Research Article

Minimality conditions for wave speed in anisotropic smectic C* liquid crystals

E.C.M. Crooks[a], M. Grinfeld[b]*and G. McKay[b]

We discuss minimality conditions for the speed of monotone travelling waves in a sample of smectic C* liquid crystal subject to a constant electric field, dealing with both isotropic and anisotropic cases. Such conditions are important in understanding the properties of domain wall switching across a smectic layer, and our focus here is on examining how the presence of anisotropy can affect the speed of this switching. We obtain an estimate of the influence of anisotropy on the minimal speed, sufficient conditions for linear and non-linear minimal speed selection mechanisms to hold in different parameter regimes, and a characterisation of the boundary separating the linear and non-linear regimes in parameter space. Copyright © 0000 John Wiley & Sons, Ltd.

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1. The Physical Set-Up

Smectics are a mesophase of liquid crystals in which the long axes of the constituent molecules align in layers with a well-defined interlayer spacing. Mathematically, liquid crystals are normally described in terms of a unit vector $\boldsymbol{n}$, the local direction of the average molecular alignment. In the smectic A (or SmA) phase, the molecules tend to align perpendicular to the layers, while in the smectic C (SmC) phase, the director $\boldsymbol{n}$ is aligned at some angle away from the smectic layer normal. This angle, here denoted by θ, is usually temperature dependent and is called the *smectic cone (or tilt) angle*. Chiral smectic C (SmC*) liquid crystals have a twist axis perpendicular to the usual smectic layers and are ferroelectric, i.e. they exhibit a spontaneous polarisation $\boldsymbol{P}$.

As discussed in Kidney et al. [16], surface tension can induce a smectic film at the surface of an isotropic droplet. We consider the behaviour of a smectic film subject to an in-plane electric field, as shown in Figure 1. The mathematical description of the smectic layer employs the unit vector $\boldsymbol{a} = (0, 0, 1)$ normal to the layer, and the unit vector $\boldsymbol{c}$ which represent the unit orthogonal projection of $\boldsymbol{n}$ onto the layer. It follows, therefore, that $\boldsymbol{n} = \boldsymbol{a}\cos\theta + \boldsymbol{c}\sin\theta$. The dynamic rotation of the director $\boldsymbol{n}$ around the layer normal $\boldsymbol{a}$ can be described via the twist angle $\phi(x, t)$ so that $\boldsymbol{c} = (\cos\phi, \sin\phi, 0)$. It is convenient to introduce a third unit vector $\boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{c} = (-\sin\phi, \cos\phi, 0)$ as this is parallel to the direction of the spontaneous polarisation $\boldsymbol{P}$, i.e. $\boldsymbol{P} = P_0\boldsymbol{b}$ for some constant P_0, typically with a magnitude in the range $10 \sim 10^3 \, \mu\text{C m}^{-2}$. We also incorporate a constant electric field $\boldsymbol{E} = E(1, 0, 0)$ of strength E applied in the x-direction, as shown in the Figure. If $P_0 > 0$ then the spontaneous polarisation will prefer to align with any applied electric field, with alignment in the opposite direction when $P_0 < 0$.

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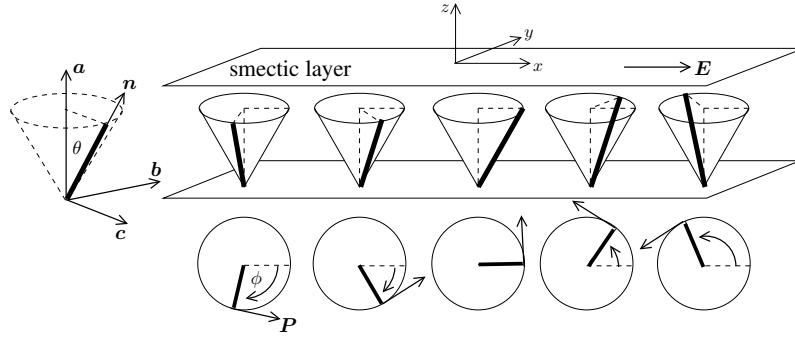*Correspondence to: E-mail: m.grinfeld@strath.ac.uk*

Figure 1. The geometry of a SmC* layer

Leslie, Stewart and Nakagawa [18] developed a continuum theory for smectic C* crystals in terms of a gradient flow of a suitable free energy density w. This energy has three components,

$$w = w_{\text{elas}} + w_{\text{pol}} + w_{\text{dielec}},$$

corresponding to elastic, spontaneous polarisation and dielectric effects. Carlsson *et al.* [8] show that the elastic energy density can be constructed from combinations of five basic distortions and their couplings, leading to a total energy involving nine terms each associated with an elastic constant. These basic elastic deformations are related to bending of the smectic layers, or the re-orientation of the c vector within or across layers. For the model considered here, in the absence of bending of the smectic layers as the vector a is assumed constant, the elastic energy density in the bulk of the liquid crystal simplifies significantly to

$$w_{\text{elas}} = \frac{1}{2} B_1 (\nabla \cdot \boldsymbol{b})^2 + \frac{1}{2} B_2 (\nabla \cdot \boldsymbol{c})^2,$$

where elastic constants B_1 and B_2 are related to the bend and splay, respectively, of the c-director in each smectic layer, i.e. within the xy-plane. Typically, the values of the elastic constants lie in the ranges [7]: $1 \le B_1 \le 10\,\text{pN}$, $5 \le B_2 \le 100\,\text{pN}$. We introduce a dimensionless measure of anisotropy in the elastic constants, ξ, such that $B_1 = B(1 - \xi)$ and $B_2 = B(1 + \xi)$ for some elastic constant B and with $|\xi| < 1$. Adopting the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ described above, we can now write the elastic energy density as

$$w_{\text{elas}} = \frac{1}{2} B \left(1 - \xi \cos 2\phi\right) \phi_x^2.$$

Note that the isotropic case $\xi = 0$ leads to a semilinear evolution equation for $\phi(x, t)$ in (3), while the anisotropic one gives rise to a quasilinear equation.

The spontaneous polarisation contribution to the free energy density is

$$w_{\text{pol}} = -P_0 \boldsymbol{b} \cdot \boldsymbol{E} = -P_0 E \sin \phi. \tag{1}$$

Finally, the dielectric energy density for the ferroelectric can be also be expressed in terms of vectors a, c and $\boldsymbol{E}$,

$$w_{\text{dielec}} = -\frac{1}{2}\epsilon_0 \epsilon_a (\boldsymbol{n} \cdot \boldsymbol{E})^2 = -\frac{1}{2}\epsilon_0 \epsilon_a (E \cos \phi \sin \theta)^2, \tag{2}$$

where ϵ_0 is the permittivity of free space and ϵ_a is the dielectric anisotropy, typically $O(1)$ and often negative. Combining the various components, we obtain

$$w(\phi) = \frac{1}{2}B(1 - \xi \cos 2\phi)\phi_x^2 + 2P_0 E \left(\frac{1}{2}\sin \phi + \frac{1}{4}\beta \cos^2 \phi\right),$$

where the dimensionless parameter

$$\beta = -\frac{\epsilon_0 \epsilon_a E}{P_0}\sin^2 \theta$$

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is a measure of the balance between dielectric and ferroelectric effects. As we are interested in understanding switching properties of a monostable device based on a smectic C* liquid crystalline material, here we restrict the range of electric field strengths in order to examine the case $|\beta| \leq 1$. This range of β is certainly achievable experimentally. For example, typical values of $\epsilon_a = -1$, $\theta = 22.5°$, $P_0 = 100\,\mu\mathrm{C\,m}^{-2}$, $E = 10\,\mathrm{V}\,\mu\mathrm{m}^{-1}$, and with $\epsilon_0 = 8.854\,\mathrm{pF\,m}^{-1}$, would lead to $\beta \approx 0.130$. We will consider $0 < \beta \leq 1$; the case of $-1 \leq \beta < 0$ is dealt with similarly.

