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Convergence rate and stability of the truncated Euler–Maruyama method for stochastic differential equations

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ABSTRACT

Recently, Mao (2015) developed a new explicit method, called the truncated Euler–Maruyama (EM) method, for the nonlinear SDE and established the strong convergence theory under the local Lipschitz condition plus the Khasminskii-type condition. In his another follow-up paper (Mao, 2016), he discussed the rates of L^q-convergence of the truncated EM method for $q \geq 2$ and showed that the order of L^q-convergence can be arbitrarily close to $q/2$ under some additional conditions. However, there are some restrictions on the truncation functions and these restrictions sometimes might force the step size to be so small that the truncated EM method would be inapplicable. The key aim of this paper is to establish the convergence rate without these restrictions. The other aim is to study the stability of the truncated EM method. The advantages of our new results will be highlighted by the comparisons with the results in Mao (2015, 2016) as well as others on the tamed EM and implicit methods.

1. Introduction

Influenced by Higham, Mao and Stuart [1], the strong convergence theory of numerical methods for nonlinear stochastic differential equations (SDEs) without the global Lipschitz condition has become more and more popular. Although the classical Euler–Maruyama (EM) method is convenient for computations and implementations, the absolute moments of its approximation for SDEs with super-linear coefficients diverge to infinite at a finite time (see, e.g., [2]). Many implicit methods were used to study the numerical solutions to SDEs with nonlinear coefficients (see, e.g., [1,3–7]). Especially, Higham, Mao and Stuart [1] proved that the implicit EM numerical solutions converge strongly to the exact solutions of SDEs with globally one-sided Lipschitz continuous drift term and globally Lipschitz diffusion term, but the explicit EM method fails to do that. For the background on the implicit methods, we refer the reader to the books [8–10]. However, it is demonstrated that the implementation of the implicit EM method requires more computational effort. Recently, due to the advantages of explicit methods, Hutzenthaler, Jentzen and Kloeden proposed an explicit method for such SDEs called tamed Euler method whose numerical solutions converge strongly to the exact solution with 1/2 order. Sabanis in [11] went a further step to propose the modified tamed Euler method approximating the SDEs with superlinearly growing drift and diffusion coefficients, moreover, recovered the strong order 1/2 in the estimation of convergence rate. Other explicit methods, such as the stopped EM method, as well as the tamed Milstein method, have been further developed (see, e.g., [12,13] for details).

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In particular, Mao [14] in 2015 proposed a new explicit method, called the truncated EM method. In his another follow-up paper [15], he investigated the convergence rates for the method under some additional conditions. We will point out that some of these additional conditions might force the step size to be so small that the truncated EM method would be inapplicable. One of our key aims in this paper is to establish the convergence rate without these restrictions so that the truncated EM method is more widely implementable. To overcome the difficulties due to removing these restrictions, some new mathematical techniques, which are significantly different from those used in [15], have been developed.

A nice numerical method should not only have an acceptable finite-time convergence rate but also have the ability to preserve the asymptotic properties of the underlying SDEs (see, e.g., [16,17]). Another aim of this paper is to show the ability of the truncated EM method to preserve the asymptotic stability of the underlying SDEs.

To show the advantages of the truncated EM method, we will compare it with other methods, e.g., the implicit EM method, the tamed Euler method and the modified tamed Euler method. We will design two numerical experiments and compute the errors between the true solution and the numerical solutions obtained by different schemes. It turns out that to achieve the same accuracy, the runtime of the truncated EM method and of the tamed Euler method are almost equivalent, but much shorter than that of the implicit EM method. However, to achieve the same accuracy, the step size for the modified tame Euler method is required to be smaller than that for the truncated EM method. These show clearly that the truncated EM method might be more efficient and is certainly suitable for the highly nonlinear SDEs.

The rest of the paper is organized as follows. Section 2 gives some notation and preliminary results on the numerical solution of the truncated EM method. Section 3 begins to demonstrate the improved convergence rate in a finite time interval. Section 4 goes further to compare our result with the previous convergence rate results. Section 5 makes use of the truncated EM method to approximate the asymptotic stability. Section 6 concludes our main results. The Appendix proves that the classical EM method cannot reproduce asymptotic stability while the truncated method does.

2. Notation and lemmas

Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (that is, it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets), and let $\mathbb{E}$ denote the probability expectation with respect to $\mathbb{P}$. Let $B(t)$ be an m-dimensional Brownian motion defined on the probability space and is $\mathcal{F}_t$-adapted. If A is a vector or matrix, its transpose is denoted by A^T. If $x \in \mathbb{R}^d$, then $|x|$ is the Euclidean norm. If A is a matrix, we let $|A| = \sqrt{\text{trace}(A^TA)}$ be its trace norm. Moreover, for two real numbers a and b, we use $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$. For a set G, its indicator function is denoted by I_G, namely $I_G(x) = 1$ if $x \in G$ and 0 otherwise.

Consider a d-dimensional nonlinear SDE

$$dx(t) = f(x(t))dt + g(x(t))dB(t), \quad t \geq 0, \tag{2.1}$$

with the initial value $x(0) = x_0 \in \mathbb{R}^d$, where $f : \mathbb{R}^d \to \mathbb{R}^d$ and $g : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are Borel measurable. We impose two standing hypotheses in this paper.

Assumption 2.1. Assume that the coefficients f and g satisfy the local Lipschitz condition: For any $R > 0$, there is a $K_R > 0$ such that

$$|f(x) - f(y)| \vee |g(x) - g(y)| \leq K_R|x - y| \tag{2.2}$$

for all $x, y \in \mathbb{R}^d$ with $|x| \vee |y| \leq R$.

Assumption 2.2. Assume that the coefficients satisfy the Khasminskii-type condition: There is a pair of constants $p > 2$ and $K > 0$ such that

$$x^Tf(x) + \frac{p-1}{2}|g(x)|^2 \leq K(1 + |x|^2) \tag{2.3}$$

for all $x \in \mathbb{R}^d$.

We state a known result (see, e.g., [18,19]) as a lemma for the use of this paper.

Lemma 2.3. *Under Assumptions 2.1 and 2.2, the SDE (2.1) has a unique global solution $x(t)$ and, moreover,*

$$\sup_{0\leq t\leq T} \mathbb{E}|x(t)|^p < \infty, \quad \forall T > 0. \tag{2.4}$$

Recall the truncated EM numerical scheme defined in [14]. We first choose a strictly increasing continuous function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\mu(u) \to \infty$ as $u \to \infty$ and

$$\sup_{|x|\leq u}\big(|f(x)| \vee |g(x)|\big) \leq \mu(u), \quad \forall u \geq 1. \tag{2.5}$$

Denote by μ^{-1} the inverse function of μ and we see that μ^{-1} is a strictly increasing continuous function from $[\mu(1),\infty)$ to $\mathbb{R}_+$. We also choose a constant $\hat{h}\geq 1\vee\mu(1)$ and a strictly decreasing function $h:(0,1]\to[\mu(1),\infty)$ such that

$$\lim_{\Delta\to 0} h(\Delta)=\infty \quad\text{and}\quad \Delta^{1/4}h(\Delta)\leq \hat{h},\quad \forall\Delta\in(0,1]. \tag{2.6}$$

We will see later that Assumption 3.2 implies (3.4), namely that both coefficients f and g grow at most polynomially, whence we can let $\mu(u)=H_3u^{1+0.5\rho}$, where H_3 is a positive constant specified in (3.4). Moreover, we can let $h(\Delta)=\hat{h}\Delta^{-\varepsilon}$ for some $\varepsilon\in(0,1/4]$. In other words, there are lots of choices for $\mu(\cdot)$ and $h(\cdot)$. Before we proceed, let us make a useful remark.

Remark 2.4. In Mao [14] where the truncated EM was originally developed, it was required to choose a number $\Delta^*\in(0,1]$ and a strictly decreasing function $h:(0,\Delta^*]\to[\mu(0),\infty)$ such that

$$h(\Delta^*)\geq\mu(2),\quad \lim_{\Delta\to0}h(\Delta)=\infty \quad\text{and}\quad \Delta^{1/4}h(\Delta)\leq 1,\ \forall\Delta\in(0,\Delta^*].$$

Here, we simply let $\Delta^*=1$ and remove condition $h(\Delta^*)\geq\mu(2)$ while we also replace condition $\Delta^{1/4}h(\Delta)\leq 1$ by a weaker one $\Delta^{1/4}h(\Delta)\leq\hat{h}$. In other words, we have made the choice of function h more flexible. We emphasize that such changes do not make any effect on the results in Mao [14,15]. In fact, condition $h(\Delta^*)\geq\mu(2)$ was only used to prove [14, Lemma 2.4]. But, in view of Lemma 2.5, we see that the constant $2K$ in [14, Lemma 2.4] is now replaced by another constant $\hat{K}$ which does not affect any other results in [14]. It is also easy to check that replacing $\Delta^{1/4}h(\Delta)\leq 1$ by $\Delta^{1/4}h(\Delta)\leq\hat{h}$ does not make any effect on the other results in [14]. Similarly, we see that these changes do not affect any results in [15] either.

