Non-Linear Dynamics of Ring World Systems

COLIN R. MCINNES
Department of Aerospace Engineering, University of Glasgow, Glasgow, G12 8QQ, UK.
Email: colinmc@aero.gla.ac.uk

The dynamics of a point mass and a thin solid ring are considered with the point mass located within the perimeter of the ring. It is demonstrated that an equilibrium configuration exists only when the center-of-mass of the ring and the point mass are collocated. However, this sole equilibrium configuration is unstable due to perturbations within the plane of the ring, but is stable due to perturbations normal to the plane of the ring. Such a ring system about a star has been envisaged in some detail in well-known works of fiction. While qualitative, or sometimes quantitative linear analysis of the problem is to be found, a full non-linear analysis of the dynamics of the problem does not appear to have been previously published.

Keywords: Ring world, instability

1. Introduction

The fabrication of a solid ring about a star has been the subject of several popular works of fiction [1,2]. It appears to be well known that such a system is dynamically unstable, destining such 'ringworlds' to be true works of fiction, or at least necessitating an active control system to compensate for the natural instability [2]. Occasionally, the instability is discussed in a qualitative form, or a linearised analysis is used to demonstrate the instability analytically [3,4]. While linear instability is in general, both a necessary and sufficient condition for non-linear instability, insight can still be obtained by completing the analysis of the problem in a fully non-linear manner.

This paper re-visits the ring world problem and formulates the full, non-linear in-plane equation of motion of the ring relative to the central mass. The sole equilibrium solution to the equation of motion is then sought and its instability established. In addition, the timescale for the instability is determined, and a simulation of the evolution of the full non-linear instability shown. Similarly, the full non-linear, out-of-plane equation of motion is formulated and the stability properties of out-of-plane perturbations investigated. While it can be shown that the out-of-plane motion is in the form of non-linear oscillations about the plane of the system, the existence of the in-plane instability renders the equilibrium configuration unstable in general.

2. Gravitational Force Model

In order to locate any equilibria, and determine their stability properties, the functional form of the gravitational force acting between the ring and point mass will be derived. For a ring of radius R and linear mass density ρ, the total mass of the ring M_R is $2\pi\rho R$. The gravitational force acting between the ring and point mass M_S can then be determined by calculating the gravitational force acting between the point mass and an infinitesimal ring element, and then integrating around the ring. It will be assumed that $M_R \prec\prec M_S$, so that the barycentre of the problem remains fixed at M_S.

Firstly, an arbitrary ring mass element dm will be located on the ring at position **R** relative to the centre-of-mass of the ring C, such that

$$\mathbf{R} = R\cos\phi\mathbf{e}_1 + R\sin\phi\mathbf{e}_2 \tag{1}$$

where ($\mathbf{e}_1$, $\mathbf{e}_2$) are the basis vectors of a Cartesian coordinate system with origin at the point mass, as shown in fig. 1. Similarly, the position of the centre-of-mass of the ring C relative to the point mass is defined by

$$\mathbf{r} = r\mathbf{e}_1 \tag{2}$$

so that the position of the mass element dm relative to the central mass $\mathbf{r}' = \mathbf{r} + \mathbf{R}$ is given by

$$\mathbf{r}' = (R\cos\phi + r)\mathbf{e}_1 + R\sin\phi\mathbf{e}_2 \tag{3}$$

Therefore, the gravitational force df between the point mass and the ring element dm can be written as

$$df = \frac{GM_S}{\|\mathbf{r}'\|^2} dm \qquad (4)$$

where $\|\mathbf{r}'\|$ is the distance from the point mass to the ring mass element, which may be obtained from (3) as

$$\|\mathbf{r}'\| = \left(r^2 + 2rR\cos\phi + R^2\right)^{1/2} \qquad (5)$$

However, as the integration is performed around the ring, it can be seen that the component of force along the e_2-axis will vanish, leaving a net radial force acting along the e_1-axis only. Therefore, the vector force component df_r acting along the radial line connecting the centre-of-mass of the ring C and the point mass is given by

$$df_r = -\mathbf{e}_1 \frac{GM_S}{\|\mathbf{r}'\|^2} \hat{\mathbf{r}}.\hat{\mathbf{r}}' dm \qquad (6)$$

where (^) donates a unit vector. From (2) and (3) it can be seen that $\hat{\mathbf{r}}.\hat{\mathbf{r}}'$ may be written as

$$\hat{\mathbf{r}}.\hat{\mathbf{r}}' = \frac{r + R\cos\phi}{\left(r^2 + 2rR\cos\phi + R^2\right)^{1/2}} \qquad (7)$$