Stewart [28, p. 312] considers a simple example of the dynamics of SmC* liquid crystals in a geometry very similar to the one examined here, albeit in the absence of elastic constant anisotropy. From the analysis in [28], we can show that the evolution equation for the director twist angle $\phi(x, t)$ can be written as

$$\eta\phi_t = -\mathrm{grad}\, w(\phi),$$

where η is a rotational viscosity, while the gradient is taken with respect to the L^2 inner product. Non-dimensionalising the x and t variables, setting $v = 1/2 - \phi/\pi$ and rearranging, we finally arrive at the mathematical object that will be our main focus:

$$v_t = \sqrt{1 + \xi \cos(2\pi v)}(\sqrt{1 + \xi \cos(2\pi v)}\, v_x)_x + f(v), \tag{3}$$

where

$$f(v) := \frac{1}{2\pi}\left[\sin(\pi v) - \frac{1}{2}\beta \sin(2\pi v)\right]. \tag{4}$$

The parameter β controls the shape of the reaction term f, whereas ξ controls the diffusion term. Clearly (3) is quasilinear in the general anisotropic case when $\xi \neq 0$, but semilinear in the special isotropic case, when $\xi = 0$.

2. Travelling Waves

Experimentally, upon electrically switching a suitably prepared sample of a smectic C* liquid crystal, one observes a wealth of propagating structures (e.g., Abduhalim *et al.* [1]). Hence, it is of interest to understand the behaviour of the travelling waves. Here we focus on only the kink solutions in the monostable regime. If $0 < \beta < 1$, the only rest points of the kinetic equation

$$v_t = f(v)$$

are at $v_k = k$ for all $k \in \mathbb{Z}$. From the graph of $f(v)$ versus v, it is clear that $v_0 = 0$ is unstable and v_1 is stable.

Setting $z = x - ct$, we will be looking for monotone travelling waves, that is, solutions $v(x, t)$ that only depend on z, which we denote by $V(z)$: $v(x, t) = V(z) = V(x - ct)$, such that in addition $\lim_{z\to-\infty} V(z) = v_1$, $\lim_{z\to+\infty} V(z) = v_0$. This means that in the (V, V') phase plane, we are looking for a monotone decreasing heteroclinic connection between the saddle at $(1, 0)$ and the node at $(0, 0)$.

Under the above assumptions on f, there is a semi-infinite interval of speeds, $[c_*(\beta, \xi), \infty)$, for which we have a monotone decreasing solution with the correct properties. It is well known that the wave with minimal speed c_* is of particular interest because it has good stability properties in the sense of attracting large sets of initial conditions in both semilinear and quasilinear cases (see, for example, the discussions in [3, 6, 13, 15, 17, 22, 24, 25, 31]); we leave the precise stability results for future work. This minimal speed satisfies

$$c_*(\beta, \xi) \geq c_l(\beta, \xi) = \sqrt{2(1 + \xi)(1 - \beta)},$$

as for $0 < c < c_l$ the rest point $(0, 0)$ is a stable spiral, so no monotone connection from $(1, 0)$ is possible.

Definition 1 *If $c_* = c_l$, we say that we are in the case of* linear selection mechanism *and if $c_* > c_l$, of* nonlinear selection *mechanism.*

Due to the stability properties of travelling waves with minimal speed, the speed $c_*(\beta, \xi)$ characterises the domain wall switching behaviour of the liquid crystalline material, i.e. the heterogeneous switching from one state to another across a smectic layer

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through rotation of the *c*-director. Roughly speaking, the larger this minimal speed, the faster the switching will be, and, clearly, faster switching materials are better candidates for use in liquid crystal devices. Hence the quantitative determination of its value is of practical importance. In the present paper, we are, in particular, interested in examining whether elastic anisotropy can lead to a boost in the switching speed for different values of the dielectric/polarisation parameter β.

The 'linear' quantity $c_l(\beta, \xi)$, and whether linear or nonlinear selection holds, are key sources of information about the minimal speed $c_*(\beta, \xi)$. The value of $c_l(\beta, \xi)$ is easily calculated, always provides a lower bound for the minimal wave speed, and when linear selection holds, gives the actual value of the minimal wave speed. Furthermore, the precise nature of the stability properties of minimal-speed waves, in terms of how large is the basin of attraction and in what sense convergence to the wave takes places as $t \to \infty$, typically differ depending on whether linear or nonlinear selection holds (see, for instance, [6, 13, 17, 25, 31, 32]). There are thus two complementary reasons for studying the question of which selection mechanism holds: (i) to try to determine the numerical value of the spreading speed; (ii) to obtain information about the precise stability properties of the minimal-speed wave. The issue of whether linear or nonlinear selection holds for a given equation is well-known to be delicate and has attracted interesting work over a number of years, mostly dealing with semilinear equations [2, 4, 5, 14, 15, 19, 20, 21, 33]. Here we address this question as a function of the parameters β and ξ for the particular quasilinear problem (3).

Introducing $V(z)$ into (3), we can write the travelling wave ODE in the form

$$\sqrt{1 + \xi \cos(2\pi V)}(\sqrt{1 + \xi \cos(2\pi V)})V')' + cV' + f(V) = 0. \tag{5}$$

Assuming that a monotone decreasing travelling wave solution exists, we can set

$$F(V) = -\sqrt{1 + \xi \cos(2\pi V)}\frac{dV}{dz} \tag{6}$$

in order to rewrite (5) as

$$F\frac{dF}{dV} - c\frac{F}{\sqrt{1 + \xi \cos(2\pi V)}} + f(V) = 0, \tag{7}$$

where $f(V)$ is given by (4).

3. The Isotropic Case

We begin with a brief discussion of the isotropic case, when $\xi = 0$ and (3) is actually a semilinear equation, for which we have a complete characterisation of minimality in (3) thanks to a β-dependent family of explicit travelling-wave solutions of (3) that exist in this special case. This isotropic characterisation is both of interest in its own right and will provide an important tool for our study of the anisotropic case in Section 4.

