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Large butterfly Cayley graphs and digraphs

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Abstract

We present families of large undirected and directed Cayley graphs whose construction is related to butterfly networks. One approach yields, for every large $k$ and for values of $d$ taken from a large interval, the largest known Cayley graphs and digraphs of diameter $k$ and degree $d$. Another method yields, for sufficiently large $k$ and infinitely many values of $d$, Cayley graphs and digraphs of diameter $k$ and degree $d$ whose order is exponentially larger in $k$ than any previously constructed. In the directed case, these are within a linear factor in $k$ of the Moore bound.

1 Introduction

The goal of the degree–diameter problem is to determine the largest possible order of a graph or digraph, perhaps restricted to some special class, with given maximum (out)degree and diameter. For an overview of progress on a wide variety of approaches to this problem, see the survey by Miller & Širáň [6].

Our concern here is with large Cayley graphs and digraphs. Recall that, for a group $G$ and a unit-free generating subset $S$ of $G$, the Cayley digraph of $G$ generated by $S$ has vertex set $G$ and a directed edge from $g$ to $gs$ for all $g \in G$ and $s \in S$. If $S$ is symmetric, i.e. $S = S^{-1}$, then the corresponding undirected simple graph is the Cayley graph of $G$ generated by $S$. The Cayley (di)graph is thus regular of (out)degree $|S|$ and vertex-transitive.

We are interested in graphs and digraphs of degree $d$ and diameter $k$, for arbitrary large $k$ and varying $d$. If a construction yields graphs of order $n_{d,k}$, we say that it has asymptotic order $f(d, k)$ if, for fixed $k$,

$$\lim_{d \to \infty} \frac{n_{d,k}}{f(d, k)} = 1.$$ 

No graph or digraph can be larger than the corresponding Moore bound. For undirected graphs, this bound is $M_{d,k} = 1 + \frac{d}{d-2}((d-1)^k - 1)$ if $d > 2$. In the directed case, it is $DM_{d,k} = \frac{1}{d-1}(d^{k+1} - 1)$ if $d > 1$. In both cases, the Moore bound has asymptotic order $d^k$.

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Previously, for arbitrary degree and diameter, the largest known directed Cayley graphs were obtained by Vetrik [7] and Abas & Vetrik [1], whose constructions have asymptotic order $k\left(\frac{4}{3}\right)^k$ for even $k$, and $2k\left(\frac{4}{3}\right)^k$ for odd $k$. Our construction yields Cayley digraphs whose order is asymptotically $kd^{k-1}$. For fixed diameter $k \geq 8$, these digraphs are larger than those in [7] and [1] for every value of $d$ in a large interval. We also construct, for fixed $k$ and infinitely many values of $d$, Cayley digraphs whose asymptotic order is $\frac{d^k}{\sqrt{e}}$, a factor of $\frac{2^{k-1}}{e^{1/2}}$ larger than those of Abas & Vetrik, and within a linear factor in $k$ of the Moore bound.

The undirected case is similar. Previously, the largest known Cayley graphs were obtained by Macbeth, Šiagiová, Širáň & Vetrik [5], whose construction has asymptotic order $k\left(\frac{4}{3}\right)^k$. For $d - k \not\equiv 3 \pmod{4}$, we construct Cayley graphs whose order is asymptotically $k\left(\frac{4}{3}\right)^{k-1}$. For sufficiently large diameter $k$, these graphs are larger than those in [5] for every suitable value of $d$ in a large interval. We also construct, for given $k$ and infinitely many values of $d$, Cayley graphs whose asymptotic order is $\frac{1}{e^3}\left(\frac{4}{3}\right)^k$, a factor of $\frac{1}{e}\left(\frac{3}{2}\right)^k$ larger than those in [5].

Our constructions are based on a two-parameter family of groups. For $t \geq 2$, let $\mathbb{Z}_t = \mathbb{Z}/t\mathbb{Z}$ be the additive group of integers modulo $t$, and for $r \geq 2$, let $\mathbb{Z}_t^r$ denote the product $\mathbb{Z}_t \times \ldots \times \mathbb{Z}_t$, where $\mathbb{Z}_t$ occurs $r$ times, considered as an additive group of vectors. Let $\alpha$ be the automorphism of $\mathbb{Z}_t^r$, defined by $\alpha(v_0, \ldots, v_{r-1}) = (v_{r-1}, v_0, \ldots, v_{r-2})$, that cyclically shifts coordinates rightwards by one, and consider the semidirect product $G = \mathbb{Z}_t^r \rtimes \mathbb{Z}_r$, of order $rt^r$, with the group operation given by $(u, s)(v, s') = (u + \alpha^s(v), s + s')$, for $u, v \in \mathbb{Z}_t^r$ and $s, s' \in \mathbb{Z}_r$. We write elements of $G$ in the form $(v_0, \ldots, v_{r-1}; s)$, where each $v_i \in \mathbb{Z}_t$ and $s \in \mathbb{Z}_r$. Using this notation, the group operation is

\[
(u_0, \ldots, u_{r-1}; s)(v_0, \ldots, v_{r-1}; s') = (u_0 + v_{r-s}, \ldots, u_{s-1} + v_{r-1}, u_s + v_0, \ldots, u_{r-1} + v_{r-1-s}; s + s'),
\]

arithmetic in the subscripts being performed modulo $r$. The group $G$ is used to create all our Cayley graphs and digraphs.

