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Almost sure exponential stability of hybrid stochastic functional differential equations

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A B S T R A C T

This paper is concerned with the almost sure exponential stability of the n-dimensional nonlinear hybrid stochastic functional differential equation (SFDE) $dx(t) = f(\psi_1(x_t,t),r(t),t)dt + g(\psi_2(x_t,t),r(t),t)dB(t)$, where $x_t = \{x(t+u) : -\tau \le u \le 0\}$ is a $C([-\tau,0];\mathbb{R}^n)$-valued process, $B(t)$ is an m-dimensional Brownian motion while $r(t)$ is a Markov chain. We show that if the corresponding hybrid stochastic differential equation (SDE) $dy(t) = f(y(t),r(t),t)dt + g(y(t),r(t),t)dB(t)$ is almost surely exponentially stable, then there exists a positive number τ^* such that the SFDE is also almost surely exponentially stable as long as $\tau < \tau^*$. We also describe a method to determine τ^* which can be computed numerically in practice.

1. Introduction

This paper is concerned with the almost sure exponential stability of the n-dimensional nonlinear hybrid stochastic functional differential equation (SFDE) of the form

$$dx(t) = f(\psi_1(x_t,t),r(t),t)dt + g(\psi_2(x_t,t),r(t),t)dB(t). \tag{1.1}$$

Here $B(t)$ is an m-dimensional Brownian motion, $r(t)$ is a Markov chain on the finite state space $\mathbb{S} = \{1,2,\cdots,N\}$, $x_t = \{x(t+s) : -\tau \le s \le 0\}$, τ is a positive number, $\psi_1,\psi_2 : C([-\tau,0];\mathbb{R}^n) \times \mathbb{R}_+ \to \mathbb{R}^n$, $f : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^{n\times m}$. The notation used will be explained in Section 2 while we refer the reader to, for example, [9–12,19,20] for the general theory on SFDEs.

To see the difficulty of this problem, let us recall some history in the area of almost sure stability of SFDEs. In 1997, Mohammed and Scheutzow [21] were first to study the almost sure exponential stability of the linear scalar stochastic differential delay equation (SDDE, a special class of SFDEs)

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$$dx(t) = \sigma x(t-\tau)dB(t), \tag{1.2}$$

where $B(t)$ is a scalar Brownian motion and σ is positive number. They showed that the SDDE (1.2) is almost surely exponentially stable provided the time delay τ is sufficiently small. Their proof for this was nontrivial. In 2005, Scheutzow [23] considered a more general scalar SFDE

$$dx(t) = \sigma\psi(x_t)dB(t), \tag{1.3}$$

where σ is positive number and ψ is a Lipschitz continuous functional from $C([-\tau,0];\mathbb{R})$ to $\mathbb{R}$ such that

$$\inf_{-\tau\le s\le 0}|\varphi(s)| \le |\psi(\varphi)| \le \sup_{-\tau\le s\le 0}|\varphi(s)|, \quad \forall\varphi\in C([-\tau,0];\mathbb{R}).$$

He also showed that equation (1.3) is almost surely exponentially stable provided τ is sufficiently small. In 2016, Guo et al. [7] considered the more general n-dimensional nonlinear SDDE with variable delays of the form

$$dx(t) = f(x(t-\delta_1(t)),t)dt + g(x(t-\delta_2(t)),t)dB(t), \tag{1.4}$$

where $B(t)$ is an m-dimensional Brownian motion, $\delta_1,\ \delta_2 : \mathbb{R}_+ \to [0,\tau]$ stand for variable delays, while $f:\mathbb{R}^n\times\mathbb{R}_+\to\mathbb{R}^n$ and $g:\mathbb{R}^n\times\mathbb{R}_+\to\mathbb{R}^{n\times m}$ are globally Lipschitz continuous. They showed that if the corresponding (non-delay) SDE

$$dx(t) = f(x(t),t)dt + g(x(t),t)dB(t) \tag{1.5}$$

is almost surely exponentially stable, so is the SDDE (1.4) provided the time delays are sufficiently small. The reason why it has taken almost 20 years to make these progresses in this area is because SFDEs (including SDDEs) are infinite-dimensional systems which are significantly different from SDEs. For example, it is straightforward to show that the linear scalar SDE $dx(t) = \sigma x(t)dB(t)$ is almost surely exponentially stable by applying the Itô formula to $\log(x(t))$ (see, e.g. [2,6]). However, it is nontrivial for Mohammed and Scheutzow [21] to show the almost sure exponential stability of the corresponding SDDE (1.2) for sufficiently small τ and they used a different approach (as one cannot apply the Itô formula to $\log(x(t))$ in this delay case).

The underlying SFDE (1.1) in this paper is more general than any of equations (1.2), (1.3) or (1.4). This is not only because of the hybrid factor modelled by the Markov chain $r(t)$ but also more general without the Markov chain. In fact, ignoring $r(t)$ and setting $\psi_1(x_t,t) = x(t-\delta_1(t))$ and $\psi_1(x_t,t) = x(t-\delta_2(t))$, we see that the SFDE (1.1) becomes equation (1.4); while if we set $f=0$, $g(x,i,t)=\sigma x$ and $\psi_2(x_t,t)=\psi(x_t)$, then the SFDE (1.1) becomes equation (1.3).

All of the above show the difficulty and generality of our proposed problem. Let us begin to develop our new theory.

2. Preliminaries

Throughout this paper, unless otherwise specified, we will use the following notation. Let $|x|$ denote the Euclidean norm of vector $x\in\mathbb{R}^n$. For a matrix A, let $|A| = \sqrt{\text{trace}(A^TA)}$ be its trace norm and $\|A\| = \max\{|Ax| : |x| = 1\}$ be the operator norm. For a vector or matrix A, its transpose is denoted by A^T. If A is a symmetric real matrix ($A = A^T$), denote by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ its smallest and largest eigenvalue, respectively.

Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\ge0},\mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying the usual conditions. Let $B(t) = (B_1(t),\cdots,B_m(t))^T$ be an m-dimensional Brownian motion with respect to the filtration. Let $r(t)$, $t \geq 0$, be a right-continuous Markov chain with respect to the filtration taking values in a finite state space $\mathbb{S} = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$. Throughout the paper, we assume that $B(t)$ and $r(t)$ are independent, and they are $\mathcal{F}_t$ adapted. It is well known that almost every sample path of $r(t)$ is a right-continuous step function with a finite number of jumps in any finite subinterval of $\mathbb{R}_+ := [0, \infty)$. As a standing hypothesis we assume in this paper that the Markov chain is *irreducible*. This is equivalent to the condition that for any $i, j \in \mathbb{S}$, one can find finite numbers $i_1, i_2, \cdots, i_k \in \mathbb{S}$ such that $\gamma_{i,i_1}\gamma_{i_1,i_2} \cdots \gamma_{i_k,j} > 0$. Note that Γ always has an eigenvalue 0. The algebraic interpretation of irreducibility is $\text{rank}(\Gamma) = N - 1$. Under this condition, the Markov chain has a unique stationary (probability) distribution $\pi = (\pi_1, \pi_2, \cdots, \pi_N) \in \mathbb{R}^{1\times N}$ which can be determined by solving the following linear equation $\pi\Gamma = 0$ subject to $\sum_{j=1}^{N} \pi_j = 1$ and $\pi_j > 0$ for all $j \in \mathbb{S}$.

