Title: **Compressive sensing for direct time of flight estimation in ultrasound-based NDT**

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ABSTRACT

This paper presents an approach for estimation of ultrasonic time-of-flight (TOF) within a Non Destructive Testing (NDT) and Structural Health Monitoring (SHM) context. The presented method leverages recent advances in the field of Compressive Sensing (CS), which makes use of sparsity in a transform domain of a signal in order to reduce the number of samples required to store it. For this, the ultrasound signals are considered to be sparse in their autocorrelation domain and a method is suggested for building suitable basis functions, based on Hankel matrices, which transform a signal into its autocorrelation domain. It is shown how this can be combined with standard CS techniques in order to achieve very low error in TOF estimates with up to one-tenth of the original ultrasound samples.

INTRODUCTION

One of the practical limiting factors in the analysis of data gathered for ultrasound Non Destructive Testing (NDT) of large structures is the high data throughput, required due to the high sample rates involved in acquiring ultrasound signals.

This work investigates the application of Compressive Sensing (CS) to this problem; this is an emerging branch of signal processing that challenges the Nyquist-Shannon theorem, which limits the maximum frequency content of a signal by its time resolution. This limit has placed a fundamental requirement for high data storage if one is to infer high frequencies, as is the case in ultrasound NDT. Compression techniques such as Fourier and wavelet decompositions have successfully been applied to the data compression problem [1, 2], and this could now be safely regarded as a standard task when considering the problem of data compression. Even though it is successful, such an approach still requires an initially large number of samples to be stored, if high frequencies are involved. There are two key ingredients in the formulation of CS: the projection of the n dimensional measurement vector x to $m < n$ dimensions through a random matrix projection, and l_1-penalised linear regression [3].

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Problem being investigated

In this paper, examples will be drawn from ultrasonic pulse-echo thickness measurements along the surface of an aerospace composite panel specimen. From a physics and signal processing point of view, an important point should be noted; only bulk waves are being launched by the excitation source, which comprises a tone burst of fixed frequency. Because bulk waves are non-dispersive (their propagation speed is independent of frequency), the structure "responds" at the same frequency as the one launched by the probe [4].

The reason that this type of signal is investigated here is due to its widespread use in ultrasound NDT. In practical terms, this means that the signal received back at the measurement probe will have the same frequency content as the original pulse launched from the excitation probe.

FEATURES OF ULTRASOUND PULSES

A typical ultrasound reflection is shown in Figure 1a, with two time indices marked as t_a and t_b. These times correspond to reflections from the front and back wall of the composite panel respectively. A pulse of this kind effectively constitutes an A-scan. The information extracted from this is the time difference $t_f = t_b - t_a$, and this is often referred to as the ultrasonic time-of-flight (TOF). This can be related to the thickness of the plate, if the propagation speed of bulk waves for the material is known. Another feature of interest is the ratio $x(t_a)/x(t_b)$ (where $x(t)$ is the measured amplitude of the ultrasound pulse), as this contains information about the attenuation of the wave as it travelled through the thickness of the plate. An A-scan thus gives information about a single physical coordinate on the plate.

A B-scan can be formed by collecting a series of A-scans along a line (illustrated in Figure 1b), while a C-scan is formed by collecting a series of B-scans, to give a two-dimensional grid of ultrasound pulse information (illustrated in Figure 1c). Note that higher times of flight in Figure 1c imply wider plate thickness. The salient features in c) are the stringers and the different layers of carbon fibre.

A good part of understanding the motivation for the CS-based data analysis proposed here involves understanding some of the signal processing hurdles one has to go through in order to extract valuable information from an ultrasound reflection, such as the one shown in Figure 1; noise presents arguably one of the biggest challenges. At the point of the front wall reflection (t_a), the Signal to Noise Ratio (SNR) is often high enough for the signal to be well defined. However, the back wall reflection has to travel twice through the thickness of the material before reaching the probe and thus contains a much lower SNR due to attenuation.

