On Pth Moment Stabilization of Hybrid Systems by Discrete-time Feedback Control

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Abstract

Since Mao initiated the study of stabilization of continuous-time hybrid stochastic differential equations (SDEs) by *feedback controls based on discrete-time state observations* in 2013, many authors have further studied and developed it. However, so far no work on the pth moment stabilization has been reported. This paper is to investigate how to stabilize a given unstable hybrid SDE by feedback controls based on discrete-time state observations, in the sense of H_∞, asymptotic and exponential stability in pth moment for all $p > 1$. The main techniques used are constructions of the Lyapunov functionals and generalizations of inequalities.

Key words: pth moment stabilization, H_∞ stability, asymptotic stability, exponential stability, feedback control based on discrete-time state observations.

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1 Introduction

In the past decades, hybrid systems have played a critical role in many applications. As an important class of hybrid systems, hybrid SDEs (also known as SDEs with Markovian switching) have attracted increasing attention in recent years. Hybrid SDEs have been widely used in various fields for modelling systems that may undergo abrupt changes in practice. An intriguing topic in the study of hybrid SDE is automatic control, with consequent emphasis being placed on the analysis of asymptotic stability [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. In particular, [13, 14] are two of most cited paper (Google citations 583 and 371,respectively) while [15] is the first book in this area (Google citation 855).

Consider an unstable hybrid SDE in the Itô sense

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dw(t), \tag{1.1}$$

where $x(t) \in R^n$ is the state, $w(t) = (w_1(t), \cdots, w_m(t))^T$ is an m-dimensional Brownian motion, $r(t)$ is a Markov chain (please see Section 2 for the formal definitions) which represents the system mode. When stabilizing the system with a feedback control, a traditional (or regular) choice of is $u(x(t), r(t), t)$ based on continuous-time observations of state $x(t)$, and the controlled stable system is

$$dx(t) = \big(f(x(t), r(t), t) + u(x(t), r(t), t)\big)dt + g(x(t), r(t), t)dw(t). \tag{1.2}$$

Nevertheless, such a regular feedback control would lead to high cost and sometimes it's unrealistic as the observations are often of discrete-time. As a result, Mao [16] investigated feedback controls based on discrete-time state observations for this problem. By choosing a positive constant τ, the controller $u(x([t/\tau]\tau), r(t), t)$, where $[t/\tau]$ is the integer part of t/τ, needs state observations only at times $0, \tau, 2\tau, \cdots$, which is more realistic and also costs less. Consequently, the controlled system becomes

$$dx(t) = \big(f(x(t), r(t), t) + u(x([t/\tau]\tau), r(t), t)\big)dt + g(x(t), r(t), t)dw(t). \tag{1.3}$$

Although the stabilization problem by feedback controls based on the discrete-time state observations for the deterministic differential equations has already been studied by many authors (see e.g. [17, 18, 19, 20, 21, 22]), Mao [16] is the first paper to study this problem for SDEs, which investigated the mean square exponential stabilization. Later, Mao et.al [23] obtained a better upper bound on observation interval τ. Recently, You et.al [24] improved the upper bound on τ again, and investigated the H_∞, asymptotic and exponential stabilization in mean square and almost surely. However, so far no work on pth moment stabilization has been reported yet.

As we know, mean square ($p = 2$) stability is not enough for some problems and a wide range of moment order p is needed. On one hand, some research problems require higher-order moment stabilities. For example, higher moment is frequently required in finance and digital image process. Moment risk premiums in finance involve the skewness swaps ($p = 3$) and kurtosis swaps ($p = 4$); pseudo-Zernike moments in image processing techniques could require, say, moment order up to 50 (see e.g. [25, 26, 27, 28]). On the other hand, some problems only require a lower moment stability. Although lower moment stability can be implied by mean square stability in some existing paper, our new theory can achieve the target under weaker conditions at lower cost. For example, mean square condition is unnecessarily too strong for almost sure exponential stability. By allowing $p < 2$, we can stabilize the system in almost surely exponential sense by weaker conditions than what [24] required. Of course, moment stability analysis of stochastic systems has been widely and deeply studied (see e.g. [29, 30, 31, 32, 33]). The difference is that, this paper will use a better controller, which is based on discrete-time state observations.

Motivated by the above discussions, the main purpose of this paper is to investigate how to control a given unstable hybrid SDE to be H_∞ stable, asymptotically stable and exponentially stable in pth moment for all $p > 1$. Our new established theory enables the readers to choose p flexibly according to their needs from a wide range $(1, \infty)$.

Unlike the mean square case ($p = 2$), a more general range of moment order brings more complexity and difficulty to the stabilization problem. For example, it involves many generalization works of inequalities, and more parameters need to be determined to choose a good τ for a fixed p.

The remainder of this paper is organised as follows. Section 2 explains the notations, presents our models and assumptions, and defines functions that will be used later. Section 3 and 4 mainly investigates the conditions for pth moment asymptotic and exponential stability respectively. Then Section 5 gives both linear and nonlinear examples to illustrate our new theory. The final conclusion is stated in Section 6.

Let us begin to develop these new techniques and to establish our new theory.

2 Notation and Stabilization Problem

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ which is increasing and right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. We write the transpose of a matrix or vector A as A^T. Denote the m-dimensional Brownian motion defined on the probability space by $w(t) = (w_1(t), \cdots, w_m(t))^T$. For a positive number a, $[a]$ means the integer part of a. For a vector x, denote by $|x|$ its Euclidean norm. For a matrix Q, its trace norm $|Q| = \sqrt{\text{trace}(Q^TQ)}$ and its operator norm $\|Q\| = \max\{|Qx| : |x| = 1\}$.

For a real symmetric matrix Q, $\lambda_{\min}(Q)$ and $\lambda_{\max}(Q)$ means its smallerst and largest eigenvalues respectively. Denote by $L^p(\Omega; R^n)$ the family of R^n-valued random variables x such that $\mathbb{E}|x|^p < \infty$.

Let $r(t)$ for $t \geq 0$ be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \cdots, N\}$ with generator matrix $\Gamma = (\gamma_{ij})_{N\times N}$ whose elements γ_{ij} are the transition rates from state i to j for $i \neq j$ and $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$. We assume that Markov chain $r(\cdot)$ is independent of the Brownian motion $w(\cdot)$.

Consider an n-dimensional hybrid SDE

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dw(t) \tag{2.1}$$

on $t \geq 0$, with initial values $x(0) = x_0 \in R^n$ and $r(0) = r_0 \in S$. Here

$$f, u : R^n \times S \times R_+ \to R^n \quad \text{and} \quad g : R^n \times S \times R_+ \to R^{n\times m},$$

The given system may not be stable and our aim is to design a feedback control $u(x(\delta_t), r(t), t)$ so that the controlled hybrid SDE

$$dx(t) = \big(f(x(t), r(t), t) + u(x(\delta_t), r(t), t)\big)dt + g(x(t), r(t), t)dw(t) \tag{2.2}$$

becomes stable, where

$$\delta_t = [t/\tau]\tau \tag{2.3}$$

for $\tau > 0$.

So our controller $u(x(\delta_t), r(t), t)$ is designed based on the discrete-time state observations $x(0), x(\tau), x(2\tau), \cdots$. Now we impose the following standing hypotheses.