Let τ be a nonnegative parameter taking values in $[0, \infty)$. Denote by $C([-\tau, 0]; \mathbb{R}^n)$ the family of continuous functions $\varphi : [-\tau, 0] \to \mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{-\tau\leq u\leq 0} |\varphi(u)|$. For $\varphi \in C([-\tau, 0]; \mathbb{R}^n)$, define

$$\mathbb{D}(\varphi) = \sup_{-\tau\leq u\leq 0} |\varphi(u) - \varphi(0)|.$$

For $t \geq 0$, denote by $L^2_{\mathcal{F}_t}(\Omega; C)$ the family of $\mathcal{F}_t$-measurable $C([-\tau, 0]; \mathbb{R}^n)$-valued random variables ξ such that $\mathbb{E}\|\xi\|^2 < \infty$, and by $L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^n)$ the family of $\mathcal{F}_t$-measurable $\mathbb{R}^n$-valued random variables η such that $\mathbb{E}|\eta|^2 < \infty$. Denote by $M_{\mathcal{F}_t}(\Omega; \mathbb{S})$ the family of $\mathcal{F}_t$-measurable $\mathbb{S}$-valued random variables. For a continuous stochastic process $x(t)$ on $[-\tau, \infty)$, define $x_t = \{x(t + s) : -\tau \leq s \leq 0\}$ for $t \geq 0$ so x_t is a $C([-\tau, 0]; \mathbb{R}^n)$-valued stochastic process on $\mathbb{R}_+$.

Consider the SFDE

$$dx(t) = f(\psi_1(x_t, t), r(t), t)dt + g(\psi_2(x_t, t), r(t), t)dB(t) \tag{2.1}$$

on $t \geq t_0(\geq 0)$ with the initial data

$$x_{t_0} = \xi \in L^2_{\mathcal{F}_{t_0}}(\Omega; C) \text{ and } r(t_0) = \zeta \in M_{\mathcal{F}_{t_0}}(\Omega; \mathbb{S}), \tag{2.2}$$

where $f : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^{n\times m}$ and $\psi_1, \psi_2 : C([-\tau, 0]; \mathbb{R}^n) \times \mathbb{R}_+ \to \mathbb{R}^n$ are all Borel measurable mappings. Please also note that ψ_1 and ψ_2 depend on the additional parameter τ. We impose some standing hypotheses on these mappings.

Assumption 2.1. Assume that there exist two nonnegative constants K_1 and K_2 such that

$$|f(x, i, t) - f(y, i, t)| \leq K_1|x - y| \quad \text{and} \quad |g(x, i, t) - g(y, i, t)| \leq K_2|x - y| \tag{2.3}$$

for all $x, y \in \mathbb{R}^n$, $i \in \mathbb{S}$ and $t \geq 0$. Assume also that $f(0, i, t) = 0$ and $g(0, i, t) = 0$ for all $i \in \mathbb{S}$ and $t \geq 0$.

Assumption 2.2. Assume that

$$|\psi_j(\varphi, t) - \psi_j(\phi, t)| \leq \|\varphi - \phi\| \quad \text{and} \quad |\psi_j(\varphi, t) - \varphi(0)| \leq \mathbb{D}(\varphi) \tag{2.4}$$

for $j = 1, 2$, $\varphi, \phi \in C([-\tau, 0]; \mathbb{R}^n)$, $t \geq 0$ and $\tau \geq 0$.

We observe that the second inequality in (2.4) forces $\psi_1(0,t) = \psi_2(0,t) = 0$ for all $t \ge 0$. We also observe that these assumptions imply that

$$|f(\psi_1(\varphi,t),i,t) - f(\psi_1(\phi,t),i,t)| \le K_1\|\varphi - \phi\|, \tag{2.5}$$

$$|g(\psi_2(\varphi,t),i,t) - g(\psi_2(\phi,t),i,t)| \le K_2\|\varphi - \phi\|, \tag{2.6}$$

and

$$|f(\psi_1(\varphi,t),i,t)| \le K_1\|\varphi\| \quad \text{and} \quad |g(\psi_2(\varphi,t),i,t)| \le K_2\|\varphi\| \tag{2.7}$$

for all $\varphi, \phi \in C([-\tau,0];\mathbb{R}^n)$, $i \in \mathbb{S}$ and $t \ge 0$. It is therefore known (see, e.g., [19, Theorem 8.3 on page 303]) that under these assumptions, the hybrid SFDE (2.1) with the initial data (2.2) has a unique solution on $t \ge t_0 - \tau$ and the solution has the property that

$$\mathbb{E}\Big(\sup_{t_0-\tau\le t\le T}|x(t)|^2\Big) < \infty, \quad \forall T > t_0. \tag{2.8}$$

We will denote the solution by $x(t;t_0,\xi,\zeta)$ in order to emphasize the initial data at time t_0, though we will often write it as $x(t)$. We also see from (2.8) that $x_t \in L^2_{\mathcal{F}_t}(\Omega;C)$ for any $t \ge t_0$. Furthermore, for any $t_0 \le s \le t < \infty$, we can regard $x(t)$ as the solution of the SFDE (2.1) on $t \ge s$ with the initial data x_s and $r(s)$ at time s. In other words, we have

$$x(t) = x(t;s,x_s,r(s)), \quad t_0 \le s \le t < \infty. \tag{2.9}$$

This shows clearly that given x_s and $r(s)$ at time s, we can determine $x(t)$ for all $t \ge s$ by solving the SFDE (2.1) but the information on how the solution reaches x_s from ξ is of no further use.

The purpose of this paper is to find sufficient conditions on the coefficients f and g as well as to obtain a positive bound τ^* such that the SFDE (2.1) is almost surely exponentially stable as long as $\tau \le \tau^*$. By the almost sure exponential stability, we mean that

$$\limsup \frac{1}{t}\log(|x(t;t_0,\xi,\zeta)|) < 0 \quad a.s.$$

for any initial data (2.2) (see, e.g., [8,11,12,15]). Let us consider a special case when $\tau = 0$. In this case, $C([-\tau,0];\mathbb{R}^n)$ becomes $\mathbb{R}^n$ and $\psi_1,\ \psi_2 : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$. Note from condition (2.4) that $\psi_1(y,t) = y$ and $\psi_2(y,t) = y$ for $(y,t) \in \mathbb{R}^n \times \mathbb{R}_+$. Hence the SFDE (2.1) becomes the corresponding hybrid SDE

$$dy(t) = f(y(t),r(t),t)dt + g(y(t),r(t),t)dB(t) \tag{2.10}$$

on $t \ge t_0$ with the initial data $(y(t_0),r(t_0)) = (\xi(0),\zeta)$. It is useful to note that $\xi(0) \in L^2_{\mathcal{F}_{t_0}}(\Omega;\mathbb{R}^n)$. Under Assumption 2.1, equation (2.10) has a unique solution (see, e.g., [14,24]) and the solution has the property that $\mathbb{E}(\sup_{t_0\le t\le T}|y(t)|^2) < \infty$ for all $T \ge t_0$. Denote the unique solution by $y(t;t_0,\xi(0),\zeta)$ on $t \ge t_0$. Let us highlight an important property provided in Mao [14, Lemma 2.1], which reads

$$\mathbb{P}\{y(t;t_0,\xi(0),\zeta) \ne 0 \text{ on } t \ge t_0 \mid \xi(0) \ne 0\} = 1. \tag{2.11}$$

That is, almost all the sample paths of any solution of equation (2.10) starting from a nonzero state will never reach the origin. Because of this property, we can choose Lyapunov functions in variety of ways. Requirements such as smoothness etc. for functions under consideration need not be imposed globally but only in a deleted neighbourhood of the origin.

3. Main results

We see clearly from the discussion in the previous section that the conditions we need to impose should at least guarantee the almost sure exponential stability of the corresponding hybrid SDE (2.10). Although there are many useful criteria on the almost sure exponential stability, we will use one established by [14]. Accordingly, we impose the following assumption.