The current practice for estimating TOF, is to place a threshold and search for the first exceedance of x of this threshold. This defines the onset of the front wall reflection. Then, a second, lower threshold is placed that is used to define the onset of the back wall reflection. The first problem with this methodology is the need for establishing suitable thresholds for the front and back wall reflections; this presents an issue due to the changing SNR associated with the attenuation of bulk waves being received. One

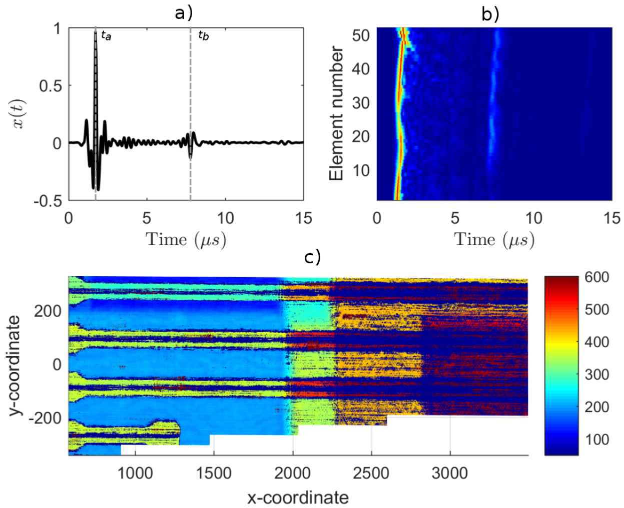

Figure 1. Illustration of a) single ultrasound reflection (A-scan), b) reflections along a phased array probe (B-scan) and c) time of flight map along the two dimensions of a composite plate (C-scan).

could argue that if the excitation and acquisition of ultrasound pulses is appropriately synchronised, only the arrival time of the back wall reflection needs to be recorded. This may be true in certain instances where the probe maintains a constant distance from the front wall; this may be easier to achieve with air-coupled than water-coupled ultrasound. In the latter case, there is physical contact between the couplant and the surface of the structure. This contact has to be maintained at an appropriate pressure, which is subject to variation depending on the control strategy being implemented [5]. The changing pressure introduces a variation in time of the first arrival, across different A and B frames. This variation can also be observed if there is a gradient in pressure applied across the ultrasound elements of the phased array (along a B-frame). The B-scan illustration presented in Figure 1b shows this clearly; the main vertical component corresponding to the front wall reflection shifts slightly through the different elements.

TOF inference from autocorrelation function

A more elegant alternative to using thresholds may be to extract the time-of-flight from the autocorrelation of ultrasound pulses. An autocorrelation is a cross correlation of a signal with itself. Correlation refers to degree of similarity of two signals, and for two signals, x_a and x_b it can be computed simply through vector multiplication $x'_a x_b$ (where $'$ denotes transposition). Autocorrelation is achieved by taking x_a as the original signal, and x_b as a lagged version of itself. Any echoes of the original pulse will be evident in the autocorrelation function. Estimating TOF for the main echo thus becomes a task of finding the location of the highest peak in the autocorrelation domain.

Establishing the ultrasonic time-of-flight in this manner is bound to be more reliable for a number of reasons. The most important is that no thresholds have to be established, which is desirable if one wants to conduct automated inspection. Furthermore, an autocorrelation may still be able to identify the similarity between the front and back wall reflections even when the back wall reflection is buried in noise, something that cannot be done via thresholding.

The concept of using an autocorrelation for this task is not as popular in the phased array ultrasound community, but it is widely used in applications that involve ranging such as radar and Global Positioning System (GPS). Here, the idea of using the ultrasound pulse autocorrelation is used as a building block in the application of compressive sensing.

COMPRESSIVE SENSING

The general goal of Compressive Sensing (CS) is to acquire a signal using a much smaller number of measurements than that required by the Nyquist-Shannon sampling theorem.

It is now a well-established principle that a signal could be efficiently compressed, or coded, using a basis such as a Discrete Cosine Transform (DCT), Fourier transform, or wavelets (if the signal contains strong transients). This statement assumes that one has some knowledge of which coefficients in the representative basis play an important role in the signal. In ultrasonic NDT applications it has already been shown that using a wavelet basis can compress a typical signal by as much as 95% in terms of compression ratio [1, 6]. However, there is a drawback to this approach, since the entire signal must be acquired, according to the Nyquist-Shannon theorem, for it to be then transformed into a sparse domain. In NDT and SHM, this presents two issues: data storage and processing. Storing an ultrasound dataset before it is compressed can be a challenge given the very high sample rates, and the potentially large scanning areas required by a high resolution C-scan. Subsequently transforming these large quantities of data into a compressed domain such as the wavelet domain can also be computationally time consuming.