Proposition 2 *If $\beta \in [0, 1/2]$, $c_*(\beta, 0) = c_l(\beta, 0) = \sqrt{2(1 - \beta)}$ and if $\beta \in (1/2, 1]$, $c_*(\beta, 0) = \sqrt{\dfrac{1}{2\beta}} > c_l(\beta, 0)$.*

Proof: In this case (7) becomes

$$F\frac{dF}{dV} - cF + f(V) = 0, \tag{8}$$

and we have

$$c_l(\beta, 0) = \sqrt{2(1 - \beta)} = 2\sqrt{f'(0)}. \tag{9}$$

In (8), taking the ansatz $F(V) = \mu \sin(\pi V)$ and matching coefficients, we obtain the result [10, 30] that (7) has a solution $F(V) = \mu \sin(\pi V)$ if

$$\mu = \frac{1}{\pi}\sqrt{\frac{\beta}{2}} \quad \text{and} \quad c = \frac{1}{\sqrt{2\beta}} := c_{nl}(\beta, 0). \tag{10}$$

We call this value of speed $c_{nl}(\beta, 0)$, as it is not obtained by a linear analysis. It is defined for all $\beta > 0$, and goes to infinity as $\beta \to 0$. Note that the linear marginal speed $c_l(\beta, 0) = \sqrt{2(1 - \beta)}$ is defined for all $\beta \leq 1$. Comparing $c_{nl}(\beta, 0)$ and $c_l(\beta, 0)$, we see that of course $c_{nl}(\beta, 0) \geq c_l(\beta, 0)$ for all $\beta \in (0, 1]$ and that the equality only holds at $\beta = 1/2$.

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Suppose first that $\beta \in [0, 1/2)$. It follows from [15, Thm 8] that

$$c_*(\beta, 0) = \inf_{F \in \Lambda} \sup_{V \in (0,1)} \left\{ F'(V) + \frac{f(V)}{F(V)} \right\}, \tag{11}$$

where

$$\Lambda = \{F \in C^1([0, 1]) : F(V) > 0 \text{ if } V \in (0, 1), \; F(0) = 0, \; F'(0) > 0\}. \tag{12}$$

Remark: Note that there is an obvious typo in [15, Thm 8]; they require that $F(1) = 0$, $F'(1) < 0$ for $F \in \Lambda$, but should instead require that $F(0) = 0$, $F'(0) > 0$, because they have forgotten to reverse their earlier change of variables $v \to 1 - v$ in their definition of Λ.

Define the family $F_\nu(V) = \nu \sin(\pi V)$, where $\nu > 0$. Then $F_\nu \in \Lambda$, and for each $V \in (0, 1)$,

$$\begin{aligned}
F_\nu'(V) + \frac{f(V)}{F_\nu(V)} &= \pi\nu \cos(\pi V) + \frac{\sin(\pi V)}{2\pi\nu \sin(\pi V)} [1 - \beta \cos(\pi V)] \\
&= \frac{1}{2\pi\nu} + \left(\pi\nu - \frac{\beta}{2\pi\nu}\right) \cos(\pi V).
\end{aligned}$$

So it follows from (11) that for each $\nu > 0$,

$$c_* \leq p(\nu, \beta), \tag{13}$$

where we define

$$p(\nu, \beta) := \frac{1}{2\pi\nu} + \sup_{V \in (0,1)} \left(\pi\nu - \frac{\beta}{2\pi\nu}\right) \cos(\pi V).$$

Now if $\nu^2 \leq \frac{\beta}{2\pi^2}$, then

$$p(\nu, \beta) = \frac{1}{2\pi\nu} + \frac{\beta}{2\pi\nu} - \pi\nu = \frac{\beta + 1}{2\pi\nu} - \pi\nu,$$

which is strictly decreasing in ν, and hence

$$\inf_{0 < \nu \leq \sqrt{\frac{\beta}{2\pi^2}}} p(\nu, \beta) = p\left(\sqrt{\frac{\beta}{2\pi^2}}, \beta\right) = \frac{1}{\sqrt{2\beta}}. \tag{14}$$

On the other hand, if $\nu^2 > \frac{\beta}{2\pi^2}$, then

$$p(\nu, \beta) = \frac{1}{2\pi\nu} + \pi\nu - \frac{\beta}{2\pi\nu} = \frac{1 - \beta}{2\pi\nu} + \pi\nu,$$

so since

$$\frac{\partial}{\partial \nu} p(\nu, \beta) = \pi - \frac{1 - \beta}{2\pi\nu^2} = 0 \text{ when } \nu^2 = \frac{1 - \beta}{2\pi^2},$$

and the fact that $\beta \in [0, 1/2)$ ensures that

$$\frac{1 - \beta}{2\pi^2} > \frac{\beta}{2\pi^2},$$

we have

$$\inf_{\nu > \sqrt{\frac{\beta}{2\pi^2}}} p(\nu, \beta) = p\left(\sqrt{\frac{1 - \beta}{2\pi^2}}, \beta\right) = \sqrt{2(1 - \beta)}. \tag{15}$$

Since $\sqrt{2(1 - \beta)} < \frac{1}{\sqrt{2\beta}}$, it then follows from (13)-(15) that

$$c_*(\beta, 0) \leq \inf_{\nu > 0} p(\nu, \beta) = \sqrt{2(1 - \beta)},$$

and hence $c_*(\beta, 0) = c_l(\beta, 0)$, because $c_l(\beta, 0) = \sqrt{2(1 - \beta)} \leq c_*(\beta, 0)$.

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Now suppose that $\beta \in [1/2, 1)$. The explicit wave solution with

$$-\frac{dV}{dz} = F(V) = \frac{1}{\pi}\sqrt{\frac{\beta}{2}} \sin(\pi V), \quad c = \frac{1}{\sqrt{2\beta}} = c_{nl}(\beta, 0),$$

satisfies

$$\frac{V'}{V} = -\sqrt{\frac{\beta}{2}} \frac{\sin(\pi V)}{\pi V}.$$

Hence $|V'/V| \le \sqrt{\frac{\beta}{2}}$ for all $V \in (0, 1)$, and

$$\frac{V'}{V} \to -\sqrt{\frac{\beta}{2}} \quad \text{as } V \to 0,$$

so that

$$\int_0^\infty e^{\frac{z}{\sqrt{2\beta}}} \left(V^2(z) + (V')^2(z)\right) \, dz < \infty,$$

because $\frac{1}{\sqrt{2\beta}} < 2\sqrt{\frac{\beta}{2}}$ since $\beta > 1/2$. Thus [19, Cor. 2.7] yields that $c_*(\beta, 0) = c_{nl}(\beta, 0)$. $\qquad\square$

Remark: In [14] and [26], the authors considered essentially the same equation as (3) with $\xi = 0$. An integral-equation framework is used to establish Proposition 2 in [14, Appl. 10.13], whereas physical arguments for Proposition 2 are given in [26].

4. The Anisotropic Case

We turn now to the anisotropic case, when $\xi \ne 0$. Here there is no explicit travelling-wave solution to make use of, and there is a difference between the cases $\xi > 0$ and $\xi < 0$. Let us first collect the tools we need.

4.1. General results

First of all, we show that, for a given wave speed c, there exists a wave front solution of the original quasilinear equation (5), which is not in divergence form, if and only if there exists a wave front solution of a certain semilinear equation. Part of the proof of this result, which allows us to obtain some results for (5) from known results for semilinear equations (*e.g.* from [15], etc), exploits work of Engler [12] that uses a front-dependent change of variables to relate wave front solutions of a divergence-form quasilinear equation to wave front solutions of a semilinear equation.

