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Brief paper

Delay dependent stability of highly nonlinear hybrid stochastic systems☆

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ABSTRACT

There are lots of papers on the delay dependent stability criteria for differential delay equations (DDEs), stochastic differential delay equations (SDDEs) and hybrid SDDEs. A common feature of these existing criteria is that they can only be applied to delay equations where their coefficients are either linear or nonlinear but bounded by linear functions (namely, satisfy the linear growth condition). In other words, there is so far no delay-dependent stability criterion on nonlinear equations without the linear growth condition (we will refer to such equations as highly nonlinear ones). This paper is the first to establish delay dependent criteria for highly nonlinear hybrid SDDEs. It is therefore a breakthrough in the stability study of highly nonlinear hybrid SDDEs.

1. Introduction

Time-delay is encountered in many real-world systems in science and industry. Differential delay equations (DDEs) (or more generally, functional differential equations) have been developed to model such time-delay systems. Time-delay often causes undesirable system transient response, or even instability. Stability of DDEs has hence been studied intensively for more than 50 years. The stability criteria are often classified into two categories: delay-dependent and delay-independent stability criteria. The delay-dependent stability criteria take into account the size of delays and hence are generally less conservative than the delay-independent ones which work for any size of delays. There is a very rich literature in this area (see, e.g., Fridman, 2014; Hale & Lunel, 1993; Kolmanovskii & Nosov, 1986).

In 1980's, stochastic differential delay equations (SDDEs) were developed in order to model real-world systems which contain some uncertainties or are subject to external noises (see, e.g., Ladde & Lakshmikantham, 1980; Mao, 1991, 1994, 2007; Mohammed, 1984). Since then, stability has been one of the most important topics in the study of SDDEs. As the literature in this area is huge and lots of papers are of open-access, there is no need to cite any reference here.

In 1990's, hybrid SDDEs (also known as SDDEs with Markovian switching) were used to model real-world systems where they may experience abrupt changes in their structure and parameters in addition to time delays and uncertainties. One of the important issues in the study of hybrid SDDEs is the automatic control, with consequent emphasis being placed on the analysis of stability. Once again, the delay-dependent stability criteria have been established by many authors (see, e.g., Mao, Lam, & Huang, 2008; Mao & Yuan, 2006; Xu, Lam, & Mao, 2007; Yue & Han, 2005). To our best knowledge, a common feature of the existing delay-dependent stability criteria is that they can only be applied to the hybrid SDDEs where their coefficients are either linear or nonlinear but bounded by linear functions (namely, satisfy the linear growth condition). In other words, there is so far no delay-dependent stability criterion on nonlinear hybrid SDDEs without the linear growth condition (we will refer to such equations as highly nonlinear ones). For example, consider the scalar highly nonlinear hybrid SDDE

$$dx(t) = f(x(t), x(t-\delta(t)), r(t), t)dt + g(x(t), x(t-\delta(t)), r(t), t)dB(t). \tag{1.1}$$

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Here $x(t) \in R$ is the state, $\delta : R \to [0, \tau]$ stands for variable time delay, $B(t)$ is a scalar Brownian motion, $r(t)$ is a Markov chain on the state space $S = \{1, 2\}$ with its generator

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 8 & -8 \end{pmatrix}, \tag{1.2}$$

and we will refer to $r(t)$ as the mode of the system. Moreover, the coefficients f and g are defined by

$$f(x, y, 1, t) = -y - 3x^3, \qquad f(x, y, 2, t) = y - 2x^3,$$
$$g(x, y, 1, t) = y^2, \qquad g(x, y, 2, t) = 0.5y^2. \tag{1.3}$$

If there is no time-delay, namely $\delta(t) = 0$, then this hybrid SDDE becomes hybrid SDE and the computer simulation shows it is asymptotically stable; while if the time-delay is large, say $\delta(t) = 2$, the computer simulation shows that the hybrid SDDE is unstable (but we here omit simulation outputs due to the page limit). In other words, whether the hybrid SDDE is stable or not depends on how small or large the time-delay is. On the other hand, both drift and diffusion coefficients of the hybrid SDDE are highly nonlinear. However, there is no delay dependent criterion which can be applied to the SDDE to derive a sufficient bound on the time-delay $\delta(t)$ for the SDDE to be stable.

We should point out that there are already some papers on the asymptotic stability of highly nonlinear hybrid SDDEs (see, e.g., Hu, Mao, & Shen, 2013; Hu, Mao, & Zhang, 2013; Liu, 2012; Luo, Mao, & Shen, 2011) but these existing results are all *delay independent*. Our paper is the first to establish *delay dependent* criteria for highly nonlinear hybrid SDDEs. It is therefore a breakthrough in the stability study of highly nonlinear hybrid SDDEs. Let us begin to establish our new theory.

