{1}{t} \log(\mathbb{E}|X(t)|^q) \leq -\varepsilon \tag{2.10}$$

and

$$\int_0^\infty \mathbb{E}H(X(t))dt \le \frac{1}{c_1 - c_2}\Big(H(X(0)) + \int_{-\tau}^0 H(\eta(s))ds\Big); \tag{2.11}$$

while it also has the sample (pathwise) properties that

$$\limsup_{t\to\infty} \frac{1}{t}\log(|X(t)|) \le -\frac{\varepsilon}{q} \quad \text{a.s.} \tag{2.12}$$

and

$$\int_0^\infty H(X(t))dt < \infty \quad \text{a.s.} \tag{2.13}$$

Proof: The whole proof is divided into three steps for three assertions.

Step 1. From Lemma A.3, we know that for the hybrid SIDDE (2.1) with the coefficients being locally Lipschitz continuous and any given initial data (2.2), there is a unique maximal local solution $X(t)$ for $\forall t \in [-\tau, \sigma_\infty)$, where σ_∞ is the explosion time. Let $m_0 > 0$ be sufficiently large for $m_0 \ge \|\eta\|$. For each integer $m \ge m_0$, define the stopping time $\tau_m = \inf\{t \in [0, \sigma_\infty) : |X(t)| \ge m\}$, where, throughout this paper, $\inf\emptyset = \infty$. Clearly, τ_m is increasing as $m \to \infty$. Set $\tau_\infty = \lim_{m\to\infty} \tau_m$, whence $\tau_\infty \le \sigma_\infty$ a.s. If we can show that $\tau_\infty = \infty$ a.s., then $\sigma_\infty = \infty$ a.s. and claim (i) follows. Next, we will show that $\tau_\infty = \infty$ a.s. By the generalized Itô formula (see, e.g., [29], Lemma 1.9 on p. 49) and condition (2.6), we can show that, for any $m \ge m_0$ and $t \ge 0$,

$$\begin{aligned}&\mathbb{E}\bar{U}(X(\tau_m \wedge t), r(\tau_m \wedge t), \tau_m \wedge t) - \bar{U}(X(0), r(0), 0)\\ &\le \mathbb{E}\int_0^{\tau_m\wedge t}\Big(c_0 - c_1H(X(s)) + c_2H(\hbar(X_s))\Big)ds,\end{aligned}$$

from which, together with condition (2.5), we have

$$\begin{aligned}\mathbb{E}|X(\tau_m \wedge t)|^q \le H(X(0)) + c_0t - c_1\mathbb{E}\int_0^{\tau_m\wedge t} H(X(s))ds\\ + c_2\mathbb{E}\int_0^{\tau_m\wedge t} H(\hbar(X_s))ds.\end{aligned}$$

By Lemma 2.2, we derive

$$\mathbb{E}\int_0^{\tau_m\wedge t} H(\hbar(X_s))ds \le \int_{-\tau}^0 H(\eta(s))ds + \mathbb{E}\int_0^{\tau_m\wedge t} H(X(s))ds,$$

which shows

$$\mathbb{E}|X(\tau_m \wedge t))|^q \le K_1 + c_0t - (c_1 - c_2)\mathbb{E}\int_0^{\tau_m\wedge t} H(X(s))ds, \tag{2.14}$$

where $K_1 = H(X(0)) + c_2\int_{-\tau}^0 H(\eta(s))ds$. Noting that $c_1 > c_2$, we get

$$\mathbb{E}|X(\tau_m \wedge t)|^q \le K_1 + c_0t.$$

Then we have $m^q\mathbb{P}(\tau_m \le t) \le K_1 + c_0 t$. Therefore, letting $m \to \infty$ in the inequality above, we have $\mathbb{P}(\tau_\infty \le t) = 0$, which shows $\mathbb{P}(\tau_\infty > t) = 1$. Due to arbitrariness of $t \ge 0$, we must have $\mathbb{P}(\tau_\infty = \infty) = 1$ as required.

Step 2. By the generalized Itô formula, we obtain that for $t \ge 0$,

$$\begin{aligned}
&\mathbb{E}\Big(e^{\varepsilon(t\wedge\tau_m)}H(X(\tau_m \wedge t))\Big) - H(X(0)) \\
&\le \mathbb{E}\int_0^{\tau_m\wedge t} e^{\varepsilon s}\Big(c_0 - (c_1-\varepsilon)H(X(s)) + c_2 H(\hbar(X_s))\Big)ds \\
&\le \frac{c_0}{\varepsilon}e^{\varepsilon t} - (c_1-\varepsilon)\mathbb{E}\int_0^{t\wedge\tau_m} e^{\varepsilon s}H(X(s))ds \\
&\quad + c_2\mathbb{E}\int_0^{t\wedge\tau_m} e^{\varepsilon s}H(\hbar(X_s))ds.
\end{aligned} \tag{2.15}$$

However, by Lemma 2.2, we derive

$$\begin{aligned}
&\mathbb{E}\int_0^{\tau_m\wedge t} e^{\varepsilon s}H(\hbar(X_s))ds \\
&\le e^{\varepsilon\tau}\Big(\int_{-\tau}^0 e^{\varepsilon s}H(\eta(s))ds + \mathbb{E}\int_0^{\tau_m\wedge t} e^{\varepsilon s}H(X(s))ds\Big),
\end{aligned}$$

which, together with (2.9) and (2.15), shows that

$$\mathbb{E}\Big(e^{\varepsilon(t\wedge\tau_m)}|X(\tau_m \wedge t)|^q\Big) \le K_2 + \frac{c_0}{\varepsilon}e^{\varepsilon t},$$

where $K_2 = H(X(0)) + c_2 e^{\varepsilon\tau}\int_{-\tau}^0 e^{\varepsilon s}H(\eta(s))ds$. Letting $m \to \infty$ we get that

$$\mathbb{E}\Big(e^{\varepsilon t}|X(t)|^q\Big) \le K_2 + \frac{c_0}{\varepsilon}e^{\varepsilon t}. \tag{2.16}$$

Thus, the desired claim (2.7) holds.

In order to show (2.8), from (2.14) we know that

$$(c_1 - c_2)\mathbb{E}\int_0^{\tau_m\wedge t} H(X(s))ds \le K_1 + c_0 t.$$

Thus, letting $m \to \infty$ and using the Fubini theorem, we have

$$(c_1 - c_2)\mathbb{E}\int_0^{t} H(X(s))ds \le K_1 + c_0 t. \tag{2.17}$$

Dividing both sides in (2.17) by t and letting $t \to \infty$ we get the claim (2.8).

Step 3. Now we consider the case when $c_0 = 0$. From the calculations in the previous steps with $c_0 \ge 0$, we know that they hold also for $c_0 = 0$. It then follows from (2.16) that

$$\mathbb{E}|X(t)|^q \le K_2 e^{-\varepsilon t},$$

which means the required claim (2.10). Moreover, from (2.17), we get

$$(c_1 - c_2)\mathbb{E}\int_0^{t} H(X(s))ds \le K_1,$$

which easily shows (2.11), which implies (2.13).

Finally, by the generalized Itô formula, we get that for any $t \geq 0$,

$$\begin{aligned} &e^{\varepsilon t}\bar{U}(X(t), r(t), t) - \bar{U}(X(0), r(0), 0) \\ &= \int_0^t e^{\varepsilon s}[\varepsilon \bar{U}(X(s), r(s), s) + \mathbb{L}\bar{U}(X(s), X_s, r(s), s)]ds + M(t), \end{aligned}$$

where $M(t)$ is a local martingale with the initial value $M(0) = 0$. Due to Assumption 2.3 with $c_0 = 0$, we easily get in the same way as in Step 2 that

$$e^{\varepsilon t}|X(t)|^q \leq K_2 + M(t).$$

Using the non-negative semi-martingale convergence theorem (see, e.g., [22], Theorem 1.45 on p. 48), we get that

$$\limsup_{t\to\infty} \left[e^{\varepsilon t}|X(t)|^q\right] < \infty \quad \text{a.s.}$$

Thus, there exists a finite positive random variable ζ such that

$$\sup_{0\leq t<\infty} \left[e^{\varepsilon t}|X(t)|^q\right] < \zeta \quad \text{a.s.},$$

which implies

$$\limsup_{t\to\infty} \frac{1}{t} \log |X(t)|^q \leq -\varepsilon \quad \text{a.s.}$$