Assumption 2.1 *Assume that the coefficients f and g are all locally Lipschitz continuous (see e.g. [15]).We also assume that they satisfy the following linear growth conditions*

$$|f(x, i, t)| \leq K_1|x| \quad \textit{and} \quad |g(x, i, t)| \leq K_2|x| \tag{2.4}$$

for all $(x, i, t) \in R^n \times S \times R_+$, *where* K_1 *and* K_2 *are both positive numbers.*

Obviously, (2.4) implies that

$$f(0, i, t) = 0, \qquad g(0, i, t) = 0 \tag{2.5}$$

for all $(i, t) \in S \times R_+$.

Assumption 2.2 *Assume the controller function* u *are globally Lipschitz continuous, i.e., there exists a positive constant* K_3 *such that*

$$|u(x, i, t) - u(y, i, t)| \leq K_3|x - y| \tag{2.6}$$

for all $(x, y, i, t) \in R^n \times R^n \times S \times R_+$. *We also assume that*

$$u(0, i, t) = 0 \tag{2.7}$$

for all $(i, t) \in S \times R_+$.

We can easily see that Assumption 2.2 implies the following linear growth condition on the controller function

$$|u(x, i, t)| \leq K_3|x| \tag{2.8}$$

for all $(x, i, t) \in R^n \times S \times R_+$.

Noticing that the controlled system (2.2) can be written as an SDDE (see [24]), then we know that under Assumptions 2.1 and 2.2, there is a unique solution $x(t)$ such that $\mathbb{E}|x(t)|^p < \infty$ for all $t \geq 0$ and $p > 1$ (see e.g. [15]).

For stabilization purpose related to the controlled system (2.2), we introduce the following Lyapunov function operator and Lyapunov functionals.

Let $V(x(t), r(t), t)$ be a Lyapunov function and we require $V \in C^{2,1}(R^n \times S \times R_+; R_+)$, i.e., the family of non-negative functions $V(x, i, t)$ is defined on $(x, i, t) \in R^n \times S \times R_+$ which are continuously twice differentiable in x and once in t. Then define an operator $\mathcal{L}V : R^n \times S \times R_+ \to R$ by

$$\begin{aligned}\mathcal{L}V(x, i, t) = V_t(x, i, t) + V_x(x, i, t)[f(x, i, t) + u(x, i, t)] \\ + \frac{1}{2}\text{trace}[g^T(x, i, t)V_{xx}(x, i, t)g(x, i, t)] \\ + \sum_{k=1}^{N} \gamma_{ik} V(x, k, t), \end{aligned} \tag{2.9}$$

where V_t, V_x and V_{xx} is the first order partial derivative with respect to t, x and the second order partial derivative with respect to x respectively.

Now we define a Lyapunov functional for a fixed moment order $p > 1$ by

$$\hat{V}(x_t, r_t, t) = \theta\tau^{\frac{p-2}{2}} \int_{t-\tau}^{t} \int_{s}^{t} \left[\tau^{\frac{p}{2}} |f(x(z), r(z), z) + u(x(\delta_z), r(z), z)|^p + \rho |g(x(z), r(z), z)|^p\right] dz ds \tag{2.10}$$

for $t \geq 0$, where $x_t := \{x(t + s) : -2\tau < s \leq 0\}$ [1], $r_t := \{r(t + s) : -\tau \leq s \leq 0\}$, θ is a positive number to be determined and

$$\rho = \begin{cases} (\frac{32}{p})^{\frac{p}{2}} & \text{for } p \in (1, 2), \\ [\frac{p(p-1)}{2}]^{\frac{p}{2}} & \text{for } p \geq 2. \end{cases} \tag{2.11}$$

For the functional to be well defined over $0 \leq t < 2\tau$, we set initial values

$$x(s) = x_0, \;\; r(s) = r_0, \;\; f(x, i, s) = f(x, i, 0), \;\; u(x, i, s) = u(x, i, 0), \;\; g(x, i, s) = g(x, i, 0)$$

for all $(x, i, s) \in R^n \times S \times [-2\tau, 0)$.

[1] For the definition of x_t, we require $s \in [-2\tau, 0]$ instead of $s \in [-\tau, 0]$. This is because $z - \tau < \delta_z \leq z$ in (2.10). At the starting point $z = s = t - \tau$, we have $t - 2\tau < \delta_z \leq t - \tau$.

In addition, we need to construct another functional by

$$U(x_t, r_t, t) = V(x(t), r(t), t) + \hat{V}(x_t, r_t, t). \tag{2.12}$$

Let's impose an assumption on the Lyapunov function.

Assumption 2.3 *Assume that there is a function $V \in C^{2,1}(R^n \times S \times R_+; R_+)$ and two positive numbers l, λ such that*

$$\mathcal{L}V(x, i, t) + l|V_x(x, i, t)|^{\frac{p}{p-1}} \leq -\lambda |x|^p \tag{2.13}$$

for all $(x, i, t) \in R^n \times S \times R_+$.

3 Asymptotic Stabilization

Theorem 3.1 *Let Assumptions 2.1, 2.2 and 2.3 hold. Choose a free parameter $\alpha \in (0, 8^{\frac{1-p}{p}})$. If $\tau > 0$ is sufficiently small for*

$$\begin{aligned} \lambda > & \frac{[2(p-1)]^{p-1} K_3^p}{p^p l^{p-1}(1 - 8^{p-1}\alpha^p)} \tau^{\frac{p}{2}} \left[2^{p-1} \tau^{\frac{p}{2}} K_1^p + \rho K_2^p + 4^{p-1} \tau^{\frac{p}{2}} K_3^p \right] \\ & \textit{and} \quad \tau \leq \frac{\alpha}{K_3}, \end{aligned} \tag{3.1}$$

then the controlled system (2.2) is H_∞-stable in $L^p(\Omega \times R_+; R^n)$ (also known as $L^p(\Omega \times R_+; R^n)$-stable) in the sense

$$\int_0^\infty \mathbb{E}|x(s)|^p ds < \infty \tag{3.2}$$

for all initial data $x_0 \in R^n$ and $r_0 \in S$.

Proof. Fix any $x_0 \in R^n$ and $r_0 \in S$. Let

$$\Phi(x_t, r_t, t) = \theta \tau^{\frac{p-2}{2}} \int_{t-\tau}^t \left[\tau^{\frac{p}{2}} |f(x(s), r(s), s) + u(x(\delta_s), r(s), s)|^p + \rho |g(x(s), r(s), s)|^p \right] ds. \tag{3.3}$$

Notice that the integrand in (2.10) is right-continuous in t, then we can use the Leibniz integral ruleto calculate the derivative of $\hat{V}(x_t, r_t, t)$ with respect to t.

$$\hat{V}_t(x_t, r_t, t) = \theta\tau^{\frac{p}{2}}\left[\tau^{\frac{p}{2}}|f(x(t), r(t), t) + u(x(\delta_t), r(t), t)|^p + \rho|g(x(t), r(t), t)|^p\right] - \Phi(x_t, r_t, t).$$