Assumption 3.1. For each $i \in \mathbb{S}$, there are constant triples α_i, ρ_i and σ_i such that

$$\begin{aligned} x^T f(x,i,t) &\le \alpha_i |x|^2, \\ |g(x,i,t)| &\le \rho_i |x|, \\ |x^T g(x,i,t)| &\ge \sigma_i |x|^2 \end{aligned} \tag{3.1}$$

for all $x \in \mathbb{R}^n$ and $t \ge 0$. Moreover,

$$\begin{vmatrix} -(\alpha_1 + 0.5\rho_1^2 - \sigma_1^2) & -\gamma_{12} & \cdots & -\gamma_{1N} \\ -(\alpha_2 + 0.5\rho_2^2 - \sigma_2^2) & -\gamma_{22} & \cdots & -\gamma_{2N} \\ \vdots & \vdots & \cdots & \vdots \\ -(\alpha_N + 0.5\rho_N^2 - \sigma_N^2) & -\gamma_{N2} & \cdots & -\gamma_{NN} \end{vmatrix} > 0. \tag{3.2}$$

It was shown in [14] that the hybrid SDE (2.10) is almost surely exponentially stable under condition (2.3) and Assumption 3.1 along with the additional condition that

$$\text{for some } u \in \mathbb{S},\ \gamma_{iu} > 0 \text{ for all } i \ne u. \tag{3.3}$$

It was also showed in [18] that under this additional condition, (3.2) is equivalent to the following simpler condition

$$\sum_{i=1}^{N} \pi_i (\alpha_i + 0.5\rho_i^2 - \sigma_i^2) < 0. \tag{3.4}$$

The reason why we do not use this simpler condition in this paper is because that we will replace condition (3.3) by a slightly weaker one which we state as another assumption.

Assumption 3.2. There is a state $u \in \mathbb{S}$ such that

$$\gamma_{iu} \vee (\sigma_i^2 - \ 0.5\rho_i^2 - \alpha_i) > 0 \quad \text{for all } i \ne u. \tag{3.5}$$

We do not know if (3.2) is equivalent to (3.4) under this assumption yet. In this paper, we will show that condition (2.3) and Assumptions 3.1 and 3.2 are sufficient to guarantee the almost sure exponential stability of the hybrid SDE (2.10), which is a slightly better result than that in [14]. Of course, our key aim is to show that under Assumptions 2.1, 2.2, 3.1 and 3.2 there is a positive bound τ^* such that the SFDE (2.1) is almost surely exponentially stable as long as $\tau \le \tau^*$. We need to present several lemmas in order to show this main result.

Lemma 3.3. *Under Assumptions 3.1 and 3.2, for any sufficiently small $p \in (0,1)$, the $N \times N$ matrix*

$$\mathcal{A}(p) := \operatorname{diag}(\theta_1(p), \cdots, \theta_N(p)) - \Gamma \tag{3.6}$$

is a nonsingular M-matrix, where

$$\theta_i(p) := \frac{p(2-p)\sigma_i^2}{2} - \frac{p\rho_i^2}{2} - p\alpha_i. \tag{3.7}$$

We defer the proof of this lemma to the Appendix. The following lemma shows that the corresponding hybrid SDE (2.10) is exponentially stable in pth moment for sufficiently small $p \in (0,1)$ and hence, by [19, Theorem 5.9 on page 167], the SDE is also almost surely exponentially stable.

Lemma 3.4. *Let Assumptions 2.1, 3.1 and 3.2 hold. Choose a (sufficiently small) number $p \in (0,1)$ for matrix $\mathcal{A}(p)$ defined by (3.6) to be a nonsingular M-matrix. Define*

$$(c_1, \cdots, c_N)^T = \mathcal{A}^{-1}(p)(1, \cdots, 1)^T \tag{3.8}$$

(so all c_i's are positive by the theory of M-matrices [4,19] or see Lemma A.1 in the Appendix) and let $c_{\min} = \min_{1\le i\le N} c_i$ and $c_{\max} = \max_{1\le i\le N} c_i$. Then for any initial data $\xi(0) \in L^2_{\mathcal{F}_{t_0}}(\Omega; \mathbb{R}^n)$ and $\zeta \in M_{\mathcal{F}_{t_0}}(\Omega; \mathbb{S})$, the solution $y(t) = y(t; t_0, \xi(0), \zeta)$ of the hybrid SDE (2.10) satisfies

$$\mathbb{E}|y(t)|^p \le M\mathbb{E}|\xi(0)|^p e^{-\gamma(t-t_0)}, \quad \forall t \ge t_0 \tag{3.9}$$

where $\gamma = 1/c_{\max}$ and $M = c_{\max}/c_{\min}$. Moreover, let $\tau_1 > 0$ be the unique root to the following equation (in τ)

$$K_1^p \tau^p + C_p K_2^p \tau^{p/2} = 1, \tag{3.10}$$

where $C_p = (32/p)^{p/2}$. Then, whenever $\tau < \tau_1$,

$$\mathbb{E}\|y_{t+\tau}\|^p \le \frac{M}{1 - (K_1^p \tau^p + C_p K_2^p \tau^{p/2})} \mathbb{E}|\xi(0)|^p e^{-\gamma(t-t_0)}, \quad \forall t \ge t_0. \tag{3.11}$$

Proof. We first assume that $\xi(0)$ is deterministic (i.e., not a random variable). If $\xi(0) = 0$, then $y(t; t_0, 0, \zeta) = 0$ a.s. for all $t \ge 0$ so the assertions hold. For $\xi(0) \ne 0$, we write $y(t; t_0, \xi(0), \zeta) = y(t)$. As pointed out in the previous section, $y(t) \ne 0$ for all $t \ge 0$ almost surely. Define the Lyapunov function

$$V(y, i, t) = c_i |y|^p e^{\gamma t} \quad \text{for } (y, i, t) \in (\mathbb{R}^n - \{0\}) \times \mathbb{S} \times \mathbb{R}_+.$$

We can therefore apply the generalised Itô formula (see, e.g., [19,24]) to obtain that

$$\mathbb{E}V(y(t), r(t), t) = \mathbb{E}V(\xi(0), \zeta, t_0) + \mathbb{E}\int_{t_0}^{t} LV(y(s), r(s), s)ds \tag{3.12}$$

for $t \ge t_0$, where $LV : (\mathbb{R}^n - \{0\}) \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$LV(y, i, t) = e^{\gamma t}\Big(\gamma c_i |y|^p + pc_i |y|^{p-2} y^T f(y, i, t) + \frac{pc_i}{2} |y|^{p-2} |g(y, i, t)|^2 - \frac{p(2-p)c_i}{2} |y|^{p-4} |y^T g(y, i, t)|^2 + \sum_{j=1}^N \gamma_{ij} c_j |y|^p\Big).$$

By Assumption 3.1 and then using definition (3.7) of $\theta_i(p)$, we have

$$LV(y, i, t) \le e^{\gamma t} |y|^p \Big(1 - c_i\theta_i(p) + \sum_{j=1}^N \gamma_{ij} c_j\Big).$$

But, by (3.8) and (3.6),

$$c_i\theta_i(p) - \sum_{j=1}^{N} \gamma_{ij} c_j = 1, \quad \forall i \in \mathbb{S}.$$