Define $h_\xi(V) := \sqrt{1 + \xi \cos(2\pi V)}$, $V \in \mathbb{R}$, and note that

$$0 < \sqrt{1 - |\xi|} \le h_\xi(V) \le \sqrt{1 + |\xi|} \quad \text{for all } V \in \mathbb{R}, \ \xi \in (-1, 1), \tag{16}$$

and also define a function $Y_\xi : [0, 1] \to \mathbb{R}$ by

$$Y_\xi(\theta) = \int_0^\theta \frac{1}{h_\xi(s)} \, ds, \quad \theta \in [0, 1]. \tag{17}$$

We have

Proposition 3 *For given $c \in \mathbb{R}$, there exists a solution $V : \mathbb{R} \to \mathbb{R}$ of*

$$h_\xi(V)(h_\xi(V)V')' + cV' + f(V) = 0 \tag{18}$$

with

$$V(z) \to 1 \ \text{as } z \to -\infty, \quad V(z) \to 0 \ \text{as } z \to +\infty, \tag{19}$$

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if and only if there exists a solution $U : \mathbb{R} \to \mathbb{R}$ of

$$U'' + cU' + k_\xi(U) = 0 \tag{20}$$

with

$$U(\eta) \to Y_\xi(1) \ \text{ as } \ \eta \to -\infty, \quad U(\eta) \to 0 \ \text{ as } \ \eta \to +\infty, \tag{21}$$

where $k_\xi : [0, Y_\xi(1)] \to \mathbb{R}$ is defined by

$$k_\xi(U) := h_\xi(Y_\xi^{-1}(U))f(Y_\xi^{-1}(U)), \quad U \in [0, Y_\xi(1)]. \tag{22}$$

Proof. First consider functions $V : \mathbb{R} \to \mathbb{R}$ and $W : \mathbb{R} \to \mathbb{R}$ related by

$$W(z) = Y_\xi(V(z)) \quad \Leftrightarrow \quad Y_\xi^{-1}(W(z)) = V(z), \quad z \in \mathbb{R},$$

with V satisfying (19) and W satisfying

$$W(z) \to Y_\xi(1) \ \text{ as } \ z \to -\infty, \quad W(z) \to 0 \ \text{ as } \ z \to +\infty. \tag{23}$$

Then, since

$$W'(z) = Y_\xi'(V(z))V'(z) = \frac{V'(z)}{h_\xi(V(z))}$$

and

$$h_\xi(V(z))V'(z) = h_\xi^2(V(z)) \cdot \frac{V'(z)}{h_\xi(V(z))} = h_\xi^2(Y_\xi^{-1}(W(z)))W'(z),$$

it follows that V is a classical solution of (18) if and only if W is a classical solution of

$$\frac{d}{dz}\left(p(W(z))W'(z)\right) + cW'(z) + q(W(z)) = 0, \quad z \in \mathbb{R}, \tag{24}$$

where

$$p(W) := h_\xi(Y_\xi^{-1}(W))^2, \qquad q(W) := \frac{f(Y_\xi^{-1}(W))}{h_\xi(Y_\xi^{-1}(W))}.$$

Then [12] yields that there exists W satisfying (23) and (24) if and only if there exists a classical solution $U : \mathbb{R} \to \mathbb{R}$ of the equation

$$U''(\eta) + cU'(\eta) + p(U(\eta)q(U(\eta)) = 0, \quad \eta \in \mathbb{R}, \tag{25}$$

that satisfies (21). Since

$$p(U)q(U) = h_\xi(Y_\xi^{-1}(U))^2 \cdot \frac{f(Y_\xi^{-1}(U))}{h_\xi(Y_\xi^{-1}(U))} = h_\xi(Y_\xi^{-1}(U))f(Y_\xi^{-1}(U)) = k_\xi(U),$$

the result follows. □

We immediately have

Theorem 4 $c_* : (0, 1) \times (-1, 1) \mapsto \mathbb{R}_+$ *is continuous.*

Proof: By the equivalence of (5) and (24), this result follows from Theorem 12 of [21]. □

From the correspondence of Proposition 3, we can also obtain convenient variational characterisations of $c_*(\beta, \xi)$, as follows.

Proposition 5 *The minimal speed $c_*(\beta, \xi)$ satisfies*

$$c_*(\beta, \xi) = \inf_{F \in \Lambda} \sup_{V \in (0,1)} h_\xi(V)\left\{F'(V) + \frac{f(V)}{F(V)}\right\}, \tag{26}$$

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and

$$c_*(\beta, \xi) = \inf_{F \in \Lambda} \sup_{V \in (0,1)} \left\{ h_\xi(V) \frac{d}{dV}(h_\xi(V)F(V)) + \frac{f(V)}{F(V)} \right\}, \tag{27}$$

where Λ is as defined in (12).

Proof. Since the function $k_\xi(U)$ clearly satisfies $k_\xi(U) > 0$ for $U \in (0, Y_\xi(1))$, $k_\xi(0) = k_\xi(Y_\xi(1)) = 0$ and $k_\xi'(0) > 0$, it follows from [15, Thm 8] that there exists a solution U of (20) and (21) if and only if $c \geq c_\xi^*$, where

$$c_\xi^* = \inf_{\rho \in \Gamma} \sup_{U \in (0, Y_\xi(1))} \left\{ \rho'(U) + \frac{k_\xi(U)}{\rho(U)} \right\},$$

and

$$\Gamma := \{ \rho \in C^1([0, Y_\xi(1)]) : \rho(U) > 0 \text{ if } U \in (0, Y_\xi(1)), \ \rho(0) = 0, \ \rho'(0) > 0 \}.$$

Then by Proposition 3, we also have

$$\begin{aligned}
c_*(\beta, \xi) &= \inf_{\rho \in \Gamma} \sup_{U \in (0, Y_\xi(1))} \left\{ \rho'(U) + \frac{k_\xi(U)}{\rho(U)} \right\} \\
&= \inf_{\rho \in \Gamma} \sup_{U \in (0, Y_\xi(1))} \left\{ \rho'(U) + \frac{h_\xi(Y^{-1}(U))f(Y^{-1}(U))}{\rho(U)} \right\} \\
&= \inf_{\hat{\rho} \in \Lambda} \sup_{V \in (0,1)} \left\{ h_\xi(V)\hat{\rho}'(V) + \frac{h_\xi(V)f(V)}{\hat{\rho}(V)} \right\}
\end{aligned}$$

(using the transformation $\hat{\rho}(V) = \rho(U)$, $U = Y_\xi(V)$), and hence

$$c_*(\beta, \xi) = \inf_{F \in \Lambda} \sup_{V \in (0,1)} h_\xi(V) \left\{ F'(V) + \frac{f(V)}{F(V)} \right\},$$

where Λ is as defined in (12). Note also that $F \in \Lambda$ if and only if $\hat{F} := h_\xi F \in \Lambda$, and

$$\begin{aligned}
h_\xi(V) \frac{d}{dV}(h_\xi(V)F(V)) + \frac{f(V)}{F(V)} &= h_\xi(V) \frac{d}{dV}\hat{F}(V) + \frac{f(V)h_\xi(V)}{\hat{F}(V)} \\
&= h_\xi(V) \left\{ \frac{d\hat{F}(V)}{dV} + \frac{f(V)}{\hat{F}(V)} \right\},
\end{aligned}$$

so that in addition to (26), we have the alternative formula

$$c_*(\beta, \xi) = \inf_{F \in \Lambda} \sup_{V \in (0,1)} \left\{ h_\xi(V) \frac{d}{dV}(h_\xi(V)F(V)) + \frac{f(V)}{F(V)} \right\}.$$

□

We also have a Benguria–Depassier type variational principle.