Thus, the claim (2.12) holds. Hence we complete the proof. □

3. Delay-Dependent Asymptotic Stability of SIDDEs

In this section, we will use the method of Lyapunov functionals to investigate the delay-dependent asymptotic stability. We define two segments $\bar{X}_t := \{X(t+s) : -2\tau \leq s \leq 0\}$ and $\bar{r}_t := \{r(t+s) : -2\tau \leq s \leq 0\}$ for $t \geq 0$. For $\bar{X}_t$ and $\bar{r}_t$ to be well defined for $0 \leq t < 2\tau$, we set $X(s) = \eta(-\tau)$ for $s \in [-2\tau, -\tau)$ and $r(s) = r_0$ for $s \in [-2\tau, 0)$. We construct the Lyapunov functional as follows

$$\begin{aligned} V(\bar{X}_t, \bar{r}_t, t) =& U(X(t), r(t), t) \\ &+ \theta_1 \int_{-\tau}^0 \int_{t+s}^t \Big[\tau(1+\theta_2)|f(X(v), r(v), v) + F(\hbar(X_v), r(v), v)|^2 \\ &+ (1 + \frac{1}{\theta_2})|g(X(v), r(v), v) + G(\hbar(X_v), r(v), v)|^2\Big]dvds \end{aligned}$$

for $t \geq 0$, where $U \in C^{2,1}(\mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+; \mathbb{R}_+)$ such that

$$\lim_{|x|\to\infty} [\inf_{(t,i)\in\mathbb{R}_+\times\mathbb{S}} U(x, t, i)] = \infty,$$

and θ_1, θ_2 are positive numbers to be determined later while we set

$$f(x,i,s) = f(x,i,0), \quad g(x,i,s) = g(x,i,0)$$
$$F(\hbar(\phi), i, s) = F(\hbar(\phi), i, 0), \quad G(\hbar(\phi), i, s) = G(\hbar(\phi), i, 0)$$

for all $x \in \mathbb{R}^d, (i,s) \in \mathbb{S} \times [-2\tau, 0), \phi \in C([-\tau, 0], \mathbb{R}^d)$. Applying the generalized Itô formula to $U(X(t), r(t), t)$, we get

$$\begin{aligned}
dU(X(t), r(t), t) = \Big(&U_t(X(t), r(t), t) \\
&+ U_x(X(t), r(t), t)(f(X(t), r(t), t) + F(\hbar(X_t), r(t), t)) \\
&+ \frac{1}{2}\text{trace}[(g(X(t), r(t), t) + G(\hbar(X_t), r(t), t))^\top \\
&\qquad \times U_{xx}(X(t), r(t), t)(g(X(t), r(t), t) + G(\hbar(X_t), r(t), t))] \\
&+ \sum_{j=1}^N \gamma_{r(t),j} U(X(t), j, t)\Big)dt + dM(t),
\end{aligned}$$

for $t \geq 0$, where $M(t)$ is a continuous local martingale with $M(0) = 0$. Denote an indicator function $\mathbb{1}_{[t-\tau,t]}(s) = 1, s \in [t-\tau, t]$, or 0. And define the functional $x\mathbb{1}_{[t-\tau,t]} := \{\phi(s); \phi(s) = x\mathbb{1}_{[t-\tau,t]}(s) = x, s \in [t-\tau, t], \text{or } 0\}$ for $x \in \mathbb{R}^d$. Thus, rearranging terms give

$$\begin{aligned}
&dU(X(t), r(t), t) \\
&= \Big(U_x(X(t), r(t), t)[F(\hbar(X_t), r(t), t) - F(\hbar(X(t)\mathbb{1}_{[t-\tau,t]}), r(t), t)] \\
&\quad + \mathcal{L}U(X(t), X_t, r(t), t)\Big) + dM(t),
\end{aligned}$$

where the function $\mathcal{L}U : \mathbb{R}^d \times C([-\tau, 0]; \mathbb{R}^d) \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$\begin{aligned}
\mathcal{L}U(x, \phi, i, t) &= U_t(x, i, t) + U_x(x, i, t)(f(x, i, t) + F(\hbar(x\mathbb{1}_{[t-\tau,t]}), i, t)) \\
&+ \frac{1}{2}\text{trace}[(g(x, i, t) + G(\hbar(\phi), i, t))^\top U_{xx}(x, i, t)(g(x, i, t) + G(\hbar(\phi), i, t))] + \sum_{j=1}^N \gamma_{ij} U(x, j, t).
\end{aligned} \tag{3.1}$$

Moreover, the fundamental theory of calculus shows

$$\begin{aligned}
&d\Big(\int_{-\tau}^0 \int_{t+s}^t \Big[\tau(1+\theta_2)|f(X(v), r(v), v) + F(\hbar(X_v), r(v), v)|^2 \\
&\quad + (1 + \frac{1}{\theta_2})|g(X(v), r(v), v) + G(\hbar(X_v), r(v), v)|^2\Big]dvds\Big) \\
&= \Big(\tau\Big[\tau(1+\theta_2)|f(X(v), r(v), v) + F(\hbar(X_v), r(v), v)|^2 \\
&\quad + (1 + \frac{1}{\theta_2})|g(X(v), r(v), v) + G(\hbar(X_v), r(v), v)|^2\Big] \\
&\quad - \int_{t-\tau}^t \Big[\tau(1+\theta_2)|f(X(v), r(v), v) + F(\hbar(X_v), r(v), v)|^2 \\
&\quad + (1 + \frac{1}{\theta_2})|g(X(v), r(v), v) + G(\hbar(X_v), r(v), v)|^2\Big]dv\Big)dt.
\end{aligned}$$

Lemma 3.1. *With the notation above, $V(\bar{x}_t, \bar{r}_t, t)$ is an Itô process on $t \geq 0$ with its Itô differential*

$$dV(\bar{x}_t, \bar{r}_t, t) = LV(\bar{x}_t, \bar{r}_t, t)dt + dM(t),$$

where $M(t)$ is a continuous local martingale with $M(0) = 0$ and

$$\begin{aligned} LV(\bar{X}_t, \bar{r}_t, t) &= U_x(X(t), r(t), t)[F(\hbar(X_t), r(t), t) \\ &- F(\hbar(X(t)\mathbb{1}_{[t-\tau,t]}), r(t), t)] \\ &+ \mathcal{L}U(X(t), X_t, r(t), t) \\ &+ \theta_1 \tau \Big[\tau(1+\theta_2)|f(X(t), r(t), t) + F(\hbar(X_t), r(t), t)|^2 \\ &+ (1+\frac{1}{\theta_2})|g(X(t), r(t), t) + G(\hbar(X_t), r(t), t)|^2\Big] \\ &- \theta_1 \int_{t-\tau}^{t} \Big[\tau(1+\theta_2)|f(X(v), r(v), v) + F(\hbar(X_v), r(v), v)|^2 \\ &+ (1+\frac{1}{\theta_2})|g(X(v), r(v), v) + G(\hbar(X_v), r(v), v)|^2\Big]dv. \end{aligned}$$

We here note that the expression for the martingale $M(t)$ in Lemma 3.1 is of no further use for our analysis, so it is not necessary to give the detailed expression. In what follows, to study the delay-dependent asymptotic stability of the SIDDE (2.1), we need to impose several new assumptions.

Assumption 3.2. *Assume that there are functions $U \in C^{2,1}(\mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+; \mathbb{R}_+)$, the nonnegative coefficient polynomial $U_1 \in C(\mathbb{R}^d; \mathbb{R}_+)$ (see Lemma 2.2), and positive numbers α_1, α_2 and β_k ($k = 1, 2, 3$) such that*

$$\alpha_2 < \alpha_1 \tag{3.2}$$

and

$$\begin{aligned} &\mathcal{L}U(x, \phi, i, t) + \beta_1 |U_x(x, i, t)|^2 \\ &\quad + \beta_2 |f(x, i, t) + F(\hbar(\phi), i, t)|^2 + \beta_3 |g(x, i, t) + G(\hbar(\phi), i, t)|^2 \\ &\leq -\alpha_1 U_1(x) + \alpha_2 U_1(\hbar(\phi)), \end{aligned} \tag{3.3}$$

for all $x \in \mathbb{R}^d, (i, t) \in \mathbb{S} \times \mathbb{R}_+, \phi \in C([-\tau, 0], \mathbb{R}^d)$.

Assumption 3.3. *Assume that there exists a positive number ϖ such that*

$$|F(\hbar(x\mathbb{1}_{[t-\tau,t]}), i, t) - F(\hbar(\phi), i, t)| \leq \varpi |x - \hbar(\phi)|$$

for all $x \in \mathbb{R}^d, (i, t) \in \mathbb{S} \times [-2\tau, \infty), \phi \in C([-\tau, 0], \mathbb{R}^d)$.