Therefore, the vector force component df_r along the radial line connecting the centre-of-mass of the ring C and the point mass can now be written as

$$df_r = -\mathbf{e}_1 GM_S \frac{\left(r + R\cos\phi\right)}{\left(r^2 + 2rR\cos\phi + R^2\right)^{3/2}} dm \qquad (8)$$

The total force is now obtained by integrating (8) around the ring using the relation $dm = \rho R d\phi$. Therefore, using the non-dimensional variable $\xi = r/R$ yields

$$f_r(\xi) = -\frac{GM_R M_S}{2\pi R^2} \int_0^{2\pi} \frac{\xi + \cos\phi}{\left(1 + 2\xi\cos\phi + \xi^2\right)^{3/2}} d\phi \qquad (9)$$

and so integrating (9) yields the total force $f_r(\xi)$ acting along the radial line connecting the centre-of-mass of the ring C and the point mass M_S as a function of the separation r between the the centre-of-mass of the ring and the point mass. It can be shown that the integral can be obtained in closed form in terms of Hypergeometric functions. Again, using the non-dimensional variable $\xi = r/R$ yields the resulting force in compact form as

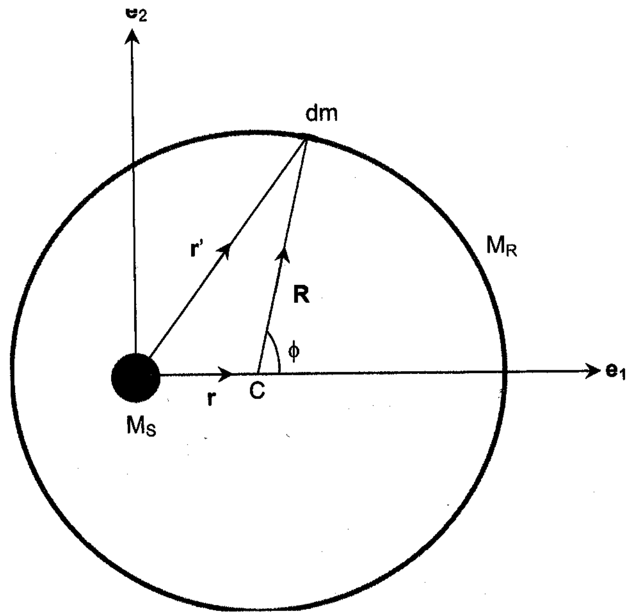

Fig. 1 Schematic in-plane displaced ring.

$$f_r(\xi) = \frac{GM_S M_R}{2R^2}\left(1+\xi^2\right)^{-5/2}\left\{3\xi\ {}_2F_1\left[\frac{7}{4},\frac{5}{4};2;\lambda\right] - 2\left(\xi + \xi^3\right){}_2F_1\left[\frac{5}{4},\frac{3}{4};1;\lambda\right]\right\} \qquad (10a)$$

$$\lambda = \frac{4\xi^2}{\left(1+\xi^2\right)^2} \qquad (10b)$$

where the Hypergeometric function ${}_2F_1[a,b;c;\lambda]$ is defined as

$${}_2F_1[a,b;c;\lambda] = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{\lambda^k}{k!} \qquad (11a)$$

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} \qquad (11b)$$

where $(a)_k$ are the Pochammer symbols and Γ the gamma function. Using this result, the in-plane dynamics of the problem can now be investigated and the instability of the sole equilibrium configuration demonstrated.