Proposition 6 *Let $A = \{ g \in C^1(0,1) \mid g > 0, \ g' < 0 \}$. Then*

$$c_*(\beta, \xi) = \max_{g \in A} \frac{2 \int_0^1 \sqrt{(-g')fg} \, dv}{\int_0^1 \frac{g}{\sqrt{1+\xi \cos(2\pi v)}} \, dv}. \tag{28}$$

Proof: We follow the ideas of Benguria and Depassier [4]. We start with the equation (7), multiply it by g/F, where g is a positive function, and integrate between 0 and 1 with respect to v. Integrating by parts, and using the fact that $F(0) = F(1) = 0$, gives

$$\int_0^1 \left[\frac{fg}{F} - g'F \right] dv = c \int_0^1 \frac{g}{\sqrt{1+\xi \cos(2\pi v)}} \, dv.$$

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Now take $g' < 0$, so that $-g'F > 0$. Then

$$\frac{fg}{F} - g'F \geq 2\sqrt{(-g')fg},$$

and hence

$$c \geq \frac{2\int_0^1 \sqrt{(-g')fg}\,dv}{\int_0^1 \frac{g}{\sqrt{1+\xi\cos(2\pi v)}}\,dv}.$$

As in [4, p. 344] and using Proposition 3, we can show that if $c_*(\beta,\xi) > c_l(\beta,\xi)$, there exists a $g \in A$ for which the equality holds, and if $c_*(\beta,\xi) = c_l(\beta,\xi)$, the maximum over A in (28) is $c_l(\beta,\xi)$. Hence the result follows. □

Finally, we have a bound for the minimal speed of the anisotropic case in terms of the minimal speed in the isotropic case,

Proposition 7 *For all β we have*

$$\sqrt{1-|\xi|}c_*(\beta,0) \ \leq \ c_*(\beta,\xi) \ \leq \ \sqrt{1+|\xi|}c_*(\beta,0).$$

Proof: For all $g \in A$ we have

$$\sqrt{1-|\xi|}\frac{2\int_0^1 \sqrt{(-g')fg}\,dv}{\int_0^1 g\,dv} \leq \frac{2\int_0^1 \sqrt{(-g')fg}\,dv}{\int_0^1 \frac{g}{\sqrt{1+\xi\cos(2\pi v)}}\,dv} \leq \sqrt{1+|\xi|}\frac{2\int_0^1 \sqrt{(-g')fg}\,dv}{\int_0^1 g\,dv},$$

and taking maxima over A, we obtain the result by the Benguria–Depassier variational principle for semi-linear equations [4]. □

4.2. The case $\xi > 0$

Proposition 8 *If $\xi > 0$, for $\beta \in [0, 1/2)$, $c_*(\beta,\xi) = c_l(\beta,\xi)$, whereas $c_*(\beta,\xi) > c_l(\beta,\xi)$ if*

$$\frac{1}{2} + \sqrt{\frac{\xi}{2(1+\xi)}} \ < \ \beta \ \leq \ 1.$$

So there is a "gap" that we cannot close at present.

Proof: Suppose first that $\beta \in [0, 1/2)$. Then since $\xi > 0$, it follows from Propositions 2 and 7 that

$$c_*(\beta,\xi) \leq \sqrt{1+\xi}\,c_*(\beta,0) = \sqrt{2(1+\xi)(1-\beta)} = c_l(\beta,\xi),$$

and hence $c_*(\beta,\xi) = c_l(\beta,\xi)$. If on the other hand $\beta \in (1/2, 1)$, Propositions 2 and 7 give that

$$c_*(\beta,\xi) \geq \sqrt{\frac{1-\xi}{2\beta}},$$

and so

$$c_*(\beta,\xi) > c_l(\beta,\xi) = \sqrt{2(1-\beta)(1+\xi)}$$

if

$$\beta > \frac{1}{2} + \sqrt{\frac{\xi}{2(1+\xi)}}.$$

The case of $\beta = 1$ is again handled by continuity. □

4.3. The case $\xi < 0$

For $\beta \in (1/2, 1]$ the situation is simple and we have

Proposition 9 *If $\xi < 0$ and $\beta \in (1/2, 1]$, $c_*(\beta,\xi) > c_l(\beta,\xi)$.*

Proof: By Proposition 7,

$$c_*(\beta,\xi) \geq \sqrt{\frac{1+\xi}{2\beta}},$$

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and hence

$$c_*(\beta, \xi) > c_l(\beta, \xi) = \sqrt{2(1-\beta)(1+\xi)}.$$

□

If $\beta < 1/2$, we have a partial result.

Proposition 10 *Assume that* $-\frac{1}{9} < \xi < 0$ *and* $0 < \beta < \frac{1+9\xi}{2(1+5\xi)}$. *Then*

$$c_*(\beta, \xi) = c_l(\beta, \xi) \tag{29}$$

(Note that $\frac{1+9\xi}{2(1+5\xi)} \to \frac{1}{2}$ as $\xi \to 0$, and if $\beta = 0$, the result (29) holds if $-1/9 \le \xi \le 0$.)
Proof. Recall from (27) that

$$c_*(\beta, \xi) = \inf_{F \in \Lambda} \sup_{V \in (0,1)} \left\{ h_\xi(V) \frac{d}{dV}(h_\xi(V)F(V)) + \frac{f(V)}{F(V)} \right\},$$

where

$$\Lambda = \{ F \in C^1([0,1]) : F(V) > 0 \text{ if } V \in (0,1), \; F(0) = 0, \; F'(0) > 0 \}.$$

Now, since $h'_\xi(V) = -\xi\pi \sin(2\pi V)/h_\xi(V)$,

$$h_\xi \frac{d}{dV}(h_\xi F) + \frac{f(V)}{F(V)} = h_\xi[h'_\xi(V)F(V) + h_\xi(V)F'(V)] + \frac{f(V)}{F(V)}$$

$$= -\xi\pi \sin(2\pi V)F(V) + (1 + \xi\cos(2\pi V))F'(V) + \frac{f(V)}{F(V)},$$

so that for the family $F_\nu(V) = \nu\sin(\pi V)$, $\nu > 0$, we have $F_\nu \in \Lambda$, and for all $V \in (0,1)$,

$$h_\xi \frac{d}{dV}(h_\xi F) + \frac{f(V)}{F(V)} = \frac{1}{2\pi\nu} + \cos(\pi V)\left[-\frac{\beta}{2\pi\nu} + \pi\nu(1 + \xi\cos(2\pi V)) - 2\nu\xi\pi\sin^2(\pi V) \right]$$

$$= \frac{1}{2\pi\nu} + \cos(\pi V)\left[-\frac{\beta}{2\pi\nu} + \pi\nu(1 - \xi + 2\xi\cos(2\pi V)) \right].$$