Theorem 3.4. *Let Assumptions 2.1, 2.3, 3.2 and 3.3 hold. Assume also that*

$$\tau \leq \sup_{\theta_2 > 0} \Big\{ \Big(\frac{4\beta_1\beta_2}{\varpi^2(1+\theta_2)} \Big)^{1/2} \bigwedge \Big(\frac{4\beta_1\beta_3}{\varpi^2(1+1/\theta_2)} \Big) \Big\}. \tag{3.4}$$

Then for any given initial data (2.2), the solution of the SIDDE (2.1) has the properties that

$$\int_0^\infty \mathbb{E} U_1(X(t)) dt < \infty \tag{3.5}$$

and

$$\sup_{0\le t<\infty} \mathbb{E}U(X(t), r(t), t) < \infty.$$

Proof: Fix the initial data $\eta \in C([-\tau, 0]; \mathbb{R}^d)$ and $r_0 \in \mathbb{S}$ arbitrarily. Let $k_0 > 0$ be a sufficiently large integer such that $\|\eta\| := \sup_{-\tau\le s\le 0} |\eta(s)| < k_0$. For each integer $k > k_0$, define the stopping time

$$\sigma_k = \inf\{t \ge 0 : |X(t)| \ge k\}.$$

It is easy to see that σ_k is increasing as $k \to \infty$ and $\lim_{k\to\infty} \sigma_k = \infty$ a.s. By the generalized Itô formula we obtain from Lemma 3.1 that

$$\mathbb{E}V(\bar{X}_{t\wedge\sigma_k}, \bar{r}_{t\wedge\sigma_k}, t \wedge \sigma_k) = V(\bar{X}_0, \bar{r}_0, 0) + \mathbb{E}\int_0^{t\wedge\sigma_k} LV(\bar{X}_s, \bar{r}_s, s)ds \tag{3.6}$$

for any $t \ge 0$ and $k \ge k_0$. Let $\theta_1 = \varpi^2/(4\beta_1)$. By Assumption 3.3 and Cauchy-Schwartz inequality, it is easy to see that

$$\begin{aligned} &U_x(X(t), r(t), t)[F(\hbar(X(t)\mathbb{1}_{[t-\tau,t]}), r(t), t) - F(\hbar(X_t), r(t), t)] \\ &\le \beta_1|U_x(X(t), r(t), t)|^2 + \frac{\varpi^2}{4\beta_1}\Big|X(t) - \frac{1}{\tau}\int_{-\tau}^0 X(t+s)ds\Big|^2 \\ &\le \beta_1|U_x(X(t), r(t), t)|^2 + \frac{\varpi^2}{4\beta_1\tau}\int_{-\tau}^0 |X(t) - X(t+s)|^2 ds. \end{aligned} \tag{3.7}$$

By condition (3.4), we also have

$$\theta_1\tau^2(1+\theta_2) \le \beta_2 \quad \text{and} \quad \theta_1\tau(1+\frac{1}{\theta_2}) \le \beta_3.$$

It then follows from Lemma 3.1 that

$$\begin{aligned} LV(\bar{X}_s, \bar{r}_s, s) \le &\mathcal{L}U(X(s), X(s), r(s), s) + \beta_1|U_x(X(s), r(s), s)|^2 \\ &+ \beta_2|f(X(s), r(s), s) + F(\hbar(X_s), r(s), s)|^2 \\ &+ \beta_3|g(X(s), r(s), s) + G(\hbar(X_s), r(s), s)|^2 \\ &+ \frac{\varpi^2}{4\beta_1\tau}\int_{-\tau}^0 |X(s) - X(s+v)|^2 dv \\ &- \frac{\varpi^2}{4\beta_1}\int_{s-\tau}^s \Big[\tau(1+\theta_2)|f(X(v), r(v), v) + F(\hbar(X_v), r(v), v)|^2 \\ &+ (1+\frac{1}{\theta_2})|g(X(v), r(v), v) + G(\hbar(X_v), r(v), v)|^2\Big]dv. \end{aligned}$$

By Assumption 3.2, we then have

$$\begin{aligned} LV(\bar{X}_s, \bar{r}_s, s) &\le -\alpha_1 U_1(X(s)) + \alpha_2 U_1(\hbar(X_s)) \\ &+ \frac{\varpi^2}{4\beta_1\tau}\int_{-\tau}^0 |X(s) - X(s+v)|^2 dv \\ &- \frac{\varpi^2}{4\beta_1}\int_{s-\tau}^s \Big[\tau(1+\theta_2)|f(X(v), r(v), v) + F(\bar{(}X_v), r(v), v)|^2 \\ &+ (1+\frac{1}{\theta_2})|g(X(v), r(v), v) + G(\hbar(X_v), r(v), v)|^2\Big]dv. \end{aligned}$$

Substituting this into (3.6) implies

$$\mathbb{E}V(\bar{X}_{t\wedge\sigma_k}, \bar{r}_{t\wedge\sigma_k}, t\wedge\sigma_k) \le V(\bar{X}_0, \bar{r}_0, 0) + I_1 + I_2 - I_3, \tag{3.8}$$

where

$$\begin{aligned}
I_1 &= \mathbb{E}\int_0^{t\wedge\sigma_k} [-\alpha_1 U_1(X(s)) + \alpha_2 U_1(\hbar(X_s))]ds,\\
I_2 &= \frac{\varpi^2}{4\beta_1\tau}\mathbb{E}\int_0^{t\wedge\sigma_k}\int_{-\tau}^0 |X(s) - X(s+v)|^2 dvds,\\
I_3 &= \frac{\varpi^2}{4\beta_1}\mathbb{E}\int_0^{t\wedge\sigma_k}\int_{s-\tau}^s \Big[\tau(1+\theta_2)|f(X(v), r(v), v) + F(\hbar(X_v), r(v), v)|^2\\
&\quad + (1+\frac{1}{\theta_2})|g(X(v), r(v), v) + G(\hbar(X_v), r(v), v)|^2\Big]dvds.
\end{aligned}$$

By Lemma 2.2 (ii), we get that

$$I_1 \le \alpha_2\int_{-\tau}^0 U_1(\eta(v))dv - \bar{\alpha}\mathbb{E}\int_0^{t\wedge\sigma_k} U_1(X(s))ds,$$

where $\bar{\alpha} = \alpha_1 - \alpha_2 > 0$ by Assumption 3.2. Substituting this into (3.8) yields

$$\bar{\alpha}\mathbb{E}\int_0^{t\wedge\sigma_k} U_1(X(s))ds \le C_1 + I_2 - I_3, \tag{3.9}$$

where C_1 is a constant defined by

$$C_1 = V(\bar{X}(0), \bar{r}_0, 0) + \alpha_2\int_{-\tau}^0 U_1(\eta(s))ds.$$

Applying the classical Fatou lemma and let $k \to \infty$ in (3.9) to obtain

$$\bar{\alpha}\mathbb{E}\int_0^t U_1(X(s))ds \le C_1 + \bar{I}_2 - \bar{I}_3, \tag{3.10}$$

where

$$\begin{aligned}
\bar{I}_2 =& \frac{\varpi^2}{4\beta_1\tau}\mathbb{E}\int_0^t\int_{-\tau}^0 |X(s) - X(s+v)|^2 dvds,\\
\bar{I}_3 =& \frac{\varpi^2}{4\beta_1}\mathbb{E}\int_0^t\int_{s-\tau}^s \Big[\tau(1+\theta_2)|f(X(v), r(v), v) + F(\hbar(X_v), r(v), v)|^2\\
&+ (1+\frac{1}{\theta_2})|g(X(v), r(v), v) + G(\hbar(X_v), r(v), v)|^2\Big]dvds.
\end{aligned} \tag{3.11}$$

By the well-known Fubini theorem, we have

$$\bar{I}_2 = \frac{\varpi^2}{4\beta_1\tau}\int_0^t \mathbb{E}\int_{-\tau}^0 |X(s) - X(s+v)|^2 dvds.$$

For $t \in [0, \tau]$, we have

$$\begin{aligned}\bar{I}_2 &\leq \frac{\varpi^2}{2\beta_1\tau}\int_0^\tau\int_{-\tau}^0(\mathbb{E}|X(s)|^2 + \mathbb{E}|X(s+v)|^2)dvds\\ &\leq \frac{\tau\varpi^2}{\beta_1}\Big(\sup_{-\tau\leq v\leq\tau}\mathbb{E}|X(v)|^2\Big).\end{aligned}$$

For $t > \tau$, we have

$$\bar{I}_2 \leq \frac{\tau\varpi^2}{\beta_1}\Big(\sup_{-\tau\leq v\leq\tau}\mathbb{E}|X(v)|^2\Big) + \frac{\varpi^2}{4\beta_1\tau}\int_\tau^t\mathbb{E}\int_{-\tau}^0|X(s) - X(s+v)|^2dvds. \tag{3.12}$$

Noting that, for $v \in [-\tau, 0]$,

$$\begin{aligned}&|X(s) - X(s+v)|\\ &= \Big|\int_{s+v}^s(f(X(u), r(u), u) + F(\hbar(X_u), r(u), u))du\\ &\quad + \int_{s+v}^s(g(X(u), r(u), u) + G(\hbar(X_u), r(u), u))dB(u)\Big|.\end{aligned}$$

By using the inequality $(a+b)^2 \leq (1+\theta_2)a^2 + (1+\frac{1}{\theta_2})b^2$ for the parameter choice, we get

$$\begin{aligned}&\mathbb{E}|X(s) - X(s+v)|^2\\ &\leq \mathbb{E}\int_{s+v}^s[\tau(1+\theta_2)|f(X(u), r(u), u) + F(\hbar(X_u), r(u), u)|^2\\ &\quad + (1+\frac{1}{\theta_2})|g(X(u), r(u), u) + G(\hbar(X_u), r(u), u)|^2]du.\end{aligned}$$

