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## NOVEL PIECEWISE TRAJECTORY SHAPING IN HILL'S CANONICAL VARIABLES

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Shape-based methods have been proven to be computationally efficient techniques to quickly estimate the cost of low-thrust interplanetary trajectories. However, in some cases the solution is far from optimal, like in the case of the exponential sinusoid,<sup>1</sup> or requires a special treatment when the motion is not completely planar. More recent developments<sup>2,3</sup> allows for a full three-dimensional representation of the trajectory but either constraints need to be imposed on the thrust direction or approximations need to be introduced on the trajectory time-evolution, causing the domain of representable trajectories to shrink. As a consequence, trajectories transferring to highly inclined or highly eccentric orbits can lead to infeasible control laws. This paper presents a new analytical framework for the quick estimation of the  $\Delta v$  and peak thrust of two-point boundary value low-thrust transfers. The novelty of this method is that it solves an inverse optimal control problem in Hill's canonical variables. The parameterisation in Hill's variables was selected so that the shaping of the in-plane and out-of-plane motions can be treated separately and the boundary conditions can be analytically satisfied. This choice leads to a computationally efficient extraction of the control profile and allows for the integration of known analytical solutions for the in-plane motion. The computation of the value of the objective function (usually the total  $\Delta v$  or the spacecraft final mass) and path constraints is reduced to computationally inexpensive quadratures. The shaping proposed in this paper is piecewise continuous and allows for a flexible full three-dimensional representation of the trajectory. In particular, the out-of-plane motion is represented by piecewise continuous functions so that one can independently maximise both the change of inclination and the variation of the longitude of the ascending node. The method is applied to some well-known test cases, a rendezvous with Mars, asteroid 1989ML and comet Tempel-1, and the results compared to the solutions obtained with exponential sinusoid, pseudoequinoctial elements and spherical shaping.

I. INTRODUCTION

The trajectory design for a low-thrust driven spacecraft has been historically tackled as an optimization problem, formulating the mathematical model as a boundary value problem with associated a performance index  $J$  to optimize, usually the total  $\Delta V$ , final mass or minimum time of flight. This approach utilizes *optimal control theory* principles to numerically compute the optimal solution able to respect the equation of motion, the boundary conditions and to optimize the performance index. However, practical methods based on this formulation are computationally expensive for complex cases such as low-thrust trajectory optimisation. Therefore, they are not suitable for the preliminary assessment of a new mission, a stage when a huge amount of possible scenarios shall be evaluated. In addition, the

majority of modern high-fidelity methods require an accurate initial guess to start with.

Shaping methods are a class of techniques which have proven to be computationally efficient for preliminary mission analysis and able to generate a quite accurate solution, very close to the optimal one for some test cases in literature.<sup>3</sup> A detailed optimality analysis has been performed on solutions computed by a shaping method,<sup>4</sup> which further justifies the employment of this approach.

The purpose of the paper is to present the development of a novel shaping framework in Hill variables (Section II), suitable to overcome some of the method's common limitations. Within the set framework, a shaping parameterization which separates the in-plane dynamics from the out-of-plane one is proposed in Section III. The developed method will be tested in Section IV against a variety of ren-

devious test cases and compared with the available results in literature, whereas the conclusions are presented in Section V.

## II. SHAPING FRAMEWORK

The intuitive idea that the perturbations of the in-plane dynamics are of a different nature than those affecting the orbital plane is well depicted by the dynamical equations written in Hill's variables.<sup>5</sup> This set of canonical variables is constituted by a mix of rapidly varying variables  $r$ ,  $v_r$  and  $u$ , respectively the radius vector, the radial velocity and the argument of latitude, and by three Delaunay integrals of motion  $h$ ,  $G$  and  $H$ , respectively the right ascension of the ascending node, the angular momentum magnitude and its  $z$ -component.

However, by definition, the parameter  $H = G \cos i$  provides redundant information and its dynamical evolution is influenced by two perturbation components. In this paper, the governing equations are written in Hill's variables, with the inclination  $I$  substituting the parameter  $H$ , as follows:

$$\begin{aligned} \dot{r} &= v_r \\ \dot{v}_r &= G^2/r^3 - \mu/r^2 + F_r \\ \dot{G} &= rF_u \\ \dot{h} &= r \sin u F_n / (G \sin I) \\ \dot{I} &= r \cos u F_n / G \\ \dot{u} &= G/r^2 - r \cos i \sin u F_n / (G \sin I) \end{aligned} \quad [1]$$

where  $F_r$ ,  $F_u$  and  $F_n$  are respectively the radial, transversal and normal components of the thrust acceleration. This parameter replacement results in a system of equations simpler to invert as any of them depends on one control component only.