We apply the generalized Itô formula (see e.g. [15]) to $U(x_t, r_t, t)$ and obtain that

$$dU(x_t, r_t, t) = LU(x_t, r_t, t)dt + dM(t)$$

for $t \geq 0$, where $M(t)$ is a continuous local martingale with $M(0) = 0$ (we do not need its explicit form here) and

$$\begin{aligned}
&LU(x_t, r_t, t) \\
=&V_t(x(t), r(t), t) + V_x(x(t), r(t), t)[f(x(t), r(t), t) + u(x(\delta_t), r(t), t)] \\
&+ \frac{1}{2}\text{trace}[g^T(x(t), r(t), t)V_{xx}(x(t), r(t), t)g(x(t), r(t), t)] \\
&+ \sum_{j=1}^{N} \gamma_{r(t),j} V(x(t), j, t) + \hat{V}_t(x_t, r_t, t).
\end{aligned} \tag{3.4}$$

Replace some terms with the operator defined in (2.9), we have

$$\begin{aligned}
&LU(x_t, r_t, t) \\
=&\mathcal{L}V(x(t), r(t), t) - V_x(x(t), r(t), t)[u(x(t), r(t), t) - u(x(\delta_t), r(t), t)] \\
&+ \theta\tau^{\frac{p}{2}}\left[\tau^{\frac{p}{2}}|f(x(t), r(t), t) + u(x(\delta_t), r(t), t)|^p + \rho|g(x(t), r(t), t)|^p\right] - \Phi(x_t, r_t, t).
\end{aligned} \tag{3.5}$$

By the Young inequality (see e.g. [15, page 52]) and Assumption 2.2, we can derive that

$$
\begin{aligned}
&-V_x(x(t),r(t),t)[u(x(t),r(t),t)-u(x(\delta_t),r(t),t)]\\
\le&|V_x(x(t),r(t),t)||u(x(t),r(t),t)-u(x(\delta_t),r(t),t)|\\
\le&\Big[\varepsilon|V_x(x(t),r(t),t)|^{\frac{p}{p-1}}\Big]^{\frac{p-1}{p}}\Big[\varepsilon^{1-p}|u(x(t),r(t),t)-u(x(\delta_t),r(t),t)|^p\Big]^{\frac{1}{p}}\\
\le&\frac{p-1}{p}\varepsilon|V_x(x(t),r(t),t)|^{\frac{p}{p-1}}+\frac{1}{p}\varepsilon^{1-p}|u(x(t),r(t),t)-u(x(\delta_t),r(t),t)|^p\\
\le&l|V_x(x(t),r(t),t)|^{\frac{p}{p-1}}+\frac{1}{p}(\frac{p-1}{pl})^{p-1}K_3^p|x(t)-x(\delta_t)|^p, \qquad (3.6)
\end{aligned}
$$

where $l=\frac{p-1}{p}\varepsilon$ for $\forall\varepsilon>0$. Moreover, by Assumptions 2.1, 2.2 and the elementary inequality $|a+b|^p\le 2^{p-1}(|a|^p+|b|^p)$ for $\forall a,b\in R$, we have

$$
\begin{aligned}
&|f(x(t),r(t),t)+u(x(\delta_t),r(t),t)|^p\\
\le&2^{p-1}\Big[K_1^p|x(t)|^p+K_3^p|x(\delta_t)|^p\Big]\\
\le&2^{p-1}(K_1^p+2^{p-1}K_3^p)|x(t)|^p+4^{p-1}K_3^p|x(t)-x(\delta_t)|^p.
\end{aligned}
\qquad (3.7)
$$

Substituting (3.6) and (3.7) into (3.5) yields that

$$
\begin{aligned}
LU(x_t,r_t,t)\le&\mathcal{L}V(x(t),r(t),t)+l|V_x(x(t),r(t),t)|^{\frac{p}{p-1}}\\
&+\theta\tau^{\frac{p}{2}}\Big[2^{p-1}\tau^{\frac{p}{2}}K_1^p+\rho K_2^p+4^{p-1}\tau^{\frac{p}{2}}K_3^p\Big]|x(t)|^p\\
&+\Big(4^{p-1}\theta\tau^p+\frac{1}{p}(\frac{p-1}{pl})^{p-1}\Big)K_3^p|x(t)-x(\delta_t)|^p-\Phi(x_t,r_t,t). \qquad (3.8)
\end{aligned}
$$

Then Assumption 2.3 implies that

$$
LU(x_t,r_t,t)\le-\beta|x(t)|^p+\Big(4^{p-1}\theta\tau^p+\frac{1}{p}(\frac{p-1}{pl})^{p-1}\Big)K_3^p|x(t)-x(\delta_t)|^p-\Phi(x_t,r_t,t). \qquad (3.9)
$$

where

$$
\beta=\beta(\theta,\tau):=\lambda-\theta\tau^{\frac{p}{2}}\Big[2^{p-1}\tau^{\frac{p}{2}}K_1^p+\rho K_2^p+4^{p-1}\tau^{\frac{p}{2}}K_3^p\Big]. \qquad (3.10)
$$

Furthermore, it's easy to see from the Itô formula that

$$
\begin{aligned}
&|x(t)-x(\delta_t)|^p \\
\le& 2^{p-1}\Big(\Big|\int_{\delta_t}^t [f(x(s),r(s),s)+u(x(\delta_s),r(s),s)]ds\Big|^p + \Big|\int_{\delta_t}^t g(x(s),r(s),s)dw(s)\Big|^p\Big).
\end{aligned}
\tag{3.11}
$$

Since $t-\delta_t \le \tau$ for all $t \ge 0$, Hölder's inequality indicates that

$$
\Big|\int_{\delta_t}^t [f(x(s),r(s),s)+u(x(\delta_s),r(s),s)]ds\Big|^p \le \tau^{p-1}\int_{\delta_t}^t |f(x(s),r(s),s)+u(x(\delta_s),r(s),s)|^p ds.
\tag{3.12}
$$

For $p \in (1,2)$, we use the Burkholder-Davis-Gundy inequality (see e.g. [14, page 40]) and Hölder's inequality to obtain that

$$
\begin{aligned}
&\mathbb{E}\Big|\int_{\delta_t}^t g(x(s),r(s),s)dw(s)\Big|^p \le \mathbb{E}\Big(\sup_{\delta_t\le z\le t}\Big|\int_{\delta_t}^z g(x(v),r(v),v)dw(v)\Big|^p\Big) \\
\le& (\frac{32}{p})^{\frac{p}{2}}\mathbb{E}\Big[\int_{\delta_t}^t |g(x(s),r(s),s)|^2 ds\Big]^{\frac{p}{2}} \le (\frac{32}{p})^{\frac{p}{2}}\tau^{\frac{p-2}{2}}\mathbb{E}\int_{\delta_t}^t |g(x(s),r(s),s)|^p ds.
\end{aligned}
\tag{3.13}
$$

For $p \ge 2$, we use [14, Theorem 7.1 on page 39] to obtain that

$$
\mathbb{E}\Big|\int_{\delta_t}^t g(x(s),r(s),s)dw(s)\Big|^p \le [\frac{p(p-1)}{2}]^{\frac{p}{2}}\tau^{\frac{p-2}{2}}\mathbb{E}\int_{\delta_t}^t |g(x(s),r(s),s)|^p ds.
\tag{3.14}
$$