3. In-Plane Dynamics

Using the non-dimensional variable ξ, the one-dimensional equation of motion for the point mass and ring may be written as

$$\frac{d^2\xi}{d\tau^2} = \frac{1}{2}\left(1+\xi^2\right)^{-5/2}\left\{3\xi\ {}_2F_1\left[\frac{7}{4},\frac{5}{4};2;\lambda\right] - 2\left(\xi+\xi^3\right){}_2F_1\left[\frac{5}{4},\frac{3}{4};1;\lambda\right]\right\} \quad (12)$$

where the time variable t has been normalised using $\tau = t\sqrt{GM_S/R^3}$. From (10) it can be shown that

$$\underset{\xi\to 0}{Limit}\ \left(1+\xi^2\right)^{-5/2}\left\{3\xi\ {}_2F_1\left[\frac{7}{4},\frac{5}{4};2;\lambda\right] - 2\left(\xi+\xi^3\right){}_2F_1\left[\frac{5}{4},\frac{3}{4};1;\lambda\right]\right\} = 0 \quad (13a)$$

$$\underset{\xi\to 1}{Limit}\ \left(1+\xi^2\right)^{-5/2}\left\{3\xi\ {}_2F_1\left[\frac{7}{4},\frac{5}{4};2;\lambda\right] - 2\left(\xi+\xi^3\right){}_2F_1\left[\frac{5}{4},\frac{3}{4};1;\lambda\right]\right\} \to \infty \quad (13b)$$

as can be seen in Fig. 2. It is therefore clear that

$$\frac{d^2\xi}{d\tau^2} = 0 \iff \xi = 0 \quad (14)$$

which yields a single equilibrium configuration, with the centre-of-mass of the ring C co-located with the point mass M_S. The stability properties of this equilibrium configuration can now be determined by expanding $f_r(\xi)$ to first order to obtain

$$\frac{d^2\xi}{d\tau^2} = \frac{1}{2}\frac{d}{d\xi}\left[\left(1+\xi^2\right)^{-5/2}\left\{3\xi\ {}_2F_1\left[\frac{7}{4},\frac{5}{4};2;\lambda\right] - 2\left(\xi+\xi^3\right){}_2F_1\left[\frac{5}{4},\frac{3}{4};1;\lambda\right]\right\}\right]_{\xi=0}\xi \quad (15)$$

Then, in the limit as $\xi \to 0$, the first order expansion yields

$$\underset{\xi\to 0}{Limit}\ \frac{d}{d\xi}\left[\left(1+\xi^2\right)^{-5/2}\left\{3\xi\ {}_2F_1\left[\frac{7}{4},\frac{5}{4};2;\lambda\right] - 2\left(\xi+\xi^3\right){}_2F_1\left[\frac{5}{4},\frac{3}{4};1;\lambda\right]\right\}\right] = 1 \quad (16)$$

so that the linearised equation of motion in a neighbourhood of the equilibrium configuration is given by

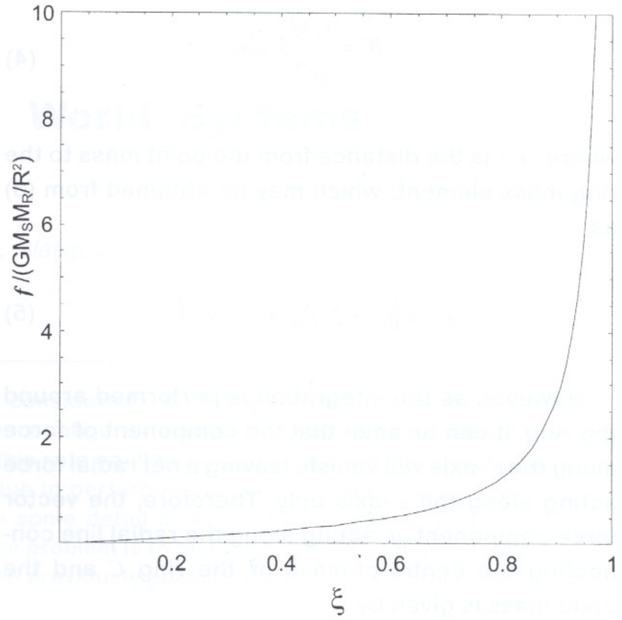

Fig. 2 Non-dimensional gravitational force acting between the point mass and ring.