Instead of the time variable, the fast angular quantity  $u$  can be used as independent variable, being more suitable to parameterize the state evolution for multi-revolution trajectories. This variable transformation holds because there is a smooth one-to-one mapping between time and the angular variable, meaning that  $u(t)$  is strictly monotonous, a condition always satisfied for low-thrust trajectories. The equations of motion are re-written in terms of

$u$  by dividing all the relations in Equation (1) by  $du/dt = G/r^2 - r \cos i \sin u F_n / (G \sin I)$ . Hence, the new system of equations, gathered in matrix form, takes the form:

$$\alpha' = \mathbf{A}(\alpha)\mathbf{F} + \mathbf{b}(\alpha) \quad [2]$$

where  $\alpha = [r, v_r, G, h, I]$ , the prime symbol indicates the derivative with respect to  $u$ ,  $\mathbf{F}$  contains the thrust acceleration components, and the quantities  $\mathbf{A}(\alpha)$  and  $\mathbf{b}(\alpha)$  are defined as:

$$\mathbf{A} = \frac{1}{du/dt} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & \frac{r \sin u}{G \sin I} \\ 0 & 0 & \frac{r \cos u}{G} \end{bmatrix}$$

$$\mathbf{b}^T = \frac{1}{du/dt} \left[ v_r \left( \frac{G^2}{r^3} - \frac{\mu}{r^2} \right) \quad 0 \quad 0 \quad 0 \right]$$

By studying the rearranged system in Equation (2), it is clear how the parameters describing the plane orientation, i.e.  $h$  and  $I$ , depend only on  $F_n$ . On the contrary, as a consequence of the independent variable transformation, the variables  $v_r$  and  $G$  depend on two thrust components, one in-plane,  $F_r$  or  $F_u$ , and the normal  $F_n$ , because of the term  $du/dt$ .

### A. Out-of-plane component inversion

Given the former premises, it follows that the first step in the inversion process is the extraction of the normal component by inversion of  $h'$  and  $I'$ , explicitly as:

$$\begin{aligned} F_n &= \frac{\mathcal{G}^2 \sin \mathcal{I}}{\mathcal{R}^3 \sin u} \frac{\mathcal{H}'}{1 + \mathcal{H}' \cos \mathcal{I}} \\ F_n &= \frac{\mathcal{G}^2}{\mathcal{R}^3} \frac{\mathcal{I}'}{\mathcal{I}' \cot \mathcal{I} \sin u + \cos u} \end{aligned} \quad [3]$$

where  $\mathcal{R}$ ,  $\mathcal{G}$ ,  $\mathcal{H}$  and  $\mathcal{I}$  are the selected functional forms of  $r$ ,  $G$ ,  $h$  and  $I$  respectively (see Section III). Because the system of equations is overdetermined in  $F_n$ , the selection of the shapes  $\mathcal{H}$  and  $\mathcal{I}$  is not arbitrary, but they shall define the normal component uniquely and respect the boundary conditions at same time. One approach could result from equaling the right-hand sides as:

$$\mathcal{H}'\mathcal{I}' \cos \mathcal{I}(1 - \sin u) + \mathcal{I}' \sin u = \mathcal{H}' \sin \mathcal{I} \cos u \quad [4]$$

and computing  $\mathcal{H}(u)$  and  $\mathcal{I}(u)$  from this differential equation. However, no function able to satisfy the aforementioned condition has been found. Furthermore, even if some functional forms were found to respect Equation (4), it is very likely that those shaped functions could result in non-optimal, or even non-feasible, trajectories.

A different approach has been employed to define uniquely  $F_n(t)$ , by exploiting the physics of the problem to generate near-optimal maneuvers at the same time. The out-of-plane equations can be written in a more compact but significant form as:

$$\begin{aligned} h' &= \frac{1}{du/dt} \frac{rF_n}{G \sin I} \sin u \\ I' &= \frac{1}{du/dt} \frac{rF_n}{G} \cos u \end{aligned} \quad [5]$$

The first expression shows that the right ascension  $h$  shall be changed preferably when  $\sin u$  is maximum or minimum, i.e. for  $u = k\frac{\pi}{2}$  with  $k = 1, 2, \dots$ , to reduce the requested  $F_n$  quantity for a prescribed change of  $h$ . On the other hand, the inclination  $i$  shall be changed when the spacecraft is near the line of nodes, i.e. when  $\cos u$  is maximum or minimum for  $u = k\pi$  with  $k = 0, 1, 2, \dots$ , in order to make the maneuver for changing  $i$  more convenient from a propellant consumption point of view. Hence, the arcs where it is more advisable to change  $h$  and  $i$  are not overlapping. In addition, in those zones the other parameter is only marginally affected, reproducing a well-known result in orbital mechanics for impulsive maneuvers.<sup>6</sup> This convenient decoupling has resulted in the definition of a novel shaping approach for the right ascension of the ascending node and the inclination, named *piecewise shaping*.

The procedure can be schematized as follows:

- When  $u \in [k\frac{\pi}{2} - \alpha, k\frac{\pi}{2} + \alpha]$  for  $k = 1, 2, \dots$ :
  - Define the shaping function  $h = \mathcal{H}(u)$  and compute its derivative  $\mathcal{H}'$ ;
  - Plug them in Equation (3-1) to obtain  $F_n(u)$  able to actually generate the prescribed  $h$ -evolution

- Plug the extracted control profile  $F_n(u)$  in Equation (5-2) to compute by iterative quadrature the inclination evolution  $i(u)$ ;

- When  $u \in [k\pi - \alpha, k\pi + \alpha]$  for  $k = 0, 1, 2, \dots$ :
  - Define the shaping function  $i = \mathcal{I}(u)$  and compute its derivative  $\mathcal{I}'$ ;
  - Plug them in Equation (3-2) to obtain  $F_n(u)$  able to actually generate the prescribed  $i$ -evolution;
  - Plug the extracted control profile  $F_n(u)$  in Equation (5-1) to compute by quadrature the right ascension evolution  $h(u)$ ;
- When  $u \notin [k\frac{\pi}{2} - \alpha, k\frac{\pi}{2} + \alpha]$  and  $u \notin [k\pi - \alpha, k\pi + \alpha]$ :
  - $h$  and  $i$  are kept constant and therefore the out-of-plane component is zero  $F_n = 0$ .