Substituting (3.12), (3.13), (3.14) and (2.11) into (3.11) yields

$$
\begin{aligned}
&\mathbb{E}|x(t)-x(\delta_t)|^p \\
\le& 2^{p-1}\tau^{\frac{p-2}{2}}\mathbb{E}\int_{\delta_t}^t \Big[\tau^{\frac{p}{2}}|f(x(s),r(s),s)+u(x(\delta_s),r(s),s)|^p + \rho|g(x(s),r(s),s)|^p\Big]ds.
\end{aligned}
\tag{3.15}
$$

Let us now choose a parameter $\alpha \in (0, 8^{\frac{1-p}{p}})$ and choose

$$
\tau \le \frac{\alpha}{K_3} \quad \text{and} \quad \theta = \frac{[2(p-1)]^{p-1}}{p^p(1-8^{p-1}\alpha^p)} l^{1-p} K_3^p.
\tag{3.16}
$$

Then

$$
2^{p-1}\tau^{\frac{p-2}{2}}[4^{p-1}\theta\tau^p + \frac{1}{p}(\frac{p-1}{pl})^{p-1}]K_3^p \le \theta\tau^{\frac{p-2}{2}}
\tag{3.17}
$$

Combining (3.3), (3.9), (3.15) and (3.17) yields

$$\mathbb{E}(LU(x_t, r_t, t)) \le -\beta \mathbb{E}|x(t)|^p, \tag{3.18}$$

and by condition (3.1) we have $\beta > 0$.

Moreover, we know from [15, Lemma 1.9 on page 49] that,

$$\mathbb{E}U(x_t, r_t, t) = U(x_0, r_0, 0) + \mathbb{E}\int_0^t LU(x_s, r_s, s)ds, \qquad \text{for } t \ge 0. \tag{3.19}$$

Denote $U(x_0, r_0, 0)$ by C_0 for simplicity, then

$$C_0 = V(x_0, r_0, 0) + 0.5\theta\tau^{\frac{p+2}{2}}\Big[\tau^{\frac{p}{2}}|f(x_0, r_0, 0) + u(x_0, r_0, 0)|^p + \rho|g(x_0, r_0, 0)|^p\Big]. \tag{3.20}$$

Clearly, C_0 is a positive number. Consequently, substituting (3.18) into (3.19) and by the Fubini theorem, we obtain that

$$0 \le \mathbb{E}U(x_t, r_t, t) \le C_0 - \beta\int_0^t \mathbb{E}|x(s)|^p ds, \tag{3.21}$$

for $t \ge 0$. Hence

$$\int_0^\infty \mathbb{E}|x(s)|^p ds \le C_0/\beta,$$

which implies the desired assertion (3.2). The proof is complete. □

Clearly, parameters θ and τ are both positive. To obtain a relatively large τ that satisfies (3.1), we need to choose a good value of α. As α increases, lower bound of λ (i.e. $\theta\tau^{\frac{p}{2}}\Big[2^{p-1}\tau^{\frac{p}{2}}K_1^p + \rho K_2^p + 4^{p-1}\tau^{\frac{p}{2}}K_3^p\Big]$) also increases. In other words, choosing a larger α could make Assumption 2.3 stronger. So we need to find a balance between the lower bound of λ and the upper bound $\frac{\alpha}{K_3}$. Moreover, we also need to choose a good value of ε. As the free positive parameter ε is positive correlated to l but negative correlated to λ. While (3.1) implies that an increasing function of τ has upper bound λl^{p-1}. So we need to find a balance between (2.13) and (3.1). These can be seen in Section 5 Examples.

Theorem 3.2 *Under the same assumptions of Theorem 3.1, the solution of the controlled system (2.2) satisfies*

$$\lim_{t\to\infty} \mathbb{E}|x(t)|^p = 0$$

for any initial data $x_0 \in R^n$ and $r_0 \in S$. In other words, the controlled system (2.2) is asymptotically stable in pth moment.

Proof. Again, fix any $x_0 \in R^n$ and $r_0 \in S$. We know from the Itô formula that for $t \geq 0$,

$$\begin{aligned}\mathbb{E}(|x(t)|^p) &= |x_0|^p + \mathbb{E}\int_0^t \Big(p|x(s)|^{p-2}x^T(s)[f(x(s),r(s),s) + u(x(\delta_s),r(s),s)]\Big)ds \\ &+ \mathbb{E}\int_0^t \Big(\frac{p}{2}|x(s)|^{p-2}|g(x(s),r(s),s)|^2 + \frac{p(p-2)}{2}|x(s)|^{p-4}|x^T(s)g(x(s),r(s),s)|^2\Big)ds.\end{aligned} \tag{3.22}$$

Since $x^Ty \leq |x||y|$ and $|x^Tg| \leq |x||g|$ for $\forall x, y \in R^n, g \in R^{n\times m}$, we have

$$\begin{aligned}\mathbb{E}(|x(t)|^p) &\leq |x_0|^p + \int_0^t p\mathbb{E}\Big[|x(s)|^{p-1}(|f(x(s),r(s),s)| + |u(x(\delta_s),r(s),s)|)\Big]ds \\ &+ \int_0^t \Big(\frac{p}{2}\mathbb{E}\Big[|x(s)|^{p-2}|g(x(s),r(s),s)|^2\Big] + \frac{p(p-2)}{2}\mathbb{E}[|x(s)|^{p-2}|g(x(s),r(s),s)|^2]\Big)ds\end{aligned}$$

for $p \geq 2$, and

$$\begin{aligned}\mathbb{E}(|x(t)|^p) \leq |x_0|^p &+ \int_0^t p\mathbb{E}\Big[|x(s)|^{p-1}(|f(x(s),r(s),s)| + |u(x(\delta_s),r(s),s)|)\Big]ds \\ &+ \int_0^t \Big(\frac{p}{2}\mathbb{E}\Big[|x(s)|^{p-2}|g(x(s),r(s),s)|^2\Big]\Big)ds\end{aligned}$$

for $1 < p < 2$. Then Assumptions 2.1 and 2.2 indicate

$$\mathbb{E}(|x(t)|^p) \leq |x_0|^p + \int_0^t \Big(pK_1\mathbb{E}|x(s)|^p + pK_3\mathbb{E}\Big[|x(s)|^{p-1}|x(\delta_s)|\Big] + \pi K_2^2\mathbb{E}|x(s)|^p\Big)ds, \tag{3.23}$$

where

$$\pi = \begin{cases} \frac{p}{2} & \text{for } p \in (1,2), \\ \frac{p(p-1)}{2} & \text{for } p \geq 2. \end{cases}$$

Moreover, the Young inequality and the elementary inequality imply that

$$
\begin{aligned}
|x(s)|^{p-1}|x(\delta_s)| &\le \Big[\frac{p-1}{p}||x(s)| + \frac{1}{p}|x(\delta_s)|\Big]^p \\
&\le \frac{2^{p-1}}{p^p}\Big[(p-1)^p|x(s)|^p + |x(\delta_s)|^p\Big] \\
&\le \frac{2^{p-1}}{p^p}\Big[((p-1)^p + 2^{p-1})|x(s)|^p + 2^{p-1}|x(s) - x(\delta_s)|^p\Big].
\end{aligned}
\tag{3.24}
$$

Substituting this into (3.23) gives

$$
\mathbb{E}(|x(t)|^p) \le |x_0|^p + C\int_0^t \mathbb{E}|x(s)|^p ds + C\int_0^t \mathbb{E}|x(s) - x(\delta_s)|^p ds, \tag{3.25}
$$

where, here and in the remaining part of this paper, C's denote positive constants that may change from line to line but we don't need their explicit forms.