Thus we get

$$\begin{aligned}&\mathbb{E}\int_{-\tau}^0|X(s) - X(s+v)|^2dv\\ &\leq \mathbb{E}\int_{-\tau}^0\int_{s+v}^s[\tau(1+\theta_2)|f(X(u), r(u), u) + F(\hbar(X_u), r(u), u)|^2\\ &\quad + (1+\frac{1}{\theta_2})|g(X(u), r(u), u) + G(\hbar(X_u), r(u), u)|^2]dudv\\ &\leq \mathbb{E}\int_{-\tau}^0\int_{s-\tau}^s[\tau(1+\theta_2)|f(X(u), r(u), u) + F(\hbar(X_u), r(u), u)|^2\\ &\quad + (1+\frac{1}{\theta_2})|g(X(u), r(u), u) + G(\hbar(X_u), r(u), u)|^2]dudv\\ &\leq \tau\mathbb{E}\int_{s-\tau}^s[\tau(1+\theta_2)|f(X(u), r(u), u) + F(\hbar(X_u), r(u), u)|^2\\ &\quad + (1+\frac{1}{\theta_2})|g(X(u), r(u), u) + G(\hbar(X_u), r(u), u)|^2]du.\end{aligned}$$

Notice also that

$$
\begin{aligned}
&\frac{1}{\tau}\int_{\tau}^{t}\mathbb{E}\int_{-\tau}^{0}|X(s)-X(s+v)|^2 dvds\\
&\le \mathbb{E}\int_{\tau}^{t}\int_{s-\tau}^{s}[\tau(1+\theta_2)|f(X(u),r(u),u)+F(\hbar(X_u),r(u),u)|^2\\
&\quad+(1+\frac{1}{\theta_2})|g(X(u),r(u),u)+G(\hbar(X_u),r(u),u)|^2]duds.
\end{aligned}
$$

Thus from (3.11) and (3.12) we get

$$
\bar{I}_2 \le \frac{\tau\varpi^2}{\beta_1}\Big(\sup_{-\tau\le v\le\tau}\mathbb{E}|X(v)|^2\Big)+\bar{I}_3. \tag{3.13}
$$

Together with (3.4), substituting (3.13) into (3.10) yields

$$
\bar{\alpha}\mathbb{E}\int_0^t U_1(X(s),s)ds \le C_1 + 4\beta_3\sup_{-\tau\le v\le\tau}\mathbb{E}|X(v)|^2 := C_2.
$$

Letting $t\to\infty$ gives

$$
\mathbb{E}\int_0^{\infty}U_1(X(s))ds \le \frac{C_2}{\bar{\alpha}}.
$$

Similarly, we see from (3.8) that

$$
\mathbb{E}U\Big(X(t\wedge\sigma_k),r(t\wedge\sigma_k),t\wedge\sigma_k\Big)\le C_1+I_2-I_3.
$$

Letting $k\to\infty$ we get

$$
\mathbb{E}U(X(t),r(t),t)\le C_2<\infty,
$$

which shows

$$
\sup_{0\le t<\infty}\mathbb{E}U(X(t),r(t),t)<\infty.
$$

Thus the proof is complete. □

Corollary 3.5. *Let the conditions of Theorem 3.4 hold. If there moreover exists a pair of positive constants c and p such that*

$$
c|x|^p\le U_1(x),\quad \forall x\in\mathbb{R}^d,
$$

then for any given initial data (2.2), the solution of the SIDDE (2.1) satisfies

$$
\int_0^{\infty}\mathbb{E}|X(t)|^p dt<\infty. \tag{3.14}
$$

That is, the SIDDE (2.1) is H_∞-stable in L^p.

This corollary follows from Theorem 3.4 obviously. However, it does not follow from (3.14) that $\lim_{t\to\infty}\mathbb{E}|X(t)|^p=0$.

Theorem 3.6. *Let the conditions of Corollary 3.5 hold. If, moreover,*

$$p \geq 2 \quad \textit{and} \quad (p + q_1 - 1) \vee (p + 2q_2 - 2) \leq q,$$

then the solution of the SIDDE (2.1) satisfies

$$\lim_{t\to\infty} \mathbb{E}|X(t)|^p = 0$$

115 *for any initial data (2.2). That is, the SIDDE (2.1) is asymptotically stable in L^p.*

Proof: Fix the initial data (2.2) arbitrarily. For any $0 \leq t_1 < t_2 < \infty$, by the generalized Itô formula, we get

$$\begin{aligned}
&\mathbb{E}|X(t_2)|^p - \mathbb{E}|X(t_1)|^p \\
=&\mathbb{E}\int_{t_1}^{t_2} \Big(p|X(t)|^{p-2}X(t)^\top(f(X(t), r(t), t) + F(\hbar(X_t), r(t), t)) \\
&+ \frac{p}{2}|X(t)|^{p-2}|g(X(t), r(t), t) + G(\hbar(X_t), r(t), t)|^2 \\
&+ \frac{p(p-2)}{2}|X(t)|^{p-4}|(X(t)^\top(g(X(t), r(t), t) + G(\hbar(X_t), r(t), t))|^2\Big)dt,
\end{aligned}$$

which, due to Assumption 2.1, implies

$$\begin{aligned}
&\Big|\mathbb{E}|X(t_2)|^p - \mathbb{E}|X(t_1)|^p\Big| \\
\leq& \mathbb{E}\int_{t_1}^{t_2} \Big(p|X(t)|^{p-1}|f(X(t), r(t), t) + F(\hbar(X_t), r(t), t)| \\
&\quad + \frac{p(p-1)}{2}|X(t)|^{p-2}|g(X(t), r(t), t) + G(\hbar(X_t), r(t), t)|^2\Big)dt \\
\leq& \mathbb{E}\int_{t_1}^{t_2} \Big(pK|X(t)|^{p-1}[1 + |X(t)|^{q_1} + |\hbar(X_t)|^{q_1}] \\
&\quad + 2p(p-1)K^2|X(t)|^{p-2}[1 + |X(t)|^{2q_2} + |\hbar(X_t)|^{2q_2}]\Big)dt.
\end{aligned}$$

By the inequalities,

$$\begin{aligned}
|X(t)|^{p-1}\|\hbar(X_t)\|^{q_1} &\leq |X(t)|^{p+q_1-1} + \|\hbar(X_t)\|^{p+q_1-1}, \\
|X(t)|^{p-1} &\leq 1 + |X(t)|^q,
\end{aligned}$$

etc., and noting that for any $1 \leq \bar{p} \leq q$, by the Hölder inequality and Theorem 2.4 we get

$$\mathbb{E}|\hbar(X_t)|^{\bar{p}} \leq \sup_{-\tau\leq s\leq 0} \mathbb{E}|X(t+s)|^{\bar{p}} \leq (1 + \sup_{-\tau\leq s<\infty} \mathbb{E}|X(s)|^q) < \infty,$$

we can obtain

$$\Big|\mathbb{E}|X(t_2)|^p - \mathbb{E}|X(t_1)|^p\Big| \leq C_3(t_2 - t_1),$$

where

$$C_3 = 4pK(1 + 2(p-1)K)(1 + \sup_{-\tau\leq t<\infty} \mathbb{E}|X(t)|^q) < \infty.$$

Thus we have $\mathbb{E}|X(t)|^p$ is uniformly continuous in t on $\mathbb{R}_+$. By (3.14), there is a sequence $\{t_l\}_{l=1}^\infty$ in $\mathbb{R}_+$ such that $\mathbb{E}|X(t_l)|^p \to 0$, which easily show the claim. Hence the proof is complete. □

Proposition 3.7. *Let the conditions of Theorem 3.4 hold. Assume that there are positive constants p and c such that*

$$c|x|^p \le U(x,i,t), \quad \forall (x,i,t) \in \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+. \tag{3.15}$$

Moreover assume there exists a function $W : \mathbb{R}^d \to \mathbb{R}_+$ *such that*

$$W(x) = 0 \text{ if and only if } x = 0$$

and

$$W(x) \le U_1(x), \forall x \in \mathbb{R}^d.$$

Then for any given initial data (2.2), the solution $X(\cdot)$ *to Eq. (2.1) obeys that*

$$\lim_{t\to\infty} X(t) = 0 \quad \text{a.s.}$$

Proof: Let $X(\cdot)$ be the solution to Eq. (2.1) with initial data η defined in (2.2). Since the conditions in Theorem 3.4 hold, by Fubini Theorem, we can show that

$$C_4 := \int_0^\infty \mathbb{E}W(X(t))dt < \infty,$$

which implies

$$\int_0^\infty W(X(t))dt < \infty \quad \text{a.s.} \tag{3.16}$$

Set $\sigma_k := \inf\{t \ge 0 : |X(t)| = k\}$. We observe from (3.16) that

$$\liminf_{t\to\infty} W(X(t)) = 0 \quad \text{a.s.} \tag{3.17}$$

Moreover, in the same way as Theorem 3.4 was proved, we can show that

$$\mathbb{E}|X(T \wedge \sigma_k)|^p \le C, \quad \forall T > 0,$$

which implies

$$k^p \mathbb{P}(\sigma_k \le T) \le C.$$

Letting $T \to \infty$ yields

$$k^p \mathbb{P}(\sigma_k < \infty) \le C. \tag{3.18}$$

We now claim that

$$\lim_{t\to\infty} W(X(t)) = 0 \quad \text{a.s.} \tag{3.19}$$

In fact, if this is false, then we can find a number $\varepsilon \in (0, 1/4)$ such that