Following this procedure, the control component  $F_n$  is uniquely defined on each sub-interval. The intervals  $[k\frac{\pi}{2} - \alpha, k\frac{\pi}{2} + \alpha]$  and  $[k\pi - \alpha, k\pi + \alpha]$  have been defined symmetrically around a center point to reduce the other element overall variation. Indeed, in the intervals where  $h$  is changing by the prescribed quantity  $\mathcal{H}(u)$ , the resulting inclination change is defined as:

$$\Delta I = \int_{k\frac{\pi}{2}-\alpha}^{k\frac{\pi}{2}+\alpha} \frac{1}{du/dt} \frac{\mathcal{R}F_n}{\mathcal{G}} \cos u \, du \quad [6]$$

The final inclination is usually approaching its initial value, i.e.  $\Delta I \approx 0$ , because the cosine is an odd-function on the symmetrical integration interval and because, if small values of  $\alpha$  are selected, the other quantities are varying marginally as attenuated by small values of  $\cos u$  near  $u = k\frac{\pi}{2}$ . A correspondent situation happens in the intervals where the inclination  $i$  is changing by the prescribed quantity  $\mathcal{I}(u)$ , leading to small changes of the right ascension of the ascending node  $\Delta h \approx 0$ . This is a valuable characteristic because it ensures that the undesired changes of one variable in the intervals where the other is changing are limited, resulting in an accurate satisfaction of the boundary conditions on both the variables.

## B In-plane components inversion

Two of the three Hill parameters describing the in-plane motion, i.e. the radius and the radial velocity, are connected by the derivative relation  $v_r = r' \cdot du/dt$  (or the equivalent integral form). For this reason, they shall not be shaped independently, but one shall be computed from the other parameterization. In the current framework development, the radius is shaped as  $r = \mathcal{R}(u)$ , leading the radial velocity to be parameterize as:

$$v_r = \frac{d\mathcal{R}}{dt} = \mathcal{R}' \frac{du}{dt} = \mathcal{R}' \left( \frac{G}{\mathcal{R}^2} - \frac{\mathcal{R} \cot i \sin u}{G} F_n \right) \quad [7]$$

This alternative allows to compute the coefficients able satisfy the boundary conditions on both the radius and velocity analytically. On the contrary, the choice of shaping directly  $v_r = \mathcal{V}_r(u)$  has been discarded because  $r$  cannot be integrated in an analytical closed-form, and the boundary conditions cannot be quickly satisfied. When a suitable  $\mathcal{R}$  is selected, the radial component is extracted analytically extracted by inversion of Equation (2-2) as:

$$F_r = \frac{\mu}{\mathcal{R}^2} - \frac{\mathcal{G}^2}{\mathcal{R}^3} + \mathcal{R}'' \left[ \frac{\mathcal{G}}{\mathcal{R}^2} - \frac{\mathcal{R} \cot \mathcal{I} \sin u}{\mathcal{G}} F_n \right] \left[ \frac{\mathcal{G}}{\mathcal{R}^2} - \frac{\mathcal{R} \cot \mathcal{I} \sin u}{\mathcal{G}} F_n \right] \quad [8]$$

where  $F_n$  has been computed as in Section II.A. The derivative of the square bracket in Equation (8) introduces limitations on the selection of the parametric functions, which shall be at least piecewise continuously differentiable.

The last in-plane element is shaped separately as  $G = \mathcal{G}(u)$ , and the transversal force component form obtained as:

$$F_u = \frac{\mathcal{G}'}{\mathcal{R}} \left[ \frac{\mathcal{G}}{\mathcal{R}^2} - \frac{\mathcal{R} \cot \mathcal{I} \sin u}{\mathcal{G}} F_n \right] \quad [9]$$

## C $\Delta V$ and Time-of-flight computation

The total cost of the transfer can be estimated in terms of  $\Delta V$  as:

$$\Delta V = \int_{u_i}^{u_f} |\mathbf{u}| \frac{1}{du/dt} du \quad [10]$$

where  $u_i$  and  $u_f$  are the initial and final values of the argument of latitude. In addition, the time-of-flight is computed as:

$$TOF = \int_{u_i}^{u_f} \frac{1}{du/dt} du \quad [11]$$

Since only the final value is of interest for both these quantities, a fast quadrature approximation can be used, e.g. a Gauss-Legendre scheme as in this paper. All the quantities in Equations (10)-(11) are known algebraically, resulting in inexpensive function evaluations for the quadratures or possible path constraints, such as the maximum thrust value.

## III. HILL SHAPING

The in-plane elements  $r$  and  $G$  can be defined by any arbitrary family of parametric functions, with the only condition of being positive definite and continuously differentiable. Thanks to the above-defined piecewise shaping for the out-of-plane parameters, the functional forms of  $h$  and  $I$  within the respective intervals can be shaped freely as well. A convenient choice is to select functions whose coefficients can be computed linearly with respect to the boundary conditions. Since the developed framework in Hill's variables allows separating the planar motion from the orbital plane's dynamics, they can be shaped and treated separately.