2. Notation and standing hypotheses

Throughout this paper, unless otherwise specified, we use the following notation. If A is a vector or matrix, its transpose is denoted by A^T. If $x \in R^n$, then $|x|$ is its Euclidean norm. If A is a matrix, we let $|A| = \sqrt{\text{trace}(A^TA)}$ be its trace norm. Let $R_+ = [0, \infty)$. For $h > 0$, denote by $C([-h, 0]; R^n)$ the family of continuous functions φ from $[-h, 0] \to R^n$ with the norm $\|\varphi\| = \sup_{-h\le u\le 0} |\varphi(u)|$. If both a, b are real numbers, then $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. If A is a subset of Ω, denote by I_A its indicator function; that is $I_A(\omega) = 1$ if $\omega \in A$ and 0 otherwise. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $B(t) = (B_1(t), \ldots, B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $r(t)$, $t \ge 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$. Here $\gamma_{ij} \ge 0$ is the transition rate from i to j if $i \ne j$ while $\gamma_{ii} = -\sum_{j\ne i} \gamma_{ij}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$. Let $\tau > 0$ and $\bar{\delta} \in [0, 1)$ be two constants. Let δ be a differentiable function from $R_+ \to [0, \tau]$ such that $\dot{\delta}(t) := d\delta(t)/dt \le \bar{\delta}$ for all $t \ge 0$. Let $f : R^n \times R^n \times S \times R_+ \to R^n$ and $g : R^n \times R^n \times S \times R_+ \to R^{n\times m}$ be Borel measurable functions. Consider an n-dimensional hybrid SDDE

$$dx(t) = f(x(t), x(t - \delta(t)), r(t), t)dt + g(x(t), x(t - \delta(t)), r(t), t)dB(t) \tag{2.1}$$

on $t \ge 0$ with initial data

$$\tilde{x}_0 = \xi \in C([-\tau, 0]; R^n) \quad \text{and} \quad r(0) = i_0 \in S, \tag{2.2}$$

where $\tilde{x}_0 := \{x(t) : -\tau \le t \le 0\}$. The classical conditions for the existence and uniqueness of the global solution are the local Lipschitz condition and the linear growth condition (see, e.g., Mao, 1991, 1994, 2007; Mao & Yuan, 2006). In this paper, we need the local Lipschitz condition. However, we will consider highly nonlinear SDDEs which, in general, do not satisfy the linear growth condition in this paper. We therefore impose the polynomial growth condition, instead of the linear growth condition. Let us state these conditions as an assumption for the use of this paper.

Assumption 2.1. Assume that for any $b > 0$, there exists a positive constant K_b such that

$$|f(x, y, i, t) - f(\bar{x}, \bar{y}, i, t)| \vee |g(x, y, i, t) - g(\bar{x}, \bar{y}, i, t)| \le K_b(|x - \bar{x}| + |y - \bar{y}|) \tag{2.3}$$

for all $x, y, \bar{x}, \bar{y} \in R^n$ with $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \le b$ and all $(i, t) \in S \times R_+$. Assume moreover that there exist three constants $K > 0$, $q_1 \ge 1$ and $q_2 \ge 1$ such that

$$|f(x, y, i, t)| \le K(1 + |x|^{q_1} + |y|^{q_1}),$$
$$|g(x, y, i, t)| \le K(1 + |x|^{q_2} + |y|^{q_2}) \tag{2.4}$$

for all $(x, y, i, t) \in R^n \times R^n \times S \times R_+$.

Of course, if $q_1 = q_2 = 1$ then condition (2.4) is the familiar linear growth condition. However, we emphasize once again that we are here interested in highly nonlinear SDDEs which have either $q_1 > 1$ or $q_2 > 1$. We will refer condition (2.4) as the polynomial growth condition. It is known that Assumption 2.1 only guarantees that the SDDE (2.1) with the initial data (2.2) has a unique maximal solution, which may explode to infinity at a finite time. To avoid such a possible explosion, we need to impose an additional condition in terms of Lyapunov functions. For this purpose, we need more notation.

Let $C^{2,1}(R^n \times S \times R_+; R_+)$ denote the family of non-negative functions $U(x, i, t)$ defined on $(x, i, t) \in R^n \times S \times R_+$ which are continuously twice differentiable in x and once in t. For such a function U, we will let $U_t = \frac{\partial U}{\partial t}$, $U_x = (\frac{\partial U}{\partial x_1}, \ldots, \frac{\partial U}{\partial x_n})$ and $U_{xx} = (\frac{\partial^2 U}{\partial x_k \partial x_l})_{n\times n}$. Let $C(R^n \times [-\tau, \infty); R_+)$ denote the family of all continuous functions from $R^n \times [-\tau, \infty)$ to R_+. We can now state another assumption.