Compressive sensing solves this using a combination of ideas, with two basic underlying assumptions. The first is that the signal has a sparse representation in some domain. The second is that some knowledge of what this domain may be is available. At the centre of CS is the concept of using l_1 regularised regression to find a coefficient set from a base dictionary that is sparse. The following sections describe in more detail the essence of sparsity in the context of ultrasound NDT, the use of l_1-regularised regression and the use of dictionaries. These are all standard techniques in the field of CS. However, some new ideas are also introduced here, namely the use of a Hankel dictionary for extracting TOF directly from the solution of the sparse linear regression problem, using only a low number of time-domain ultrasound measurements.

Sparsity

A signal, x, with a high number of (time-domain) measurements n, is sparse in a transform domain if a very small number of coefficients, $m \ll n$, in such domain are sufficient to accurately represent the signal. Such a transform could be represented as

$$x = \Psi \boldsymbol{\beta} \tag{1}$$

where Ψ is a basis function set, or dictionary, and $\boldsymbol{\beta}$ is the coefficient vector that represents the signal in the transform domain[1]. A good example of a sparse representation would be a sinusoid at a fixed frequency, which may contain a high number of points in the time domain, but may be fully represented by one complex coefficient in the Fourier domain.

Data gathered from ultrasound pulses is dense in the time domain, but it is bound to be sparse in some other domain considering that the pulses are sent at a fixed frequency, and bulk waves are being excited, which are non-dispersive. This means that the information content will be focused in, or around, a very narrow frequency range. From this point of view, it should be easy to see how this type of signal could be modelled in a sparse domain with a Short-Time-Fourier Transform (STFT) or a discrete wavelet decomposition. This is not illustrated here in the interest of brevity, and since it could be argued that this is standard practice in signal processing. In this work, the authors focus on developing a dictionary that directly captures the information of interest, which is the ultrasonic time of flight difference between front and back walls. This information is captured very well by the autocorrelation function, as discussed above. One could say that this type of signal is sparse in its autocorrelation domain. Each "echo" of the front wall reflection will cause one peak in the autocorrelation function. Even in the presence of multiple echoes, as a result of reflection from through-thickness layers of the material, there will still be significantly fewer echoes than there are time domain measurements. In an autocorrelation domain, each major peak corresponds to one echo of the main pulse. The autocorrelation function will also contain coefficients related to the carrier frequency of the pulse; these are normally visible in the first few lags, and can be safely discarded as they are not relevant to TOF estimation (again, this is not illustrated here). Hence, in this paper, the ultrasound signals will be treated as sparse in an autocorrelation domain, given the disparity between the number of reflections, and the number of time domain measurements.

Random matrix projection

There is particular interest in the problem of dimensionality reduction, for the purposes of algorithm design, in SHM and many other areas; this is also central to the idea of CS and so it is worth a brief discussion. A way of "compressing" a dataset is to project the n-dimensional measurement vector x to a lower, m-dimensional space using a linear or nonlinear transformation. One popular approach is to use transformations, such as Principal Component Analysis (PCA), Independent Component Analysis (ICA) or factor analysis [7]. These particular examples project the measurements into spaces with certain constraints. For example, PCA is designed to rotate a data set such that the resulting vectors are forced to explain as much of the variance as possible. Such a linear transformation could be written down as

$$z = \Phi x \tag{2}$$

where z is now a low dimensional representation of x. An interesting projection results if the rotation matrix, Φ, is set to be a random matrix. Johnson and Lindenstrauss [8]

[1]Note that in this paper all vectors are assumed to be column vectors unless otherwise specified.

have shown that if Φ is distributed according to a Gaussian, or Bernoulli distribution, this linear dimensionality reduction preserves, with low error, certain features of x, such as pairwise distances. A Gaussian or Bernoulli random matrix also yields an orthogonal transformation. This is a central result within research in metric embedding [8]. This random matrix transform is a key ingredient in the formulation of the CS problem. In this paper, the elements of Φ have been drawn from a Gaussian distribution: $\mathcal{N}(0, 1)$, and used in order to project the original measurement vector x into a lower dimensions, thus compressing it.