Note that for any $s \ge 0$, there is a unique integer $v \ge 0$ for $s \in [v\tau, (v+1)\tau)$, and $\delta_z = v\tau$ for any $z \in [v\tau, s]$.

Recall (3.15) as well as the Assumptions 2.1 and 2.2, we derive that

$$
\begin{aligned}
&\mathbb{E}|x(s) - x(\delta_s)|^p = \mathbb{E}|x(s) - x(v\tau)|^p \\
\le& 2^{p-1}\tau^{\frac{p-2}{2}}\mathbb{E}\int_{v\tau}^s \tau^{\frac{p}{2}}|f(x(z), r(z), z) + u(x(\delta_z), r(z), z)|^p + \rho|g(x(z), r(z), z)|^p dz \\
\le& 2^{p-1}\tau^{\frac{p-2}{2}}\mathbb{E}\int_{v\tau}^s 2^{p-1}\tau^{\frac{p}{2}}\Big[K_1^p|x(z)|^p + K_3^p|x(v\tau)|^p\Big] + \rho K_2^p|x(z)|^p dz \\
\le& 2^{p-1}\tau^{\frac{p-2}{2}}\Big[2^{p-1}\tau^{\frac{p}{2}}K_1^p + \rho K_2^p\Big]\int_{v\tau}^s \mathbb{E}|x(z)|^p dz + 4^{p-1}\tau^p K_3^p \mathbb{E}|x(v\tau)|^p \\
\le& 2^{p-1}\tau^{\frac{p-2}{2}}\Big[2^{p-1}\tau^{\frac{p}{2}}K_1^p + \rho K_2^p\Big]\int_{v\tau}^s \mathbb{E}|x(z)|^p dz + 8^{p-1}\tau^p K_3^p\Big(\mathbb{E}|x(s)|^p + \mathbb{E}|x(s) - x(v\tau)|^p\Big).
\end{aligned}
$$

Note that the condition (3.1) implies $8^{p-1}\tau^p K_3^p < 1$, then we can rearrange it and obtain that

$$\mathbb{E}|x(s)-x(\delta_s)|^p \le \frac{2^{p-1}\tau^{\frac{p-2}{2}}\left[2^{p-1}\tau^{\frac{p}{2}}K_1^p+\rho K_2^p\right]}{1-8^{p-1}\tau^p K_3^p}\int_{\delta_s}^{s}\mathbb{E}|x(z)|^p dz+\frac{8^{p-1}\tau^p K_3^p}{1-8^{p-1}\tau^p K_3^p}\mathbb{E}|x(s)|^p. \tag{3.26}$$

Substituting this into (3.25) yields

$$\mathbb{E}|x(t)|^p \le |x_0|^p + C\int_0^t \mathbb{E}|x(s)|^p ds + C\int_0^t\int_{\delta_s}^{s}\mathbb{E}|x(z)|^p dzds. \tag{3.27}$$

Besides, it's easy to show that for a non-negative function $F(t)$,

$$\begin{aligned}&\int_0^t\int_{\delta_s}^{s}F(z)dzds \le \int_0^t\int_{s-\tau}^{s}F(z)dzds\\ \le&\int_{-\tau}^{t}F(z)\int_z^{z+\tau}dsdz \le \tau\int_{-\tau}^{t}F(z)dz.\end{aligned}$$

Applying this to $\mathbb{E}|x(z)|^p$ gives

$$\int_0^t\int_{\delta_s}^{s}\mathbb{E}|x(z)|^p dzds \le \tau\int_{-\tau}^{t}\mathbb{E}|x(z)|^p dz \le \tau^2|x_0|^p+\tau\int_0^t\mathbb{E}|x(z)|^p dz,$$

then we can rewrite (3.27) as

$$\mathbb{E}|x(t)|^p \le C|x_0|^p + C\int_0^t \mathbb{E}|x(s)|^p ds. \tag{3.28}$$

So by Theorem 3.1, we have

$$\mathbb{E}|x(t)|^p \le C \quad \forall t\ge 0. \tag{3.29}$$

Furthermore, it's easy to see from the Itô formula that

$$\begin{aligned}&\mathbb{E}|x(t_2)|^p-\mathbb{E}|x(t_1)|^p\\ =&\mathbb{E}\int_{t_1}^{t_2}\Big(p|x(t)|^{p-2}x^T(t)[f(x(t),r(t),t)+u(x(\delta_t),r(t),t)]\Big)dt\\ &+\mathbb{E}\int_{t_1}^{t_2}\Big(\frac{p}{2}|x(t)|^{p-2}|g(x(t),r(t),t)|^2+\frac{p(p-2)}{2}|x(t)|^{p-4}|x^T(t)g(x(t),r(t),t)|^2\Big)dt.\end{aligned}$$

After similar calculations to (3.22) and(3.23), we derive that

$$\mathbb{E}|x(t_2)|^p-\mathbb{E}|x(t_1)|^p \le \int_{t_1}^{t_2}\Big(pK_1\mathbb{E}|x(t)|^p+pK_3\mathbb{E}\Big[|x(t)|^{p-1}|x(\delta_t)|\Big]+\pi K_2^2\mathbb{E}|x(t)|^p\Big)dt.$$

Then by (3.29), we get that for any $0 \le t_1 < t_2 < \infty$,

$$\Big| \mathbb{E}|x(t_2)|^p - \mathbb{E}|x(t_1)|^p \Big| \le C(t_2 - t_1).$$

According to Barbalat's lemma (see e.g. [34, page 123]), combining this uniform continuity with Theorem 3.1 yields that $\lim_{t\to\infty} \mathbb{E}|x(t)|^p = 0$. The proof is complete. $\Box$

4 Exponential Stabilization

In last section, we discussed the asymptotic stability and proved that eventually (as $t \to \infty$), $\mathbb{E}|x(t)|^p$ goes to 0, but we don't know its speed. To explore the rate at which the solution tends to zero, let us discuss the exponential stabilization in this section. We need to impose the following condition at first.

Assumption 4.1 *Assume that there is a pair of positive numbers c_1 and c_2 such that*

$$c_1|x|^p \le V(x,i,t) \le c_2|x|^p \tag{4.1}$$

for all $(x,i,t) \in R^n \times S \times R_+$.