For a given step size $\Delta\in(0,1]$, let us define the truncated mapping $\pi_\Delta:\mathbb{R}^d\to\{x\in\mathbb{R}^d:|x|\leq\mu^{-1}(h(\Delta))\}$ by

$$\pi_\Delta(x)=\big(|x|\wedge\mu^{-1}(h(\Delta))\big)\frac{x}{|x|},$$

where we set $x/|x|=0$ when $x=0$. That is, π_Δ maps x to itself if $|x|\leq\mu^{-1}(h(\Delta))$ and to $\mu^{-1}(h(\Delta))x/|x|$ if $|x|>\mu^{-1}(h(\Delta))$. Define the truncated functions

$$f_\Delta(x)=f(\pi_\Delta(x))\quad\text{and}\quad g_\Delta(x)=g(\pi_\Delta(x)) \tag{2.7}$$

for $x\in\mathbb{R}^d$. It is easy to see that

$$|f_\Delta(x)|\vee|g_\Delta(x)|\leq\mu(\mu^{-1}(h(\Delta)))=h(\Delta),\quad\forall x\in\mathbb{R}^d. \tag{2.8}$$

The discrete-time truncated EM numerical solutions $X_\Delta(t_k)\approx x(t_k)$ for $t_k=k\Delta$ are formed by setting $X_\Delta(0)=x_0$ and computing

$$X_\Delta(t_{k+1})=X_\Delta(t_k)+f_\Delta(X_\Delta(t_k))\Delta+g_\Delta(X_\Delta(t_k))\Delta B_k, \tag{2.9}$$

for $k=0,1,\ldots$, where $\Delta B_k=B(t_{k+1})-B(t_k)$. There are two versions of the continuous-time truncated EM solutions. The first one is defined by

$$\bar{x}_\Delta(t)=\sum_{k=0}^{\infty}X_\Delta(t_k)I_{[t_k,t_{k+1})}(t),\quad t\geq0. \tag{2.10}$$

This is a simple step process so its sample paths are not continuous. We will refer to it as the continuous-time step-process truncated EM solution. The other one is defined by

$$x_\Delta(t)=x_0+\int_0^t f_\Delta(\bar{x}_\Delta(s))ds+\int_0^t g_\Delta(\bar{x}_\Delta(s))dB(s) \tag{2.11}$$

for $t\geq0$. We will refer to it as the continuous-time continuous-sample truncated EM solution. We observe that $x_\Delta(t_k)=\bar{x}_\Delta(t_k)=X_\Delta(t_k)$ for all $k\geq0$. Moreover, $x_\Delta(t)$ is an Itô process with its Itô differential

$$dx_\Delta(t)=f_\Delta(\bar{x}_\Delta(t))dt+g_\Delta(\bar{x}_\Delta(t))dB(t). \tag{2.12}$$

The following lemma shows that the truncated functions f_Δ and g_Δ preserve Assumption 2.2 very well.

Lemma 2.5. *Let Assumption 2.2 hold. Then, for all $\Delta\in(0,1]$, we have*

$$x^Tf_\Delta(x)+\frac{p-1}{2}|g_\Delta(x)|^2\leq\hat{K}(1+|x|^2),\quad\forall x\in\mathbb{R}^d, \tag{2.13}$$

where $\hat{K}=2K\big(1\vee[1/\mu^{-1}(h(1))]\big)$.

Proof. This lemma was essentially proved in [14] but we here do not need condition $h(\Delta^*) \ge \mu(2)$ as we already pointed out in Remark 2.4.

Fix any $\Delta \in (0, 1]$. For $x \in \mathbb{R}^d$ with $|x| \le \mu^{-1}(h(\Delta))$, the required assertion (2.13) holds clearly. For $x \in \mathbb{R}^d$ with $|x| > \mu^{-1}(h(\Delta))$, the proof of [14, Lemma 2.4] shows that

$$x^T f_\Delta(x) + \frac{p-1}{2}|g_\Delta(x)|^2 \le \frac{|x|}{\mu^{-1}(h(\Delta))} K(1 + [\mu^{-1}(h(\Delta))]^2).$$

Noting that $\mu^{-1}(h(\Delta)) \ge \mu^{-1}(h(1))$, we then derive

$$\begin{aligned} x^T f_\Delta(x) + \frac{p-1}{2}|g_\Delta(x)|^2 &\le K|x|\Big(\frac{1}{\mu^{-1}(h(1))} + |x|\Big) \\ &\le K\big(1 \vee [1/\mu^{-1}(h(1))]\big)(|x| + |x|^2) \le 0.5\hat{K}(1 + 2|x|^2) \le \hat{K}(1 + |x|^2) \end{aligned}$$

as required. □

Recalling Remark 2.4, we can then cite two lemmas from [14] on the continuous-time truncated EM solutions defined by (2.10) and (2.11) for the use of this paper.

Lemma 2.6. *For any $\Delta \in (0, 1]$ and any $\hat{p} > 0$, we have*

$$\mathbb{E}|x_\Delta(t) - \bar{x}_\Delta(t)|^{\hat{p}} \le c_{\hat{p}} \Delta^{\hat{p}/2}(h(\Delta))^{\hat{p}}, \quad \forall t \ge 0, \tag{2.14}$$

where $c_{\hat{p}}$ is a positive constant dependent only on $\hat{p}$. Consequently

$$\lim_{\Delta \to 0} \mathbb{E}|x_\Delta(t) - \bar{x}_\Delta(t)|^{\hat{p}} = 0, \quad \forall t \ge 0. \tag{2.15}$$

Lemma 2.7. *Let Assumptions 2.1* and *2.2 hold. Then*

$$\sup_{0<\Delta\le 1} \Big(\sup_{0 \le t \le T} \mathbb{E}|x_\Delta(t)|^p \Big) \le C, \quad \forall T > 0, \tag{2.16}$$

where, and from now on, C stands for generic positive real constants dependent on T, p, K, x_0, etc. but independent of Δ and its values may change between occurrences.

3. Convergence rates

Mao [14] established the theory of L^q-convergence for $2 \le q < p$ for the truncated EM method, where p is a parameter in Assumption 2.2. However, the convergence was in the asymptotic form without the convergence rate. Recently, Mao [15] investigated the convergence rates for the method under some additional conditions. However, there are some restrictions on the functions $\mu(\cdot)$ and $h(\cdot)$ and these restrictions sometimes force the step size to be so small that the truncated EM method is inapplicable. We are now going to establish the convergence rates without these restrictions. We need some additional conditions.

Assumption 3.1. Assume that there is a pair of constants $q > 2$ and $H_1 > 0$ such that

$$(x-y)^T(f(x) - f(y)) + \frac{q-1}{2}|g(x) - g(y)|^2 \le H_1|x-y|^2 \tag{3.1}$$

for all $x, y \in \mathbb{R}^d$.

Assumption 3.2. Assume that there is a pair of positive constants ρ and H_2 such that

$$|f(x) - f(y)|^2 \vee |g(x) - g(y)|^2 \le H_2(1 + |x|^\rho + |y|^\rho)|x-y|^2 \tag{3.2}$$

for all $x, y \in \mathbb{R}^d$.

It is useful to observe that the truncated functions f_Δ and g_Δ preserve Assumption 3.2 perfectly. In fact, we derive that

$$\begin{aligned} |f_\Delta(x) - f_\Delta(y)|^2 \vee |g_\Delta(x) - g_\Delta(y)|^2 &= |f(\pi_\Delta(x)) - f(\pi_\Delta(y))|^2 \vee |g(\pi_\Delta(x)) - g(\pi_\Delta(y))|^2 \\ &\le H_2(1 + |\pi_\Delta(x)|^\rho + |\pi_\Delta(y)|^\rho)\,|\pi_\Delta(x) - \pi_\Delta(y)|^2 \end{aligned}$$

for all $x, y \in \mathbb{R}^d$. Noting

$$|\pi_\Delta(x)| \le |x|, \quad |\pi_\Delta(y)| \le |y|, \quad |\pi_\Delta(x) - \pi_\Delta(y)|^2 \le |x-y|^2,$$

we get

$$|f_\Delta(x) - f_\Delta(y)|^2 \vee |g_\Delta(x) - g_\Delta(y)|^2 \le H_2(1 + |x|^\rho + |y|^\rho)|x - y|^2. \tag{3.3}$$

Moreover, we also observe from Assumption 3.2 that

$$|f(x)| \vee |g(x)| \le H_3|x|^{(2+\rho)/2}, \quad \forall |x| \ge 1, \tag{3.4}$$

where $H_3 = \sqrt{2H_2} + |f(0)| + |g(0)|$.

To point out the restrictive condition imposed in [15], we cite its main result on the convergence rate.

Theorem 3.3 (*[15]*). *Let Assumptions 2.1, 2.2, 3.1* and *3.2 hold with $p > q$ and $2p > q\rho$. Let $\bar{q} \in [2, q)$ (we have $q > 2$ in Assumption 3.1). If*

$$h(\Delta) \ge \mu\big([\Delta^{\bar{q}/2}(h(\Delta))^{\bar{q}}]^{-1/(p-\bar{q})}\big) \tag{3.5}$$

for all sufficiently small $\Delta \in (0, 1]$, then, for every such small Δ,

$$\mathbb{E}|x(T) - x_\Delta(T)|^{\bar{q}} \le C\Delta^{\bar{q}/2}(h(\Delta))^{\bar{q}} \quad \text{and} \quad \mathbb{E}|x(T) - \bar{x}_\Delta(T)|^{\bar{q}} \le C\Delta^{\bar{q}/2}(h(\Delta))^{\bar{q}}. \tag{3.6}$$

In particular, if we choose $h(u) = \Delta^{-\varepsilon}$ for $\varepsilon \in (0, 0.25]$, it then follows from (3.6) that

$$\mathbb{E}|x(T) - x_\Delta(T)|^{\bar{q}} = O(\Delta^{\bar{q}(1-2\varepsilon)/2}) \quad \text{and} \quad \mathbb{E}|x(T) - \bar{x}_\Delta(T)|^{\bar{q}} = O(\Delta^{\bar{q}(1-2\varepsilon)/2}). \tag{3.7}$$

This theorem shows that the truncated EM method has the order of $L^{\bar{q}}$-convergence close to $\bar{q}/2$. This is almost optimal in theory if we recall that the classical EM method has order $\bar{q}/2$ of $L^{\bar{q}}$-convergence. However, condition (3.5) could sometimes make the truncated EM method impracticable. For example, consider the case where $p = 6$, $\bar{q} = 2$, $\mu(u) = 100u^{4/3}$ and $h(u) = u^{-0.25}$. Then (3.5) becomes $\Delta \le 10^{-24}$. In other words, the step size is required to be extremely small. The key aim of this paper is to remove condition (3.5) and still to be able to establish the theory of the strong convergence rates. To overcome the difficulties without imposing condition (3.5), we develop some new mathematical techniques, which are significantly different from those used in [15], and get the following result on the $L^{\bar{q}}$-convergence rate. From now on, we will fix $T > 0$ arbitrarily.