Linear regression with the Lasso

In order for the compressed version of x, through the random transformation of (2), to be of immediate practical use, there needs to be an algorithm that is able to recover the original measurement. This is where the *Least absolute shrinkage and selection operator* (Lasso) comes into play. The Lasso solves the classical linear regression problem of $\mathbf{X}\boldsymbol{\beta} = y$, where $\mathbf{X}$ is a matrix with column-wise vectors of inputs, y is an output and $\boldsymbol{\beta}$ holds the regression coefficients. The Lasso encourages sparse solutions for $\boldsymbol{\beta}$ through a penalty term based on an l_1 norm [3]. The optimisation problem can be formulated as:

$$\text{minimise: } \{\frac{1}{2N}||y - \mathbf{X}\boldsymbol{\beta}||_2^2 + \lambda||\boldsymbol{\beta}||_1\} \tag{3}$$

The l_1 penalty is regularised by the term λ. A general l_q penalty could be computed using the sum $||\boldsymbol{\beta}|| = \sum_{j=1}^{N} |\boldsymbol{\beta}|^q$, and the Lasso is the special case when $q = 1$. This constraint ensures that the optimisation problem remains convex [3]. If $q = 0$, the resulting constrain would yield a subset selection problem that is non-convex and combinatorially hard, thus not computationally efficient. The Lasso is thus an attractive method for recovering sparse solutions in high dimensions whilst maintaining efficient computation.

The regularisation parameter, λ, dictates the degree of sparsity in the solution. A high value of λ encourages a low number of non-zero coefficients, and vice-versa. Therefore, an appropriate value of λ needs to be chosen for each problem in particular. The authors of the Lasso suggest using cross-validation [3] in order to estimate the best, but Bayesian approaches have also been suggested [9], as well as bootstrap-based methodologies [10].

This brings the discussion of CS to the last step [11], which is concerned with finding a sparse set of coefficients $\boldsymbol{\beta}$ that best describe the random matrix projection Φx (the compressed signal representation). This is where the power of the Lasso is unleashed. What is available to the regression problem is not the full signal, but rather a projection of it through Φ. The coefficient set can be inferred if the basis dictionary is also projected through the sensing matrix to yield the following regression problem:

$$\Phi\Psi\boldsymbol{\beta} = \Phi x \tag{4}$$

where, as before, Φ is a random matrix projection, Ψ is a basis function set, and x is the (uncompressed n-dimensional) signal of interest. The Lasso minimisation of (3) can now be used in order to obtain a sparse solution for $\boldsymbol{\beta}$.

Several algorithms can be used to obtain a Lasso solution. Because the shape of the l_1 constraint is convex, gradient-based methods are well suited to this problem. Most methods use an iterative technique to find solutions across the entire regularisation path (from high to low λ) efficiently. For example, Friedman has suggested cyclical coordinate descent as an efficient approach [12]. However, the authors have found this to be slow when $m > 200$, and have opted for an approach based on an older technique, the Least Angle Regression (LAR) [13] implemented through the open-source code, **spasm** [14]. The details of these implementations and comparisons are omitted here.

Hankel Matrix Dictionary

As discussed above, the dictionary, Ψ, plays an important role in the reconstruction process of the original signal. Some investigations into appropriate dictionaries for sparse represntations of ultrasound signals have already been carried out [2, 15]; however, these largely focus on the use of parametric dictionaries such as wavelet functions or variations of Fourier transforms.

Here, a different approach is taken, motivated mainly by the fact that the main piece of information one wishes to extract from the pulse is the TOF. As discussed above, this information is encoded well in the autocorrelation function of the ultrasound pulse. It has also been discussed that these type of signals are sparse in their autocorrelation coefficients. This motivates the idea of a dictionary that transforms the original signal into its autocorrelation domain; this can be achieved by letting Ψ be a Hankel matrix of the original fully sampled signal. This is nothing more than a matrix where each column is a lagged version of the first column, which contains the original signal. Using the Lasso to reconstruct x using such a dictionary is trivial problem, as it implies that all measurements are known, and therefore no compression is achieved. However, the interesting information lies in the cross-correlation between the first pulse arrival (front wall reflection) and the rest of the signal. More specifically, what is of interest is the lag index at which there is a good degree of correlation between the front and back wall reflections. With this observation in mind, a Hankel matrix can be built using only the front wall reflection. This is straightforward to capture as it *should* be the reflection with the greatest amplitude (though this may not always be true). In this paper, Hankel matrices are assembled using the first k points of the signal, where k is assumed to be a low number compared to the overall length of the signal (n). This Hankel matrix could be defined more formally as