Assumption 2.2. Assume that there exists a pair of functions $\bar{U} \in C^{2,1}(R^n \times S \times R_+; R_+)$ and $G \in C(R^n \times [-\tau, \infty); R_+)$, as well as positive numbers c_1, c_2, c_3 and $q \ge 2(q_1 \vee q_2)$ (where q_1 and q_2 are the same as in Assumption 2.1), such that

$$c_3 < c_2(1 - \bar{\delta}); \tag{2.5}$$

$$|x|^q \le \bar{U}(x, i, t) \le G(x, t) \tag{2.6}$$

for $(x, i, t) \in R^n \times S \times R_+$; and

$$\begin{aligned}\mathbb{L}\bar{U}(x, y, i, t) &:= \bar{U}_t(x, i, t) + \bar{U}_x(x, i, t)f(x, y, i, t) \\ &\quad + \frac{1}{2}\text{trace}[g^T(x, y, i, t)\bar{U}_{xx}(x, i, t)g(x, y, i, t)] \\ &\quad + \sum_{j=1}^{N} \gamma_{ij}\bar{U}(x, j, t) \\ &\le c_1 - c_2G(x, t) + c_3G(y, t - \delta(t))\end{aligned} \tag{2.7}$$

for $(x, y, i, t) \in R^n \times R^n \times S \times R_+$.

We now cite a theorem from Hu, Mao, and Shen (2013, Theorem 4.3), which shows the unique global solution of the SDDE (2.1) and its qth moment property under the above assumptions.

Theorem 2.3. *Under Assumptions 2.1 and 2.2, the SDDE (2.1) with the initial data (2.2) has the unique global solution $x(t)$ on $t \ge -\tau$ and the solution has the property that* $\sup_{-\tau\le t<\infty} \mathbb{E}|x(t)|^q < \infty$.

Theorem 2.3 implies a number of nice properties of the solution. For example, for any $t \geq 0$, $x(t)$ is in L^p for any $p \in (0, q]$ while both $f(x(t), x(t-\delta(t)), r(t), t)$ and $g(x(t), x(t-\delta(t)), r(t), t)$ are in L^2. These properties will play their fundamental roles when we discuss the asymptotic stability of the SDDE (2.1) in the next section.

3. Delay-dependent asymptotic stability

In this section, we will use the method of Lyapunov functionals to investigate the delay-dependent asymptotic stability. To define a Lyapunov functional for the use of this paper, we define two segments $\hat{x}_t := \{x(t+s) : -2\tau \leq s \leq 0\}$ and $\hat{r}_t := \{r(t+s) : -2\tau \leq s \leq 0\}$ for $t \geq 0$. For $\hat{x}_t$ and $\hat{r}_t$ to be well defined for $0 \leq t < 2\tau$, we set $x(s) = \xi(-\tau)$ for $s \in [-2\tau, -\tau)$ and $r(s) = r_0$ for $s \in [-2\tau, 0)$. The Lyapunov functional used in this paper will be of the form

$$V(\hat{x}_t, \hat{r}_t, t) = U(x(t), r(t), t) + \theta \int_{-\tau}^{0} \int_{t+s}^{t} \Big[\tau |f(x(v), x(v-\delta(v)), r(v), v)|^2 + |g(x(v), x(v-\delta(v)), r(v), v)|^2\Big] dv ds \tag{3.1}$$

for $t \geq 0$, where $U \in C^{2,1}(R^n \times S \times R_+; R_+)$ and θ is a positive number to be determined later while we set

$$f(x, y, i, s) = f(x, y, i, 0), \qquad g(x, y, i, s) = g(x, y, i, 0)$$

for $(x, y, i, s) \in R^n \times R^n \times S \times [-2\tau, 0)$.