$$\frac{d^2\xi}{d\tau^2} - \frac{\xi}{2} \approx 0 \quad (17)$$

again, in non-dimensional form. The solution to the linearised equation of motion is therefore of the form

$$\xi(\tau) \approx \xi_o \exp\left(\tau/\sqrt{2}\right) \quad (18)$$

for some initial displacement ξ_o, so that the equilibrium configuration is unstable as expected. In dimensional variables, the instability timescale T can be determined from (18) as

$$T = \sqrt{\frac{2R^3}{GM_S}} \quad (19)$$

While the instability of the sole equilibrium configuration has been demonstrated from the linear analysis, which is both a necessary and sufficient condition for non-linear instability, it is clear that since $f_r(\xi) > 0\ \forall \xi \in [0,1)$, the equilibrium configuration is unstable in general. The evolution of the instability obtained by numerically integrating (12) is shown in fig. 3 for a range of in-plane perturbations. It can be seen that after a slow radial drift, there is a rapid acceleration as the ring approaches the point mass, as expected from fig. 2. The instability timescale defined in (19) is related to the circular orbit period at orbit radius R, since the circular orbit period $T_o = 2\pi\sqrt{R^3/GM_S}$. Therefore, the instability timescale is just $T = T_o/\sqrt{2}\pi$. For example, a ring about the Sun at 1 AU has an instability timescale of

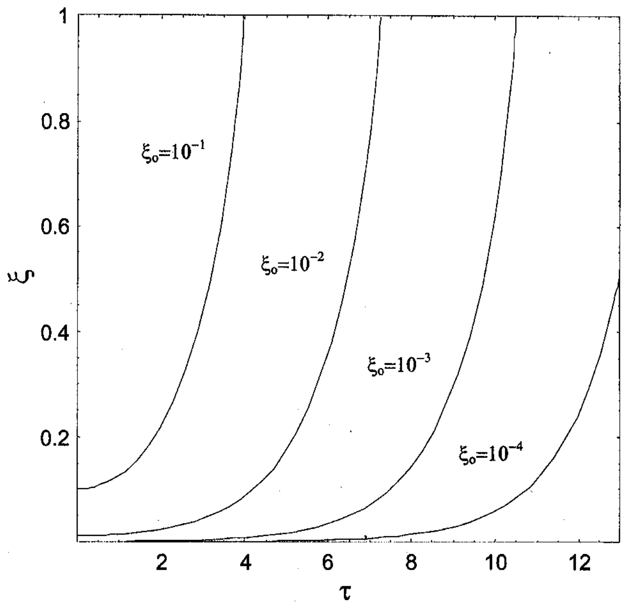

Fig. 3 Evolution of in-plane motion.

approximately 0.23 years. It can be seen from Fig. 3 that for an initial displacement ξ_o of 10^{-3}, the ring will ultimately collide with Sun after approximately 1.6 years, indicating the inherent, and rather fast, instability of the system.

4. Out-of-Plane Dynamics

Now that the in-plane dynamics of the problem have been investigated, the stability of out-of-plane motion will be determined. Again, the gravitational force acting between the ring and point mass M_S can be determined by integrating around the ring. It will now be assumed that the centre-of-mass of the ring C is displaced relative to the point mass M_S along the axis of symmetry of the ring, as shown in fig. 4. Again, the mass element dm can be located on the ring at position R relative to the centre-of-mass of the ring C, such that

$$\mathbf{R} = R\cos\phi\mathbf{e}_1 + R\sin\phi\mathbf{e}_2 \tag{20}$$

Similarly, the position of the centre-of-mass of the ring C relative to the point mass is now defined by

$$\mathbf{r} = z\mathbf{e}_3 \tag{21}$$

where $\mathbf{e}_3$ is a the unit vector normal to the $(\mathbf{e}_1, \mathbf{e}_2)$ plane, so that the position of the mass element dm relative to the central mass $\mathbf{r}' = \mathbf{r} + \mathbf{R}$ is given by

$$\mathbf{r}' = R\cos\phi\mathbf{e}_1 + R\sin\phi\mathbf{e}_2 + z\mathbf{e}_3 \tag{22}$$