$$\mathbb{P}(\Omega_1) \ge 4\varepsilon, \tag{3.20}$$

where $\Omega_1 = \{\limsup_{t\to\infty} W(X(t)) > 2\varepsilon\}$. Recalling (3.18), we can find an integer m sufficiently large for $\mathbb{P}(\sigma_m < \infty) \le \varepsilon$. This means that

$$P(\Omega_2) \ge 1 - \varepsilon. \tag{3.21}$$

where $\Omega_2 := \{|X(t)| < m \text{ for } \forall t \geq -\tau\}$. By (3.20) and (3.21) we get

$$\mathbb{P}(\Omega_1 \cap \Omega_2) \geq \mathbb{P}(\Omega_1) - \mathbb{P}(\Omega_2^c) \geq 3\varepsilon, \tag{3.22}$$

where Ω_2^c is the complement of Ω_2. Let us now define the stopped process $\zeta(t) = X(t \wedge \sigma_m)$ for $t \geq -\tau$. Clearly, $\zeta(t)$ is a bounded Itô process with its differential

$$d\zeta(t) = \varphi(t)dt + \psi(t)dB(t),$$

where

$$\varphi(t) = (f(X(t), r(t), t) + F(\hbar(X_t), r(t), t)I_{[0,\sigma_m)}(t),$$
$$\psi(t) = (g(X(t), r(t), t) + G(\hbar(X_t), r(t), t)I_{[0,\sigma_m)}(t),$$

where f, g, F, G are defined by (2.1). Recalling the polynomial growth condition (2.3), we know that $\varphi(t)$ and $\psi(t)$ are bounded processes, say

$$|\varphi(t)| \vee |\psi(t)| \leq C_5 \quad \text{a.s.} \tag{3.23}$$

for all $t \geq 0$ and some $C_5 > 0$. Moreover, we also observe that $|\zeta(t)| \leq m$ for all $t \geq -\tau$. Define a sequence of stopping times

$$\rho_1 = \inf\{t \geq 0 : W(\zeta(t)) \geq 2\varepsilon\},$$
$$\rho_{2j} = \inf\{t \geq \rho_{2j-1} : W(\zeta(t)) \leq \varepsilon\}, \quad j = 1, 2, \cdots,$$
$$\rho_{2j+1} = \inf\{t \geq \rho_{2j} : W(\zeta(t)) \geq 2\varepsilon\}, \quad j = 1, 2, \cdots.$$

From (3.17) and the definition of Ω_1 and Ω_2, we have

$$\Omega_1 \cap \Omega_2 \subset \{\sigma_m = \infty\} \bigcap \Big(\cap_{j=1}^{\infty} \{\rho_j < \infty\} \Big).$$

We also note that for all $\omega \in \Omega_1 \cap \Omega_2$, and $j \geq 1$,

$$W(\zeta(\rho_{2j-1})) - W(\zeta(\rho_{2j})) = \varepsilon \quad \text{and}$$
$$W(\zeta(t)) \geq \varepsilon \quad \text{when } t \in [\rho_{2j-1}, \rho_{2j}]. \tag{3.24}$$

Since $W(\cdot)$ is uniformly continuous in the close ball $\bar{S}_m = \{x \in \mathbb{R}^d : |x| \leq m\}$. We can choose $\delta = \delta(\varepsilon) > 0$ small sufficiently for which

$$|W(\zeta) - W(\bar{\zeta})| < \varepsilon, \zeta, \bar{\zeta} \in \bar{S}_m, \text{ with } \quad |\zeta - \bar{\zeta}| < \delta. \tag{3.25}$$

We emphasize that for $\omega \in \Omega_1 \cap \Omega_2$, if $|\zeta(\rho_{2j-1} + u) - \zeta(\rho_{2j-1})| < \delta$ for all $u \in [0, \lambda]$ and some $\lambda > 0$, then $\rho_{2j} - \rho_{2j-1} \geq \lambda$. Choose a sufficiently small positive number λ and then a sufficiently large positive integer j_0 such that

$$2C_5^2\lambda(\lambda + 4) \leq \varepsilon\delta^2 \text{ and } C_4 < \varepsilon^2\lambda j_0. \tag{3.26}$$

By (3.20) and (3.22), we can further choose a sufficiently large number T for

$$\mathbb{P}(\rho_{2j_0} \leq T) \geq 2\varepsilon. \tag{3.27}$$

In particular, if $\rho_{2j_0} \leq T$, then $|\zeta(\rho_{2j_0})| < m$, and hence $\rho_{2j_0} < \sigma_m$ by the definition of $\zeta(t)$. We hence have

$$\zeta(t,\omega) = X(t,\omega) \text{ for all } 0 \leq t \leq \rho_{2j_0} \text{ and } \omega \in \{\rho_{2j_0} \leq T\}.$$

By the Burkholder-Davis-Gundy inequality, we can have that, for $1 \leq j \leq j_0$,

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\leq t\leq\lambda} |\zeta(\rho_{2j-1} \wedge T + t) - \zeta(\rho_{2j-1} \wedge T)|^2\Big)\\
&\quad\leq 2\mathbb{E}\Big(\lambda \int_{\rho_{2j-1}\wedge T}^{\rho_{2j-1}\wedge T+\lambda} |\varphi(s)|^2 ds + 8\mathbb{E}\int_{\rho_{2j-1}\wedge T}^{\rho_{2j-1}\wedge T+\lambda} |\psi(s)|^2 ds\Big)\\
&\quad\leq 2C_5^2\lambda(\lambda + 4),
\end{aligned}$$

which, along with (3.26) and the Markov inequality, we can obtain that

$$\mathbb{P}\Big(\sup_{0\leq t\leq\lambda} |\zeta(\rho_{2j-1} \wedge T + t) - \zeta(\rho_{2j-1} \wedge T)| \geq \delta\Big) \leq \varepsilon.$$

Noting that $\rho_{2j-1} \leq T$ if $\rho_{2j_0} \leq T$, we can derive from (3.27) and the above inequality that

$$\begin{aligned}
&\mathbb{P}\Big(\{\rho_{2j_0} \leq T\} \cap \Big\{\sup_{0\leq t\leq\lambda} |\zeta(\rho_{2j-1} + t) - \zeta(\rho_{2j-1})| < \delta\Big\}\Big)\\
&\quad= \mathbb{P}(\rho_{2j_0} \leq T)\\
&\qquad- \mathbb{P}\Big(\{\rho_{2j_0} \leq T\} \cap \Big\{\sup_{0\leq t\leq\lambda} |\zeta(\rho_{2j-1} + t) - \zeta(\rho_{2j-1})| \geq \delta\Big\}\Big)\\
&\quad\geq \mathbb{P}(\rho_{2j_0} \leq T) - \mathbb{P}\Big(\sup_{0\leq t\leq\lambda} |\zeta(\rho_{2j-1} + t) - \zeta(\rho_{2j-1})| \geq \delta\Big)\\
&\quad\geq \varepsilon.
\end{aligned}$$

This, together with (3.25) , implies easily that

$$\mathbb{P}\Big(\{\rho_{2j_0} \leq T\} \cap \{\rho_{2j} - \rho_{2j-1} \geq \lambda\}\Big) \geq \varepsilon. \tag{3.28}$$

By (3.24) and (3.28), we derive

$$\begin{aligned}
C_4 &\geq \sum_{j=1}^{j_0} \mathbb{E}\Big(I_{\{\rho_{2j_0}\leq T\}} \int_{\rho_{2j-1}}^{\rho_{2j}} W(X(t))dt\Big)\\
&\geq \varepsilon \sum_{j=1}^{j_0} \mathbb{E}\Big(I_{\{\rho_{2j_0}\leq T\}}(\rho_{2j} - \rho_{2j-1})\Big)\\
&\geq \varepsilon\lambda \sum_{j=1}^{j_0} \mathbb{P}\Big(\{\rho_{2j_0} \leq T\} \cap \{\rho_{2j} - \rho_{2j-1} \geq \lambda\}\Big)\\
&\geq \varepsilon^2\lambda j_0.
\end{aligned}$$

This contradicts the second inequality in (3.26). Thus (3.19) must hold.

We now claim $\lim_{t\to\infty} X(t) = 0$ a.s. If this were not true, then

$$\varepsilon_1 := \mathbb{P}(\Omega_3) > 0,$$

where $\Omega_3 = \{\limsup_{t\to\infty} |X(t)| > 0\}$. On the other hand, by (3.18), we can find a positive integer m_0 large enough for $\mathbb{P}(\sigma_{m_0} < \infty) \le 0.5\varepsilon_1$. Let $\Omega_4 = \{\sigma_{m_0} = \infty\}$. Then

$$\mathbb{P}(\Omega_3 \cap \Omega_4) \ge \mathbb{P}(\Omega_3) - \mathbb{P}(\Omega_4^c) \ge 0.5\varepsilon_1.$$

For any $\omega \in \Omega_3 \cap \Omega_4$, $X(t,\omega)$ is bounded on $t \in \mathbb{R}_+$. We can then find a sequence $\{t_j\}_{j\ge1}$ such that $t_j \to \infty$ and $X(t_j,\omega) \to \bar{X}(\omega) \neq 0$ as $j \to \infty$. This, together with the continuity of W, implies

$$\lim_{j\to\infty} W(X(t_j,\omega)) = W(\bar{X}(\omega)) > 0,$$

which show

$$\limsup_{t\to\infty} W(X(t,\omega)) > 0 \ \text{ for all } \omega \in \Omega_3 \cap \Omega_4.$$

But this contradicts (3.19). We therefore must have the assertion $\lim_{t\to\infty} X(t) = 0$ a.s. Hence, the proof is complete. □

4. An Example for SIDDEs

Let us now discuss an example to illustrate our theory.