Theorem 3.4. *Let Assumptions 2.1, 2.2, 3.1* and *3.2 hold and assume that $2p > (2+\rho)q$. Then, for any $\bar{q} \in [2, q)$ and $\Delta \in (0, 1]$,*

$$\mathbb{E}|x(T) - x_\Delta(T)|^{\bar{q}} \le C\Big((\mu^{-1}(h(\Delta)))^{-(2p-(2+\rho)\bar{q})/2} + \Delta^{\bar{q}/2}(h(\Delta))^{\bar{q}}\Big) \tag{3.8}$$

and

$$\mathbb{E}|x(T) - \bar{x}_\Delta(T)|^{\bar{q}} \le C\Big((\mu^{-1}(h(\Delta)))^{-(2p-(2+\rho)\bar{q})/2} + \Delta^{\bar{q}/2}(h(\Delta))^{\bar{q}}\Big). \tag{3.9}$$

In particular, recalling (3.4), we may define

$$\mu(u) = H_3u^{(2+\rho)/2}, \quad u \ge 1, \tag{3.10}$$

and let

$$h(\Delta) = \Delta^{-\varepsilon} \quad \text{for some } \varepsilon \in (0, 1/4] \text{ and } \hat{h} \ge 1, \tag{3.11}$$

to get

$$\mathbb{E}|x(T) - x_\Delta(T)|^{\bar{q}} = O\big(\Delta^{[\varepsilon(2p-(2+\rho)\bar{q})/(2+\rho)]\wedge[\bar{q}(1-2\varepsilon)/2]}\big) \tag{3.12}$$

and

$$\mathbb{E}|x(T) - \bar{x}_\Delta(T)|^{\bar{q}} = O\big(\Delta^{[\varepsilon(2p-(2+\rho)\bar{q})/(2+\rho)]\wedge[\bar{q}(1-2\varepsilon)/2]}\big). \tag{3.13}$$

Proof. Fix $\bar{q} \in [2, q)$ and $\Delta \in (0, 1]$ arbitrarily. Let $e_\Delta(t) = x(t) - x_\Delta(t)$ for $t \ge 0$. For each integer $n > |x_0|$, define the stopping time

$$\theta_n = \inf\{t \ge 0 : |x(t)| \vee |x_\Delta(t)| \ge n\},$$

where we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). By the Itô formula, we have that for any $0 \le t \le T$,

$$\mathbb{E}|e_\Delta(t \wedge \theta_n)|^{\bar{q}} \le \mathbb{E}\int_0^{t\wedge\theta_n} \bar{q}|e_\Delta(s)|^{\bar{q}-2}\Big(e_\Delta^T(s)[f(x(s)) - f_\Delta(\bar{x}_\Delta(s))] + \frac{\bar{q}-1}{2}|g(x(s)) - g_\Delta(\bar{x}_\Delta(s))|^2\Big)ds. \tag{3.14}$$

Noting

$$\begin{aligned}&\frac{\bar q-1}{2}|g(x(s))-g_\Delta(\bar x_\Delta(s))|^2\\ \le\ &\frac{\bar q-1}{2}\Big[\Big(1+\frac{q-\bar q}{\bar q-1}\Big)|g(x(s))-g(x_\Delta(s))|^2+\Big(1+\frac{\bar q-1}{q-\bar q}\Big)|g(x_\Delta(s))-g_\Delta(\bar x_\Delta(s))|^2\Big]\\ =\ &\frac{q-1}{2}|g(x(s))-g(x_\Delta(s))|^2+\frac{(\bar q-1)(q-1)}{2(q-\bar q)}|g(x_\Delta(s))-g(\bar x_\Delta(s))|^2,\end{aligned}$$

we get from (3.14) that

$$\mathbb{E}|e_\Delta(t\wedge\theta_n)|^{\bar q}\le J_1+J_2, \tag{3.15}$$

where

$$J_1=\mathbb{E}\int_0^{t\wedge\theta_n}\bar q|e_\Delta(s)|^{\bar q-2}\Big(e_\Delta^T(s)[f(x(s))-f(x_\Delta(s))]+\frac{q-1}{2}|g(x(s))-g(x_\Delta(s))|^2\Big)ds \tag{3.16}$$

and

$$J_2=\mathbb{E}\int_0^{t\wedge\theta_n}\bar q|e_\Delta(s)|^{\bar q-2}\Big(e_\Delta^T(s)[f(x_\Delta(s))-f_\Delta(\bar x_\Delta(s))]+\frac{(\bar q-1)(q-1)}{2(q-\bar q)}|g(x_\Delta(s))-g_\Delta(\bar x_\Delta(s))|^2\Big)ds. \tag{3.17}$$

By Assumption 3.1, we have

$$J_1\le\bar qH_1\mathbb{E}\int_0^{t\wedge\theta_n}|e_\Delta(s)|^{\bar q}ds. \tag{3.18}$$

Rearranging J_2, we get

$$\begin{aligned}J_2\le\ &\mathbb{E}\int_0^{t\wedge\theta_n}\bar q|e_\Delta(s)|^{\bar q-2}\Big(e_\Delta^T(s)[f(x_\Delta(s))-f_\Delta(x_\Delta(s))]\\ &\qquad+\frac{(\bar q-1)(q-1)}{(q-\bar q)}|g(x_\Delta(s))-g_\Delta(x_\Delta(s))|^2\Big)ds\\ &+\mathbb{E}\int_0^{t\wedge\theta_n}\bar q|e_\Delta(s)|^{\bar q-2}\Big(e_\Delta^T(s)[f_\Delta(x_\Delta(s))-f_\Delta(\bar x_\Delta(s))]\\ &\qquad+\frac{(\bar q-1)(q-1)}{(q-\bar q)}|g_\Delta(x_\Delta(s))-g_\Delta(\bar x_\Delta(s))|^2\Big)ds\\ &=:J_{21}+J_{22}.\end{aligned} \tag{3.19}$$

We estimate J_{21} first. By the Young inequality $a^{\bar q-2}b\le(\bar q-2)a^{\bar q}/\bar q+2b^{\bar q/2}/\bar q$ for any $a,b\ge0$ and $0\le t\wedge\theta_n\le t\le T$, we can show that

$$\begin{aligned}J_{21}\le\ &\mathbb{E}\int_0^{t\wedge\theta_n}\bar q|e_\Delta(s)|^{\bar q-2}\Big(0.5|e_\Delta(s)|^2+0.5|f(x_\Delta(s))-f_\Delta(x_\Delta(s))|^2\\ &\qquad+\frac{(\bar q-1)(q-1)}{(q-\bar q)}|g(x_\Delta(s))-g_\Delta(x_\Delta(s))|^2\Big)ds\\ \le\ &\frac{(\bar q-1)^2(q-2)}{(q-\bar q)}\mathbb{E}\int_0^{t\wedge\theta_n}|e_\Delta(s)|^{\bar q}ds+\mathbb{E}\int_0^{t\wedge\theta_n}|f(x_\Delta(s))-f_\Delta(x_\Delta(s))|^{\bar q}ds\\ &+\frac{2(\bar q-1)(q-1)}{(q-\bar q)}\mathbb{E}\int_0^{t\wedge\theta_n}|g(x_\Delta(s))-g_\Delta(x_\Delta(s))|^{\bar q}ds\\ \le\ &C_1\mathbb{E}\int_0^{t\wedge\theta_n}|e_\Delta(s)|^{\bar q}ds+J_{23},\end{aligned} \tag{3.20}$$

where

$$J_{23}=C_1\,\mathbb{E}\int_0^{t\wedge\theta_n}\Big(|f(x_\Delta(s))-f_\Delta(x_\Delta(s))|^{\bar q}+|g(x_\Delta(s))-g_\Delta(x_\Delta(s))|^{\bar q}\Big)ds$$

and

$$C_1=\max\left\{\frac{(\bar q-1)^2(q-2)}{(q-\bar q)},1,\frac{2(\bar q-1)(q-1)}{(q-\bar q)}\right\}.$$

Due to $t \wedge \theta_n \le T$ and Assumption 3.2, we derive that

$$J_{23} \le C_1 \mathbb{E} \int_0^T \Big(|f(x_\Delta(s)) - f(\pi_\Delta(x_\Delta(s)))|^{\bar q} + |g(x_\Delta(s)) - g(\pi_\Delta(x_\Delta(s)))|^{\bar q} \Big) ds$$
$$\le 2 \times 3^{\bar q/2} H_2 C_1 \int_0^T \mathbb{E}\Big((1 + |x_\Delta(s)|^{\rho \bar q/2} + |\pi_\Delta(x_\Delta(s))|^{\rho\bar q/2})\, |x_\Delta(s) - \pi_\Delta(x_\Delta(s))|^{\bar q} \Big) ds$$
$$\le 4 \times 3^{\bar q/2} H_2 C_1 \int_0^T \mathbb{E}\Big((1 + |x_\Delta(s)|^{\rho \bar q/2})\, |x_\Delta(s) - \pi_\Delta(x_\Delta(s))|^{\bar q} \Big) ds.$$