### A Out-of-plane shaping

The boundary conditions are trivially formulated as:

$$\begin{cases} \mathcal{I}(u_0) = I_0, & \mathcal{I}(u_f) = I_f \\ \mathcal{H}(u_0) = h_0, & \mathcal{H}(u_f) = h_f \end{cases} \quad [12]$$

However, because low-thrust trajectories usually involve a continuous thrust over a large number of revolutions, there could be many intervals such that  $u \in [k\frac{\pi}{2} - \alpha, k\frac{\pi}{2} + \alpha]$  and  $u \in [k\pi - \alpha, k\pi + \alpha]$ . It is therefore convenient to spread the total  $\Delta I$  and  $\Delta h$  variations over those intervals through intermediate increments, leading to smaller peaks of the out-of-plane control component. On the other hand, two issues arise from this piecewise shaping:

- The strategy for dividing the total  $\Delta h$  and  $\Delta I$  changes into smaller increments  $\Delta h_j$  and  $\Delta I_j$ ;
- The definition of multiple  $\mathcal{H}_j$  and  $\mathcal{I}_j$  shaping functions over the different intervals.

For the former, two strategies have been implemented in the Hill shaping method. The first strategy implies a uniform division of the total change. Therefore, if the number of trajectory's intervals satisfying the relation  $u \in [k\frac{\pi}{2} - \alpha, k\frac{\pi}{2} + \alpha]$  is labeled as  $n_h$ , the increments within each interval are defined as:

$$\Delta h_j = \frac{\Delta h}{n_h} \quad \text{for } j = 1, \dots, n_h \quad [13]$$

The same procedure is applied for the  $n_i$  intervals where the relation  $u \in [k\pi - \alpha, k\pi + \alpha]$  is satisfied. The second strategy exploits the functional form of Equations (3). In those expressions it is highlighted that, for a fixed  $h$  or  $I$  change, the  $F_n$  control component is proportional to  $\mathcal{G}^2/\mathcal{R}^3$ . Since these quantities are already known at this stage, being the first to be guessed, the evolution of that ratio is known along the trajectory. Therefore, the increments  $\Delta h_j$  and  $\Delta I_j$  can be defined as proportional to an averaged value of the inverse ratio  $\mathcal{R}^3/\mathcal{G}^2$  within each interval. This procedure exploits the known part of the trajectory evolution for optimizing further the out-of-plane control, which now results approximately equal within each interval. On the contrary, for the first strategy, the  $F_n$  magnitude is dissimilar in different intervals, resulting proportional to  $\mathcal{G}^2/\mathcal{R}^3$ . These characteristics can be seen in Figure 1, where the two strategies have been run for the same test case, using a quite big  $\alpha$ -value to better depict this difference. As a general characteristic for the run simulations, the thrust peak and the total  $\Delta V_n$  for normal maneuvers are lower for the second strategy.

The problem of shape selection for  $h$  and  $I$  evolution can be addressed now that the parameters' increments have been defined for every interval, and consequently the intermediate boundary conditions for the shaping functions  $\mathcal{H}_j$  and  $\mathcal{I}_j$  within each interval have been determined. Because the number of intervals is usually high in multi-revolution low-thrust trajectories, these functions will not involve free parameters but will be defined completely by the

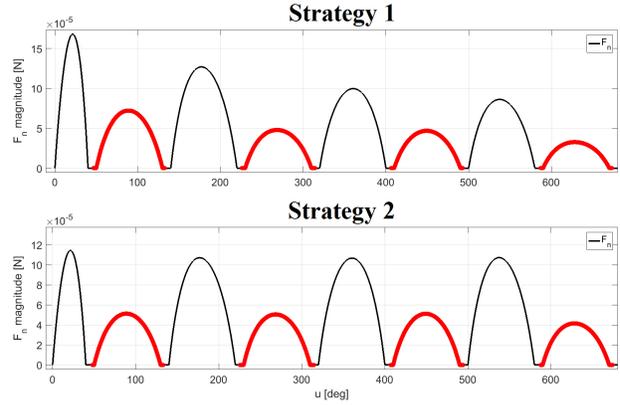


Fig. 1: Approach comparison for dividing total  $\Delta h$  and  $\Delta I$ ; red-line where  $h$  changes, black-line where  $I$  varies.

boundary conditions. Any of them shall have at least two parameters to satisfy the initial and final values, hence the linear evolution is a logic choice. However, a linear evolution does not exploit the information that the central zone of the defined intervals, i.e. where  $|\sin u|$  and  $|\cos u|$  assume the maximum values, is the most convenient for varying the parameters. Another disadvantage is that a linear shape generates a discontinuous out-of-plane control profile, which make less precise the validation of the computed trajectory by control propagation. To overcome both these issues, a 3rd-order polynomial function was selected. In order to eliminate the free parameters from the selected shape and to make the resulting out-of-plane control profile  $F_n(u)$  a continuous function, two extra boundary conditions have been added on the initial and final first-derivative values to be zero.