Example 4.1. Let us consider the SIDDE (1.1) with the generator (1.2), we consider two case: $\tau = 0.01$ and $\tau = 2$ for all $t \ge 0$. In case with $\tau = 0.01$, let the initial data $x(u) = 2 + \sin(u)$ for $u \in [-0.01, 0]$, $r(0) = 2$, the sample paths of the Markov chain and the solution of the SIDDE are shown in Figure 4.1, which indicates that the SIDDE is asymptotically stable. In the case with $\tau = 2$, let the initial data $x(u) = 2 + \sin(u)$ for $u \in [-3, 0]$, $r(0) = 2$, the sample paths of the Markov chain and the solution of the affine delay SDE (1.1) are plotted in Figure 4.2, which indicates that the SIDDE is asymptotically unstable. From the example we can see SIDDE (1.1) is stable or not depends on how long or short the time-delay is.

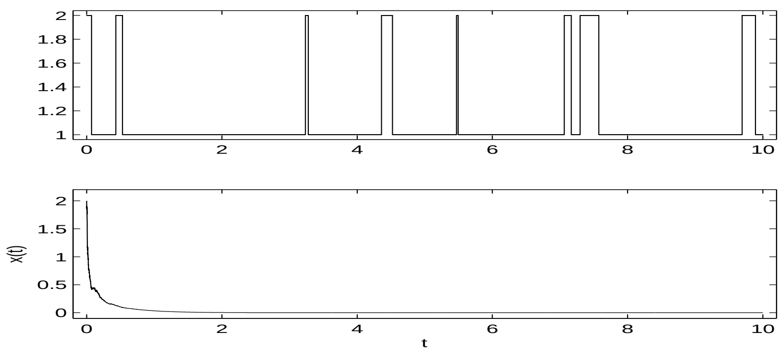

Figure 4.1: The computer simulation of the sample paths of the Markovian chain and the SIDDE (1.1) with $\tau = 0.01$ using the Euler–Maruyama method with step size 10^{-3}.

We can see coefficients defined by (1.1) satisfy Assumption 2.1 with $q_1 = 3$ and $q_2 = 2$. Define $\bar{U}(x,i,t) = |x|^6$ for $(x,i,t) \in \mathbb{R} \times \mathbb{S} \times \mathbb{R}_+$. It is easy to show that

$$\mathbb{L}\bar{U}(x,\phi,i,t) = 6x^5(f(x,i,t) + F(\hbar(\phi),i,t)) + 15x^4|g(x,i,t) + G(\hbar(\phi),i,t)|^2$$

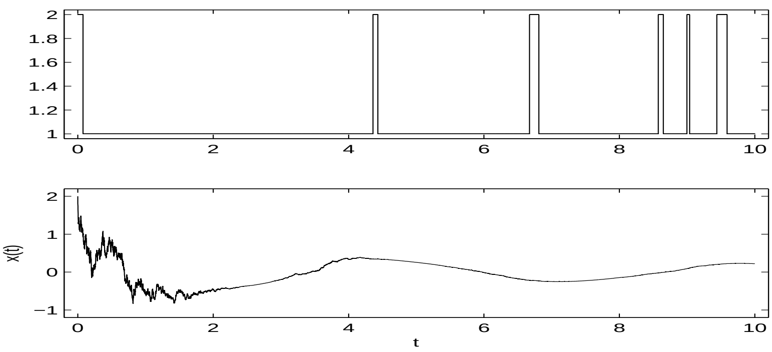

Figure 4.2 : The computer simulation of the sample paths of the Markovian chain and the SIDDE (1.1) with $\tau = 2$ using the Euler–Maruyama method with step size 10^{-3}.

for $(x, i, t) \in \mathbb{R} \times \mathbb{S} \times \mathbb{R}_+, \phi \in C([-\tau, 0]; \mathbb{R})$. Thus we get

$$\begin{aligned}\mathbb{L}\bar{U}(x, \phi, 1, t) =& 6x^5(-10x^3 - 2\hbar(\phi)) + 15x^4|\hbar(\phi)|^4 \\ &\leq 10x^6 + 2|\hbar(\phi)|^6 - 52.5x^8 + 7.5|\hbar(\phi)|^8\end{aligned}$$

and

$$\begin{aligned}\mathbb{L}\bar{U}(x, \phi, 2, t) =& 6x^5(-5x^3 + \hbar(\phi)) + 15x^4|\hbar(\phi)|^4 \\ &\leq 5x^6 + |\hbar(\phi)|^6 - 22.5x^8 + 7.5|\hbar(\phi)|^8.\end{aligned}$$

Thus, we can obtain

$$\begin{aligned}\mathbb{L}\bar{U}(x, \phi, i, t) \leq& 10x^6 + 2|\hbar(\phi)|^6 - 22.5x^8 + 7.5|\hbar(\phi)|^8 \\ =& (13 + 10x^6 + 2|\hbar(\phi)|^6 - 1.5x^8 - 0.5|\hbar(\phi)|^8) - 21(1 + x^8) + 8(1 + |\hbar(\phi)|^8) \\ \leq& c_0 - c_1(1 + x^8) + c_2(1 + |\hbar(\phi)|^8),\end{aligned}$$

where

$$0 < c_0 = \sup_{x\in\mathbb{R}, \phi\in C([-\tau,0];\mathbb{R})} \{13 + 10x^6 + 2|\hbar(\phi)|^6 - 1.5x^8 - 0.5|\hbar(\phi)|^8\} < \infty$$

and $H(x) = 1 + x^8$. Due to $c_2 = 8 < c_1 = 21$, we know Assumption 2.3 holds. Therefore, Assumption 2.3 is satisfied. From Theorem 2.4, solution of the SIDDE (1.1) satisfies

$$\sup_{-\tau\leq t<\infty} \mathbb{E}|X(t)|^6 < \infty.$$

To verify Assumption 3.2, we define

$$U(x, i, t) = \begin{cases} x^2 + x^4, & \text{if } i = 1, \\ 2x^2 + 2x^4, & \text{if } i = 2 \end{cases} \tag{4.1}$$

which shows

$$U_x(x,i,t) = \begin{cases} 2x + 4x^3, & \text{if } i = 1, \\ 4x + 8x^3, & \text{if } i = 2 \end{cases}$$

for $(x, i, t) \in \mathbb{R} \times \mathbb{S} \times \mathbb{R}_+$. By the equation (3.1), we have

$$\begin{aligned} \mathcal{L}U(x,\phi,1,t) =& (2x + 4x^3)(-10x^3 - 2x) + \frac{1}{2}|\hbar(\phi)|^4(2 + 12x^2) + x^2 + x^4 \\ &\leq -3x^2 - 27x^4 - 38x^6 + |\hbar(\phi)|^4 + 4|\hbar(\phi)|^6 \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}U(x,\phi,2,t) =& (4x + 8x^3)(-5x^3 + x) + \frac{1}{2}|\hbar(\phi)|^4(4 + 24x^2) - 8x^2 - 8x^4 \\ &\leq -4x^2 - 20x^4 - 36x^6 + 2|\hbar(\phi)|^4 + 8|\hbar(\phi)|^6. \end{aligned}$$

Moreover

$$|U_x(x,i,t)|^2 = \begin{cases} 4x^2 + 16x^4 + 16x^6, & \text{if } i = 1, \\ 16x^2 + 64x^4 + 64x^6, & \text{if } i = 2, \end{cases} \tag{4.2}$$

$$|f(x,i,t) + F(\hbar(\phi),i,t)|^2 = \begin{cases} |10x^3 + 2\hbar(\phi)|^2 \leq 200x^6 + 8|\hbar(\phi)|^2, & \text{if } i = 1, \\ |5x^3 + \hbar(\phi)|^2 \leq 50x^6 + 2|\hbar(\phi)|^2, & \text{if } i = 2, \end{cases} \tag{4.3}$$

$$|g(x,i,t) + G(\hbar(\phi),i,t)|^2 = \begin{cases} |\hbar(\phi)|^4, & \text{if } i = 1, \\ |\hbar(\phi)|^4, & \text{if } i = 2. \end{cases} \tag{4.4}$$

Setting $\beta_1 = 0.1, \beta_2 = 0.1, \beta_3 = 2$, using (4.2)-(4.4), we obtain that

$$\begin{aligned} &\mathcal{L}U(x,\phi,i,t) + \beta_1|U_x(x,i,t)|^2 + \beta_2|f(x,i,t) + F(\hbar(\phi),i,t)|^2 + \beta_3|g(x,i,t) + G(\hbar(\phi),i,t)|^2 \\ &\leq \begin{cases} -2.6x^2 - 25.4x^4 - 16.4x^6 + 0.8|\hbar(\phi)|^2 + 3|\hbar(\phi)|^4 + 4|\hbar(\phi)|^6, & \text{if } i = 1, \\ -2.4x^2 - 13.6x^4 - 24.6x^6 + 0.2|\hbar(\phi)|^2 + 4|\hbar(\phi)|^4 + 8|\hbar(\phi)|^6, & \text{if } i = 2. \end{cases} \end{aligned}$$