Using the Hölder inequality and Lemma 2.7 yields

$$J_{23} \le 4 \times 3^{\bar q/2} H_2 C_1 \int_0^T \Big(\mathbb{E}(1 + |x_\Delta(s)|^p) \Big)^{\frac{\rho \bar q}{2p}} \Big(\mathbb{E}|x_\Delta(s) - \pi_\Delta(x_\Delta(s))|^{\frac{2p\bar q}{2p-\rho\bar q}} \Big)^{\frac{2p-\rho\bar q}{2p}} ds$$
$$\le 4 \times 3^{\bar q/2} H_2 C_1 (C+1)^{\frac{\rho\bar q}{2p}} \int_0^T \Big(\mathbb{E}\Big[I_{\{|x_\Delta(s)| > \mu^{-1}(h(\Delta))\}} |x_\Delta(s)|^{\frac{2p\bar q}{2p-\rho\bar q}} \Big] \Big)^{\frac{2p-\rho\bar q}{2p}} ds$$
$$\le 4 \times 3^{\bar q/2} H_2 C_1 (C+1)^{\frac{\rho\bar q}{2p}} \int_0^T \Big(\Big[\mathbb{P}\{|x_\Delta(s)| > \mu^{-1}(h(\Delta))\} \Big]^{\frac{2p-(2+\rho)\bar q}{2p-\rho\bar q}} \Big[\mathbb{E}|x_\Delta(s)|^p \Big]^{\frac{2\bar q}{2p-\rho\bar q}} \Big)^{\frac{2p-\rho\bar q}{2p}} ds$$
$$\le 4 \times 3^{\bar q/2} H_2 C_1 (C+1)^{\frac{\bar q(\rho+2)}{2p}} \int_0^T \Big(\frac{\mathbb{E}|x_\Delta(s)|^p}{(\mu^{-1}(h(\Delta)))^p} \Big)^{\frac{2p-(2+\rho)\bar q}{2p}} ds$$
$$\le 4 \times 3^{\bar q/2} H_2 C_1 (C+1) (\mu^{-1}(h(\Delta)))^{-\frac{2p-(2+\rho)\bar q}{2}}.$$

Substituting this into (3.20) gives

$$J_{21} \le C_1 \mathbb{E} \int_0^{t\wedge\theta_n} |e_\Delta(s)|^{\bar q} ds + 4 \times 3^{\bar q/2} C_1 (C+1) (\mu^{-1}(h(\Delta)))^{-\frac{2p-(2+\rho)\bar q}{2}}. \tag{3.21}$$

Similarly, we can show

$$J_{22} \le C_3 \mathbb{E} \int_0^{t\wedge\theta_n} |e_\Delta(s)|^{\bar q} ds + C_4 \mathbb{E} \int_0^T \Big(\mathbb{E}|x_\Delta(s) - \bar x_\Delta(s)|^{2p\bar q/(2p-\rho\bar q)} \Big)^{(2p-\rho\bar q)/2p} ds, \tag{3.22}$$

where C_3, C_4 and the following C_5, etc. are generic constants independent of Δ. By Lemma 2.6, we then have

$$J_{22} \le C_2 \mathbb{E} \int_0^{t\wedge\theta_n} |e_\Delta(s)|^{\bar q} ds + C_5 \Delta^{\bar q/2} (h(\Delta))^{\bar q}. \tag{3.23}$$

Combining (3.15), (3.18), (3.19), (3.21) and (3.23) together, we get

$$\mathbb{E}|e_\Delta(t\wedge\theta_n)|^{\bar q}$$
$$\le C_6 \Big(\mathbb{E} \int_0^{t\wedge\theta_n} |e_\Delta(s)|^{\bar q} ds + (\mu^{-1}(h(\Delta)))^{-(2p-(2+\rho)\bar q)/2} + \Delta^{\bar q/2}(h(\Delta))^{\bar q} \Big)$$
$$\le C_6 \Big(\int_0^t \mathbb{E}|e_\Delta(s\wedge\theta_n)|^{\bar q} ds + (\mu^{-1}(h(\Delta)))^{-(2p-(2+\rho)\bar q)/2} + \Delta^{\bar q/2}(h(\Delta))^{\bar q} \Big). \tag{3.24}$$

An application of the Gronwall inequality yields that

$$\mathbb{E}|e_\Delta(T\wedge\theta_n)|^{\bar q} \le C_7 \Big((\mu^{-1}(h(\Delta)))^{-(2p-(2+\rho)\bar q)/2} + \Delta^{\bar q/2}(h(\Delta))^{\bar q} \Big).$$

Using the well-known Fatou lemma, we can let $n \to \infty$ to obtain the desired assertion (3.8). The other assertion (3.9) follows from (3.8) and Lemma 2.6. Finally, when μ is defined by (3.10), then $\mu^{-1}(u) = (u/H_3)^{2/(2+\rho)}$. Substituting this and (3.11) into (3.8) we get

$$\mathbb{E}|x(T) - x_\Delta(T)|^{\bar q} \le C\Big(\Delta^{\varepsilon(2p-(2+\rho)\bar q)/(2+\rho)} + \Delta^{\bar q(1-2\varepsilon)/2} \Big),$$

which is the required assertion (3.12). Similarly, we can show (3.13). The proof is therefore complete. □

The following theorem shows that the order of $L^{\bar q}$-convergence could be close to $\bar q/2$ arbitrarily.

Theorem 3.5. *Let Assumptions 2.1, 3.1 and 3.2 hold and let Assumption 2.2 hold for any $p > 2$. Let $\mu(\cdot)$ and $h(\cdot)$ be defined by (3.10) and (3.11). Then, for any $\bar q \in [2, q)$ and any $\varepsilon \in (0, 1/4)$,*

$$\mathbb{E}|x(T) - x_\Delta(T)|^{\bar q} \le O\big(\Delta^{\bar q(1-2\varepsilon)/2}\big) \tag{3.25}$$

and

$$\mathbb{E}|x(T)-\bar{x}_\Delta(T)|^{\bar{q}} \le O\big(\Delta^{\bar{q}(1-2\varepsilon)/2}\big). \tag{3.26}$$

Proof. Choosing p sufficiently large for

$$\varepsilon(2p-(2+\rho)\bar{q})/(2+\rho) > \bar{q}(1-2\varepsilon)/2,$$

we can get the assertions from (3.12) and (3.13) easily. □

This theorem shows that the order of $L^{\bar{q}}$-convergence can be close to $\bar{q}/2$ arbitrarily. This is almost optimal if we recall that the classical EM method has order $\bar{q}/2$ of $L^{\bar{q}}$-convergence under the global Lipschitz condition. Let us discuss an example to illustrate our theory before we make some comparisons to highlight the advantages of our new results on the convergence rates.

Example 3.6. Consider the scalar stochastic Ginzburgh–Landau equation (see, e.g., [8,20,21])

$$dx(t) = (ax(t) - bx^3(t))dt + cx(t)dB(t), \tag{3.27}$$

with $x(0) = x_0$, where $B(t)$ is a scalar Brownian motion and a, b, c are three positive numbers. Clearly, its coefficients $f(x) = ax - bx^3$ and $g(x) = cx$ are locally Lipschitz continuous for $x \in \mathbb{R}$, namely, satisfy Assumption 2.1. Also, for any $p > 2$, we have

$$xf(x) + \frac{p-1}{2}|g(x)|^2 = ax^2 - bx^4 + \frac{(p-1)c^2}{2}x^2 \le \frac{1}{16b}(2a+(p-1)c^2)^2.$$

That is, Assumption 2.2 is satisfied for any $p > 2$. Moreover, for any $q > 2$,

$$(x-y)(f(x)-f(y)) + \frac{q-1}{2}|g(x)-g(y)|^2 \le \big(a+0.5c^2(q-1)\big)(x-y)^2, \quad \forall x, y \in \mathbb{R}.$$

This means that Assumption 3.1 is satisfied for any $q > 2$ with $H_1 = a + 0.5c^2(q-1)$. Furthermore, we can show

$$|f(x)-f(y)|^2 \vee |g(x)-g(y)|^2 \le H_2(1+|x|^4+|y|^4)|x-y|^2,$$

where $H_2 = 2a^2 + 9b^2 + c^2$. So, Assumption 3.2 is also satisfied with $\rho = 4$. To apply Theorem 3.4, we still need to design functions μ and h. Noting that

$$\sup_{|x|\le u}(|f(x)| \vee |g(x)|) \le \alpha u^3, \quad \forall u \ge 1,$$

where $\alpha = a+b+c$, we can have $\mu(u) = \alpha u^3$ and its inverse function $\mu^{-1}(u) = (u/\alpha)^{1/3}$ for $u \ge \alpha$. For $\varepsilon \in (0, 1/4]$, we define $h(\Delta) = \alpha(1+|x_0|^3)\Delta^{-\varepsilon}$ for $\Delta > 0$. Now, for any $\bar{q} \ge 2$, we can choose p sufficiently large for

$$\varepsilon(2p-(2+\rho)\bar{q})/(2+\rho) > \bar{q}(1-2\varepsilon)/2.$$

We can therefore conclude by Theorem 3.5 that the truncated EM solutions of the SDE (3.27) satisfy

$$\mathbb{E}|x(T)-x_\Delta(T)|^{\bar{q}} = O(\Delta^{\bar{q}(1-2\varepsilon)/2}) \quad \text{and} \quad \mathbb{E}|x(T)-\bar{x}_\Delta(T)|^{\bar{q}} = O(\Delta^{\bar{q}(1-2\varepsilon)/2}).$$

That is, the order of $L^{\bar{q}}$-convergence can be arbitrarily close to $\bar{q}/2$.