$$\Psi_{ij} = \begin{cases} y_{i+j} & \text{if } i \leq k \\ 0 & \text{if } j > k \end{cases}. \tag{5}$$

This is illustrated in Figure 2a, where the greyed-out part of the signal indicates the portion of the signal used to assemble the Hankel matrix, which is shown on the right, in Figure 2b.

INVESTIGATION ON NDT DATASET

So far, the main ingredients for CS have been outlined, and a procedure has been suggested for assembling a dictionary that would capture TOF from a reduced number of samples. This section presents some results from this approach, applied to an NDT data set. The focus is not on visualisation, but on an analysis of the quality of predictions as m (the reduced number of samples) and the regularisation value λ are varied.

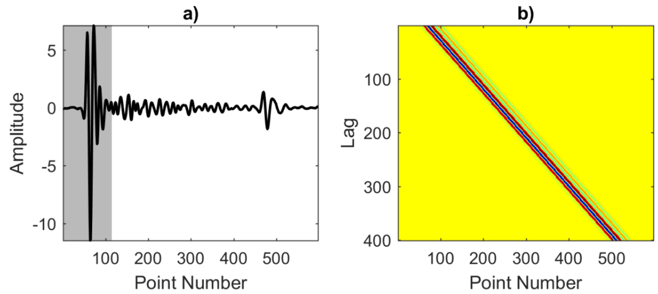

Figure 2. a) Original pulse, where the grey area shows the portion of $x(t)$ identified as the front wall reflection $x(a:b)$ and b) Hankel matrix assembled using lagged versions of $x(a:b)$ and containing zeroes elsewhere

Experimental set-up

A $1.2\text{m} \times 3\text{m}$ composite panel was scanned using a six-axis robotic head, with a water coupled ultrasound probe. The probe consists of 64 transducers, each of which fire a MHz tone burst, and also act as receivers. The resolution of the scan can be adjusted, but for these results, the speed of the probe was adjusted to yield a spatial resolution of 0.8mm in the direction of the probe travel. The C-scan shown in Figure 1c was generated using this data set, using a maximum autocorrelation to estimate the TOF. Further details of this experimental technique have been published in [5], where the interested reader is referred to for further details.

Results

As an illustration, Figure 3a demonstrates a Lasso solution, $\boldsymbol{\beta}$, for increasing m with its corresponding autocorrelation function for a fully sampled $\mathbf{x}$, with $n = 943$ samples, in Figure 3b. An initial sample of 125 samples was taken to assemble the Hankel matrix from the front wall reflections. For these trials, the LAR algorithm was run until a value of λ was reached that yielded 50 non-zero coefficients. The values of $\boldsymbol{\beta}$ shown in Figure 3a show the 10^{th} iteration before the final one, showcasing a low value of λ, with a number of non-zero coefficients between 40 to 50. Note that $\boldsymbol{\beta}$ as well as the full autocorrelation function have been squared to show the dominant component values more clearly. When comparing Figure 3a and b, one should be interested in the location of the highest peak, as this contains the location of the main echo. Note that the sparse solution tends to to assign higher values at higher lags and this is the main source of error in this procedure. This is due to the fact that a Hankel matrix is computing a biased cross-correlation and this tends to magnify the coefficient values at high lags. Also, note that the peak locations of the Lasso solutions and the true autocorrelation function are not perfectly aligned. There will always be some error associated with a slight loss of resolution from the random matrix projection from n to m samples. However, *on average*, even a very low m produces an acceptable error.