Theorem 4.2 *Let Assumptions 2.1, 2.2, 2.3 and 4.1 hold. Choose a parameter $\alpha \in (0, 8^{\frac{1-p}{p}})$, let $\tau > 0$ be sufficiently small for (3.1) to hold, set parameters θ as (3.16) and β as (3.10), so $\beta > 0$. Then the solution of the controlled system (2.2) satisfies*

$$\limsup_{t\to\infty} \frac{1}{t} \log(\mathbb{E}|x(t)|^p) \le -\gamma \tag{4.2}$$

and

$$\limsup_{t\to\infty} \frac{1}{t} \log(|x(t)|) \le -\frac{\gamma}{2} \quad a.s. \tag{4.3}$$

for all initial data $x_0 \in R^n$ and $r_0 \in S$, where $\gamma > 0$ is the unique root to the following equation

$$2\tau\gamma e^{2\tau\gamma}(H_1 + \tau H_2) + \gamma c_2 = \beta, \tag{4.4}$$

in which

$$H_1 = \theta\tau^{\frac{p}{2}}\Big[2^{p-1}\tau^{\frac{p}{2}}K_1^p + \rho K_2^p + 4^{p-1}\tau^{\frac{p}{2}}K_3^p + \frac{32^{p-1}\tau^{\frac{3p}{2}}K_3^{2p}}{1-8^{p-1}\tau^p K_3^p}\Big]$$

$$\textit{and} \qquad H_2 = \frac{8^{p-1}\theta\tau^{\frac{3p-2}{2}}K_3^p\Big[2^{p-1}\tau^{\frac{p}{2}}K_1^p + \rho K_2^p\Big]}{1-8^{p-1}\tau^p K_3^p}. \tag{4.5}$$

Proof. It's easy to see from the generalized Itô formula that

$$\mathbb{E}\big[e^{\gamma t}U(x_t, r_t, t)\big] = U(x_0, r_0, 0) + \mathbb{E}\int_0^t e^{\gamma s}[\gamma U(x_s, r_s, s) + LU(x_s, r_s, s)]ds \tag{4.6}$$

for $t \geq 0$. By (4.1) and (2.12), we have

$$c_1 e^{\gamma t}\mathbb{E}|x(t)|^p \leq e^{\gamma t}\mathbb{E}V(x(t), r(t), t) \leq e^{\gamma t}\mathbb{E}U(x_t, r_t, t)$$

Then combining (4.6), (3.18) and (3.20) gives

$$c_1 e^{\gamma t}\mathbb{E}|x(t)|^p \leq C_0 + \int_0^t e^{\gamma s}[\gamma\mathbb{E}U(x_s, r_s, s) - \beta\mathbb{E}|x(s)|^p]ds. \tag{4.7}$$

Moreover, substutiting (2.10) and (4.1) into (2.12) gives

$$\mathbb{E}U(x_s, r_s, s) \leq c_2\mathbb{E}|x(s)|^p + \mathbb{E}\hat{V}(x_s, r_s, s). \tag{4.8}$$

Since for a function $F(v)$, we have

$$\begin{aligned}\int_{s-\tau}^s\int_z^s F(v)dvdz &= \int_{s-\tau}^s\int_{s-\tau}^v F(v)dzdv = \int_{s-\tau}^s F(v)\int_{s-\tau}^v dzdv \\ &= \int_{s-\tau}^s F(v)(v-s+\tau)dv < \tau\int_{s-\tau}^s F(v)dv.\end{aligned}$$

Applying this to $\mathbb{E}\hat{V}(x_s, r_s, s)$ yields that

$$
\begin{aligned}
&\mathbb{E}\hat{V}(x_s, r_s, s) \\
\leq &\theta\tau^{\frac{p}{2}}\mathbb{E}\int_{s-\tau}^{s}\Big[\tau^{\frac{p}{2}}|f(x(v), r(v), v) + u(x(\delta_v), r(v), v)|^p + \rho|g(x(v), r(v), v)|^p\Big]dv \\
\leq &\theta\tau^{\frac{p}{2}}\int_{s-\tau}^{s}\Big[2^{p-1}\tau^{\frac{p}{2}}K_1^p + \rho K_2^p + 4^{p-1}\tau^{\frac{p}{2}}K_3^p\Big]\mathbb{E}|x(v)|^p + 4^{p-1}\tau^{\frac{p}{2}}K_3^p\mathbb{E}|x(v) - x(\delta_v)|^p dv.
\end{aligned}
$$

To make $\delta_v > 0$, we need $v \geq \tau$ and so $s \geq 2\tau$. Then by (3.26), we have

$$
\mathbb{E}\hat{V}(x_s, r_s, s) \leq H_1\int_{s-\tau}^{s}\mathbb{E}|x(v)|^p dv + H_2\int_{s-\tau}^{s}\int_{\delta_v}^{v}\mathbb{E}|x(y)|^p dydv. \tag{4.9}
$$

where both H_1 and H_2 have been defined by (4.5).

Since for a non-negative function $F(y)$,

$$
\begin{aligned}
&\int_{s-\tau}^{s}\int_{\delta_v}^{v}F(y)dydv \leq \int_{s-\tau}^{s}\int_{v-\tau}^{v}F(y)dydv \\
&< \int_{s-2\tau}^{s}\int_{s-\tau}^{s}F(y)dvdy = \tau\int_{s-2\tau}^{s}F(y)dy.
\end{aligned}
$$

Thus, $\int_{s-\tau}^{s}\int_{\delta_v}^{v}\mathbb{E}|x(y)|^p dydv \leq \tau\int_{s-2\tau}^{s}\mathbb{E}|x(y)|^p dy$. Hence we have

$$
\mathbb{E}(\hat{V}(x_s, r_s, s)) \leq (H_1 + \tau H_2)\int_{s-2\tau}^{s}\mathbb{E}|x(y)|^p dy. \tag{4.10}
$$

Combining (4.7), (4.8) and (4.10), we obtain that

$$
\begin{aligned}
&c_1 e^{\gamma t}\mathbb{E}|x(t)|^p \\
\leq &C_0 - (\beta - \gamma c_2)\int_0^t e^{\gamma s}\mathbb{E}|x(s)|^p ds + \gamma(H_1 + \tau H_2)\int_0^t e^{\gamma s}\Big(\int_{s-2\tau}^{s}\mathbb{E}|x(y)|^p dy\Big)ds
\end{aligned} \tag{4.11}
$$

for $\forall t \geq 2\tau$. Obviously,

$$
\int_0^{2\tau}e^{\gamma s}\int_{s-2\tau}^{s}\mathbb{E}|x(y)|^p dyds \leq \int_{-2\tau}^{2\tau}\int_0^{2\tau}e^{\gamma s}\mathbb{E}|x(y)|^p dsdy = \frac{e^{2\tau\gamma - 1}}{\gamma}\int_{-2\tau}^{2\tau}\mathbb{E}|x(y)|^p dy. \tag{4.12}
$$

Besides, it can be easily seen that

$$\int_{2\tau}^{t} e^{\gamma s}\Big(\int_{s-2\tau}^{s} \mathbb{E}|x(y)|^p dy\Big) ds \le \int_{0}^{t} \mathbb{E}|x(y)|^p \Big(\int_{y}^{y+2\tau} e^{\gamma s} ds\Big) dy$$
$$\le 2\tau e^{2\tau\gamma}\int_0^t e^{\gamma y}\mathbb{E}|x(y)|^p dy. \tag{4.13}$$

Substituting (4.12) and (4.13) into (4.11) gives

$$c_1 e^{\gamma t}\mathbb{E}|x(t)|^p \le C + \Big(2\tau\gamma e^{2\tau\gamma}(H_1+\tau H_2) + \gamma c_2 - \beta\Big)\int_0^t e^{\gamma s}\mathbb{E}|x(s)|^p ds. \tag{4.14}$$

The condition (4.4) implies that for $\forall t \ge 2\tau$,

$$c_1 e^{\gamma t}\mathbb{E}|x(t)|^p \le C. \tag{4.15}$$

Hence we obtain the assertion (4.2). Finally by [15, Theorem 8.8 on page 309], we obtain the assertion (4.3) as well. The proof is complete. □

In practice, a common choice of Lyapunov functions is quadratic functions, for example, $V(x(t), r(t), t) = (x^T(t)Q_{r(t)}x(t))^{\frac{p}{2}}$ where $Q_{r(t)}$ are positive-definite $n\times n$ matrices. So we propose the following corollaries to state how to use this kind of Lyapunov functions to help exponentially stabilize an unstable hybrid system.