Then

$$\frac{d}{dV}\left\{ h_\xi \frac{d}{dV}(h_\xi F_\nu) + \frac{f(V)}{F_\nu(V)} \right\} = -\pi\sin(\pi V)$$

$$\times \left[-\frac{\beta}{2\pi\nu} + \pi\nu(1 - \xi + 2\xi\cos(2\pi V)) + 8\nu\xi\pi\cos^2(\pi V) \right]$$

$$= -\pi\sin(\pi V)\left[-\frac{\beta}{2\pi\nu} + \pi\nu(1 - 3\xi + 12\xi\cos^2(\pi V)) \right]$$

$$\le -\pi\sin(\pi V)\left[-\frac{\beta}{2\pi\nu} + \pi\nu(1 + 9\xi) \right] \quad \text{for all } V \in (0,1),$$

so that if $1 + 9\xi > 0$, the mapping $V \mapsto h_\xi \frac{d}{dV}(h_\xi F_\nu) + \frac{f(V)}{F_\nu(V)}$ is strictly decreasing on $(0,1)$ provided

$$\nu > \sqrt{\frac{\beta}{2\pi^2(1 + 9\xi)}},$$

in which case

$$\sup_{V \in (0,1)}\left[h_\xi \frac{d}{dV}(h_\xi F_\nu) + \frac{f(V)}{F_\nu(V)} \right] = h_\xi \frac{d}{dV}(h_\xi F_\nu) + \frac{f(V)}{F_\nu(V)}\Big|_{V=0}$$

$$= \frac{1}{2\pi\nu} + \left[-\frac{\beta}{2\pi\nu} + \pi\nu(1 + \xi) \right]$$

$$= \frac{1}{2\pi\nu}(1 - \beta) + \pi\nu(1 + \xi).$$

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Now

$$\inf_{\nu>0}\left[\frac{1}{2\pi\nu}(1-\beta)+\pi\nu(1+\xi)\right]=\sqrt{2(1-\beta)(1+\xi)}$$

$$=\left.\frac{1}{2\pi\nu}(1-\beta)+\pi\nu(1+\xi)\right|_{\nu=\sqrt{\frac{1-\beta}{2\pi^2(1+\xi)}}}.$$

So if $\beta < \frac{1+9\xi}{2(1+5\xi)}$,

$$\frac{1-\beta}{2\pi^2(1+\xi)}>\frac{\beta}{2\pi^2(1+9\xi)},$$

and hence

$$\inf_{\nu>0}\sup_{V\in(0,1)}\left[h_\xi\frac{d}{dV}(h_\xi F_\nu)+\frac{f(V)}{F_\nu(V)}\right]\leq\inf_{\nu\geq\sqrt{\frac{\beta}{2\pi^2(1+9\xi)}}}\sup_{V\in(0,1)}\left[h_\xi\frac{d}{dV}(h_\xi F_\nu)+\frac{f(V)}{F_\nu(V)}\right]$$

$$=\inf_{\nu\geq\sqrt{\frac{\beta}{2\pi^2(1+9\xi)}}}\left[\frac{1}{2\pi\nu}(1-\beta)+\pi\nu(1+\xi)\right]$$

$$=\sqrt{2(1-\beta)(1+\xi)}$$

$$=c_l(\beta,\xi).$$

□

Remark: An argument using Jensen's inequality as in [3] furnishes a smaller region in the (ξ,β) plane where the linear selection mechanism holds; interestingly, for $\beta = 0$ it also suggests that the linear selection mechanism holds for $\xi \geq -1/9$.

Proposition 11 *Suppose that $-1 < \xi < 0$ and $\beta \in [0,1)$. Then $c_*(\beta,\xi) > c_l(\beta,\xi)$ provided*

$$\beta > 1-\frac{(-\pi\xi+\sqrt{\xi^2\pi^2+8(1+\xi)})^2}{8\pi^2(1+\xi)}.\tag{30}$$

In particular, there exists $\xi^* \in (-1,0)$ such that for each $\beta \in [0,1)$, $c_*(\beta,\xi) > c_l(\beta,\xi)$ if $-1 < \xi < \xi^*$.

Proof. We adapt some ideas from [5]. Let V be a decreasing travelling-wave profile with $V(z) \to 1, 0$ as $z \to -\infty, \infty$. First note that multiplying the equation

$$h_\xi(h_\xi V')'+cV'+f(V)=0,\quad z\in\mathbb{R},\tag{31}$$

by V' and integrating over $\mathbb{R}$ yields that

$$c\int_\mathbb{R}(V')^2\,dz=\frac{1}{\pi^2},\tag{32}$$

since $\int_\mathbb{R} h_\xi V'(h_\xi V')'\,dz=0$ and

$$\int_\mathbb{R} f(V)V'\,dz=-\int_0^1 f(V)\,dV=-\int_0^1\left\{\frac{1}{2\pi}\sin(\pi V)-\frac{\beta}{4\pi}\sin(2\pi V)\right\}\,dV=-\frac{1}{\pi^2}.$$

On the other hand, multiplying (31) by $1-V$ and then integrating over $\mathbb{R}$ gives

$$\int_\mathbb{R}(h_\xi V')^2\,dz-\frac{\xi}{2}-\frac{c}{2}=-\int_\mathbb{R} f(V)(1-V)\,dz,\tag{33}$$

because

$$\int_\mathbb{R} c(1-V)V'\,dz=-\frac{c}{2},$$

and

$$\int_{\mathbb{R}} h_\xi(1-V)(h_\xi V')'\,dz = \int_{\mathbb{R}} (h_\xi V')^2\,dz - \int_{\mathbb{R}} h_\xi h_\xi'(1-V)V'\,dz$$

$$= \int_{\mathbb{R}} (h_\xi V')^2\,dz + \xi\pi \int_{\mathbb{R}} \sin(2\pi V)(1-V)V'\,dz$$

$$= \int_{\mathbb{R}} (h_\xi V')^2\,dz - \xi\pi \int_0^1 \sin(2\pi V)(1-V)\,dV$$

$$= \int_{\mathbb{R}} (h_\xi V')^2\,dz - \frac{\xi}{2}.$$

Since $f(V)(1-V) > 0$ and $c > 0$, (33) implies that

$$c^2 > 2c \int_{\mathbb{R}} (h_\xi V')^2\,dz - c\xi, \tag{34}$$

and since

$$h_\xi^2 = 1 + \xi\cos(2\pi V) \geq 1 - |\xi| = 1 + \xi,$$

we have

$$\int_{\mathbb{R}} (h_\xi V')^2\,dz \geq (1+\xi) \int_{\mathbb{R}} (V')^2\,dz. \tag{35}$$

It then follows from (32), (34) and (35) that

$$c^2 > \frac{2(1+\xi)}{\pi^2} - c\xi \quad\Leftrightarrow\quad c^2 + c\xi - \frac{2(1+\xi)}{\pi^2} > 0. \tag{36}$$

Now the quadratic function $y(c) := c^2 + c\xi - \frac{2(1+\xi)}{\pi^2}$ has $y(0) < 0$ and tends to ∞ as $|c| \to \infty$, so $y(c) = 0$ has two real roots, one of each sign. Since we know that $c > 0$, it thus follows from (36) that c is bounded below by the positive root of $y(c) = 0$, and hence

$$c \geq \frac{-\xi\pi + \sqrt{\xi^2\pi^2 + 8(1+\xi)}}{2\pi} =: q(\xi). \tag{37}$$