Let us now compare the simulations by the implicit EM method, the tamed Euler Method and the truncated EM method. For this purpose, we set $T = a = b = c = 1$ and $x(0) = 2$. The SDE (3.27) thus reads as

$$dx(t) = (x(t) - x^3(t))dt + x(t)dB(t), \quad x(0) = 2 \tag{3.28}$$

for $t \in [0, 1]$. Let $\Delta = 1/N$, $t_k = \Delta k$, $\Delta B_k = B(t_{k+1}) - B(t_k)$.

In order to compute the approximation errors for the different schemes, we will make use of the explicit solution $x(t)$ of SDE (3.28). Taking the Bernoulli transformation $u(t) = 1/x^2(t)$ yields

$$du(t) = (u(t)+2)dt - 2u(t)dB(t), \quad u(0) = 1/4,$$

which is a linear SDE and its solution has a closed form. Thus, by solving the above SDE we get the explicit solution

$$x(t) = \frac{1}{\left[\frac{1}{4}e^{-t-B(t)} + 2\int_0^t e^{s-t+B(s)-B(t)}ds\right]^2}.$$

To approximate the mean square error, for example, of the implicit EM method, we run $M = 1000$ independent trajectories $[x(T)]^{(i)}$ and $[\tilde{Y}_N^\Delta]^{(i)}$:

$$(\mathbb{E}|x(T)-\tilde{Y}_N^\Delta|^2)^{1/2} \approx \left(\frac{1}{M}\sum_{i=1}^{M} |[x(T)]^{(i)} - [\tilde{Y}_N^\Delta]^{(i)}|^2\right)^{1/2},$$

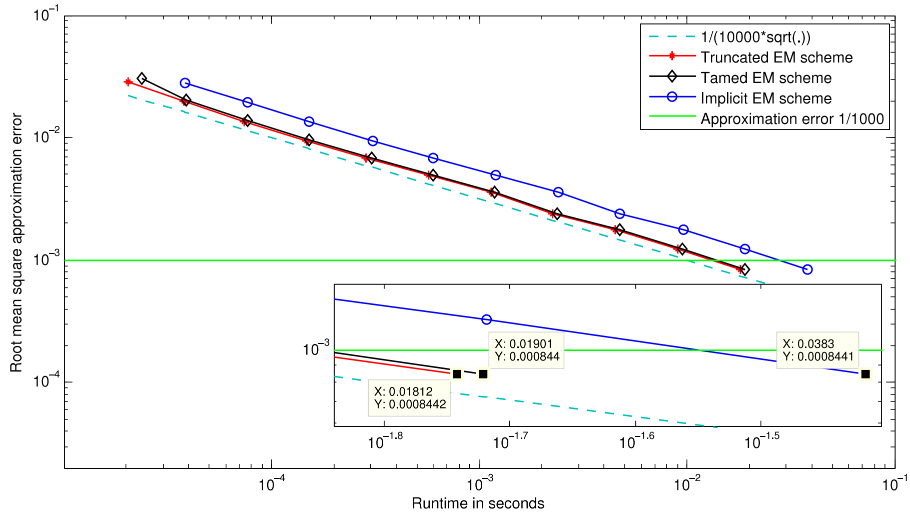

Fig. 1. The root mean square approximation errors between the exact solution $x(T)$ of the SDE (3.28) and the numerical solutions: $\tilde{Y}_N^{\Delta}$ by the implicit EM scheme; $\check{Y}_N^{\Delta}$ by the tamed Euler scheme; X_N^{Δ} by the truncated EM scheme with $\varepsilon = 1/4$, respectively, as functions of the runtime when $\Delta \in \{2^{-7}, 2^{-8}, \dots, 2^{-17}\}$.

where M denotes the number of running independent trajectories. Fig. 1 depicts the root mean square approximation errors between the exact solution $x(T)$ of the SDE (3.28) and the numerical solutions: $\tilde{Y}_N^{\Delta}$ by the implicit EM scheme; $\check{Y}_N^{\Delta}$ by the tamed Euler scheme; X_N^{Δ} by the truncated EM scheme with $\varepsilon = 1/4$, respectively, as functions of the runtime when $\Delta \in \{2^{-7}, 2^{-8}, \dots, 2^{-17}\}$. When $\Delta = 2^{-17}$, the runtime of $\tilde{Y}_N^{\Delta}$, $\check{Y}_N^{\Delta}$ and X_N^{Δ} achieving the accuracy 0.000844 on our computer running at Intel Core i3-4170 CPU 3.70 GHz, is about 0.0383 s, 0.01901 s and 0.01812 s, respectively. In this case the runtime of the truncated EM method is the shortest.

The MATLAB codes of simulating the implicit EM approximation and the tamed Euler approximation are from [21]. Our MATLAB codes for simulating the truncated EM approximation X_N^{Δ} for SDE (3.28) are:

```
clear all;
Y=2; Delta=2^(-17); N=2^17; v=9*Delta^(-1/12);
for n=1:N
    if abs(Y)>=v
        Y=Y+(v*Y/abs(Y)-(v*Y/abs(Y))^3)*Delta+...
          (v*Y/abs(Y))*randn*sqrt(Delta);
    else
        Y=Y+(Y-Y^3)*Delta+Y*randn*sqrt(Delta);
    end
end
```

4. Comparisons with known results

First of all, let us make a comparison between our new Theorem 3.4 and one of the main results in [15], namely Theorem 3.3, in order to highlight the significant contribution of our new result. Although the assumptions imposed in both theorems are almost the same, we observe the following key differences:

- The key feature of Theorem 3.4 is that it does not require the restrictive condition (3.5).
- The assertions of Theorem 3.4 hold for any $\Delta \in (0, 1]$ while the assertions of Theorem 3.3 hold only for sufficiently small Δ which satisfies condition (3.5).
- Theorem 3.4 needs a slightly stronger condition on the parameters, namely $2p > (2+\rho)q$, which implies that $p > q$ and $2p > q\rho$ imposed in Theorem 3.3.
- The assertions of Theorem 3.4 look slightly worse than those of Theorem 3.3 but could be the same when p is large as demonstrated in Theorem 3.5.

The key advantage of our new Theorem 3.4 lies in that it does not need condition (3.5). Let us now explain, via the following example, that condition (3.5) could sometimes make Theorem 3.3 inapplicable and hence our new Theorem 3.4 without condition (3.5) is particularly useful in this situation.

Example 4.1. Consider the scalar SDE

$$dx(t) = -10x^3(t)dt + x^2(t)dB(t), \tag{4.1}$$

where $B(t)$ is a scalar Brownian motion. Its coefficients $f(x) = -10x^3$ and $g(x) = x^2$ are clearly locally Lipschitz continuous for $x \in \mathbb{R}$. For $p = 21$, we have

$$xf(x) + \frac{p-1}{2}|g(x)|^2 = 0$$

so Assumption 2.2 is satisfied with $p = 21$. Moreover, for $q = 3$, we have

$$\begin{aligned}
&(x-y)(f(x)-f(y)) + \frac{q-1}{2}|g(x)-g(y)|^2 \\
&= -10(x^2+xy+y^2)|x-y|^2 + (x+y)^2|x-y|^2 \\
&= -(9x^2+8xy+9y^2)|x-y|^2 \\
&\le 0.
\end{aligned}$$

Thus Assumption 3.1 holds with $q = 3$. Furthermore, it is easy to show that

$$|f(x)-f(y)|^2 \vee |g(x)-g(y)|^2 \le 800(1+x^4+y^4)|x-y|^2.$$

That is, Assumption 3.2 is satisfied with $\rho = 4$.

We first apply Theorem 3.3 to see what we can get. Obviously, we have $p > q$ and $2p > q\rho$ and we choose $\bar{q} = 2$. Noting

$$|f(x)| \vee |g(x)| \le 10|x|^3, \quad \forall |x| \ge 1,$$

we can then choose $\mu(u) = 10u^3$ and $h(\Delta) = \Delta^{-1/4}$ to define the truncated EM solution $x_\Delta(t)$ to the SDE (4.1). It is easy to see that condition (3.5) becomes

$$\Delta^{-1/4} \ge 10\Delta^{-3/38}, \quad \text{i.e.,} \quad \Delta \le 10^{-76/13} = 1.425103 \times 10^{-6}.$$

For such a small step size, Theorem 3.3 shows

$$\mathbb{E}|x(T) - x_\Delta(T)|^2 \vee \mathbb{E}|x(T) - \bar{x}_\Delta(T)|^2 \le C\Delta^{0.5}. \tag{4.2}$$

The key issue here is that step size is required to be very small, namely less than 1.425103×10^{-6}, due to condition (3.5).

Let us now apply our new Theorem 3.4 to see if we can get a better result. Clearly, $2p > (2+\rho)q$. We let $\bar{q} = 2$, $\mu(u) = 10u^3$ and $h(\Delta) = \Delta^{-1/4}$ as before. Noting that $\mu^{-1}(u) = (u/10)^{1/3}$ and

$$(\mu^{-1}(h(\Delta)))^{-(2p-(2+\rho)\bar{q})/2} + \Delta^{\bar{q}/2}(h(\Delta))^{\bar{q}} = 10^5\Delta^{5/4} + \Delta^{1/2} = O(\Delta^{0.5}),$$

we conclude by Theorem 3.4 that for any $\Delta \in (0, 1]$,

$$\mathbb{E}|x(T) - x_\Delta(T)|^2 \vee \mathbb{E}|x(T) - \bar{x}_\Delta(T)|^2 \le C\Delta^{0.5}. \tag{4.3}$$

This is the same as (4.2) but the step size Δ can now be any number in $(0, 1]$ rather than $\Delta \le 1.425103 \times 10^{-6}$.