We hence have

$$LV(y,t,i) \le 0.$$

Substituting this into (3.12) yields

$$\mathbb{E}V(y(t), r(t), t) \le \mathbb{E}V(\xi(0), \zeta, t_0).$$

This implies

$$\mathbb{E}|y(t)|^p \le M|\xi(0)|^p e^{-\gamma(t-t_0)}.$$

That is, we have shown that assertion (3.9) holds when $\xi(0)$ is deterministic. Now, for general $\xi(0) \in L^2_{\mathcal{F}_{t_0}}(\Omega; \mathbb{R}^n)$, we have

$$\mathbb{E}|y(t)|^p = \mathbb{E}\Big(\mathbb{E}\big(|y(t)|^p \big| \mathcal{F}_{t_0}\big)\Big) \le \mathbb{E}\Big(M|\xi(0)|^p e^{-\gamma(t-t_0)}\Big) = M\mathbb{E}|\xi(0)|^p e^{-\gamma(t-t_0)}$$

which is the first assertion (3.9). To show the second assertion, we see from equation (2.10) that

$$\mathbb{E}\|y_{t+\tau}\|^p \le \mathbb{E}|y(t)|^p + \mathbb{E}\Big(\sup_{0\le u\le\tau} \Big| \int_t^{t+u} f(y(s), r(s), s)ds\Big|^p\Big)$$
$$+ \mathbb{E}\Big(\sup_{0\le u\le\tau} \Big| \int_t^{t+u} g(y(s), r(s), s)dB(s)\Big|^p\Big),$$

using the elementary inequality $(a+b)^p \le a^p + b^p$ (for any $a, b \ge 0$). By condition (2.3) and the Burkholder–Davis–Gundy inequality (see, e.g., [19]), we can then easily show that

$$\mathbb{E}\|y_{t+\tau}\|^p \le \mathbb{E}|y(t)|^p + (K_1^p\tau^p + C_p K_2^p \tau^{p/2})\mathbb{E}\|y_{t+\tau}\|^p.$$

This, together with (3.9), implies the other assertion (3.11). □

It is known that the solution of the SDDE (2.1) has property (2.8). However, we need a more precise bound, as described in the following lemma, for the use of this paper.

Lemma 3.5. *Let Assumptions 2.1 and 2.2 hold. Let the initial data (2.2) be arbitrary and write $x(t; t_0, \xi, \zeta) = x(t)$. Then*

$$\mathbb{E}\Big(\sup_{t_0-\tau\le t\le T} |x(t)|^2\Big) \le 3e^{(4K_1+38K_2^2)(T-t_0)}\mathbb{E}\|\xi\|^2, \quad \forall T \ge t_0 \tag{3.13}$$

and

$$\mathbb{E}|\mathbb{D}(x_{T+\tau})|^2 \le 12\tau(2\tau K_1^2 + 5K_2^2)e^{(4K_1+38K_2^2)(T+\tau-t_0)}\mathbb{E}\|\xi\|^2, \quad \forall T \ge t_0. \tag{3.14}$$

Proof. By the Itô formula and (2.7), it is easy to show that

$$\mathbb{E}\Big(\sup_{t_0\le t\le T}|x(t)|^2\Big) \le \mathbb{E}|x(t_0)|^2 + (2K_1+K_2^2)\int_{t_0}^{T}\mathbb{E}\|x_t\|^2dt + \mathbb{E}\Big(\sup_{t_0\le t\le T}\Big|\int_{t_0}^{t}2x(s)g(\psi_2(x_s,s),r(s),s)dB(s)\Big|\Big). \tag{3.15}$$

But, by the Burkholder–Davis–Gundy inequality (see, e.g., [5]),

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{t_0\le t\le T}\Big|\int_{t_0}^{t}2x(s)g(\psi_2(x_s,s),r(s),s)dB(s)\Big|\Big)\\
&\le 3\mathbb{E}\Big(\int_{t_0}^{T}4K_2^2|x(t)|^2\|x_t\|^2dt\Big)^{1/2}\\
&\le 6\mathbb{E}\Big\{\Big(\sup_{t_0\le t\le T}|x(t)|\Big)\Big(\int_{t_0}^{T}K_2^2\|x_t\|^2dt\Big)^{1/2}\Big\}\\
&\le 0.5\mathbb{E}\Big(\sup_{t_0\le t\le T}|x(t)|^2\Big) + 18K_2^2\int_{t_0}^{T}\mathbb{E}\|x_t\|^2dt.
\end{aligned}$$

Substituting this into (3.15) yields

$$\mathbb{E}\Big(\sup_{t_0\le t\le T}|x(t)|^2\Big) \le 2\mathbb{E}|x(t_0)|^2 + (4K_1+38K_2^2)\int_{t_0}^{T}\mathbb{E}\|x_t\|^2dt.$$

Consequently

$$\mathbb{E}\Big(\sup_{t_0-\tau\le t\le T}|x(t)|^2\Big) \le 3\mathbb{E}\|\xi\|^2 + (4K_1+38K_2^2)\int_{t_0}^{T}\mathbb{E}\Big(\sup_{t_0-\tau\le s\le t}|x(s)|^2\Big)dt.$$

The Gronwall inequality gives the desired assertion (3.13). Now, by the Hölder inequality, the Doob martingale inequality as well as (3.13), we can easily show that

$$\mathbb{E}\Big(|x(T+\tau)-x(T)|^2\Big) \le 6\tau(\tau K_1^2+K_2^2)e^{(4K_1+38K_2^2)(T+\tau-t_0)}\mathbb{E}\|\xi\|^2$$

and

$$\mathbb{E}\Big(\sup_{0\le u\le\tau}|x(T+u)-x(T)|^2\Big) \le 6\tau(\tau K_1^2+4K_2^2)e^{(4K_1+38K_2^2)(T+\tau-t_0)}\mathbb{E}\|\xi\|^2. \tag{3.16}$$

But

$$\begin{aligned}
|\mathbb{D}(x_{T+\tau})|^2 &= \sup_{0\le u\le\tau}|x(T+u)-x(T+\tau)|^2\\
&\le 2|x(T+\tau)-x(T)|^2 + 2\Big(\sup_{0\le u\le\tau}|x(T+u)-x(T)|^2\Big).
\end{aligned}$$

We hence have the other assertion (3.14). □

Lemma 3.6. *Let Assumptions 2.1 and 2.2 hold and $p \in (0,1)$. Let the initial data (2.2) be arbitrary and write $x(t;t_0,\xi,\zeta) = x(t)$. Then*

$$\mathbb{E}|y(t)-x(t)|^p \le (J(\tau,t-t_0))^{p/2}\mathbb{E}\|\xi\|^p, \quad \forall t \ge t_0+\tau, \tag{3.17}$$

where $y(t) = y(t;t_0+\tau, x(t_0+\tau), r(t_0+\tau))$ and

$$\begin{aligned} J(\tau,z) = {} & \frac{12\tau(2\tau K_1^2+5K_2^2)(K_1+2K_2^2)}{4K_1+38K_2^2} \\ & \times e^{(3K_1+2K_2^2)(z-\tau)}\left[e^{(4K_1+38K_2^2)z} - e^{(4K_1+38K_2^2)\tau}\right] \quad \text{for } z \ge \tau. \end{aligned} \tag{3.18}$$

Proof. We first show the lemma for the case when $\xi \in C([-\tau,0];\mathbb{R}^n)$. By the Itô formula and Assumption 2.1, it is easy to show that for $t \ge t_0+\tau$,

$$\mathbb{E}|x(t)-y(t)|^2 \le \mathbb{E}\int_{t_0+\tau}^{t}\left[2K_1|x(s)-y(s)||\psi_1(x_s,s)-y(s)| + K_2^2|\psi_2(x_s,s)-y(s)|^2\right]ds.$$

But, by (2.4),

$$|\psi_1(x_s,s)-y(s)| \le |\psi_1(x_s,s)-x(s)| + |x(s)-y(s)| \le \mathbb{D}(x_s) + |x(s)-y(s)|.$$

Hence

$$\begin{aligned} \mathbb{E}|x(t)-y(t)|^2 \le {} & (3K_1+2K_2^2)\int_{t_0+\tau}^{t}\mathbb{E}|x(s)-y(s)|^2 ds \\ & + (K_1+2K_2^2)\int_{t_0+\tau}^{t}\mathbb{E}|\mathbb{D}(x_s)|^2 ds. \end{aligned}$$

The Gronwall inequality gives

$$\mathbb{E}|y(t)-x(t)|^2 \le (K_1+2K_2^2)e^{(3K_1+2K_2^2)(t-t_0-\tau)}\int_{t_0+\tau}^{t}\mathbb{E}|\mathbb{D}(x_s)|^2 ds.$$