Lemma 3.1. *With the notation above, $V(\hat{x}_t, \hat{r}_t, t)$ is an Itô process on $t \geq 0$ with its Itô differential*

$$dV(\hat{x}_t, \hat{r}_t, t) = LV(\hat{x}_t, \hat{r}_t, t)dt + dM(t), \tag{3.2}$$

where $M(t)$ is a continuous local martingale with $M(0) = 0$ (the explicit form of $M(t)$ is of no use in this paper so we do not state it here but it can be found in Mao & Yuan, 2006, Theorem 1.45 on page 48), and

$$\begin{aligned} LV(\hat{x}_t, \hat{r}_t, t) &= U_x(x(t), r(t), t) \\ &\quad \times [f(x(t), x(t-\delta(t)), r(t), t) - f(x(t), x(t), r(t), t)] \\ &\quad + \mathcal{L}U(x(t), x(t-\delta(t)), r(t), t) \\ &\quad + \theta\tau\Big[\tau |f(x(t), x(t-\delta(t)), r(t), t)|^2 \\ &\quad + |g(x(t), x(t-\delta(t)), r(t), t)|^2\Big] \\ &\quad - \theta \int_{t-\tau}^{t} \Big[\tau |f(x(v), x(v-\delta(v)), r(v), v)|^2 \\ &\quad + |g(x(v), x(v-\delta(v)), r(v), v)|^2\Big] dv, \end{aligned} \tag{3.3}$$

in which $\mathcal{L}U : R^n \times R^n \times S \times R_+ \to R$ is defined by

$$\begin{aligned} \mathcal{L}U(x, y, i, t) &= U_t(x, i, t) + U_x(x, i, t) f(x, x, i, t) \\ &\quad + \frac{1}{2}\text{trace}[g^T(x, y, i, t) U_{xx}(x, i, t) g(x, y, i, t)] \\ &\quad + \sum_{j=1}^{N} \gamma_{ij} U(x, j, t). \end{aligned} \tag{3.4}$$

This lemma can be proved easily by the generalized Itô formula (see, e.g., Mao, 2002; Mao & Yuan, 2006) and the fundamental theory of calculus but the details are omitted due to the page limit. To study the delay-dependent asymptotic stability of the SDDE (2.1), we need to impose a couple of new assumptions.

Assumption 3.2. Assume that there are functions $U \in C^{2,1}(R^n \times S \times R_+; R_+)$, $U_1, U_2 \in C(R^n \times [-\tau, \infty); R_+)$, and positive numbers α_k $(k = 1, 2, 3, 4)$ and β_j $(j = 1, 2, 3)$ such that

$$\alpha_2 < \alpha_1(1-\bar{\delta}), \qquad \alpha_4 \leq \alpha_3(1-\bar{\delta}), \tag{3.5}$$

and

$$\begin{aligned} &\mathcal{L}U(x, y, i, t) + \beta_1 |U_x(x, i, t)|^2 \\ &\quad + \beta_2 |f(x, y, i, t)|^2 + \beta_3 |g(x, y, i, t)|^2 \\ &\leq -\alpha_1 U_1(x, t) + \alpha_2 U_1(y, t-\delta(t)) \\ &\quad - \alpha_3 U_2(x, t) + \alpha_4 U_2(y, t-\delta(t)), \end{aligned} \tag{3.6}$$

for all $(x, y, i, t) \in R^n \times R^n \times S \times R_+$.

Assumption 3.3. Assume that there exists a positive number β_4 such that

$$|f(x, x, i, t) - f(x, y, i, t)| \leq \beta_4 |x-y| \tag{3.7}$$

for all $(x, y, i, t) \in R^n \times R^n \times S \times R_+$.

Theorem 3.4. *Let Assumptions 2.1, 2.2, 3.2 and 3.3 hold. Assume also that*

$$\tau \leq \frac{\sqrt{2\beta_1\beta_2}}{\beta_4} \wedge \frac{2\beta_1\beta_3}{\beta_4^2}. \tag{3.8}$$

Then for any given initial data (2.2), the solution of the SDDE (2.1) has the properties that

$$\int_0^\infty \mathbb{E}U_1(x(t), t)dt < \infty \tag{3.9}$$

and

$$\sup_{0 \leq t < \infty} \mathbb{E}U(x(t), r(t), t) < \infty. \tag{3.10}$$

Proof. Fix the initial data $\xi \in C([-\tau, 0]; R^n)$ and $r_0 \in S$ arbitrarily. Let $k_0 > 0$ be a sufficiently large integer such that $\|\xi\| < k_0$. For each integer $k \geq k_0$, define the stopping time

$$\sigma_k = \inf\{t \geq 0 : |x(t)| \geq k\},$$

where throughout this paper we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). It is easy to see that σ_k is increasing as $k \to \infty$ and, by Theorem 2.3, $\lim_{k\to\infty} \sigma_k = \infty$ a.s. By the generalized Itô formula, we obtain from Lemma 3.1 that

$$\mathbb{E}V(\hat{x}_{t\wedge\sigma_k}, \hat{r}_{t\wedge\sigma_k}, t\wedge\sigma_k) = V(\hat{x}_0, \hat{r}_0, 0) + \mathbb{E}\int_0^{t\wedge\sigma_k} LV(\hat{x}_s, \hat{r}_s, s)ds \tag{3.11}$$

for any $t \geq 0$ and $k \geq k_0$.