Thus, we get

$$\begin{aligned} &\mathcal{L}U(x,\phi,i,t) + \beta_1|U_x(x,i,t)|^2 + \beta_2|f(x,i,t) + F(\hbar(\phi),i,t)|^2 + \beta_3|g(x,i,t) + G(\hbar(\phi),i,t)|^2 \\ &\leq -2.4x^2 - 13.6x^4 - 16.4x^6 + 0.8|\hbar(\phi)|^2 + 4|\hbar(\phi)|^4 + 8|\hbar(\phi)|^6 \\ &\leq -1.6(x^2 + 5x^4 + 10x^6) + 0.8(|\hbar(\phi)|^2 + 5|\hbar(\phi)|^4 + 10|\hbar(\phi)|^6) \\ &\leq -1.6(x^2 + 5x^4 + 10x^6) + 0.8(|\hbar(\phi)|^2 + 5|\hbar(\phi)|^4 + 10|\hbar(\phi)|^6). \end{aligned}$$

Letting $U_1(x) = x^2 + 5x^4 + 10x^6$. Due to $\alpha_1 = 1.6$, $\alpha_2 = 0.8$, we get condition (3.2). In the case with $\tau = 0$, the conditions in Theorem 3.4 obviously hold. Thus for each $\tau > 0$, the conditions in Theorem 3.4 hold as well. Noting that $\varpi = 2$ and taking $\theta_2 = 1$, we can obtain a upper bound of lag time $\tau \leq 0.07$. Thus, by Theorem 3.4, we can therefore conclude that the solution of the SIDDE (1.1) has the properties that

$$\int_0^\infty (X^2(t) + X^4(t) + X^6(t))dt < \infty \ a.s. \ \text{ and } \ \int_0^\infty \mathbb{E}(X^2(t) + X^4(t) + X^6(t))dt < \infty.$$

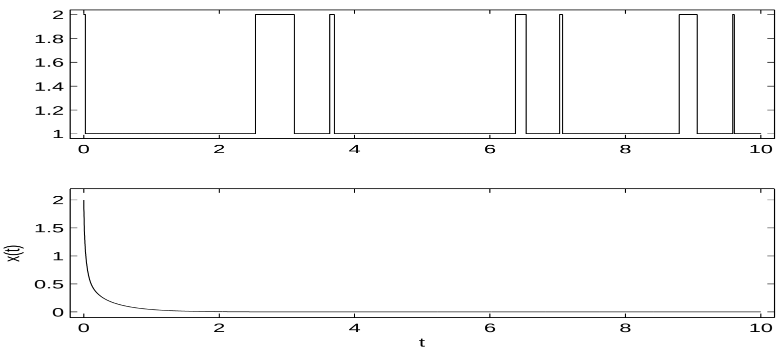

Figure 4.3 : The computer simulation of the sample paths of the Markovian chain and the SIDDE (1.1) with $\tau = 0.07$ using the Euler–Maruyama method with step size 10^{-3}.

Moreover, as $|X(t)|^p \leq x^2(t) + x^4(t) + x^6(t)$ for any $p \in [2, 6]$, we have

$$\int_0^\infty \mathbb{E}|X(t)|^p dt < \infty.$$

Recalling $q_1 = 3$, $q_2 = 2$ and $q = 6$, we see that for $p = 4$, all conditions of Theorem 3.6 are satisfied and hence we have

$$\lim_{t\to\infty} \mathbb{E}|X(t)|^4 = 0.$$

We perform a computer simulation with the time-delay $\tau = 0.07$ for all $t \geq 0$ and the initial data $x(u) = 2 + \sin(u)$ for $u \in [-0.07, 0]$ and $r(0) = 2$. The sample paths of the Markovian chain and the solution of the SIDDE (1.1) are plotted in Figure 4.3. The simulation supports our theoretical results.

5. Conclusion

In real applications, we are often faced with the stochastic integro-differential delay equations (SIDDEs). The boundedness and the stability of solutions to SIDDEs are the important topics. In this paper, we try to give the criteria of the stability and boundedness of the solutions to the hybrid highly nonlinear SIDDEs. To this end, we investigate the hybrid highly nonlinear hybrid SIDDEs. In fact, the stability of hybrid SDDEs have been studied for many years, most of the results in this topic require that the coefficients of equations are linear or nonlinear but bounded by linear functions. Recently, without the linear growth condition, [6] has established delay-dependent stability criteria for the highly nonlinear SDDEs by the method of Lyapunov function. In this paper, we first obtain the stability and boundedness of the hybrid highly nonlinear SIDDE in Section 2. In Section 3, by constructing a Lyapunov functional we further establish the delay-dependent stability criteria of the highly nonlinear SIDDEs, the H_∞ stability in L^p, and the asymptotic stability in L^p. Moreover, the almost surely asymptotic stability is also discussed. Finally, an illustrative example is given.

Besides, it is noteworthy that (i) we give the Assumptions 2.1, 2.3, 3.2 and 3.3 only are the sufficient conditions which contain a kind of equations with their coefficients satisfying these assumptions; (ii) the condition (3.4) in Theorem 3.4 only tell us that the highly nonlinear integro-differential system is stable as long as the system time lag τ verifies condition (3.4), but it cannot tell us if the system is stable as condition (3.4) is not verified. Thus, our delay boundary is conservative, and the less conservative delay boundary might be obtained. Moreover, we believe that there exist other sufficient conditions such that the concerned systems are stable, which can be explored further in future.

Appendix

A. The existence and uniqueness of the maximal solution

To show the existence and uniqueness of the global solution to the SIDDE (2.1) with initial data (2.2), we first provide the following lemmas.

Lemma A.1. *Assume that the condition (2.3) with $q_1 = q_2 = 1$ holds, i.e. the coefficients of the SIDDE (2.1) satisfy the linear growth condition. If $X(\cdot)$ is a solution of the SIDDE (2.1), then we have*

$$\mathbb{E}\Big(\sup_{0\le t\le T}|X(t)|^2\Big) \le (1+3\|\eta\|^2)\exp\Big(3K^2(3T+4)\max\{2, 1+\|\eta\|^2\}T\Big). \tag{A.1}$$

In particular, $X(\cdot) \in \mathcal{M}^2([0,T];\mathbb{R}^d)$.

Proof: For each integer $k \ge 1$, define the stopping time

$$\tau_k = T \wedge \inf\{t \in [0,T] : |X(t)| \ge k\},$$

which shows $\tau_k \uparrow T$ a.s. Thus $X_k(t) := X(t\wedge\tau_k)$ verifies

$$\begin{aligned} X_k(t) =& \eta(0) + \int_0^{t\wedge\tau_k}[f(X_k(s), r(s), s) + F(\hbar(X_{k,s}), r(s), s)]ds \\ &+ \int_0^{t\wedge\tau_k}[g(X_k(s), r(s), s) + G(\hbar(X_{k,s}), r(s), s)]dB(s), \end{aligned}$$

where $X_{k,s} := \{X_k(r) : r \in [s-\tau, s]\}$. By condition (2.3) with $q_1 = q_2 = 1$ and the definition of $\hbar(\phi)$, we have

$$\begin{aligned} &|f(X_k(s), r(s), s) + F(\hbar(X_{k,s}), r(s), s)|^2 \le 3K^2\Big(1 + |X_k(s)|^2 + \sup_{s-\tau\le r\le s}|X_k(r)|^2\Big), \\ &|g(X_k(s), r(s), s) + G(\hbar(X_{k,s}), r(s), s)|^2 \le 3K^2\Big(1 + |X_k(s)|^2 + \sup_{s-\tau\le r\le s}|X_k(r)|^2\Big), \end{aligned}$$

where we have used $|\hbar(X_{k,s})| \le \|X_{k,s}\| = \sup_{s-\tau\le r\le s}|X_k(r)|$. Thus, by employing the Hölder inequality and the Doob martingale inequality, we get that

$$\begin{aligned} \mathbb{E}\Big(\sup_{0\le s\le t}|X_k(s)|^2\Big) \le& 3\|\eta\|^2 + 3t\mathbb{E}\int_0^t |f(X_k(v), r(v), v) + F(\hbar(X_{k,v}), r(v), v)|^2 dv \\ &+ 3\mathbb{E}\sup_{0\le s\le t}\Big|\int_0^{s\wedge\tau_k}[g(X_k(v), r(v), v) + G(\hbar(X_{k,v}), r(v), v)]dB(v)\Big|^2 \\ \le& 3\|\eta\|^2 + 3K^2(3t+4)\mathbb{E}\int_0^t(1 + |X_k(v)|^2 + \sup_{v-\tau\le r\le v}|X_k(r)\|^2)dv. \end{aligned}$$