This, together with Lemma 3.5, yields

$$\mathbb{E}|y(t)-x(t)|^2 \le J(\tau,t-t_0)\|\xi\|^2.$$

An application of the Hölder inequality implies

$$\mathbb{E}|y(t)-x(t)|^p \le (J(\tau,t-t_0))^{p/2}\|\xi\|^p.$$

Now, for general $\xi \in L^2_{\mathcal{F}_{t_0}}(\Omega;C)$, we have

$$\begin{aligned} \mathbb{E}|y(t)-x(t)|^p &= \mathbb{E}\Big(\mathbb{E}\big(|y(t)-x(t)|^p\big|\mathcal{F}_{t_0}\big)\Big) \\ &\le \mathbb{E}\Big((J(\tau,t-t_0))^{p/2}\|\xi\|^p\Big) = (J(\tau,t-t_0))^{p/2}\mathbb{E}\|\xi\|^p \end{aligned} \tag{3.19}$$

as desired. □

Theorem 3.7. *Let Assumptions 2.1, 2.2, 3.1 and 3.2 hold. Then there exists a positive number τ^* such that the hybrid SFDE (2.1) is almost surely exponentially stable provided $\tau < \tau^*$. In practice, we can choose $p \in (0,1)$ sufficiently small for matrix $\mathcal{A}(p)$ defined by (3.6) to be a nonsingular M-matrix and choose another free parameter $\varepsilon \in (0,1)$, and let $\tau^* > 0$ be the unique root to the equation (in τ)*

$$\varepsilon e^{p\tau(2K_1+19K_2^2)} + (J(\tau,\tau+h))^{p/2} + \left[12\tau(2\tau K_1^2 + 5K_2^2)\right]^{p/2} e^{p(2K_1+19K_2^2)(\tau+h)} = 1, \tag{3.20}$$

where $h = \log(3^{p/2}M/\varepsilon)/\gamma$ while γ, M and $J(\tau, z)$ have been defined in Lemmas 3.4 and 3.6, respectively.

Proof. We first observe that once p and ε are chosen, the sum of the left-hand-side terms in equation (3.20) is a continuously increasing function of $\tau \geq 0$ and is equal to ε when $\tau = 0$ but tends to infinity as $\tau \to \infty$, whence equation (3.20) must have a unique root $\tau^* > 0$. We also note from the definition of h that

$$3^{p/2} M e^{-\gamma h} = \varepsilon. \tag{3.21}$$

Fix $\tau \in (0, \tau^*)$ and the initial data (2.2). Write $x(t; t_0, \xi, \zeta) = x(t)$ for $t \geq t_0$ and $y(t_0+\tau+h; t_0+\tau, x(t_0+\tau), r(t_0+\tau)) = y(t_0+\tau+h)$. By Lemma 3.4, we have

$$\mathbb{E}|y(t_0+\tau+h)|^p \leq M\mathbb{E}|x(t_0+\tau)|^p e^{-\gamma h}.$$

But, by the technique of conditional expectation (as (3.19) was proved), we can show using Lemma 3.5 that

$$\mathbb{E}|x(t_0+\tau)|^p \leq 3^{p/2} e^{p\tau(2K_1+19K_2^2)} \mathbb{E}\|\xi\|^p.$$

Thus

$$\mathbb{E}|y(t_0+\tau+h)|^p \leq \varepsilon e^{p\tau(2K_1+19K_2^2)} \mathbb{E}\|\xi\|^p, \tag{3.22}$$

where (3.21) has been used. By the elementary inequality $(a+b)^p \leq a^p + b^p$ (for any $a, b \geq 0$), we have

$$\mathbb{E}|x(t_0+\tau+h)|^p \leq \mathbb{E}|y(t_0+\tau+h)|^p + \mathbb{E}|x(t_0+\tau+h) - y(t_0+\tau+h)|^p.$$

Using (3.22) as well as Lemma 3.6, we get

$$\mathbb{E}|x(t_0+\tau+h)|^p \leq \left(\varepsilon e^{p\tau(2K_1+19K_2^2)} + (J(\tau,\tau+h))^{p/2}\right) \mathbb{E}\|\xi\|^p. \tag{3.23}$$

On the other hand,

$$\mathbb{E}\|x_{t_0+\tau+h}\|^p \leq \mathbb{E}|x(t_0+\tau+h)|^p + \mathbb{E}|\mathbb{D}(x_{t_0+\tau+h})|^p. \tag{3.24}$$

But, again by the technique of conditional expectation, we can show using Lemma 3.5 that

$$\mathbb{E}|\mathbb{D}(x_{t_0+\tau+h})|^p \leq \left[12\tau(2\tau K_1^2 + 5K_2^2)\right]^{p/2} e^{p(2K_1+19K_2^2)(\tau+h)} \mathbb{E}\|\xi\|^p. \tag{3.25}$$

Substituting (3.23) and (3.25) into (3.24) gives

$$\mathbb{E}\|x_{t_0+\tau+h}\|^p \leq \bar{J}(\tau)\mathbb{E}\|\xi\|^p, \tag{3.26}$$

where

$$\bar{J}(\tau) = \varepsilon e^{p\tau(2K_1+19K_2^2)} + (J(\tau,\tau+h))^{p/2} + \left[12\tau(2\tau K_1^2 + 5K_2^2)\right]^{p/2} e^{p(2K_1+19K_2^2)(\tau+h)}.$$

But, as $\tau < \tau^*$, we see from (3.20) that $\bar{J}(\tau) < 1$. We may therefore write $\bar{J}(\tau) = e^{-\lambda(\tau+h)}$ for some $\lambda > 0$. It then follows from (3.26) that

$$\mathbb{E}\|x_{t_0+\tau+h}\|^p \le e^{-\lambda(\tau+h)}\mathbb{E}\|\xi\|^p. \tag{3.27}$$

Let us now consider the solution $x(t)$ on $t \ge t_0 + \tau + h$. By property (2.9), this can be regarded as the solution of the SFDE (2.1) with the initial data $x_{t_0+\tau+h}$ and $r(t_0 + \tau + h)$ at time $t_0 + \tau + h$. In the same way as (3.27) was proved, we can show

$$\mathbb{E}\|x_{t_0+2(\tau+h)}\|^p \le e^{-\lambda(\tau+h)}\mathbb{E}\|x_{t_0+\tau+h}\|^p.$$

This, together with (3.27), implies

$$\mathbb{E}\|x_{t_0+2(\tau+h)}\|^p \le e^{-2\lambda(\tau+h)}\mathbb{E}\|\xi\|^p.$$

Repeating this procedure, we have

$$\mathbb{E}\|x_{t_0+k(\tau+h)}\|^p \le e^{-k\lambda(\tau+h)}\mathbb{E}\|\xi\|^p \tag{3.28}$$

for all $k = 1, 2, \cdots$. But this holds for $k = 0$ obviously so (3.28) holds for all $k = 0, 1, 2, \cdots$. On the other hand, by Lemma 3.5, we can show, in the same way as (3.19) was proved, that

$$\mathbb{E}\Big(\sup_{t_0+k(\tau+h)\le t\le t_0+(k+1)(\tau+h)} |x(t)|^p\Big) \le K\mathbb{E}\|x_{t_0+k(\tau+h)}\|^p \tag{3.29}$$

for all $k = 0, 1, 2, \cdots$, where $K = 3^{p/2}e^{pk(\tau+h)(2K_1+19K_2^2)}$. This, together with (3.28), implies

$$\mathbb{E}\Big(\sup_{t_0+k(\tau+h)\le t\le t_0+(k+1)(\tau+h)} |x(t)|^p\Big) \le Ke^{-k\lambda(\tau+h)}\mathbb{E}\|\xi\|^p.$$