The left side of Figure 3 corresponds to a single ultrasound pulse. On the right side, Figure 3c shows the error in TOF for a set of 6700 pulses taken at random from the composite panel data set described above, and shown in Figure 1. The error here is defined as the difference between the lag corresponding to the highest autocorrelation coefficient of $\mathbf{x}$ and to the greatest value of the sparse solution, $\boldsymbol{\beta}$. In other words, the error in TOF estimation between a compressed and uncompressed signal is being computed. The four columns of the right side of Figure 3 show this error, for $m = \{25, 50, 100, 150\}$, on each row, while the horizontal axis represents the regularisation path of $\log \lambda$ values. High values of λ correspond to very sparse solutions (high number of non-zero $\boldsymbol{\beta}$) and vice-versa. Note that the mean error is shown in solid line, while 3σ bounds are plotted with dashed lines.

The LAR algorithm starts at a λ value that yields one non-zero coefficient, and moves towards the left (low λ). In this case, the algorithm was set-up to stop when it reaches 50 non-zero values, for all values of m.

There are various interesting aspects in the error curves of Figure 3c. The first is the trend that at low values of λ (not-so-sparse solutions), the average error and its standard deviation are all within 10 samples across the range of m. The key difference between high and low compression values of m is that a low error is achieved quicker throughout the regularisation path for high m. In other words, at lower compression ratios, more sparse solutions can be used, while at higher compression ratios one is forced to solve for a much higher number of non-zero coefficients before the major peaks start to predict the TOF well. It could be argued that using a higher compression ratio (low m) yields estimates that are just as good (in terms of error) provided one adjusts λ accordingly. In fact, the computation time required to achieve the regularisation path shown in Figure 3 is approximately 10 times greater for $m = 150$ than for $m = 25$. This is important given that ultrasound NDT data sets involve processing millions of pulses, and if one is to reconstruct them, the computation time required to achieve this must be low.

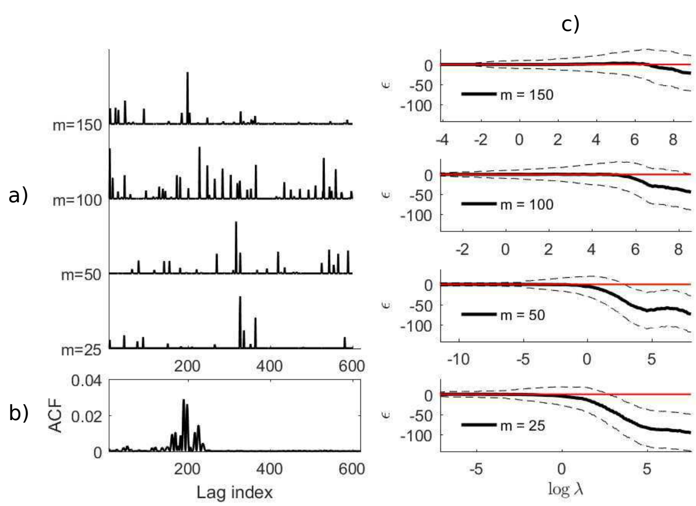

Figure 3. a) Lasso solutions to $\boldsymbol{\beta}$, for a single pulse, with increasing m, b) autocorrelation function of fully sampled signal and c) error in TOF estimates for a sample of 6700 pulses with changing λ

Conclusions

An approach to ultrasound TOF detection has been presented using Compressive Sensing (CS). The used techniques (the random matrix projection and the Lasso) are standard in the statistical inference literature. The use of a Hankel matrix of the front wall reflection has been suggested as a basis dictionary in order to yield a solution that can be interpreted as the ultrasonic TOF without further post-processing. A set of results has been presented which shows that even with very low m (high compression ratios), the a low error on the TOF estimates can be achieved.

Several aspects have been omitted in this paper, namely methods for estimating an appropriate λ that produces low error whilst maintaining quick computation, learning of appropriate dictionaries from large samples of data, and the use of the coefficient vectors, $\boldsymbol{\beta}$ as statistical features themselves. These aspects are motivated as items for further investigation.

ACKNOWLEDGEMENTS

This work is part of the Autonomous Inspection in Manufacturing and Re-Manufacturing (AIMaReM) project, funded by the UK Engineering and Physical Science Research Council (EPSRC). The authors also wish to thank Spirit Aerosystems for providing the composite panel used in this study.

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