$$\mathcal{H}_i = \sum_{k=0}^3 \mathcal{H}_{ik}(u - u_{i0})^k \quad \text{for } i = 1, \dots, n_h \quad [14]$$

$$\mathcal{I}_j = \sum_{k=0}^3 \mathcal{I}_{jk}(u - u_{j0})^k \quad \text{for } j = 1, \dots, n_i$$

where  $u_{i0}$  and  $u_{j0}$  are the initial argument of latitude values of the correspondent thrusting interval, while  $\mathcal{H}_{ik}$  and  $\mathcal{I}_{jk}$  are the coefficients analytically determined to satisfy the 4 boundary conditions. This functional form with the imposed conditions also ensures that the parameters derivatives  $h'$  and  $I'$  are

greater at the center of the interval and smaller near its extrema, leading to an optimal evolution of the parameters from the propellant consumption point of view. Furthermore, it imposes the normal control  $F_n$  to be zero at the departure and arrival conditions, resulting in the independent variable time-derivative to be equal to its unperturbed value as

$$\left. \frac{du}{dt} \right|_{u_0} = \frac{\mathcal{G}(u_0)}{\mathcal{R}(u_0)^2} \quad \left. \frac{du}{dt} \right|_{u_f} = \frac{\mathcal{G}(u_f)}{\mathcal{R}(u_f)^2} \quad [15]$$

automatically satisfying the boundary constraints on the angular velocity when the in-plane boundary conditions for  $\mathcal{R}$  and  $\mathcal{G}$  are met.

The overall defined shaping and procedure allow to define a continuous  $F_n$  control profile able to satisfy, when propagated, the boundary conditions on both the right ascension of the ascending node and on the inclination up to a very low threshold, by exploiting optimally the functional relations within the equations of the out-of-plane motion.

## B In-plane shaping

The radius parametrization shall satisfy four boundary conditions, two on initial and final distance and two on initial and final radial velocity, while the angular momentum have two conditions on its extrema. These conditions can be summarized as follows:

$$\begin{cases} \mathcal{R}(u_i) = R_i & \mathcal{R}(u_f) = R_f \\ \mathcal{R}'(u_0) = \frac{v_{r_i}}{du/dt|_{u_i}} & \mathcal{R}'(u_f) = \frac{v_{r_f}}{du/dt|_{u_f}} \\ \mathcal{G}(u_i) = G_i & \mathcal{G}(u_f) = G_f \end{cases} \quad [16]$$

Physically,  $\mathcal{R}$  and  $\mathcal{G}$  shall be selected as positive functions because they define respectively the radius and the angular momentum magnitude, two non-negative quantities. Again, another valuable characteristics would be to have expressions for which the boundary conditions can be satisfied analytically, avoiding an expensive iterative procedure.

In the two-body problem, the radius evolution as function of the argument of latitude  $u$  is given by:<sup>6</sup>

$$\frac{1}{r} = \frac{1}{a(1-e^2)} + e \cos \omega \cos u + e \sin \omega \sin u \quad [17]$$

where the quantities  $a$ ,  $e$  and  $\omega$  are the orbital semi-major axis, eccentricity and argument of periapsis, constants of motion for the unperturbed problem. When the perturbed two-body motion is considered, following the logic of the variation of parameters method, we can suppose that the functional form of the solution remains unaltered. However, the parameters, which were constant in the reference motion, are now evolving as function of the independent variable. Following this reasoning, the radius evolution is shaped as:

$$\frac{1}{\mathcal{R}} = \mathcal{R}_0 + \mathcal{R}_1 u + (\mathcal{R}_2 + \mathcal{R}_3 u) \cos u + (\mathcal{R}_4 + \mathcal{R}_5 u) \sin u \quad [18]$$

A similar radius shaping resulted in feasible and convenient trajectories from the propellant consumption point of view in the spherical shaping.<sup>3</sup> In the novel Hill shaping, this radius shape provides flexibility as two free parameters can be used to optimize the trajectory and reduce the constraints' violation. The coefficients  $\mathcal{R}_0$ ,  $\mathcal{R}_1$ ,  $\mathcal{R}_2$  and  $\mathcal{R}_4$  are used to satisfy analytically the boundary conditions as above discussed.

De Pascale and Vasile in 2006<sup>2</sup> explained how the numerical integration of a tangential thrust with constant magnitude profile leads to an exponential evolution of the parameter  $p = a(1 - e^2)$ , while trigonometric terms are negligible for its secular evolution. Because the angular momentum is directly connected to the semilatus rectum as  $G = \sqrt{\mu p}$ , the latter is shaped and then  $\mathcal{G}$  defined accordingly as:

$$\mathcal{G} = \sqrt{\mu (\mathcal{P}_0 + \mathcal{P}_1 e^{\mathcal{P}_2(u-u_0)})} \quad [19]$$

The parameters  $\mathcal{P}_0$  and  $\mathcal{P}_1$  are employed to respect analytically the boundary conditions, while  $\mathcal{P}_2$  is a free parameter.

## IV. TEST CASES

Three test cases, already studied in literature,<sup>3</sup> have been analyzed to test the performance of the developed Hill piecewise shaping, specifically rendezvous departing from Earth and approaching Mars, the asteroid 1989ML, and the comet Tempel-1. The orbital elements used for the test cases are summarized in Table 1. The initial mass of the spacecraft is set to

1000 kg with a low-thrust engine of specific impulse of 3000 s.

	Mars	1989ML	Tempel-1
a [AU]	1.524	1.272	3.124
e [-]	0.093	0.137	0.517
I [deg]	1.850	4.378	10.527
h [deg]	49.557	104.411	68.933
$\omega$ [deg]	286.502	183.267	178.926

Table 1: Reference Keplerian elements of Mars, 1989ML and Tempel-1.