So $c_*(\beta, \xi) \geq q(\xi)$ for each $\beta \in [0, 1)$. Thus $c_*(\beta, \xi) > c_l(\beta, \xi) = \sqrt{2(1+\xi)(1-\beta)}$ whenever

$$q(\xi) > \sqrt{2(1+\xi)(1-\beta)},$$

which holds if and only if

$$\left(\frac{-\xi\pi + \sqrt{\xi^2\pi^2 + 8(1+\xi)}}{2\pi}\right)^2 > 2(1+\xi)(1-\beta)$$

$$\Leftrightarrow \quad \beta > 1 - \frac{(-\pi\xi + \sqrt{\xi^2\pi^2 + 8(1+\xi)})^2}{8\pi^2(1+\xi)} =: p(\xi),$$

as required. Also, $p : (-1, 0] \to \mathbb{R}$ is strictly increasing, tends to $-\infty$ as $\xi \to -1$, and $p(0) = 1 - \frac{1}{\pi^2}$. So there exists a unique value $\xi^* \in (-1, 0)$ such that

$$p(\xi^*) = 0, \quad p(\xi) < 0 \text{ for } \xi \in (-1, \xi^*), \quad \text{and} \quad p(\xi) > 0 \text{ for } \xi \in (\xi^*, 0],$$

from which it follows, in particular, that $c_*(\beta, \xi) > c_l(\beta, \xi)$ for all $\beta \in [0, 1)$ if $-1 < \xi < \xi^*$. $\qquad\square$

Remarks: 1. Numerically, $\xi^* \approx -0.698$.

2. A qualitatively similar result, giving a poorer estimate of ξ^*, can be obtained using (5.3) of [19] and explicitly calculating $Y_\xi(1)$.

3. It is instructive to compare Proposition 11 with Propositions 2 and 7. By Proposition 11, the minimal speed $c_*(\beta, \xi)$ is nonlinear for all $\beta \in [0, 1)$ if $-1 < \xi < \xi^*$. On the other hand, the bounds derived in Proposition 7 and the formulae in Proposition

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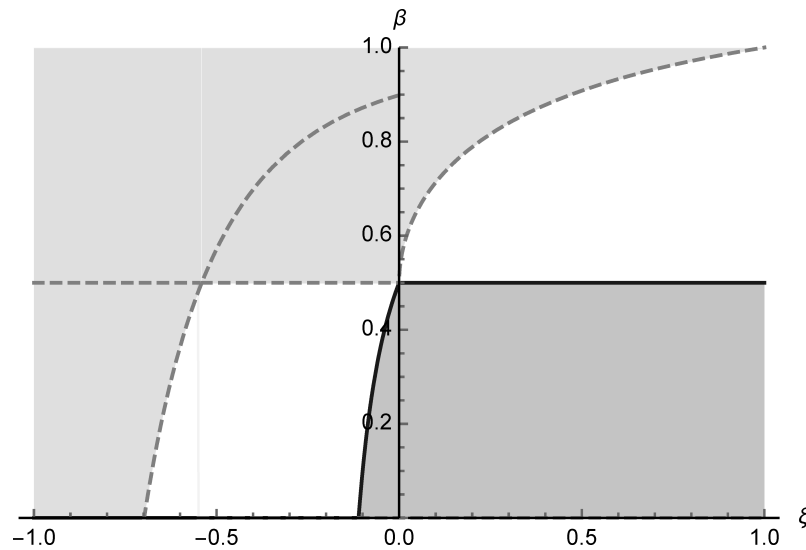
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Figure 2. Zones in the (ξ, β) plane where linear (dark grey) and nonlinear (light grey) selection has been shown to hold.

2 for the isotropic minimal speeds $c_*(\beta, 0)$ together imply that the minimal speeds $c_*(\beta, \xi)$ are uniformly bounded independently of $(\beta, \xi) \in [0, 1) \times (-1, 1)$. Thus for small β and $-1 < \xi < \xi^*$, the nonlinear minimal speeds $c_*(\beta, \xi)$ are clearly unrelated to the existence of the explicit travelling wave solution of the isotropic semilinear equation with speed $1/\sqrt{2\beta}$, which tends to ∞ as $\beta \to 0$.

Figure 2 illustrates the results of Propositions 2, 8, 9 and 11. The linear and nonlinear selection mechanisms hold in the dark grey and light grey regions respectively. The white regions are where we have not yet been able to determine which mechanism holds.

In fact, there is a curve $\gamma(t) = (\xi(t), \beta(t))$, $t \in [0, 1]$ in the (ξ, β) plane that is monotone in ξ and β, passes through $(\xi, \beta) = (0, \frac{1}{2})$, and separates the linear selection regime from the nonlinear selection regime. This claim is proved in the following two propositions, which illustrate the complementary character of the Hadeler–Rothe and Benguria–Depassier variational principles.

Proposition 12 If $c_*(\beta^*, \xi^*) = c_l(\beta^*, \xi^*)$, then $c_*(\beta^*, \xi) = c_l(\beta^*, \xi)$ if $\xi > \xi^*$.

Proof. Since there exists a monotone decreasing travelling wave solution of (5) of speed $c = c_*(\beta^*, \xi^*)$, there exists $\hat{F} \in \Lambda$ such that

$$c_*(\beta^*, \xi^*) = h_{\xi^*}(V) \left\{ \hat{F}'(V) + \frac{f(V)}{\hat{F}(V)} \right\} \quad \text{for all } V \in (0, 1).$$

Then by (26),

$$
\begin{aligned}
c_*(\beta^*, \xi) &= \inf_{F \in \Lambda} \sup_{V \in (0,1)} h_\xi(V) \left\{ F'(V) + \frac{f(V)}{F(V)} \right\} \\
&\leq \sup_{V \in (0,1)} \frac{h_\xi(V)}{h_{\xi^*}(V)} h_{\xi^*}(V) \left\{ \hat{F}'(V) + \frac{f(V)}{\hat{F}(V)} \right\} \\
&= c_*(\beta^*, \xi^*) \sup_{V \in (0,1)} \frac{h_\xi(V)}{h_{\xi^*}(V)} \\
&= \sqrt{2(1-\beta)(1+\xi^*)} \sup_{V \in (0,1)} \frac{h_\xi(V)}{h_{\xi^*}(V)}.
\end{aligned}
\tag{38}
$$

Now

$$\frac{h_\xi(V)}{h_{\xi^*}(V)} = \sqrt{\frac{1 + \xi \cos(2\pi V)}{1 + \xi^* \cos(2\pi V)}}, \quad \frac{h_\xi(0)}{h_{\xi^*}(0)} = \sqrt{\frac{1+\xi}{1+\xi^*}},$$

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and, since $\xi^* < \xi$, we have

$$\frac{1 + \xi \cos(2\pi V)}{1 + \xi^* \cos(2\pi V)} - \frac{1 + \xi}{1 + \xi^*} = \frac{(\xi^* - \xi)(1 - \cos(2\pi V))}{(1 + \xi^*)(1 + \xi^* \cos(2\pi V))} < 0.$$

So

$$\frac{h_\xi(V)}{h_{\xi^*}(V)} \le \frac{1 + \xi}{1 + \xi^*} \quad \text{for all } V \in (0, 1),$$

and hence

$$c_*(\beta^*, \xi) \le \sqrt{2(1 - \beta)(1 + \xi^*)}\sqrt{\frac{1 + \xi}{1 + \xi^*}} = \sqrt{2(1 - \beta)(1 + \xi)} = c_l(\beta^*, \xi),$$

as required. □

We also have

Proposition 13 *If $c_*(\beta^*, \xi^*) = c_l(\beta^*, \xi^*)$, then $c_*(\beta, \xi^*) = c_l(\beta, \xi^*)$ if $\beta < \beta^*$.*