We now let $\theta = \beta_4^2/(2\beta_1)$. (Please recall that θ is the free parameter in the definition of the Lyapunov functional.) By Assumption 3.2 and condition (3.8), it is easy to show from Lemma 3.1 that

$$\begin{aligned} &LV(\hat{x}_s, \hat{r}_s, s) \\ &\leq -\alpha_1 U_1(x(s), s) + \alpha_2 U_1(x(s-\delta(s)), s-\delta(s)) \\ &\quad - \alpha_3 U_2(x(s), s) + \alpha_4 U_2(x(s-\delta(s)), s-\delta(s)) \\ &\quad + \frac{\beta_4^2}{4\beta_1}|x(s) - x(s-\delta(s))|^2 \\ &\quad - \frac{\beta_4^2}{2\beta_1}\int_{s-\tau}^{s}\Big[\tau |f(x(v), x(v-\delta(v)), r(v), v)|^2 \\ &\quad + |g(x(v), x(v-\delta(v)), r(v), v)|^2\Big] dv. \end{aligned}$$

Substituting this into (3.11) implies

$$\mathbb{E}V(\hat{x}_{t\wedge\sigma_k},\hat{r}_{t\wedge\sigma_k},t\wedge\sigma_k) \le V(\hat{x}_0,\hat{r}_0,0)+J_1+J_2+J_3-J_4, \tag{3.12}$$

where

$$J_1=\mathbb{E}\int_0^{t\wedge\sigma_k}\big[-\alpha_1U_1(x(s),s) + \alpha_2U_1(x(s-\delta(s)),s-\delta(s))\big]ds,$$

$$J_2=\mathbb{E}\int_0^{t\wedge\sigma_k}\big[-\alpha_3U_2(x(s),s) + \alpha_4U_2(x(s-\delta(s)),s-\delta(s))\big]ds,$$

$$J_3=\frac{\beta_4^2}{4\beta_1}\mathbb{E}\int_0^{t\wedge\sigma_k}|x(s)-x(s-\delta(s))|^2ds,$$

$$J_4=\frac{\beta_4^2}{2\beta_1}\mathbb{E}\int_0^{t\wedge\sigma_k}\Big(\int_{s-\tau}^{s}\Big[\tau|f(x(v),x(v-\delta(v)),r(v),v)|^2 + |g(x(v),x(v-\delta(v)),r(v),v)|^2\Big]dv\Big)ds.$$

It is straightforward to show that

$$J_1\le\frac{\alpha_2}{1-\bar{\delta}}\int_{-\tau}^{0}U_1(\xi(v),v)dv - \bar{\alpha}\mathbb{E}\int_0^{t\wedge\sigma_k}U_1(x(s),s)ds, \tag{3.13}$$

and

$$J_2\le\frac{\alpha_4}{1-\bar{\delta}}\int_{-\tau}^{0}U_2(\xi(v),v)dv, \tag{3.14}$$

where $\bar{\alpha}=\alpha_1-\alpha_2/(1-\bar{\delta})>0$ by condition (3.5). Substituting these two inequalities into (3.12) yields

$$\bar{\alpha}\mathbb{E}\int_0^{t\wedge\sigma_k}U_1(x(s),s)ds\le C_1+J_3-J_4, \tag{3.15}$$

where C_1 is a positive constant dependent on the initial data. Applying the well-known Fatou lemma (see, e.g., Loeve, 1955) and recalling the paragraph below Theorem 2.3, we can let $k\to\infty$ in (3.15) to obtain

$$\bar{\alpha}\mathbb{E}\int_0^{t}U_1(x(s),s)ds\le C_1+\bar{J}_3-\bar{J}_4, \tag{3.16}$$

where

$$\bar{J}_3=\frac{\beta_4^2}{4\beta_1}\mathbb{E}\int_0^{t}|x(s)-x(s-\delta(s))|^2ds,$$

$$\bar{J}_4=\frac{\beta_4^2}{2\beta_1}\mathbb{E}\int_0^{t}\Big(\int_{s-\tau}^{s}\Big[\tau|f(x(v),x(v-\delta(v)),r(v),v)|^2 + |g(x(v),x(v-\delta(v)),r(v),v)|^2\Big]dv\Big)ds.$$

But, by the well-known Fubini theorem (see, e.g., Loeve, 1955),

$$\bar{J}_3=\frac{\beta_4^2}{4\beta_1}\int_0^{t}\mathbb{E}|x(s)-x(s-\delta(s))|^2ds.$$