Noting $\sup_{v-\tau\le r\le v}|X_k(r)\|^2 \le \|\eta\|^2 + \sup_{0\le r\le v}|X_k(r)\|^2$, we have

$$1+\mathbb{E}\Big(\sup_{0\le s\le t}|X_k(s)|^2\Big) \le 1+3\|\eta\|^2+3K^2(3t+4)\max\{2,1+\|\eta\|^2\}\int_0^t (1+\mathbb{E}\sup_{0\le r\le v}|X_k(r)\|^2)dv,$$

From the Gronwall inequality, we know

$$1+\mathbb{E}\Big(\sup_{0\le s\le t}|X_k(s)|^2\Big) \le (1+3\|\eta\|^2)\exp\Big(3K^2(3t+4)\max\{2,1+\|\eta\|^2\}t\Big),$$

which shows (A.1) by letting $k\to\infty$. Thus the proof is complete. □

Lemma A.2. *Let Assumption 2.1 with $q_1=q_2=1$ hold. Then there exists a unique solution $X(\cdot)$ to the SIDDE (2.1) and the solution belongs to $\mathcal{M}^2([0,T];\mathbb{R}^d)$.*

Proof: We first give the proof of uniqueness. Let $X(\cdot)$ and $\tilde X(\cdot)$ be two solutions of the SIDDE (2.1) with the initial data (2.2). From Lemma A.1, we know $X(\cdot),\tilde X(\cdot)\in\mathcal{M}^2([0,T];\mathbb{R}^d)$. Moreover, we have

$$X(t)-\tilde X(t) = \int_0^t [(f(X(s),r(s),s)-f(\tilde X(s),r(s),s))+(F(\hbar(X_s),r(s),s)-F(\hbar(\tilde X_s),r(s),s)]ds$$
$$+\int_0^t [(g(X(s),r(s),s)-g(\tilde X(s),r(s),s))+(G(\hbar(X_s),r(s),s)-G(\hbar(\tilde X_s),r(s),s))]dB(s).$$

For each $h\ge 1$, define the stopping time $\tilde\tau_h=\inf\{t\in[0,T]:|X(t)|\vee|\tilde X(t)|\ge h\}$. Obviously, $\tilde\tau_h\to T$ as $h\to\infty$. By the Hölder inequality, the Doob martingale inequality and the local Lipshcitz condition, we can show as in the proof of Lemma A.1 that

$$\mathbb{E}\Big(\sup_{0\le s\le t}|X(s)-\tilde X(s)|^2\Big)\le C(K_h,\eta)\int_0^{t\wedge\tilde\tau_h}\mathbb{E}\Big(\sup_{0\le r\le v}|X(r)-\tilde X(r)|^2\Big)dv. \tag{A.2}$$

From the Gronwall inequality, we get

$$\mathbb{E}\Big(\sup_{0\le s\le t\wedge\tilde\tau_h}|X(s)-\tilde X(s)|^2\Big)=0.$$

Letting $h\to\infty$, we know that $X(t)=\tilde X(t)$ for all $0\le t\le T$ almost surely. Thus the uniqueness has been proved.

Next the proof of existence is similar to that of [29, Theorem 3.13, page 89]. Due to page limit, we omit the details. □

The following assertion shows the existence of unique maximal local solution under the local Lipschitz condition without the linear growth condition similar to the discussion in [29, Theorem 3.15, page 91].

Lemma A.3. *Let Assumption 2.1 hold. Then there exists a unique maximal local solution to the SIDDE (2.1) with the initial data (2.2).*

Proof: By a truncation procedure, we show our claim. For each integer $m\ge 1$, define the truncation functions

$$f_m(x,i,t)=\begin{cases} f(x,i,t) & \text{if} \quad |x|\le m,\\ f(mx/|x|,i,t) & \text{if} \quad |x|>m,\end{cases}$$
$$F_m(\hbar(\phi),i,t)=\begin{cases} F(\hbar(\phi),i,t) & \text{if} \quad \|\phi\|\le m,\\ F(m\hbar(\phi)/\|\phi\|,i,t) & \text{if} \quad \|\phi\|>m,\end{cases}$$

where $x \in \mathbb{R}^d$ and $\phi \in C([-\tau, 0]; \mathbb{R}^d)$. Similarly, we can define the functions $g_m(x, i, t)$ and $G_m(\hbar(\phi), i, t)$. Due to $|\hbar(\phi)| \le \|\phi\|$, we can know that the functions f_m, F_m, g_m and G_m satisfy Lipschitz condition and the linear growth condition. Hence from Lamma A.2, there exists a unique solution $X_m(\cdot)$ in $\mathcal{M}^2([0, T]; \mathbb{R}^d)$ to the equation

$$\begin{aligned} dX_m(t) =& [f_m(X_m(t), r(t), t) + F_m(\hbar(X_{m,t}), r(t), t)]dt \\ &+ [g_m(X_m(t), r(t), t) + G_m(\hbar(X_{m,t}), r(t), t)]dB(t) \end{aligned} \tag{A.3}$$

on $t \ge 0$ with initial data (2.2). Define the stopping time

$$\nu_m = T \wedge \inf\{t \in [0, T] : |X_m(t)| \ge m\}.$$

We easily know

$$X_m(t) = X_{m+1}(t) \quad \text{if} \quad 0 \le t \le \nu_m, \tag{A.4}$$

which means that ν_m is increasing, and $\nu_\infty = \lim_{m\to\infty} \nu_m$. Define now the process $\{X(t) : 0 \le t < \infty\}$ by

$$X(t) = X_m(t), \;\; \nu_{m-1} \le t < \nu_m, m \ge 1,$$

where $\nu_0 = 0$. In virtue of (A.4), we have $X(t \wedge \nu_m) = X_m(t \wedge \nu_m)$. Thus from (A.3), we get

$$\begin{aligned} dX(t \wedge \nu_m) = \eta(0) &+ \int_0^{t\wedge\nu_m} [f_m(X(s), r(s), s) + F_m(\hbar(X_s), r(s), s)]ds \\ &+ \int_0^{t\wedge\nu_m} [g_m(X(s), r(s), s) + G_m(\hbar(X_s), r(s), s)]dB(s) \\ &= \eta(0) + \int_0^{t\wedge\nu_m} [f(X(s), r(s), s) + F(\hbar(X_s), r(s), s)]ds \\ &+ \int_0^{t\wedge\nu_m} [g(X(s), r(s), s) + G(\hbar(X_s), r(s), s)]dB(s) \end{aligned}$$

for each $t \in [0, T]$ and $m \ge 1$. It is easy to verify that if $\nu_\infty < T$, then

$$\limsup_{t\to\nu_\infty} |X(t)| \ge \limsup_{m\to\infty} X(\nu_m) = \limsup_{m\to\infty} X_m(\nu_m) = \infty.$$

Therefore, $\{X(t) : 0 \le t < \nu_\infty\}$ is a maximal local solution. Now we show the uniqueness. To this end, let $\{\tilde{X}(t) : 0 \le t < \tilde{\nu}_\infty\}$ be another maximal local solution. Define $\tilde{\nu}_m = \tilde{\nu}_\infty \wedge \inf\{t \in [[0, \tilde{\nu}_\infty[[: |\tilde{X}(t)| \ge m\}$. We can show that $\tilde{\nu}_m \to \tilde{\nu}_\infty$ a.s. and

$$\mathbb{P}\{X(t) = \tilde{X}(t) \; \forall t \in [[0, \nu_m \wedge \tilde{\nu}_m[[\} = 1 \quad \forall m \ge 1.$$

Letting $m \to \infty$, we get

$$\mathbb{P}\{X(t) = \tilde{X}(t) \; \forall t \in [[0, \nu_\infty \wedge \tilde{\nu}_\infty[[\} = 1.$$

Next, we need to prove that $\nu_\infty = \tilde{\nu}_\infty$ a.s. Indeed, for almost any $\omega \in \{\nu_\infty < \tilde{\nu}_\infty\}$, we can obtain

$$|\tilde{X}(\nu_\infty, \omega)| = \lim_{m\to\infty} |\tilde{X}(\nu_m, \omega)| = \lim_{m\to\infty} |X(\nu_m, \omega)| = \infty$$

which contracts the fact that $\tilde{X}(t, \omega)$ is continuous on $t \in [0, \tilde{\nu}_\infty(\omega))$. Thus we must have $\nu_\infty \ge \tilde{\nu}_\infty$ a.s. Similarly, we can show $\nu_\infty \le \tilde{\nu}_\infty$ a.s. Hence we get $\nu_\infty = \tilde{\nu}_\infty$ a.s. Thus the proof is complete. □

Acknowledgements

Many helpful comments and suggestions from the editor and referees are gratefully acknowledged. The authors would like to thank Liangjian Hu and Wei Liu for their helpful discussions. This paper is supported by the Natural Science Foundation of China (71571001), China Scholarship Council (201806630049), the Promoting Plan of Higher Education of Anhui Province (T-SKJ2016B11), the Royal Society (WM160014, Royal Society Wolfson Research Merit Award), the Royal Society and the Newton Fund (NA160317, Royal Society- Newton Advanced Fellowship) and the EPSRC (EP/K503174/1) for their financial support.

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