The shaping approach described in the previous sections has been implemented in MATLAB and coupled with the local NLP solver *fmincon* to properly tune the free parameters to optimize the trajectory  $\Delta V$  and respect the imposed constraints. The computations have been performed with a processor Intel Core i5-2410M on a laptop running Windows 7.

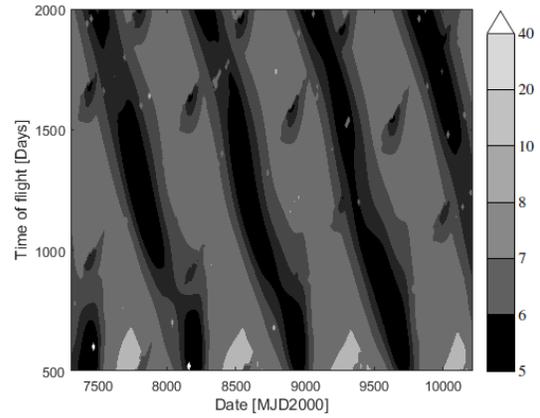
#### A Earth-Mars rendezvous

This mission scenario involves a 3D trajectory from Earth to Mars which has been already investigated with both the pseudo-equinoctial and the spherical shaping methods. Therefore, it represents a reliable basis for methods comparison. The search-space of the global variables in this test-case was discretized with a uniform grid as:<sup>3</sup>

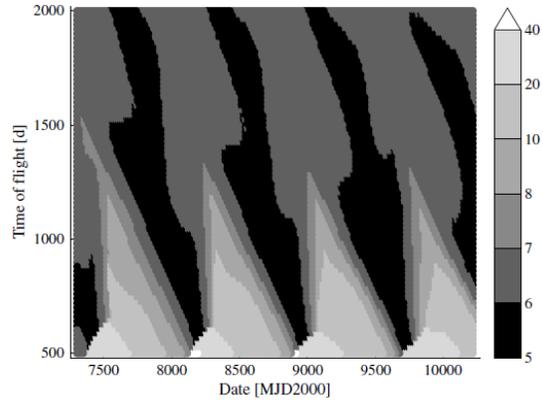
	Range	Step
T <sub>dep</sub>	01/01/2020-31/12/2027	15-day
TOF	500-2000	20-day
N <sub>rev</sub>	0-4	1-rev

Table 2: Earth-Mars discretization grid for global search.

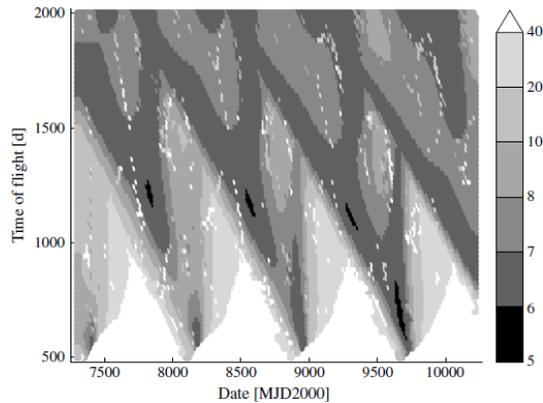
On top of the time of flight constraint, a maximum value of 0.2 N has been imposed on the peak thrust for the sake of comparison. Indeed, this value has been chosen as the best trajectory computed by the spherical shaping, which imposes no constraint on the thrust, has a peak value of 0.22 N (see Table 3). Figure 2 illustrates the  $\Delta V$  required for the rendezvous when computed by the Hill shaping, the spherical shaping and the pseudoequinoctial



(a) Hill shaping.



(b) Spherical Shaping.<sup>3</sup>



(c) Pseudoequinoctial Shaping.<sup>3</sup>

Fig. 2: Comparison of  $\Delta V$  estimation as function of departure date and time of flight when computed by different shaping methods for Earth-Mars rendezvous.

one, while Table 3 resumes the percentage of TOF feasible trajectories as well as the best-found ren-

dezzvous solution. The Hill shaping allows a wider

	Pseudo-Eq. <sup>3</sup>	Spherical <sup>3</sup>	Hill
$\Delta v$ [km/s]	5.83	5.74	5.67
Max Thrust [N]	0.16	0.22	0.14
Feasible traj.	89.1%	100%	99.7%

Table 3: Earth-Mars rendezvous mission results comparison.

set of low-cost mission when compared to the pseudo-equinoctial approach. While the spherical shaping computes the highest number of trajectories below 6 km/s, also because of the lack of thrust peak constraints, the Hill approach opens new inexpensive departure-duration combinations or it shapes differently the already cheap ones, still computing almost only feasible solutions. In terms of the single best trajectory, among all the shaping approaches, the Hill method finds the cheapest transfer with the lowest peak thrust, departing the June 28<sup>th</sup> of 2026 (9674.5 MJD2000) with a time of flight of 640 days. If the thrust peak constraint is removed as for the other two methods, the best trajectory found has a  $\Delta V$  of 5.61 km/s with maximum thrust of 0.32 N.