Consequently, for any $\bar{\varepsilon} \in (0, \lambda)$,

$$\mathbb{P}\Big(\sup_{t_0+k(\tau+h)\le t\le t_0+(k+1)(\tau+h)} |x(t)|^p \ge e^{-k(\lambda-\bar{\varepsilon})(\tau+h)}\Big) \le Ke^{-k\bar{\varepsilon}(\tau+h)}\mathbb{E}\|\xi\|^p.$$

By the Borel–Cantelli lemma (see, e.g., [15, Lemma 2.4 on page 7]), we obtain that for almost all $\omega \in \Omega$, there is an integer $k_0 = k_0(\omega)$ such that

$$\sup_{t_0+k(\tau+h)\le t\le t_0+(k+1)(\tau+h)} |x(t)|^p < e^{-k(\lambda-\bar{\varepsilon})(\tau+h)} \quad \forall k \ge k_0(\omega).$$

This implies easily that

$$\limsup_{t\to\infty} \frac{1}{t}\log(|x(t)|) \le -\frac{\lambda - \bar{\varepsilon}}{p} \quad a.s.$$

As $\bar{\varepsilon}$ is arbitrary, we have

$$\limsup_{t\to\infty} \frac{1}{t}\log(|x(t)|) \le -\frac{\lambda}{p} \quad a.s.$$

The proof is hence complete. □

In the statement of Theorem 3.7, we describe a method to determine τ^* by choosing two parameters p and ε. Unfortunately, we do not know how to determine them in order to get the optimal τ^* yet. Our bound on τ^* is therefore conservative but it is a challenge to get the optimal bound. Let us make a useful remark to close this section.

Remark 3.8. We observe from the proof of Lemma 3.2 that conditions (3.2) and (3.5) are only used to guarantee that

$$\text{there is a number } p \in (0,1) \text{ for matrix } \mathcal{A}(p) \text{ defined by (3.6) to be a nonsingular M-matrix.} \tag{3.30}$$

We therefore see that Theorem 3.7 still holds if the sentence "Let Assumptions 2.1, 2.2, 3.1 and 3.2 hold" there is replaced by "Let Assumptions 2.1 and 2.2 as well as conditions (3.1) and (3.30) hold".

4. Special SFDEs

In this section we will discuss a number of special but important classes of hybrid SFDEs. We will show more clearly from these discussions that our new theory established in this paper is a generalisation of the earlier papers, e.g., [7,21,23] in this area. As before, $B(t)$ is an m-dimensional Brownian motion unless otherwise specified. We will omit mentioning the initial data as they are obvious.

4.1. Scalar hybrid SFDEs

Let us first consider the scalar hybrid SFDE

$$dx(t) = b_{r(t)}\psi(x_t)dB(t), \tag{4.1}$$

where $B(t)$ is a scalar Brownian motion, b_i ($i \in \mathbb{S}$) are all non-zero real numbers and $\psi : C[-\tau,0];\mathbb{R}) \to \mathbb{R}$ satisfying

$$|\psi(\varphi) - \psi(\phi)| \le \|\varphi - \phi\| \quad \text{and} \quad |\psi(\varphi) - \varphi(0)| \le \mathbb{D}(\varphi)$$

for $\varphi, \phi \in C[-\tau,0];\mathbb{R})$. This is a special case of the SFDE (2.1) with $f(x,i,t) = 0$, $g(x,i,t) = \sigma_i x$, $\psi_1(\varphi,t) = 0$ and $\psi_2(\varphi,t) = \psi(\varphi)$. It is easy to see that Assumptions 2.1 and 2.2 are satisfied with $K_1 = 0$ and $K_2 = \max_{i\in\mathbb{S}} |b_i|$. It is also easy to see that condition (3.1) holds with $\alpha_i = 0$ and $\rho_i = \sigma_i = |b_i|$. Hence, for $p \in (0,1)$, matrix $\mathcal{A}(p)$ defined by (3.6) becomes

$$\mathcal{A}(p) = 0.5p(1-p)\mathrm{diag}(b_1^2, \cdots, b_N^2) - \Gamma.$$

By the property of Γ, we have $\Gamma\vec{1} = 0$ and hence

$$\mathcal{A}(p)\vec{1} = 0.5p(1-p)(b_1^2, \cdots, b_N^2)^T > 0,$$

where $\vec{1} = (1, \cdots, 1)^T \in \mathbb{R}^N$. It then follows from Lemma A.1 that $\mathcal{A}(p)$ is a nonsingular M-matrix for any $p \in (0,1)$. By Remark 3.8, we can then conclude that there exists a positive number τ^* such that the hybrid SFDE (4.1) is almost surely exponentially stable provided $\tau < \tau^*$.

4.2. Hybrid SDDEs

Let δ_1 and δ_2 be two Borel measurable functions from $\mathbb{R}_+$ to $[0,\tau]$. Define $\psi_1, \psi_2 : C([-\tau,0];\mathbb{R}^n) \times \mathbb{R}_+ \to \mathbb{R}^n$ by $\psi_1(\varphi,t) = \varphi(-\delta_1(t))$ and $\psi_2(\varphi,t) = \varphi(-\delta_2(t))$. Then

$$|\psi_1(\varphi,t)-\psi_1(\phi,t)| = |\varphi(-\delta_1(t))-\phi(-\delta_1(t))| \le \|\varphi-\phi\|$$

and

$$|\psi_1(\varphi,t)-\varphi(0)| = |\varphi(-\delta_1(t))-\varphi(0)| \le \mathbb{D}(\varphi)$$

for all $\varphi,\phi \in C([-\tau,0];\mathbb{R}^n)$ and $t \ge 0$, and similarly for ψ_2. That is, ψ_1 and ψ_2 satisfy Assumption 2.2. The SFDE (2.1) becomes the hybrid SDDE

$$dx(t) = f(x(t-\delta_1(t)), r(t), t)dt + g(x(t-\delta_2(t)), r(t), t)dB(t). \tag{4.2}$$

By Theorem 3.7, we can then conclude that under Assumptions 2.1, 3.1 and 3.2, there exists a positive number τ^* such that the hybrid SDDE (4.2) is almost surely exponentially stable provided $\tau < \tau^*$.

4.3. Hybrid SFDEs with distributed delays

Denote by $\mathcal{P}([0,\tau])$ the family of non-decreasing and right-continuous functions μ from $\mathbb{R}$ to $[0,1]$ satisfying $\mu(u)=1$ for $u \ge \tau$ and $\mu(u)=0$ for $u<0$. It is easy to see that

$$\int_0^\tau d\mu(u) = 1, \quad \forall \mu \in \mathcal{P}([0,\tau]).$$

In other words, $\mathcal{P}([0,\tau])$ is in fact a space of probability measures on $[0,\tau]$. Let $\mu_1,\mu_2 \in \mathcal{P}([0,\tau])$ and define $\psi_1,\psi_2 : C([-\tau,0];\mathbb{R}^n)\times\mathbb{R}_+ \to \mathbb{R}^n$ by

$$\psi_1(\varphi,t) = \int_0^\tau \varphi(-u)d\mu_1(u) \quad \text{and} \quad \psi_2(\varphi,t) = \int_0^\tau \varphi(-u)d\mu_2(u), \tag{4.3}$$

where the integrals are of Stieltjes-type while

$$T_1 = \int_0^\tau u d\mu_1(u) \quad \text{and} \quad T_2 = \int_0^\tau u d\mu_2(u)$$

are known as the average time delays. When ψ_1 and ψ_2 are defined by (4.3), equation (2.1) is known as a hybrid SFDE with distributed delays. It includes hybrid SDDEs with several time delays where, for example,

$$\psi_1(\varphi,t) = \sum_{k=1}^{\kappa} w_k \varphi(-\tau_k)$$

in which $0<\tau_1<\cdots<\tau_\kappa \le \tau$ and $w_k \in (0,1)$ with $w_1+\cdots+w_\kappa = 1$. For $\varphi,\phi \in C([-\tau,0];\mathbb{R}^n)$, we have

$$|\psi_1(\varphi,t)-\psi_1(\phi,t)| \le \int_0^\tau |\varphi(-u)-\phi(-u)|d\mu_1(u) \le \|\varphi-\phi\|$$

and

$$|\psi_1(\varphi,t)-\varphi(0)| = \Big|\int_0^{\tau}(\varphi(-u)-\varphi(0))d\mu_1(u)\Big| \le \mathbb{D}(\varphi),$$

and similarly for ψ_2. That is, ψ_1 and ψ_2 satisfy Assumption 2.2. By Theorem 3.7, we can then conclude that under Assumptions 2.1, 3.1 and 3.2, there exists a positive number τ^* such that the hybrid SFDE (2.1) with (4.3) is almost surely exponentially stable provided $\tau < \tau^*$.