For $t\in[0,\tau]$, we clearly have

$$\bar{J}_3\le\frac{\tau\beta_4^2}{\beta_1}\Big(\sup_{-\tau\le v\le\tau}\mathbb{E}|x(v)|^2\Big)=:C_2,$$

where, as usual, =: means 'denoted by'. For $t>\tau$, we have

$$\bar{J}_3\le C_2+\frac{\beta_4^2}{4\beta_1}\int_\tau^{t}\mathbb{E}|x(s)-x(s-\delta(s))|^2ds.$$

But, it follows from the SDDE (2.1) that, for $s\ge\tau$,

$$\mathbb{E}|x(s)-x(s-\delta(s))|^2 \le 2\mathbb{E}\int_{s-\tau}^{s}\Big(\tau|f(x(v),x(v-\delta(v)),r(v),v)|^2 + |g(x(v),x(v-\delta(v)),r(v),v)|^2\Big)dv.$$

We hence easily obtain that $\bar{J}_3=C_2+\bar{J}_4$ for $t>\tau$. In other words, we always have

$$\bar{J}_3\le C_2+\bar{J}_4,\quad\forall t\ge0. \tag{3.17}$$

Substituting this into (3.16) yields

$$\bar{\alpha}\mathbb{E}\int_0^{t}U_1(x(s),s)ds\le C_1+C_2.$$

Letting $t\to\infty$ gives

$$\mathbb{E}\int_0^{\infty}U_1(x(s),s)ds\le(C_1+C_2)/\bar{\alpha}.$$

This, along with the Fubini theorem, implies the assertion (3.9). Similarly, we see from (3.12) that

$$\mathbb{E}U(x(t\wedge\sigma_k),r(t\wedge\sigma_k),t\wedge\sigma_k)\le C_1+J_3-J_4. \tag{3.18}$$

Letting $k\to\infty$ and using (3.17), we get

$$\mathbb{E}U(x(t),r(t),t)\le C_1+C_2,$$

which implies the other assertion (3.10) as the above inequality holds for any $t\ge0$. The proof is therefore complete. □

The assertions of Theorem 3.4 are in terms of the Lyapunov functions U and U_1. If we have a slightly more information on these Lyapunov functions, we can get some familiar stability results, e.g., H_∞-stability, as described in the following corollary.

Corollary 3.5. *Let the conditions of Theorem 3.4 hold. If there also exist positive constants c and p such that*

$$c|x|^p\le U_1(x,t),\quad\forall(x,t)\in R^n\times R_+, \tag{3.19}$$

then for any given initial data (2.2), the solution of the SDDE (2.1) satisfies

$$\int_0^\infty\mathbb{E}|x(t)|^pdt<\infty. \tag{3.20}$$

That is, the SDDE (2.1) is H_∞-stable in L^p.

This corollary follows from Theorem 3.4 obviously. In general, it does not follow from (3.20) that $\lim_{t\to\infty}\mathbb{E}(|x(t)|^p)=0$. However, this is true provided $\mathbb{E}|x(t)|^p$ is uniformly continuous in t. We can achieve this with an additional condition on the parameters p, q_1, q_2 and q as described in the following theorem.

Theorem 3.6. *Let the conditions of Corollary 3.5 hold. If, moreover,*

$$p\ge2\quad\text{and}\quad(p+q_1-1)\vee(p+2q_2-2)\le q,$$

(please recall that q_1, q_2 and q were specified in Assumptions 2.1 and 2.2, respectively), then the solution of the SDDE (2.1) satisfies that $\lim_{t\to\infty}\mathbb{E}|x(t)|^p=0$ for any initial data (2.2). That is, the SDDE (2.1) is asymptotically stable in L^p.

Proof. Again, fix the initial data (2.2) arbitrarily. For any $0 \le t_1 < t_2 < \infty$, by the Itô formula, we have

$$\begin{aligned}\mathbb{E}|x(t_2)|^p - \mathbb{E}|x(t_1)|^p &= \mathbb{E}\int_{t_1}^{t_2}\Big(p|x(t)|^{p-2}x^T(t)f(x(t),x(t-\delta(t)),r(t),t) \\ &\quad + \frac{p}{2}|x(t)|^{p-2}|g(x(t),x(t-\delta(t)),r(t),t)|^2 \\ &\quad + \frac{p(p-2)}{2}|x(t)|^{p-4}|x^T(t)g(x(t),x(t-\delta(t)),r(t),t)|^2\Big)dt.\end{aligned}$$