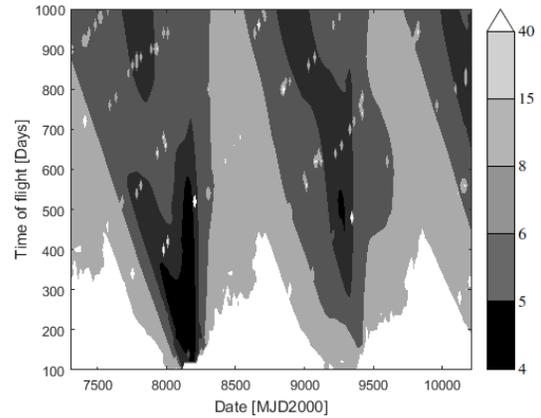
### B Earth-1989ML rendezvous

The systematic search has been performed on a grid defined as:

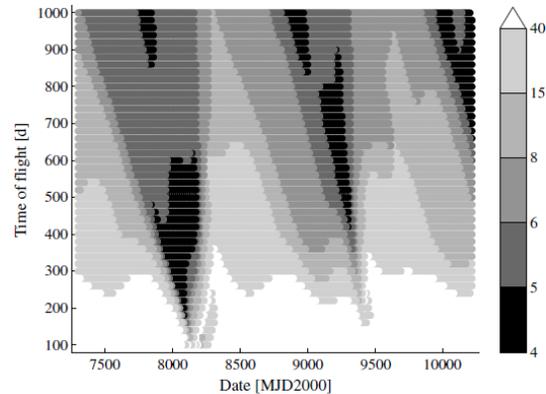
	Range	Step
$T_{dep}$	01/01/2020-31/12/2027	15-day
TOF	100-1000	20-day
$N_{rev}$	0-2	1-rev

Table 4: Earth-1989ML discretization grid for global search.

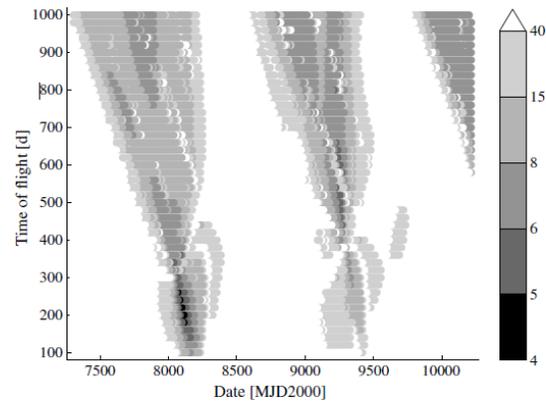
Within this discretization, the comparison of the global investigation with different methods are shown in Figure 3. All the three shapings catch the same areas where the transfer is more propellant convenient. Indeed, because the synodic period of the Earth-1989ML system is approximately 3.3 years, two quasi-periodic zones can be noticed. On a general basis the Hill shaping performs slightly worse than the spherical approach, probably because of the simpler radius shaping selected for this work. On the



(a) Hill shaping.



(b) Spherical Shaping.<sup>3</sup>



(c) Pseudoequinoctial Shaping.<sup>3</sup>

Fig. 3: Comparison of  $\Delta V$  estimation as function of departure date and time of flight when computed by different shaping methods for Earth-1989ML rendezvous.

contrary, the improvements with respect to the pseudo-equinoctial approach are widely distributed for all

the range of time-of-flight.

The best solutions computed by each shaping method are reported in Table 5.

	Pseudo-Eq. <sup>3</sup>	Spherical <sup>3</sup>	Hill
$\Delta v$ [km/s]	4.82	4.47	4.55
Max Thrust [N]	0.33	0.31	0.52
Feasible traj.	75.5%	83.7%	74.33%

Table 5: Earth-1989ML rendezvous mission results comparison.

The Hill solution for this test case requires a high peak thrust and higher  $\Delta V$  with respect to the spherical shaping. In addition, the percentage of feasible solutions decreases, in particular when short transfers are considered, probably due to the inability of the selected shaping for  $p$  to represent shorter oscillations.

### C Earth-Tempel 1 rendezvous

Tempel-1 is a Jupiter-family comet with a very eccentric and quite inclined orbit. This test case has been studied by McConaghy et Al.<sup>7</sup> with the exponential sinusoid, and by Novak and Vasile<sup>3</sup> with both the pseudoequinoctial and the spherical shaping methods. To run the Hill shaping and compare the performances, the search-space of the global variables is defined as:

	Range	Step
$T_{\text{dep}}$	01/01/2000-03/01/2016	15-day
TOF	400-1500	20-day
$N_{\text{rev}}$	0-2	1-rev

Table 6: Earth-Tempel 1 discretization grid for global search.

The information on the day-step for grid discretization was not available from previous literature, therefore the combinations investigated in this research could be slightly different from those studied with different shapings.

The best results' comparison is reported in Table 7, also resuming the percentage of TOF feasible trajectories. The Hill shaping provides a worse solution when compared to the spherical shaping in terms of  $\Delta V$ , but better in terms of peak thrust. This

result is probably caused by the secular shape chosen for The opposite happens when the Hill and pseudo-equinoctial methods are compared, while the exponential sinusoid solution requires the highest propellant ratio among all the approaches.

	Expsin <sup>7</sup>	Ps.-Eq. <sup>3</sup>	Spherical <sup>3</sup>	Hill
$\Delta v$ [km/s]	-	13.44	11.13	13.06
Prop. ratio	50%	36.7%	31.5%	35.8%
Max Thr. [N]	-	1.13	1.40	1.30
Feasible traj.	-	43.2%	68.1%	90.3%

Table 7: Earth-Tempel rendezvous mission results comparison.