4.4. Stochastic stabilised systems

Suppose that we are given an unstable hybrid differential equation

$$dx(t)/dt = f(x(t), r(t), t) \tag{4.4}$$

and we need to design a stochastic delay feedback control $g(x(t-\tau), r(t), t)dB(t)$ so that the controlled system

$$dx(t) = f(x(t), r(t), t)dt + g(x(t-\tau), r(t), t)dB(t) \tag{4.5}$$

becomes almost surely exponentially stable. The reader can find more information on the stochastic stabilisation from, for example, [1,3,13,16,17,22]. We assume that f and g satisfy Assumption 2.1 and condition (3.1). We also assume that one of the following items is satisfied:

- Conditions (3.2) and (3.5) hold.
- Conditions (3.3) and (3.4) hold.
- There is a number $p \in (0, 1)$ such that matrix $\mathcal{A}(p)$ defined by (3.6) is a nonsingular M-matrix.

By Theorem 3.7 or Remark 3.8, we can then conclude that there exists a positive number τ^* such that the controlled system (4.5) is almost surely exponentially stable provided $\tau < \tau^*$.

Example 4.1. Consider the unstable system (4.4) under the situation where the space $\mathbb{S}$ of the Markov chain is divided into two proper subspaces $\mathbb{S}_1$ and $\mathbb{S}_2$ (namely $\mathbb{S} = \mathbb{S}_1 \cup \mathbb{S}_2$ and $\mathbb{S}_1 \cap \mathbb{S}_2 = \emptyset$) such that the state $x(t)$ is not observable when the system is in any mode $i \in \mathbb{S}_1$ but is fully observable in any mode $i \in \mathbb{S}_2$. Let us now design our stochastic delay feedback control. To make it simple, we only use a scalar Brownian motion $B(t)$ and design the linear delay feedback control

$$g(x(t-\tau), r(t), t)dB(t) = A_{r(t)}x(t-\tau)dB(t).$$

Namely, the stochastically controlled system has the form

$$dx(t) = f(x(t), r(t), t)dt + A_{r(t)}x(t-\tau)dB(t). \tag{4.6}$$

Given that the system is not controllable in any mode $i \in \mathbb{S}_1$, we must have $A_i = 0$ for all $i \in \mathbb{S}_1$ (so the parameters $\rho_i = \sigma_i = 0$ in (3.1)). Our aim here is to design A_i, $i \in \mathbb{S}_2$, for the controlled system (4.6) to be almost surely exponentially stable provided τ is sufficiently small. Let us discuss two cases.

Case 1. There is some $u \in \mathbb{S}$ such that $\gamma_{iu} > 0$ for all $i \neq u$.

In other words, condition (3.3) holds. This means that the Markov chain can jump to state u directly from any other state in very short time with positive probability. On the other hand, as the Markov chain is irreducible, it can also jump to some other state directly from state u in very short time with positive probability. In other words, the system modes will switch among themselves sufficiently frequently so that the corresponding delay feedback control based on the information observed in S_2 modes could influence the system in S_1 modes as well. As a result, the controlled system (4.6) could be stabilised. Let us now explain how to design A_i to achieve this goal. For each $i \in \mathbb{S}_2$, we choose matrix $A_i \in \mathbb{R}^{n\times n}$ such that

$$\lambda_{\min}(A_i + A_i^T) > \sqrt{2}\|A_i\|. \tag{4.7}$$

Noting that

$$|A_i x| \le \|A_i\||x| \quad \text{and} \quad |x^T A_i x| = 0.5|x^T(A_i + A_i^T)x| \ge 0.5\lambda_{\min}(A_i + A_i^T)|x|^2$$

for $x \in \mathbb{R}^n$, we see the parameters in (3.1) are

$$\rho_i = \|A_i\| \quad \text{and} \quad \sigma_i = 0.5\lambda_{\min}(A_i + A_i^T), \quad i \in \mathbb{S}_2.$$

Accordingly, condition (3.4) becomes

$$\sum_{i\in\mathbb{S}} \pi_i\alpha_i < 0.5 \sum_{i\in\mathbb{S}_2} \pi_i \Big(0.5\big(\lambda_{\min}(A_i + A_i^T)\big)^2 - \|A_i\|^2\Big). \tag{4.8}$$

There are lots of matrices A_i which satisfy conditions (4.7) and (4.8). For example, for each i, choose a matrix $\bar{A}_i \in \mathbb{R}^{n\times n}$ such that

$$\|\bar{A}_i\| = 1 \quad \text{and} \quad \lambda_{\min}(\bar{A}_i + \bar{A}_i^T) \ge \sqrt{3}. \tag{4.9}$$

Let $\beta > 0$ and $A_i = \sqrt{\beta/\pi_i}\bar{A}_i$. Then (4.7) holds and (4.8) becomes

$$\sum_{i\in\mathbb{S}} \pi_i\alpha_i < 0.25\beta N_2, \tag{4.10}$$

where N_2 is the number of the states in $\mathbb{S}_2$, and this holds provided $\beta > (4/N_2)\sum_{i\in\mathbb{S}}\pi_i\alpha_i$. We can therefore conclude that if we let $A_i = 0$ for all $i \in \mathbb{S}_1$ and choose A_i for $i \in \mathbb{S}_2$ for (4.7) and (4.8) to hold, then there exists a positive number τ^* such that the controlled system (4.6) is almost surely exponentially stable provided $\tau < \tau^*$.

Case 2. For each $i \in \mathbb{S}_1$, there is a $j_i \in \mathbb{S}_2$ such that $\gamma_{i,j_i} > 0$.

In layman's terms, this case means that the Markov chain can jump to a state $j_i \in \mathbb{S}_2$ directly from (every) state $i \in \mathbb{S}_1$ in very short time with a positive probability. In other words, the system will return to (controllable) S_2 modes frequently from (uncontrollable) S_1 modes. To explain how to design matrices A_i ($i \in \mathbb{S}_2$), let us assume, without loss of generality, that $\mathbb{S}_1 = \{1, \cdots, \bar{N}\}$ and $\mathbb{S}_2 = \{\bar{N}+1, \cdots, N\}$ for some $1 \le \bar{N} < N$. Note that

$$\sum_{j=N_1+1}^{N} \gamma_{ij} \ge \gamma_{i,j_i} > 0, \quad \forall i \in \mathbb{S}_1.$$

We can first choose a pair of numbers $p \in (0, 2/3)$ and $\beta \in (0, 1)$ such that

$$(1-\beta)\sum_{j=\bar{}+1}^{N} \gamma_{ij} > p\alpha_i, \quad \forall i \in \mathbb{S}_1. \tag{4.11}$$

We then, for each $i \in \mathbb{S}_2$, find a nonnegative number δ_i such that

$$\frac{\beta p \delta_i^2 (2-3p)}{8} > (1-\beta)\sum_{j=1}^{N_1} \gamma_{ij} + \beta p \alpha_i. \tag{4.12}$$