This implies

$$\begin{aligned}\big|\mathbb{E}|x(t_2)|^p - \mathbb{E}|x(t_1)|^p\big| &\le \mathbb{E}\int_{t_1}^{t_2}\Big(p|x(t)|^{p-1}|f(x(t),x(t-\delta(t)),r(t),t)| \\ &\quad + \frac{p(p-1)}{2}|x(t)|^{p-2}|g(x(t),x(t-\delta(t)),r(t),t)|^2\Big)dt.\end{aligned}$$

By the polynomial growth condition (2.4), we then have

$$\begin{aligned}\big|\mathbb{E}|x(t_2)|^p - \mathbb{E}|x(t_1)|^p\big| &\le \mathbb{E}\int_{t_1}^{t_2}\Big(pK|x(t)|^{p-1}\big[1+|x(t)|^{q_1}+|x(t-\delta(t))|^{q_1}\big] \\ &\quad + \frac{3p(p-1)K^2}{2}|x(t)|^{p-2}\big[1+|x(t)|^{2q_2}+|x(t-\delta(t))|^{2q_2}\big]\Big)dt.\end{aligned}$$

Using the inequalities

$$|x(t)|^{p-1}|x(t-\delta(t))|^{q_1} \le |x(t)|^{p+q_1-1} + |x(t-\delta(t))|^{p+q_1-1},$$
$$|x(t)|^{p-1} \le 1 + |x(t)|^q$$

etc., we further obtain

$$|\mathbb{E}|x(t_2)|^2 - \mathbb{E}|x(t_1)|^2| \le C_3(t_2-t_1),$$

where

$$C_3 = 4\Big(pK + \frac{3p(p-1)K^2}{2}\Big)\Big(1+\sup_{-\tau\le t<\infty}\mathbb{E}|x(t)|^q\Big) < \infty.$$

That is, $\mathbb{E}|x(t)|^p$ is uniformly continuous in t on R_+. It then follows from (3.20) that $\lim_{t\to\infty}\mathbb{E}|x(t)|^p = 0$ as required. □

4. An example

In this section we will only be able to discuss an example to illustrate our theory due to the page limit. Although our example is a scalar hybrid SDDE, it will illustrate our theory very well.

Example 4.1. Let us return to the SDDE (1.1) with the coefficients defined by (1.3). Recall that $r(t)$ is a Markov chain with its state space $S = \{1,2\}$ and the generator Γ given by (1.2). Let us now consider the general variable delay $\delta(t)$ that satisfies the conditions imposed in Section 2 and assume $\bar\delta = 0.2$. Clearly, the coefficients defined by (1.3) satisfy Assumption 2.1 with $q_1 = 3$ and $q_2 = 2$. To verify Assumption 2.2, we set $q = 6$ and define $\bar U(x,i,t) = |x|^6$ for $(x,i,t) \in R\times S\times R_+$. It is easy to show that

$$\begin{aligned}\mathbb{L}\bar U(x,y,1,t) &= 6x^5(-y-3x^3) + 15x^4y^4 \\ &\le 5x^6 + y^6 - 10.5x^8 + 7.5y^8 \qquad (4.1)\end{aligned}$$

and

$$\begin{aligned}\mathbb{L}\bar U(x,y,2,t) &= 6x^5(y-2x^3) + (15/4)x^4y^4 \\ &\le 5x^6 + y^6 - (12-15/32)x^8 + 7.5y^8. \qquad (4.2)\end{aligned}$$

We hence always have

$$\begin{aligned}\mathbb{L}\bar U(x,y,i,t) &\le 5x^6 + y^6 - 10.5x^8 + 7.5y^8 \\ &\le c_1 - 10(1+x^8) + 8(1+y^8) \qquad (4.3)\end{aligned}$$

for $(x,y,i,t) \in R\times R\times S\times R_+$, where

$$c_1 = \sup_{x,y\in R}\big[2 + 5x^6 + y^6 - 0.5(x^8+y^8)\big] < \infty.$$

Therefore, Assumption 2.2 is satisfied with $G(x,t) = 1 + x^8$, $c_2 = 10$ and $c_3 = 8$. By Theorem 2.3, we can first conclude that the SDDE (1.1) with the initial data (2.2) (replace R^n there by R of course) has the unique global solution $x(t)$ on $t \ge -\tau$ and the solution has the property that $\sup_{-\tau\le t<\infty}\mathbb{E}|x(t)|^6 < \infty$.