Proof. Denote the function $f(v)$ corresponding to a particular value z of β by f_z. Suppose that for some β^* we have the linear case. Consider $\beta < \beta^*$. We have the Benguria and Depassier variational principle (28), and if we assume that $c_*(\beta, \xi^*) > c_l(\beta^*, \xi^*)$, there exists $g_\beta \in A$ such that

$$c_*(\beta, \xi^*) = \frac{2 \int_0^1 \sqrt{(-g_\beta')f_\beta g_\beta}\, dv}{\int_0^1 \frac{g_\beta}{\sqrt{1 + \xi^* \cos(2\pi v)}}\, dv} = \frac{2 \int_0^1 \sqrt{(-g_\beta')f_\beta g_\beta \frac{f_{\beta^*}}{f_{\beta^*}}}\, dv}{\int_0^1 \frac{g_\beta}{\sqrt{1 + \xi^* \cos(2\pi v)}}\, dv}$$

$$\le \sup_{v \in [0,1]} \sqrt{\frac{f_\beta}{f_{\beta^*}}} \frac{2 \int_0^1 \sqrt{(-g_\beta')f_\beta^* g_\beta}\, dv}{\int_0^1 \frac{g_\beta}{\sqrt{1 + \xi^* \cos(2\pi v)}}\, dv} \le \sup_{v \in [0,1]} \sqrt{\frac{f_\beta}{f_{\beta^*}}}\, c_*(\beta^*, \xi^*)$$

$$= \sup_{v \in [0,1]} \sqrt{\frac{f_\beta}{f_{\beta^*}}} \sqrt{2(1 - \beta^*)(1 + \xi^*)}.$$

However, as before,

$$\sup_{v \in [0,1]} \frac{f_\beta}{f_{\beta^*}} = \sup_{v \in [0,1]} \frac{1 - \beta \cos(\pi v)}{1 - \beta^* \cos(\pi v)} = \frac{1 - \beta}{1 - \beta^*},$$

and hence

$$c_*(\beta, \xi^*) \le \sqrt{2(1 - \beta^*)(1 + \xi^*)}\sqrt{\frac{1 - \beta}{1 - \beta^*}} = \sqrt{2(1 - \beta)(1 + \xi^*)} = c_l(\beta, \xi^*),$$

which is impossible by our assumption, this concluding the proof. □

5. Conclusions

In this paper we have considered monotone travelling wave solutions of Equation (3), which governs the dynamics of the director twist angle of an elastically anisotropic smectic C^* liquid crystalline material. We have characterised the influence of anisotropy on the minimal switching speed of the material for certain ranges of parameters (β, ξ) related to the dielectric and ferroelectric effects (β) and the anisotropy of the elastic constants in the model (ξ), with some regions of parameter space remaining as open questions at present. We have also given a description of the boundary in parameter space between the linear and the nonlinear selection mechanisms. Understanding the difference between the linear and nonlinear mechanisms is important practically because the linear minimal speed c_l, which is easy to evaluate, does not necessarily give a good lower bound for the practically observed speed, and because typically the stability properties of minimal speed waves, which characterise the nature of domain wall switching, depend on which selection mechanism holds. The precise stability properties of travelling waves for the quasilinear equation (3) are left for future work.

The mathematical interest in the current paper is in showing how a variety of ideas previously developed for semilinear parabolic equations can be exploited, adapted and extended to obtain information about a novel quasilinear equation (not in divergence form) (3). In particular, we emphasise the complementary nature of information obtained from the Hadeler–Rothe type and the

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Benguria–Depassier type variational principles given in Propositions 5 and 6, respectively. Note that the change-of-variable ideas of Engler [12], which are employed in the proof of Proposition 3, yield some preliminary information and tools, such as existence of fronts and the variational characterisation in Proposition 5, about the anisotropic case with $\xi \neq 0$. However, careful application of these tools combined with other ideas, such as the Benguria-Depassier type variational principle and the Berestycki-Nirenberg approach used in the proof of Proposition 11, is needed to establish our rigorous results about selection mechanisms in the anisotropic case.

As seen from Figure 2, there are two regions in (β, ξ) parameter space where there still are gaps in our knowledge. It is an interesting question what additional tools need to be employed or developed to close these two gaps. It is also tempting to conjecture that for all $\xi \in (-1, 1)$ the minimal speed $c_*(\beta, \xi)$ is a monotone decreasing function of β, which we know to be the case if $\xi = 0$ or when $c_*(\beta, \xi) = c_l(\beta, \xi)$.

Note that in the isotropic case, we have only considered electric field orientation in the x-direction (see Figure 1). The influence of the orientation angle on the minimal speed is considered numerically in [27]; it would be interesting to reproduce their results using the variational methods of [3, 4, 15, 19], and extend this to include anisotropy in the system.

Another interesting point is that in the proof of Proposition 2 we used an explicit solution. A priori it is not clear that an explicit solution should provide the minimal speed solution for any value of parameter β. A similar example occurs in Theorem 11 of [15], and a discussion of how explicit solutions can help to determine minimal wave speed is given on page 97 of [14].

Note that the value $\beta = 1/2$ at which the explicit wave V_β, given by (10), switches from giving the minimal-speed wave to being just one of the many waves of super-critical speed can be interpreted in terms of the linearisation of the travelling wave equation (5) about the wavefront, in the following sense. For each $\beta \in [0, 1)$, the derivative of the explicit wave profile dV_β/dz, with respect to the travelling-wave variable z, is a solution of the linearisation of the travelling wave equation about the explicit wave V_β. A wave profile that travels at a speed c that is strictly larger than the linear speed $c_l(\beta, 0)$ can converge to 0 at $+\infty$ at one of two possible rates, which are $\left(-c \pm \sqrt{c^2 - 2(1-\beta)}\right)/2$; the derivative of the profile will converge to 0 at the same rate as the profile itself. Whether the explicit solution is the wave of minimal speed ($1/2 < \beta < 1$) or has a super-critical wave speed ($0 \leq \beta < 1/2$) corresponds precisely to whether the wave, and its derivative decay to 0 at ∞ at the faster or slower of these rates; see also [19, Cor. 2.7] and [14, Thm. 10.12]. This can be viewed as whether or not the derivative dV_β/dz belongs to the weighted space Ξ_c of functions $y : \mathbb{R} \to \mathbb{R}$ with $\sup_{z \in \mathbb{R}} |(1 + \exp(\frac{cz}{2}))y(z)| < \infty$, and thus whether or not the linearisation about the explicit travelling wave V_β has a zero eigenvalue in this weighted space. Whenever the speed c is strictly larger than the linear speed $c_l(\beta, 0)$, local stability properties of the wave in the weighted space Ξ_c hold (see [32, Chp. 5, Sec. 2.1], [11, Lem. 4.4, Thm. 4.8]) and qualitatively depend on whether this linearisation has a zero eigenvalue in Ξ_c or not.

It is possible that the structural stability considerations [23] and renormalization group arguments [9] would also be useful in clarifying the rôle of explicit solutions in determining minimal wave speeds.

Acknowledgement

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