Since $V_x(x,i,t) = p(x^TQ_ix)^{\frac{p}{2}-1}x^TQ_i$, then we have $|V_x(x,i,t)| \le p\lambda_{\max}^{\frac{p}{2}-1}(Q_i)\|Q_i\||x|^{p-1}$. So we only need to require $\mathcal{L}U(x,i,t) \le -b|x|^p$ for $b>0$ to satisfy Assumption 2.3. This leads us to the following alternative assumption. By calculating the derivatives $V_t(x,i,t) = 0$ and $V_{xx}(x,i,t) = p(p-2)[x^TQ_ix]^{\frac{p}{2}-2}Q_ixx^TQ_i + p[x^TQ_ix]^{\frac{p}{2}-1}Q_i$, we can easily obtain $\mathcal{L}U(x,i,t)$, which is the left-hand-side of (4.16) below.

Assumption 4.3 *Assume that there exist positive-definite symmetric matrices $Q_i \in R^{n\times n}$ $(i\in S)$ and a constant $b>0$ such that*

$$p(x^TQ_ix)^{\frac{p}{2}-1}\Big(x^TQ_i[f(x,i,t)+u(x,i,t)]+\frac{1}{2}\text{trace}[g^T(x,i,t)Q_ig(x,i,t)]\Big)$$
$$+p(\frac{p}{2}-1)[x^TQ_ix]^{\frac{p}{2}-2}|g^TQ_ix|^2+\sum_{j=1}^N\gamma_{ij}[x^TQ_jx]^{\frac{p}{2}}\le -b|x|^p, \tag{4.16}$$

for all $(x,i,t)\in R^n\times S\times R_+$.

Corollary 4.4 *Let Assumptions 2.1, 2.2 and 4.3 hold. Set the parameters in (4.1) as*

$$c_1=\min_{i\in S}\lambda_{\min}^{\frac{p}{2}}(Q_i),\ \ c_2=\max_{i\in S}\lambda_{\max}^{\frac{p}{2}}(Q_i)\quad \textit{and}\quad d=\big[p\max_{i\in S}\lambda_{\max}^{\frac{p-2}{2}}(Q_i)\max_{i\in S}\|Q_i\|\big]^{\frac{p}{p-1}}.$$

Choose $l<b/d$ *and* $\alpha\in(0,8^{\frac{1-p}{p}})$. *Set parameters* $\lambda=b-ld$, θ *as (3.16) and* β *as (3.10). Let* $\tau>0$ *be sufficiently small for (3.1) to hold. Then (4.2) holds, i.e., the controlled system is pth moment exponentially stable.*

5 Examples

Now we illustrate our theory with two examples.

Example 5.1 Now we consider a nonlinear hybrid SDE

$$dx(t)=f(x(t),r(t),t)dt+g(x(t),r(t),t)dw(t) \tag{5.1}$$

on $t\ge 0$. Here

$$x(t)=\begin{bmatrix} x_1(t)\\ x_2(t)\end{bmatrix};$$

$w(t)$ is a scalar Brownian motion; $r(t)$ is a Markov chain on the state space $S=\{1,2\}$ with the generator matrix

$$\Gamma=\begin{bmatrix} -4 & 4\\ 1 & -1\end{bmatrix};$$

and the coefficients are

$$f(x(t),1,t) = \begin{bmatrix} x_2(t)\cos(x_1(t)) \\ x_1(t)\sin(x_2(t)) \end{bmatrix}, \quad f(x(t),2,t) = \begin{bmatrix} x_2(t)\sin(x_1(t)) \\ x_1(t)\cos(x_2(t)) \end{bmatrix},$$

$$g(x(t),1,t) = \begin{bmatrix} 0.2\sqrt{3x_1^2(t)+x_2^2(t)} \\ 0.2\sqrt{x_1^2(t)+3x_2^2(t)} \end{bmatrix}, \quad g(x(t),2,t) = \begin{bmatrix} 0.1 & -0.1 \\ -0.2 & 0.4 \end{bmatrix} x(t).$$

Figure 5.1 below shows simulated paths and obviously this system is not stable in the sense of 3rd moment exponential stability.

Figure 5.1: One simulated path of Markov chain $r(t)$ and state $x(t)$, as well as sample mean of $|x(t)|^3$ from 2000 simulated paths, all by the Euler–Maruyama method with step size 1e-6 and random initial values.

Note that this system satisfies the Assumption 2.1 with $K_1 = 1$ and $K_2 = 0.4671$. We will design a feedback control of the form $u(x,i,t) = A_i(x)x$ and find the observation interval τ to make the controlled system

$$dx(t) = \big(f(x(t),r(t),t) + u(x(\delta_t),r(t),t)\big)dt + g(x(t),r(t),t)dw(t), \tag{5.2}$$

become 3rd moment exponentially stable. In the controller, $A_i(x) : R^2 \to R^{2\times 2}$ and Assumption 2.2 will hold with $K_3 = \max_{i\in S, x\in R^2} \|A_i(x)\|$.

Now we can start designing $A_i(x)$ by choosing our auxiliary Lyapunov functions. We choose Lyapunov functions of the form $V(x,i,t) = (x^T Q_i x)^{1.5}$ where $Q_i = q_i I$ (I is the $2\kappa 2$ identity matrix), so Corollary 4.4 can be applied.

Let $V(x,i,t) = q_i^{1.5}|x|^3$ where $q_1 = 2, q_2 = 1$. Then the left-hand-side of (4.16) becomes

$$
\begin{aligned}
&3q_i^{0.5}|x|[q_i x^T(f(x,i,t)+u(x,i,t))+0.5q_i|g(x,i,t)|^2]+1.5q_i|x|^{-1}|g^Tx|^2+\sum_{j=1}^{N}\gamma_{ij}q_j^{1.5}|x|^3 \\
\le& 1.5q_i^{1.5}|x|[(2K_1+K_2^2)|x|^2+2x^TA_i(x)x]+1.5q_iK_2^2|x|^{1.5}+\sum_{j=1}^{N}\gamma_{ij}q_j^{1.5}|x|^3 \\
\le& |x|x^T\tilde{Q}_i x \le \lambda_{\max}(\tilde{Q}_i)|x|^3 \qquad (5.3)
\end{aligned}
$$

for all $i \in S$, where

$$
\tilde{Q}_i = 1.5q_i^{1.5}(2K_1+K_2^2)I + 1.5q_i^{1.5}(A_i(x)+A_i^T(x)) + 1.5q_iK_2^2 I + \sum_{j=1}^{N}\gamma_{ij}q_j^{1.5}I.
$$