To apply our theorems established in the previous section, we need to verify assumptions imposed there. Let us do so one by one. To verify Assumption 3.2, we define

$$U(x,i,t) = \begin{cases} x^2 + x^4 & \text{if } i = 1, \\ 2x^2 + 3x^4 & \text{if } i = 2 \end{cases} \qquad (4.4)$$

for $(x,i,t) \in R\times S\times R_+$. By definition (3.4), it is straightforward to show

$$\mathcal{L}U(x,y,i,t) \le \begin{cases} -x^2 - 8x^4 + y^4 - 10x^6 + 4y^6 & \text{if } i = 1, \\ -4x^2 - 12x^4 + 0.5y^4 - 22x^6 + 4y^6 & \text{if } i = 2. \end{cases} \qquad (4.5)$$

Moreover,

$$|U_x(x,i,t)|^2 = \begin{cases} 4x^2 + 16x^4 + 16x^6 & \text{if } i = 1, \\ 16x^2 + 96x^4 + 144x^6 & \text{if } i = 2; \end{cases} \qquad (4.6)$$

$$|f(x,y,i,t)|^2 = \begin{cases} |y+3x^3|^2 \le 2y^2 + 18x^6 & \text{if } i = 1, \\ |y-2x^3|^2 \le 2y^2 + 8x^6 & \text{if } i = 2; \end{cases} \qquad (4.7)$$

$$|g(x,y,i,t)|^2 = \begin{cases} y^4 & \text{if } i = 1, \\ 0.25y^4 & \text{if } i = 2. \end{cases} \qquad (4.8)$$

Setting

$$\beta_1 = 0.1, \qquad \beta_2 = 0.2, \qquad \beta_3 = 1, \qquad (4.9)$$

and using (4.5)–(4.8), we can then show that

$$\begin{aligned}&\mathcal{L}U(x,y,i,t) + \beta_1|U_x(x,i,t)|^2 + \beta_2|f(x,y,i,t)|^2 + \beta_3|g(x,y,i,t)|^2 \\ &\le \begin{cases} -0.6x^2 + 0.4y^2 - 6.4x^4 + 2y^4 - 4.8x^6 + 4y^6 \\ \quad \text{if } i = 1, \\ -2.4x^2 + 0.4y^2 - 2.4x^4 + 0.75y^4 - 6x^6 + 4y^6 \\ \quad \text{if } i = 2. \end{cases} \qquad (4.10)\end{aligned}$$

This implies

$$\begin{aligned}&\mathcal{L}U(x,y,i,t) + \beta_1|U_x(x,i,t)|^2 + \beta_2|f(x,y,i,t)|^2 + \beta_3|g(x,y,i,t)|^2 \\ &\le -0.6x^2 + 0.4y^2 - 2.4x^4 + 2y^4 - 4.8x^6 + 4y^6. \\ &\le -4.8(0.1x^2 + x^6) + 4(0.1y^2 + y^6) - 2.4x^4 + 2y^4. \qquad (4.11)\end{aligned}$$

Letting

$$\begin{aligned}&U_1(x,t) = 0.1x^2 + x^6, \qquad U_2(x,t) = x^4, \\ &\alpha_1 = 4.8, \quad \alpha_2 = 4, \quad \alpha_3 = 2.4, \quad \alpha_4 = 2, \qquad (4.12)\end{aligned}$$

we get condition (3.6). Moreover, it is easy to check that condition (3.5) holds as well. In other words, Assumption 3.2 is satisfied. Noting that

$$|f(x,x,t,i) - f(x,y,t,i)| \le |x-y|, \qquad (4.13)$$

we see that Assumption 3.3 is satisfied with $\beta_4 = 1$. Furthermore, condition (3.8) becomes

$$\tau \leq 0.2. \tag{4.14}$$

By Theorem 3.4, we can therefore conclude that the solution of the SDDE (1.1) has the properties that $\int_0^\infty (x^2(t) + x^6(t))dt < \infty$ a.s. and $\int_0^\infty \mathbb{E}(x^2(t) + x^6(t))dt < \infty$. Moreover, as $|x(t)|^p \leq x^2(t) + x^6(t)$ for any $p \in [2, 6]$, we have

$$\int_0^\infty \mathbb{E}|x(t)|^p dt < \infty. \tag{4.15}$$

Recalling $q_1 = 3$, $q_2 = 2$ and $q = 6$, we see that for $p = 4$, all the conditions of Theorem 3.6 are satisfied and hence we have

$$\lim_{t\to\infty} \mathbb{E}|x(t)|^4 dt = 0. \tag{4.16}$$

5. Conclusion

In this paper we have established the new theory on the delay-dependent stability criteria for highly nonlinear hybrid SDDEs. The stabilities discussed in this paper include the H_∞ stability in L^p and asymptotic stability in L^p. The key feature of our paper is that the coefficients of the underlying SDDEs are no longer bounded by linear functions while all the existing delay-dependent stability criteria could only be applied to the hybrid SDDEs satisfying the linear growth condition. Our new theory is therefore a breakthrough in the stability study of highly nonlinear hybrid SDDEs.

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