As the integration is performed around the ring, it can be seen that the component of force in the $(\mathbf{e}_1, \mathbf{e}_2)$

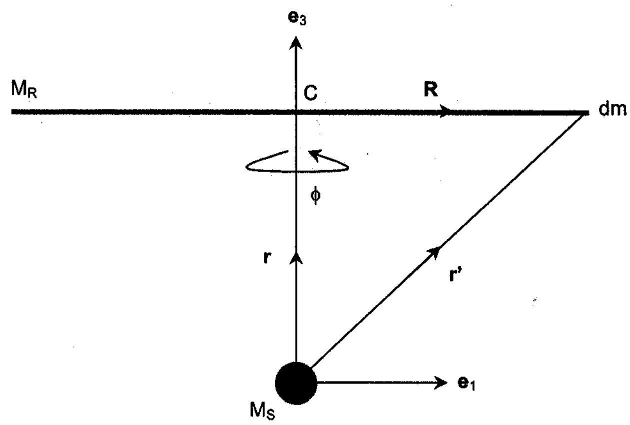

Fig. 4 Schematic out-of-plane displaced ring.

plane will vanish, leaving a net force along the $\mathbf{e}_3$-axis. Therefore, the vector force component df_z acting between the point mass and ring mass element dm along the $\mathbf{e}_3$-axis can be written as

$$df_z = -\mathbf{e}_3 \frac{GM_S}{\|\mathbf{r}'\|^2} \hat{\mathbf{r}}.\hat{\mathbf{r}}' dm \tag{23}$$

where $\|\mathbf{r}'\|$ is the distance from the central mass to the ring element, which may be obtained from (22) as

$$\|\mathbf{r}'\| = \left(z^2 + R^2\right)^{1/2} \tag{24}$$

Similarly, from (21) and (22) it can be seen that

$$\hat{\mathbf{r}}.\hat{\mathbf{r}}' = \frac{z}{\left(z^2 + R^2\right)^{1/2}} \tag{25}$$

so that total force is now obtained by integrating (23) around the ring using the relation $dm = \rho R d\phi$. Again, using a non-dimensional variable $\eta = z/R$ yields

$$f_z(\eta) = -\frac{GM_S M_R}{2\pi R^2} \int_0^{2\pi} \frac{\eta}{\left(1+\eta^2\right)^{3/2}} d\phi \tag{26}$$

which integrates to

$$f_z(\eta) = -\frac{GM_S M_R}{R^2} \frac{\eta}{\left(1+\eta^2\right)^{3/2}} \tag{27}$$

Using the non-dimensional variable η, the out-of-plane equation of motion for the central mass and ring may now be written as

$$\frac{d^2\eta}{d\tau^2} + \frac{\eta}{\left(1+\eta^2\right)^{3/2}} = 0 \tag{28}$$

where the time variable t has again been normalised using $\tau = t\sqrt{GM_S/R^3}$. It can seen that the out-of-plane problem, and hence (28), is not unlike the Sitnikov problem [5]. For small displacements such that $\eta << 1$ (28) reduces to

$$\frac{d^2\eta}{d\tau^2} + \eta \approx 0 \tag{29}$$

which has an oscillatory solution of the form

$$\eta(\tau) \approx \eta_o \cos(\tau) \tag{30}$$

for some initial displacement η_o. It is therefore concluded that the out-of-plane motion is linearly stable with, in dimensional variables, a linear oscillation of period of $T = \sqrt{R^3/GM_S}$. The non-linear nature of the large amplitude oscillations can be seen in fig. 5, where it is evident that there is a dependence between the amplitude and period of the oscillations.