The advantage of our new Theorem 3.4 is even more clear if we choose $h(\Delta) = \Delta^{-1/8}$ while still use $\mu(u) = 10u^3$ to define the truncated EM solution $x_\Delta(t)$. Let $\bar{q} = 2$ as before. In this case, condition (3.5) becomes

$$\Delta^{-1/8} \ge 10\Delta^{-9/76}, \text{ namely } \Delta \le 1.425103 \times 10^{-154}.$$

This is almost impossible so Theorem 3.3 is inapplicable. However, our new Theorem 3.4 can still be applied. In fact, noting

$$(\mu^{-1}(h(\Delta)))^{-(2p-(2+\rho)\bar{q})/2} + \Delta^{\bar{q}/2}(h(\Delta))^{\bar{q}} = 10^5\Delta^{5/8} + \Delta^{6/8} = O(\Delta^{5/8}),$$

we can then conclude, by Theorem 3.4, that for any $\Delta \in (0, 1]$,

$$\mathbb{E}|x(T) - x_\Delta(T)|^2 \vee \mathbb{E}|x(T) - \bar{x}_\Delta(T)|^2 \le C\Delta^{5/8}, \tag{4.4}$$

where $\mu(u) = 10u^3$ and $h(\Delta) = \Delta^{-1/8}$ are used to define the truncated EM solution $x_\Delta(t)$ to the SDE (4.1). In other words, our new Theorem 3.4 is not only applicable in this situation but also shows that the truncated EM solution $x_\Delta(t)$ defined by using $\mu(u) = 10u^3$ and $h(\Delta) = \Delta^{-1/8}$ has a better strong convergence rate to the true solution of the SDE (4.1) than that using $\mu(u) = 10u^3$ and $h(\Delta) = \Delta^{-1/4}$.

Let us now compare the truncated EM method with the modified tamed Euler scheme (see, e.g., [11]) numerically. Consider the SDE (4.1) with $x(0) = 1$ and $T = 1$. Since there is no explicit solution, we use the modified tamed Euler solution (see, e.g., [11]) with $\Delta = 2^{-17}$ as a good approximation of the exact solution. Fig. 2 depicts the root mean square approximation error $(\mathbb{E}|x(T) - \bar{Y}_N^\Delta|^2)^{1/2}$ between the exact solution of the SDE (3.28) and the numerical solution by the modified tamed EM scheme, and the error $(\mathbb{E}|x(T) - X_N^\Delta|^2)^{1/2}$ between the exact solution and that by the truncated EM scheme with $\varepsilon = 1/4$, $M = 1000$, as functions of the runtime when $\Delta \in \{2^{-7}, 2^{-8}, \dots, 2^{-14}\}$. When $\Delta = 2^{-14}$, the runtime

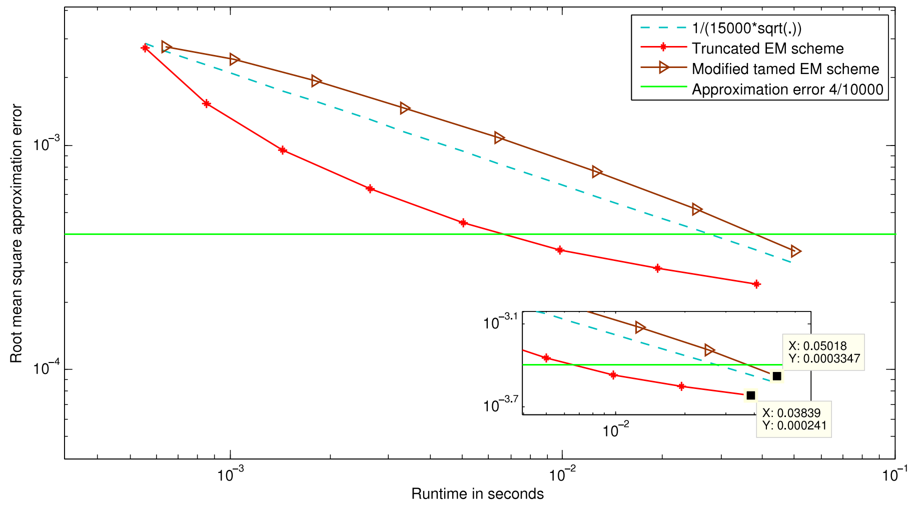

Fig. 2. The root mean square approximation errors between the exact solution $x(T)$ of the SDE (3.28) and the numerical solutions: $\bar{Y}_N^{\Delta}$ by the modified tamed Euler scheme; X_N^{Δ} by the truncated EM scheme, respectively, as functions of runtime for $\Delta \in \{2^{-7}, 2^{-8}, \dots, 2^{-14}\}$.

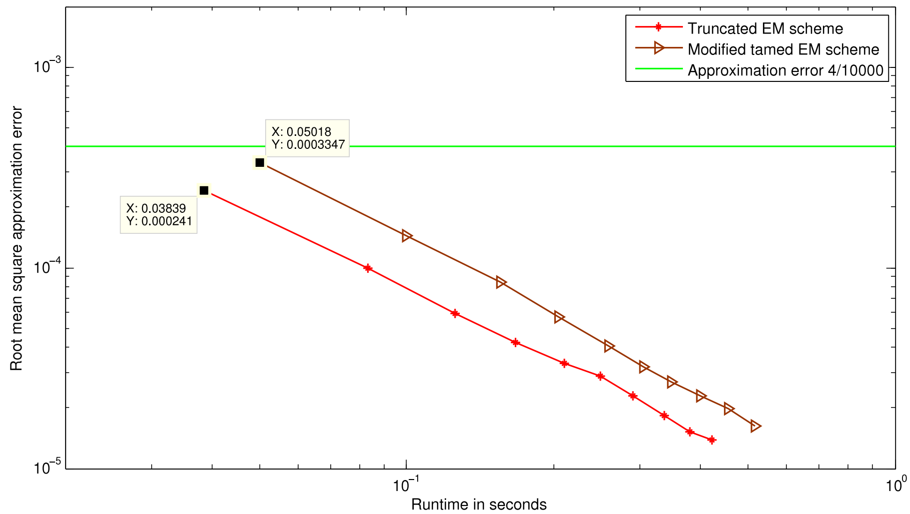

Fig. 3. The root mean square approximation errors between the exact solution $x(T)$ of the SDE (3.28) and the numerical solutions: $\bar{Y}_N^{\Delta}$ by the modified tamed Euler scheme, X_N^{Δ} by the truncated EM scheme, respectively, as functions of runtime for $T \in \{1, 2, \dots, 10\}$ with the same step size $\Delta = 2^{-14}$.

of $\bar{Y}_N^{\Delta}$ achieving the accuracy 0.0003347 on our computer with Intel Core i3-4170 CPU 3.70 GHz, is about 0.05018 s while the runtime of X_N^{Δ} achieving the accuracy 0.000241 is about 0.03839 s (see the enlargement in Fig. 2). Thus, the speed of the truncated Euler scheme for the SDE (4.1) is 1.3 times faster than that of the modified tamed Euler scheme, while the accuracy achieved by the truncated Euler scheme is almost 1.4 times better than that of the modified tamed Euler scheme. Moreover, for step size $\Delta = 2^{-14}$, we go further to simulate the root mean square approximation error for different values of time T. Fig. 3 depicts the root mean square approximation error $\left(\mathbb{E}\left|x(T) - \bar{Y}_N^{2^{-14}}\right|^2\right)^{1/2}$ between the exact solution of the SDE (3.28) and the numerical solution by the modified tamed Euler scheme, and the error $\left(\mathbb{E}\left|x(T) - X_N^{2^{-14}}\right|^2\right)^{1/2}$ between the exact solution and that of the truncated EM scheme with $\varepsilon = 1/4, M = 1000$, as functions of running time, for $T \in \{1, 2, \dots, 10\}$.

The MATLAB codes for simulating the modified tamed Euler approximation $\bar{Y}_k^{\Delta}$ are:

```
clear all;
Y=1; Delta=2^(-14); N=2^14; alpha=0.5; l=2;
for n=1:N
    Y=Y+1/(1+n^(-alpha)*abs(Y)^l)*(-10*Y^3)*Delta+...
      1/(1+n^(-alpha)*abs(Y)^l)*Y^2*randn*sqrt(Delta);
end
```

5. Asymptotic stability

In this section we will discuss if the truncated EM method can preserve the asymptotic stability of the underlying SDE (2.1). We will let the assumptions imposed in the previous sections as the standing hypotheses so we will not mention them explicitly in the theorem in this section. Moreover, for the stability purpose (see, e.g., [18]), we also assume in this section that

$$f(0) = 0, \quad g(0) = 0. \tag{5.1}$$

We need an additional assumption to guarantee the asymptotic stability of the underlying SDE (2.1). We need one more notation. Let $\mathcal{K}$ denote the family of continuous non-decreasing functions $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\kappa(0) = 0$ and $\kappa(u) > 0$ for all $u > 0$.

Assumption 5.1. Assume that there is a function $\kappa \in \mathcal{K}$ such that

$$2x^T f(x) + |g(x)|^2 \le -\kappa(|x|) \tag{5.2}$$

for all $x \in \mathbb{R}^d$.