Choose a matrix $\bar{A}_i$ satisfying condition (4.9) and let $A_i = \delta_i \bar{A}_i$. We therefore see that the second and third inequality in (3.1) hold with $\rho_i = \delta_i$ and $\sigma_i = \sqrt{3/4}\delta_i$ for $i \in \mathbb{S}_2$ while (recall $A_i = 0$ so) $\rho_i = \sigma_i = 0$ for $i \in \mathbb{S}_1$. Define

$$\xi = (\overbrace{1, \cdots, 1}^{\bar{N} \text{ times}}, \overbrace{\beta, \cdots, \beta}^{N-\bar{N} \text{ times}})^T,$$

and set

$$(\zeta_1, \cdots, \zeta_N)^T := \mathcal{A}(p)\xi.$$

Then, for $i \in \mathbb{S}_1$,

$$\zeta_i = -p\alpha_i - \sum_{j=1}^{N_1} \gamma_{ij} - \beta \sum_{j=N_1+1}^{N} \gamma_{ij} = -p\alpha_i + (1-\beta) \sum_{j=N_1+1}^{N} \gamma_{ij} > 0$$

by (4.11), while for $i \in \mathbb{S}_2$,

$$\begin{aligned}
\zeta_i &= \beta p \Big(\frac{(2-3p)\delta_i^2}{8} - \alpha_i \Big) - \sum_{j=1}^{N_1} \gamma_{ij} - \beta \sum_{j=N_1+1}^{N} \gamma_{ij} \\
&= \beta p \Big(\frac{(2-3p)\delta_i^2}{8} - \alpha_i \Big) - (1-\beta) \sum_{j=1}^{N_1} \gamma_{ij} \\
&> 0
\end{aligned}$$

by (4.12). By Lemma A.1, $\mathcal{A}(p)$ is a nonsingular M-matrix. In other words, we have design A_i to meet Assumption 3.2 in this case. We can therefore conclude by Theorem 3.7 that if we design A_i as described above, then there exists a positive number τ^* such that the stochastic controlled hybrid system (4.6) is almost surely exponentially stable provided $\tau \le \tau^*$.

5. Conclusion

In this paper we investigated the almost sure exponential stability of the n-dimensional nonlinear hybrid SFDE (2.1). Under the Lipschitz condition, we showed that if the corresponding hybrid SDE (2.10) is almost surely exponentially stable, then there exists a positive number τ^* such that the SFDE (2.1) is also almost surely exponentially stable as long as $\tau < \tau^*$. We also provided the reader with a method to determine τ^* which can be computed numerically in practice. Several special classes of hybrid SFDEs were discussed to demonstrate that our new theory established in this paper is a generalisation of the existing papers, e.g., [7,21,23], in this area.

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Appendix A

In this appendix, we will prove Lemma 3.3. For this purpose, we need the theory of M-matrices. For the convenience of the reader, let us cite some useful results on M-matrices. For more detailed information please see, e.g., [4,19]. If A is a vector or matrix, by $A > 0$ we mean all elements of A are positive. Moreover, a square matrix $A = [a_{ij}]_{N\times N}$ is called a Z-matrix if it has non-positive off-diagonal entries, namely

$$a_{ij} \le 0 \text{ for all } i \ne j.$$

Lemma A.1 *(see, e.g., [4,19]). If $A = [a_{ij}]_{N\times N}$ is a Z-matrix, then the following statements are equivalent:*

(1) A is a nonsingular M-matrix.
(2) A is semi-positive; that is, there exists $x > 0$ in $\mathbb{R}^N$ such that $Ax > 0$.
(3) A^{-1} exists and its elements are all nonnegative.
(4) All the leading principal minors of A are positive; that is

$$\begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{vmatrix} > 0 \quad \text{for every } k = 1, 2, \cdots, N.$$

We also need another result.

Lemma A.2. *If a Z-matrix $A = [a_{ij}]_{N\times N}$ has all of its row sums positive, that is*

$$\sum_{j=1}^{N} a_{ij} > 0, \quad \forall i = 1, 2, \cdots, N,$$

then $\det A > 0$.

This lemma is an immediate consequence of Lemma A.1. In fact, it is easy to see that $Ax > 0$ for $x = (1, 1, \cdots, 1)^T \in \mathbb{R}^N$. By statement (2) of Lemma A.1, A is a nonsingular M-matrix. Consequently, by statement (4) of Lemma A.1, $\det A > 0$ as desired. We can now prove Lemma 3.3.

Proof of Lemma 3.3. Without loss of generality, we may assume that the state $u = N$ in Assumption 3.2, namely

$$\gamma_{iN} \vee (\sigma_i^2 - \ 0.5\rho_i^2 - \alpha_i) > 0 \quad \text{for all } 1 \le i \le N - 1. \tag{A.1}$$

If not, we can simply reorder the states of the Markov chain $r(t)$ by switching state u with N, that is, rename state u as N while N as u. Consequently, the determinant in the left hand side of (3.2) will switch the uth row with the Nth row and then switch the uth column with the Nth column but these do not change the value of the determinant, namely the determinant remains positive. Moreover, given a nonsingular M-matrix, if we switch the uth row with the Nth row and then switch the uth column with the Nth column, the new matrix is still a nonsingular M-matrix.

By [19, Lemma 5.2 on page 173], the derivative $d(\det \mathcal{A}(0))/dp =$ the determinant in the left hand side of (3.2), whence $d(\det \mathcal{A}(0))/dp > 0$. It is also easy to see $\det \mathcal{A}(0) = 0$. Consequently, for all $p \in (0,1)$ sufficiently small, we have

$$\det \mathcal{A}(p) > 0. \tag{A.2}$$

On the other hand, for each $i = 1, 2, \cdots, N-1$, either $\gamma_{iN} > 0$ or $\gamma_{iN} = 0$. In the case when $\gamma_{iN} > 0$, we clearly have

$$\theta_i(p) > -\gamma_{iN} \quad \text{for all sufficiently small } p \in (0,1);$$

while in the case when $\gamma_{iN} = 0$, condition (A.1) implies $\sigma_i^2 - \ 0.5\rho_i^2 - \alpha_i > 0$ whence

$$\theta_i(p) > 0 = -\gamma_{iN} \quad \text{for all sufficiently small } p \in (0,1).$$

In other words, we always have

$$\theta_i(p) > -\gamma_{iN}, \quad i = 1, 2, \cdots, N-1 \tag{A.3}$$

for all $p \in (0,1)$ sufficiently small. Fix any $p \in (0,1)$ sufficiently small for both (A.2) and (A.3) to hold. For each $k = 1, 2, \cdots, N-1$, consider the leading principal sub-matrix

$$\mathcal{A}_k(p) = \begin{bmatrix} \theta_1(p) - \gamma_{11} & -\gamma_{12} & \cdots & -\gamma_{1k} \\ -\gamma_{21} & \theta_2(p) - \gamma_{22} & \cdots & -\gamma_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ -\gamma_{k1} & -\gamma_{k2} & \cdots & \theta_k(p) - \gamma_{kk} \end{bmatrix}$$

of $\mathcal{A}(p)$. Obviously, $\mathcal{A}_k(p)$ is a Z-matrix. Moreover, for every $i = 1, 2, \cdots, k$, the ith row of this sub-matrix has its sum

$$\theta_i(p) - \sum_{j=1}^{k} \gamma_{ij} = \theta_i(p) + \sum_{j=k+1}^{N} \gamma_{ij} \geq \theta_i(p) + \gamma_{iN} > 0$$

by (A.3). By Lemma A.2, $\det \mathcal{A}_k(p) > 0$. In other words, we have shown that all the leading principal minors of $\mathcal{A}(p)$ are positive. By Lemma A.1, $\mathcal{A}(p)$ is a nonsingular M-matrix as desired. The proof is therefore complete.

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