Substituting the constant coefficients gives

$$
\tilde{Q}_1 = 2.7517I + 4.2426(A_1(x)+A_1^T(x))
$$

$$
\text{and}\quad \tilde{Q}_2 = 5.4829I + 1.5(A_2(x)+A_2^T(x)).
$$

Thus, we need to design $A_i(x)$ such that $\tilde{Q}_i$ is negative-definite for $i \in S$. Of course there are many choices of $A_i(x)$, here we use

$$
A_1(x(t)) = \begin{bmatrix} 0.5\sin(x_1(t))-1 & -1 \\ 1 & 0.5\cos(x_2(t))-1 \end{bmatrix}
$$

$$
\text{and}\quad A_2(x(t)) = \begin{bmatrix} -2.3 & 0.2\cos(x_1(t)x_2(t)) \\ -0.2\cos(x_1(t)x_2(t)) & -2.3 \end{bmatrix}.
$$

Substituting the coefficient matrices gives $\lambda_{\max}(\tilde{Q}_1) = -1.491$ and $\lambda_{\max}(\tilde{Q}_2) = -1.417$. That is, Assumption 4.3 holds with $b = 1.417$. Assumption 2.2 holds with $K_3 = 2.309$. Then we calculate parameters in Corollary 4.4 and get $c_1 = 1$, $c_2 = 2.828$ and $d = 24.717$. To obtain a relatively large observation interval τ, we choose $\alpha = 0.008$ and $l = 0.038$. This gives $\lambda = 0.4777$ and condition (3.1) requires $\tau \le 0.003$.

Corollary 4.4 indicates that the controlled system (5.2) with feedback control defined as above and $\tau \leq 0.003$ is 3rd moment exponentially stable, which is indeed in accordance with the Figure 5.2.

Figure 5.2: One simulated path of Markov chain $r(t)$ and the corresponding state $x(t)$, as well as sample mean of $|x(t)|^3$ from 2000 simulated paths, all by the Euler–Maruyama method with step size 1e-6, observation interval $\tau = 0.003$ and random initial values.

Example 5.2 Let us consider the same linear hybrid system as the Example 6.1 in [24]. [24] achieved the almost sure exponential stability by stabilizing the unstable system in mean square ($p = 2$) sense. We can achieve the almost sure exponential stability by weaker conditions: a smaller moment order p. Using the same way of control as in [24], i.e., using the same F_is and G_is for feedback control $u(x, i, t) = F_i G_i x$, we can apply a bigger observe interval for the controlled system to be almost surely exponentially stable.

Let $p = 1.5$. Now we calculate the observe interval τ.

Recall that the original system is

$$dx(t) = A_{r(t)}x(t)dt + B_{r(t)}x(t)dw(t) \tag{5.4}$$

on $t \geq t_0$ with coefficients

$$A_1 = \begin{bmatrix} 1 & -1 \\ 1 & -5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -5 & -1 \\ 1 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Here $w(t)$ is a scalar Brownian motion and $r(t)$ is a Markov chain on the state space $S = \{1, 2\}$ generated by

$$\Gamma = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Figure 5.3 shows simulated paths and obviously this system is not stable in the sense of 1.5th moment exponential stability.

Figure 5.3: One simulated path of Markov chain $r(t)$ and the corresponding state $x(t)$, as well as sample mean of $|x(t)|^{1.5}$ from 2000 simulated paths, all by the Euler–Maruyama method with step size 1e-6 and random initial values.

Recall that the controller coefficients are

$$F_1 = \begin{bmatrix} -10 \\ 0 \end{bmatrix}, \quad G_1 = (1, 0), \quad F_2 = \begin{bmatrix} 0 \\ -10 \end{bmatrix}, \quad G_2 = (0, 1),$$

and the controlled system

$$dx(t) = [A(r(t))x(t) + F(r(t))G(r(t))x(\delta_t)]dt + B_{r(t)}x(t)dw(t); \tag{5.5}$$

satisfies the Assumption 2.1 and 2.3 with $K_1 = \max_{i \in S} \|A_i\| = 5.236$, $K_2 = \max_{i \in S} \|B_i\| = \sqrt{2}$ and $K_3 = 10$.

Choosing the same Lyapunov functions as in [24]: $Q_1 = Q_2 = I$ (the 2×2 identity matrix), the left-hand-side of (??) for $p = 1.5$ becomes

$$\begin{aligned}
&\mathcal{L}U(x,i,t)\\
=&1.5|x|^{-0.5}[x^T(f(x,i,t)+u(x,i,t))+0.5|g(x,i,t)|^2]-0.375|x|^{-2.5}|g^Tx|^2+\sum_{j=1}^{N}\gamma_{ij}|x|^{1.5}\\
\le&0.75|x|^{-0.5}\big(2x^T[A_i+F_iG_i]x+|B_ix|^2\big)-0.375\lambda_{\min}^2(B_i)|x|^{1.5}+\sum_{j=1}^{N}\gamma_{ij}|x|^{1.5}\\
\le&|x|^{-0.5}x^T\tilde{Q}_ix\\
\le&\lambda_{\max}(\tilde{Q}_i)|x|^{1.5}
\end{aligned} \tag{5.6}$$

for all $i \in S$, where

$$\tilde{Q}_i = 0.75\big[(A_i+F_iG_i)+(A_i^T+F_i^TG_i^T)+B_i^TB_i\big]-0.375\lambda_{\min}^2(B_i)I+\sum_{j=1}^{N}\gamma_{ij}I. \tag{5.7}$$

Substituting the coefficient matrices gives

$$\tilde{Q}_1 = \begin{bmatrix} -12.75 & 0 \\ 0 & -6.75 \end{bmatrix}, \quad \tilde{Q}_2 = \begin{bmatrix} -6.75 & 0 \\ 0 & -12.75 \end{bmatrix}.$$

So the Assumption 4.3 holds with $b = 6.75$.

Then we calculate parameters in Corollary 4.4 and get $c_1 = c_2 = 1$, $d = 3.375$. To maximize observation interval τ, we choose $\alpha = 0.15$ and $l = 0.66$. This gives $\lambda = 4.5225$, and finally the condition (3.1) requires $\tau \le 0.01456$.

By Corollary 4.4, the controlled system (5.5) with controllers u_1, u_2 defined as above and $\tau \le 0.01456$ is exponentially stable in 1.5th moment and almost surely as well. Figure 5.4 shows the computer simulation supports our results clearly.

When we choose the same controller coefficients and Lyapunov functions but different moment order p, the almost sure exponential stability of the controlled system requires

Figure 5.4: One simulated path of Markov chain $r(t)$ and the corresponding state $x(t)$, as well as sample mean of $|x(t)|^{1.5}$ from 2000 simulated paths, all by the Euler–Maruyama method with step size 1e-6, observation interval $\tau = 0.0145$ and random initial values.

different upper bounds on observation interval τ. This is shown in Table 1 below. If we only need the almost surely exponential stability (no requirement on moment stability), then we can choose $p = 1.01$. This could reduce the state observation frequency to around one thrid of what was required in [24] for the mean square case. Hence we reduce the cost of control.

Table 1: Moment order p vs. Observation interval upper bound τ^*.

6 Conclusion

In this paper we have discussed the stabilization of continuous-time hybrid stochastic differential equations by *feedback controls based on discrete-time state observations*. The stabilities analysed include pth moment H_∞ stability and asymptotic stability, pth moment and almost sure exponential stabilities. The main contributions of this paper are expanding from the sense of mean square to pth moment for all $p > 1$, and improving the upper bound of observation interval τ to some extent.

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