Let us state a theorem which follows easily from [22].

Theorem 5.2. *Let Assumption 5.1 hold. Then for any initial value $x_0 \in \mathbb{R}^d$, the solution of the SDE (2.1) satisfies*

$$\lim_{t\to\infty} x(t) = 0 \quad a.s. \tag{5.3}$$

The following theorem shows that the truncated EM method can preserve this almost surely asymptotical stability with an additional condition (5.4). We will see from the example below that this additional condition is not restrictive.

Theorem 5.3. *Let Assumption 5.1 hold. Assume also that*

$$\limsup_{|x|\downarrow 0} \frac{|f(x)|^2}{\kappa(|x|)} < \infty. \tag{5.4}$$

Set

$$H = \sup_{0<|x|\le\mu^{-1}(h(1))} \frac{|f(x)|^2}{\kappa(|x|)} \tag{5.5}$$

and

$$\hat{\Delta} = \min\Big(1,\ 0.5/H,\ 0.25(\kappa(\mu^{-1}(h(1)))/\hat{h})^2\Big). \tag{5.6}$$

Then for every $\Delta \in (0, \hat{\Delta}]$ and any initial value $x_0 \in \mathbb{R}^d$, the solution of the truncated EM method (2.9) satisfies

$$\lim_{k\to\infty} X_\Delta(t_k) = 0 \quad a.s. \tag{5.7}$$

Proof. We first observe that $H < \infty$ from condition (5.4) and the continuity of $f(\cdot)$ as well as the property of $\kappa(\cdot)$ and hence we have $\hat{\Delta} \in (0, 1]$.

We next show that the truncated functions f_Δ and g_Δ preserve property (5.2) perfectly in the sense that, for any $\Delta \in (0, 1]$,

$$2x^T f_\Delta(x) + |g_\Delta(x)|^2 \le -\kappa(|\pi_\Delta(x)|), \quad x \in \mathbb{R}^d. \tag{5.8}$$

In fact, this holds obviously for $x \in \mathbb{R}^d$ with $|x| \le \mu^{-1}(h(\Delta))$. For $x \in \mathbb{R}^d$ with $|x| > \mu^{-1}(h(\Delta))$, we derive, by Assumption 5.1,

$$\begin{aligned} &2x^T f_\Delta(x) + |g_\Delta(x)|^2 \\ &= 2(x - \pi_\Delta(x))^T f(\pi_\Delta(x)) + 2(\pi_\Delta(x))^T f(\pi_\Delta(x)) + |g(\pi_\Delta(x))|^2 \\ &\le 2(x - \pi_\Delta(x))^T f(\pi_\Delta(x)) - \kappa(|\pi_\Delta(x)|). \end{aligned} \tag{5.9}$$

But, by Assumption 5.1 again,

$$2(x - \pi_\Delta(x))^T f(\pi_\Delta(x)) = 2[|x|/\mu^{-1}(h(\Delta)) - 1](\pi_\Delta(x))^T f(\pi_\Delta(x)) \le 0.$$

Substituting this into (5.9) yields (5.8) as desired.

Let us now fix any $\Delta \in (0, \hat{\Delta}]$ and $x_0 \in \mathbb{R}^d$. It is easy to derive from (2.9) and (5.8) that

$$\begin{aligned} |X_\Delta(t_{k+1})|^2 = &\ |X_\Delta(t_k)|^2 + 2X_\Delta(t_k)^T f_\Delta(X_\Delta(t_k))\Delta + |g_\Delta(X_\Delta(t_k))|^2\Delta \\ &+ |f_\Delta(X_\Delta(t_k))|^2\Delta^2 + \Delta M_k \\ \le &\ |X_\Delta(t_k)|^2 - \kappa(|\pi_\Delta(X_\Delta(t_k))|)\Delta + |f_\Delta(X_\Delta(t_k))|^2\Delta^2 + \Delta M_k \end{aligned} \tag{5.10}$$

for $k = 0, 1, \ldots$, where

$$\begin{aligned} \Delta M_k = &\ 2(X_\Delta(t_k) + f_\Delta(X_\Delta(t_k))\Delta)^T g_\Delta(X_\Delta(t_k))\Delta B_k \\ &+ |g_\Delta(X_\Delta(t_k))\Delta B_k|^2 - |g_\Delta(X_\Delta(t_k))|^2\Delta. \end{aligned} \tag{5.11}$$

Note

$$\begin{aligned} \mathbb{E}\big(|g_\Delta(X_\Delta(t_k))\Delta B_k|^2 \big| \mathcal{F}_{t_k}\big) &= \mathbb{E}\Big(\text{trace}\big[g_\Delta(X_\Delta(t_k))\Delta B_k \Delta B_k^T g_\Delta(X_\Delta(t_k))^T\big]\Big|\mathcal{F}_{t_k}\Big) \\ &= \text{trace}\big[g_\Delta(X_\Delta(t_k))\mathbb{E}\big(\Delta B_k \Delta B_k^T \big| \mathcal{F}_{t_k}\big) g_\Delta(X_\Delta(t_k))^T\big] \\ &= \text{trace}\big[g_\Delta(X_\Delta(t_k))\Delta \mathbb{I}_m g_\Delta(X_\Delta(t_k))^T\big] \\ &= \Delta|g_\Delta(X_\Delta(t_k))|^2, \end{aligned}$$

where $\mathbb{I}_m$ denotes the $m \times m$ identity matrix. It is then easy to show that

$$\mathbb{E}(\Delta M_k | \mathcal{F}_{t_k}) = 0.$$

This implies immediately that

$$M_k := \sum_{i=0}^{k} \Delta M_i, \quad k = 0, 1, 2, \ldots, \tag{5.12}$$

is a martingale. Using (5.5) and recalling that $\mu^{-1}(h(\Delta)) \ge \mu^{-1}(h(1))$, we hence have

$$|f_\Delta(x)|^2 = |f(x)|^2 \le H\kappa(|x|) = H\kappa(|\pi_\Delta(x)|) \quad \text{if } 0 \le |x| \le \mu^{-1}(h(1)).$$

On the other hand, if $|x| > \mu^{-1}(h(1))$, we have

$$|f_\Delta(x)|^2 \le (h(\Delta))^2 \le \frac{(h(\Delta))^2}{\kappa(\mu^{-1}(h(1)))}\kappa(|\pi_\Delta(x)|).$$

Consequently, for all $x \in \mathbb{R}^d$,

$$\begin{aligned} |f_\Delta(x)|^2\Delta &\le \kappa(|\pi_\Delta(x)|)\max\Big\{H\Delta, \frac{(h(\Delta))^2\Delta}{\kappa(\mu^{-1}(h(1)))}\Big\} \\ &\le \kappa(|\pi_\Delta(x)|)\max\Big\{H\Delta, \frac{\hat{h}^2\sqrt{\Delta}}{\kappa(\mu^{-1}(h(1)))}\Big\} \\ &\le 0.5\kappa(|\pi_\Delta(x)|), \end{aligned}$$

where (2.6) and (5.6) have been used. Substituting this into (5.10), we get

$$|X_\Delta(t_{k+1})|^2 \le |X_\Delta(t_k)|^2 - 0.5\kappa(|\pi_\Delta(X_\Delta(t_k))|)\Delta + \Delta M_k, \quad k \ge 0. \tag{5.13}$$

This implies

$$|X_\Delta(t_{k+1})|^2 \le |x_0|^2 - 0.5\Delta\sum_{i=0}^{k}\kappa(|\pi_\Delta(X_\Delta(t_i))|) + M_k, \quad k \ge 0. \tag{5.14}$$

Applying the nonnegative semi-martingale convergence theorem (see, e.g., [23, Theorem 7 on page 139] or [9, Theorem 1.10 on page 18]), we get

$$\sum_{i=0}^{\infty}\kappa(|\pi_\Delta(X_\Delta(t_i))|) < \infty \quad a.s.$$

This implies

$$\lim_{i\to\infty}\kappa(|\pi_\Delta(X_\Delta(t_i))|) = 0 \quad a.s.$$

Consequently, we must have

$$\lim_{i\to\infty}\pi_\Delta(X_\Delta(t_i)) = 0 \quad a.s.$$

and the desired assertion (5.7) follows. The proof is complete. □

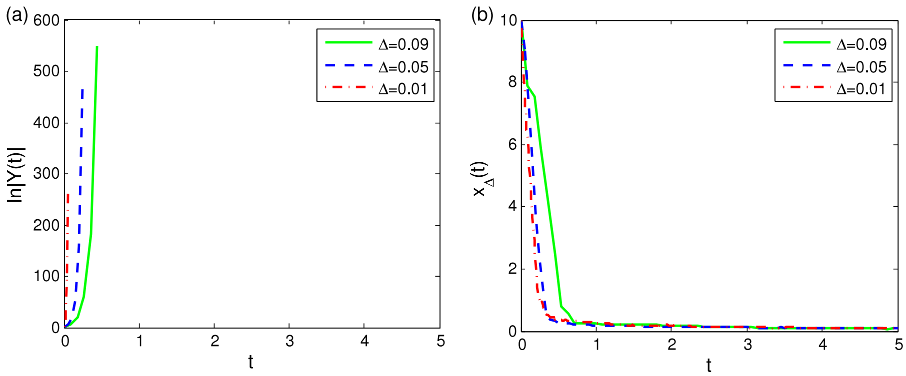

Fig. 4. (a) Sample paths of the classical EM solution $\ln|Y(t)|$; (b) Sample paths of the truncated EM solution $x_\Delta(t)$ with the same initial value $x_0 = 10$ for different values of step size Δ and $t \in [0, 90]$.