The Hill's best trajectory, for which the spacecraft leaves the Earth the 14<sup>th</sup> April 2006 (2295 MJD2000) to reach Tempel-1 in 1500 days, is showed in Figure 4 as well as the corresponding  $F_r$  (blue),  $F_u$  (black) and  $F_n$  (red) profiles.

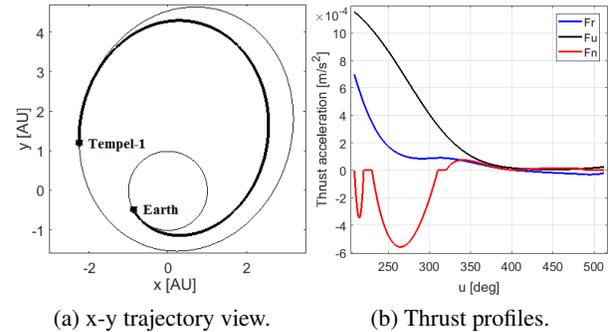


Fig. 4: Trajectory and thrust profile of Earth-Tempel rendezvous.

It is worth noting how, near the departure date, the spacecraft thrusts mainly in transversal and radial direction to raise the semi-major axis and increase the orbital eccentricity. Once the latter value is high enough, the radial component is reduced and mainly the orbit scale is changed by  $F_u$ . There are only two main out-of-plane thrust arcs because the whole trajectory takes less than one revolution.

Figure 5 depicts the  $\Delta V$  associated to each rendezvous on the discretized grid for different shaping methods. The three plots show that there is a periodic pattern following the synodic period of the departure and arrival orbits, and that trajectories with higher time-of-flight are favourable. The Hill method leads

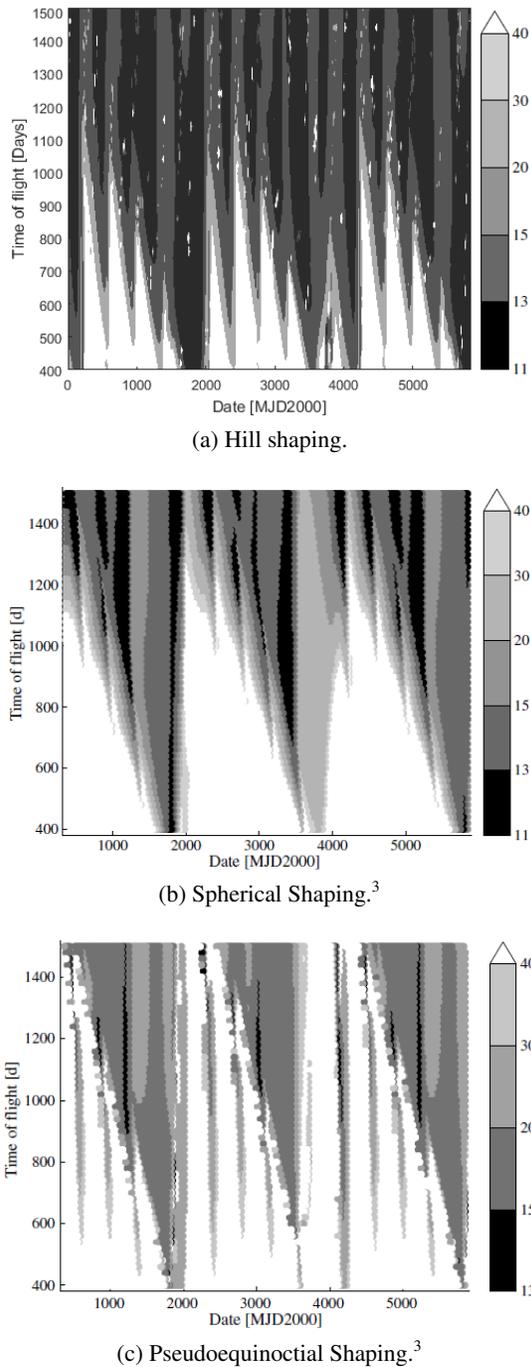


Fig. 5: Comparison of  $\Delta V$  estimation as function of departure date and time of flight when computed by different shaping methods for Earth-Tempel rendezvous.

to a widening of the launch and time-of-flight windows where the trajectory is achievable with a convenient consumption of propellant. Therefore, even

if the Hill's best solution is worse than the spherical shaping one, on average the method performs better on the majority of global variables' combinations, leading to 90% of time-of-flight feasible solutions.

## V. CONCLUSIONS

The paper presents the development of a novel shaping framework in Hill variables to quickly estimate the  $\Delta V$  and control profile of an orbital transfer. It has been shown how this formulation has the major novel advantage of separating the treatment of the in-plane and out-of-plane dynamics. Within this framework, it has been designed a specific piecewise shaping able to describe complete 3D trajectories thanks to the out-of-plane parameterization, and to respect constraints on time-of-flight and peak thrust thanks to the free parameters of the in-plane shaping. This method has been tested against three rendezvous cases and its strengths and weaknesses highlighted in comparison to other shaping approaches. The Hill shaping resulted in  $\Delta V$  optimal transfers on wider departure and time-of-flight windows, and it showed high flexibility in respecting the time-of-flight constraints.

As a future research idea, different shapes shall be tested for the in-plane trajectory as the selected parameterization did not perform optimally for some short transfers, while the out-of-plane motion can remain unaltered as it behaved properly on all the test cases. Again, it is possible to change only the in-plane shaping because of the dynamics separation presented in the framework development, the main advantage of the Hill formulation.

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