Since linear stability is only a necessary condition for non-linear stability, the boundness of the large amplitude behaviour of (28) can be investigated by calculating an energy integral. Integrating (28) yields

$$\frac{1}{2}\left(\frac{d\eta}{d\tau}\right)^2 - \frac{2}{\sqrt{1+\eta^2}} = C \tag{31}$$

for some constant C, which is a function of the initial conditions. Level curves of C are shown on a phase-plane in fig. 6, with a separatrix delineating bound oscillatory motion from unbound motion defined by C=0. The condition C=0 corresponds to $d\eta/d\tau \to 0$ as $\eta \to \pm\infty$. The out-of-plane motion is therefore stable provided the initial conditions are chosen such that C<0. In dimensional variables, the initial vertical speed V_z at some initial displacement z_o must be

$$V_z \prec \sqrt{\frac{2GM_S}{R}} \frac{1}{\sqrt{1+(z_o/R)^2}} \tag{32}$$

which reduces to a quasi-parabolic escape speed $V_z \prec \sqrt{2GM_S/R}$ when the centre-of-mass of the ring C is initially collocated with the point mass.

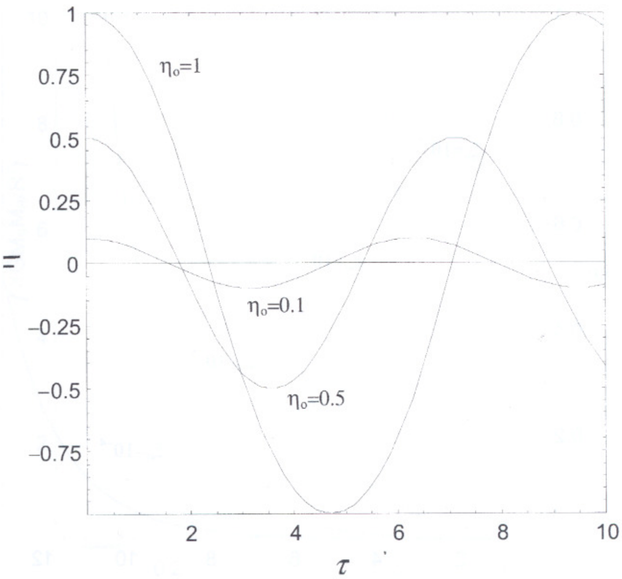

Fig. 5 Evolution of out-of-plane motion.

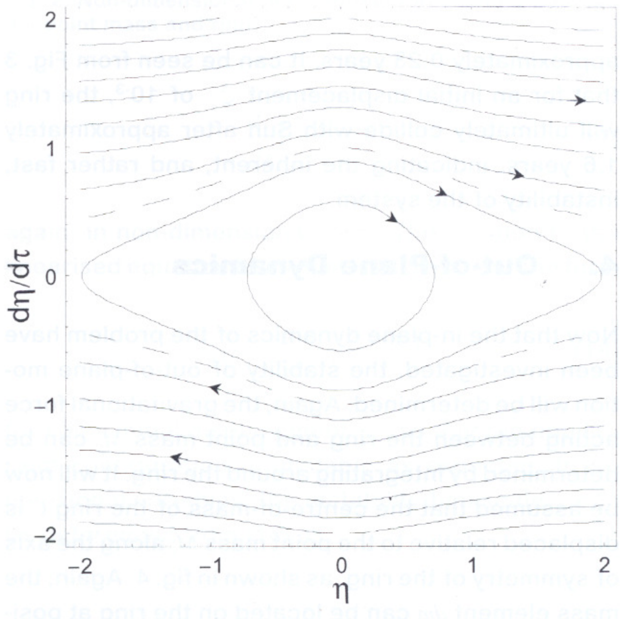

Fig. 6 Phase-plane for out-of-plane motion (- - - separatrix C=0).

4. Conclusions

The dynamics of a point mass and solid ring have been investigated by deriving the full, non-linear equation of motion for the in-plane and out-of-plane dynamics. It has been shown that there is a single equilibrium configuration with the centre-of-mass of the ring co-located with the point mass. This equilibrium configuration is unstable to in-plane perturbations, but is stable to out-of-plane perturbations, where the out-of-plane motion is bounded by a separatrix in phase-space. The analysis of the full non-linear problem allows the instability timescale to be investigated and the time to collision between the ring and central mass can be found.

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