Example 5.4. Consider the scalar SDE (4.1). Noting that

$$2xf(x) + |g(x)|^2 = -19|x|^4, \quad \forall x \in \mathbb{R},$$

we see that Assumption 5.1 is satisfied with $\kappa(u) = 19u^4$. Theorem 5.2 shows that for any initial value $x(0) = x_0 \in \mathbb{R}$, the solution of the SDE (4.1) will tend to 0 almost surely. In other words, the SDE (4.1) is almost surely asymptotically stable.

In the Appendix below, we will show that the classical EM method will not be able to reproduce this asymptotic stability. However, we now show that the truncated EM method can reproduce this stability very well. We observe that

$$\limsup_{|x|\downarrow 0} \frac{|f(x)|^2}{\kappa(|x|)} = 0,$$

which implies condition (5.4). Let us choose $h(\Delta) = 10\Delta^{-1/4}$ and $\mu(u) = 10u^3$ to define the truncated EM solution $x_\Delta(t)$. Noting that $\mu^{-1}(h(1)) = 1$, we can easily compute by (5.5) and (5.6) that

$$H = 100/19 \text{ and } \hat{\Delta} = 0.095.$$

Applying Theorem 5.3, we can then conclude that for every $\Delta \in (0, 0.095]$ and any initial value $x_0 \in \mathbb{R}$, the truncated EM solution of the SDE (4.1) satisfies

$$\lim_{k\to\infty} X_\Delta(t_k) = 0 \quad a.s. \tag{5.15}$$

Fig. 4 gives sample paths of the classical EM solution $Y(t)$ and of the truncated EM solution $x_\Delta(t)$ with the same initial value $x_0 = 10$ for different values of step size Δ and $t \in [0, 90]$. Fig. 4(a) displays that the classical EM solution blows up quickly, so it cannot capture the stability behavior of SDE (4.1). Fig. 4(b) displays clearly that the truncated EM solution reproduces the almost sure stability of SDE (4.1).

6. Conclusion

In this paper we made a quick review on the main result of [14,15] on the truncated EM method and pointed out that condition (3.5) imposed there to obtain the strong convergence rate is somehow restrictive. We then established our new results on the convergence rate without condition (3.5). The advantages of our new results were showed by the comparison with the main result of [15] as well as by the numerical comparisons with the implicit EM and the tamed Euler method in two examples. The mathematical techniques developed here were significantly different from those used in [15] in order to overcome the difficulties without imposing condition (3.5). The stability of the truncated EM method was also studied. It was showed that the truncated EM method is able to preserve the asymptotic stability of the underlying SDEs.

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Appendix

In this appendix, we will show that the classical EM method will not be able to reproduce the almost sure asymptotic stability of the SDE (4.1). We will show that for any given step size $\Delta \in (0, 1]$ and any initial value $x_0 \neq 0$, the EM solution will tend to infinity super-exponentially with a positive probability.

The classical EM method applied to the SDE (4.1) produces approximations $Y_k \approx x(t_k)$ for $k = 0, 1, 2, \ldots$, where $t_k = k\Delta$, $Y_0 = x_0$ and

$$Y_{k+1} = Y_k(1 - 10Y_k^2\Delta + Y_k\Delta B_k). \tag{A.1}$$

To obtain the continuous-time approximation, we define $Y(t)$ by $Y(t) = Y_k$ for any $t \in [t_k, t_{k+1})$. We first note that

$$|Y_1| \geq |x_0|^2|\Delta B_0| - |x_0|(1 + 10\Delta|x_0|^2) \geq \frac{e}{\sqrt{\Delta}}$$

if

$$|\Delta B_0| \geq \frac{e/\sqrt{\Delta} + |x_0|(1 + 10\Delta|x_0|^2)}{|x_0|^2}.$$

In other words, we have

$$\mathbb{P}\Big(|Y_1| \geq \frac{e}{\sqrt{\Delta}}\Big) \geq \mathbb{P}\Big(|\Delta B_0| \geq \frac{e/\sqrt{\Delta} + |x_0|(1 + 10\Delta|x_0|^2)}{|x_0|^2}\Big) := \xi \tag{A.2}$$

and ξ is positive as $\Delta B_0 \sim N(0, \Delta)$. We observe that, for $k \geq 1$, if $|Y_k| \geq \frac{\exp(3^{k-1})}{\sqrt{\Delta}}$ and $|\Delta B_k| \leq 8\sqrt{\Delta}\exp(3^{k-1})$ hold, then

$$|Y_{k+1}| \geq \frac{\exp(3^k)}{\sqrt{\Delta}}. \tag{A.3}$$

In fact,

$$\begin{aligned} |Y_{k+1}| &= |Y_k|\,|1 - 10Y_k^2\Delta + Y_k\Delta B_k| \\ &\geq |Y_k|\big(10Y_k^2\Delta - 1 - |Y_k||\Delta B_k|\big) \\ &\geq |Y_k|^2\sqrt{\Delta}\big(10|Y_k|\sqrt{\Delta} - 1 - \frac{|\Delta B_k|}{\sqrt{\Delta}}\big) \\ &\geq \frac{\exp(2\times 3^{k-1})}{\sqrt{\Delta}}\big(10\exp(3^{k-1}) - 1 - 8\exp(3^{k-1})\big) \\ &\geq \frac{\exp(3^k)}{\sqrt{\Delta}}. \end{aligned} \tag{A.4}$$

We therefore have, for any $\bar{k} \geq 1$,

$$\begin{aligned} &\mathbb{P}\Big(|Y_{k+1}| \geq \frac{\exp(3^k)}{\sqrt{\Delta}}\ \forall 1 \leq k \leq \bar{k}\ \Big|\ |Y_1| \geq \frac{e}{\sqrt{\Delta}}\Big) \\ &\geq \mathbb{P}\Big(|\Delta B_k| \leq 8\sqrt{\Delta}\exp(3^{k-1})\ \forall 1 \leq k \leq \bar{k}\ \Big|\ |Y_1| \geq \frac{e}{\sqrt{\Delta}}\Big) \\ &= \prod_{k=1}^{\bar{k}}\mathbb{P}\Big(|\Delta B_k| \leq 8\sqrt{\Delta}\exp(3^{k-1})\Big). \end{aligned} \tag{A.5}$$

But

$$\begin{aligned} \mathbb{P}\Big(\frac{|\Delta B_k|}{\sqrt{\Delta}} > 8\exp(3^{k-1})\Big) &= \frac{2}{\sqrt{2\pi}}\int_{8\exp(3^{k-1})}^{\infty} e^{-x^2/2}dx \\ &\leq \frac{2}{\sqrt{2\pi}}\int_{8\exp(3^{k-1})}^{\infty} e^{-x}dx \leq \exp(-8\exp(3^{k-1})). \end{aligned} \tag{A.6}$$

Hence, in (A.5),

$$\mathbb{P}\Big(|Y_{k+1}| \geq \frac{\exp(3^k)}{\sqrt{\Delta}}\ \forall 1 \leq k \leq \bar{k}\ \Big|\ |Y_1| \geq \frac{e}{\sqrt{\Delta}}\Big) \geq \prod_{k=1}^{\bar{k}}\Big(1 - \exp(-8\exp(3^{k-1}))\Big). \tag{A.7}$$

But, by the elementary inequality $\log(1-u) \geq -2u$ for $0 \leq u < 0.5$, we derive

$$\log\Big(\prod_{k=1}^{\bar{k}}\Big(1-\exp(-8\exp(3^{k-1}))\Big)\Big) = \sum_{k=1}^{\bar{k}}\log\Big(1-\exp(-8\exp(3^{k-1}))\Big) \geq -2\sum_{k=1}^{\bar{k}}\exp(-8\exp(3^{k-1})). \tag{A.8}$$

Noting that $\exp(3^{k-1}) \geq 1+3^{k-1} \geq 1+3(k-1)$, we then get

$$\log\Big(\prod_{k=1}^{\bar{k}}\Big(1-\exp(-8\exp(3^{k-1}))\Big)\Big) \geq -2\sum_{k=1}^{\bar{k}}\exp(-8-24(k-1)) \geq -2\sum_{k=1}^{\infty}\exp(-8-24(k-1)) = -\frac{2e^{-8}}{1-e^{-24}}. \tag{A.9}$$

Combining (A.7)–(A.9) together implies

$$\mathbb{P}\Big(|Y_{k+1}| \geq \frac{\exp(3^k)}{\sqrt{\Delta}}\ \forall 1 \leq k \leq \bar{k}\ \Big|\ |Y_1| \geq \frac{e}{\sqrt{\Delta}}\Big) \geq \exp\Big(-\frac{2e^{-8}}{1-e^{-24}}\Big). \tag{A.10}$$

Since $\bar{k} \geq 1$ is arbitrary, we must have

$$\mathbb{P}\Big(|Y_{k+1}| \geq \frac{\exp(3^k)}{\sqrt{\Delta}}\ \forall 1 \leq k < \infty\Big|\ |Y_1| \geq \frac{e}{\sqrt{\Delta}}\Big) \geq \exp\Big(-\frac{2e^{-8}}{1-e^{-24}}\Big). \tag{A.11}$$

It then follows from (A.2) and (A.11) that

$$\mathbb{P}\Big(|Y_{k+1}| \geq \frac{\exp(3^k)}{\sqrt{\Delta}}\ \forall 1 \leq k < \infty\Big) \geq \xi\exp\Big(-\frac{2e^{-8}}{1-e^{-24}}\Big). \tag{A.12}$$

That is, $|Y_{k+1}|$ will tend to infinity faster than $\exp(3^k)/\sqrt{\Delta}$ with a positive probability.

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