The Cayley digraph generated by elements of $G$ of the form $(a, 0, \ldots, 0; 1)$, $a \in \mathbb{Z}_t$ is isomorphic to the base-$t$ order-$r$ (wrapped) butterfly network, $B_t(r)$, so called because it is composed of $rt^{r-1}$ edge-disjoint $t$-butterflies (copies of the complete bipartite graph $K_{t,t}$); see [2, Figure 2]. Butterfly networks are closely related to the de Bruijn graphs [3], the directed base-$t$ order-$r$ de Bruijn graph being a coset graph of $B_t(r)$ [2, Theorem 4.4].

Cayley graphs and digraphs of $G$ were used previously by Macbeth, Šiagiová, Širáň & Vetrik [5] and Vetrik [7] in the constructions mentioned above, though in neither case is the connection to the butterfly networks made explicit. Each of our results is a consequence of choosing an appropriate set of generators for $G$. We make use of two distinct constructions.
2 The first construction

We present the directed case first, since it is slightly simpler.

**Theorem 1.** For any \( k \geq 4 \) and \( d \geq k - 1 \), there exist Cayley digraphs that have diameter \( k \), outdegree \( d \), and order \((k - 1)(d - k + 3)^{k-1}\).

**Proof.** Let \( r = k - 1 \) and \( t = d - k + 3 \), and let the underlying group of the Cayley digraph be \( G = \mathbb{Z}_r^* \times \mathbb{Z}_r \). The order of \( G \) is \( rt^r = (k - 1)(d - k + 3)^{k-1} \).

As generators for the Cayley digraph we use the \( t \) shifts and add elements \((a, 0, \ldots, 0; 1), \) for each \( a \in \mathbb{Z}_t \), together with the remaining \( r - 2 \) nonzero cyclic shift elements \((0, \ldots, 0; s)\), for \( 2 \leq s \leq r - 1 \). Thus the digraph has outdegree \( t + r - 2 = d \).

It also has diameter \( r + 1 = k \). Every element is the product of \( r \) shift and add operations (establishing the vector) and possibly a single cyclic shift (to establish the final shift value if it is nonzero). On the other hand, if \( s \neq 0 \) then \((1, \ldots, 1; s)\) cannot be obtained as a product of fewer than \( k \) generators.

Clearly, the butterfly network \( B_t(r) \) is a subdigraph of the Cayley digraph of Theorem 1. The additional edges in our construction, corresponding to the cyclic shift elements, consist of \( t^r \) vertex-disjoint copies of the complete digraph on \( r \) vertices with a directed \( r \)-cycle removed.

Vetrík [7] presents, for any \( k \geq 3 \) and \( d \geq 4 \), a family of Cayley digraphs of diameter \( k \), degree \( d \), and order \( k|\frac{d}{2}|^k \). For odd diameters, Abas & Vetrík [1] improve this result by a factor of two, constructing Cayley digraphs of diameter at most \( k \) and degree \( d \) of order \( 2k|\frac{d}{2}|^k \). Clearly, for large enough \( d \), these digraphs are bigger than those of Theorem 1. However, for any given diameter \( k \geq 8 \), it can be confirmed (using a computer algebra system, or otherwise) that the digraphs of Theorem 1 are larger than those of Vetrík and Abas & Vetrík if

\[
2k + 2\ln k < d < 2^{k-1}(1 - \frac{1}{k}) - k^2.
\]

For specific values of the degree, we can do much better. If we set \( d = k^2 - 3k \), then the digraphs of Theorem 1 have orders at least \( DK_{d,k}/ek \), within a linear factor of the Moore bound, and exceeding those of Abas & Vetrík by a factor of at least \( 2^{k-1}/ek^2 \), which exceeds 1 for \( k \geq 9 \).

For the undirected case, we simply add elements to the generating set to make it symmetric.

**Theorem 2.** For any \( k \geq 5 \) and \( d \geq k \) such that \( d - k \not\equiv 3 \) (mod 4), there exist Cayley graphs that have diameter \( k \), degree \( d \), and order \((k - 1)(\lfloor \frac{d-k}{2} \rfloor + 2)^{k-1}\).

**Proof.** Let \( r = k - 1 \) and \( t = \lfloor \frac{d-k}{2} \rfloor + 2 \), and let \( G = \mathbb{Z}_r^* \times \mathbb{Z}_r \). As generators for the Cayley graph of \( G \) we use the \( t \) elements \((a, 0, \ldots, 0; 1), \) along with their inverses \((0, \ldots, 0, -a; -1)\), and the remaining \( r - 3 \) nonzero elements \((0, \ldots, 0; s)\) for \( 2 \leq s \leq r - 2 \). In addition, if \( d - k \equiv 1 \) (mod 4), in which case \( t \) is even, then the involution \((0, \ldots, 0, \frac{k}{2}; 0)\) is also included as a generator.
Thus the graph has degree $2t + r - 3 + (d - k \mod 2) = d$. As in the directed case, it has diameter $r + 1 = k$. Every element is the product of $k - 1$ shift and add operations and possibly a single cyclic shift. On the other hand, if $s \not\in \{-1, 0, 1\}$ then $\{1, \ldots, 1; s\}$ cannot be obtained as a product of fewer than $k$ generators, and $G$ has such an element since $t \geq 4$. \hfill \Box

Macbeth, Šiagiová, Širáň & Vetřík [5] present, for any $k \geq 3$ and $d \geq 5$, a family of Cayley graphs with diameter at most $k$, degree $d$, and order no greater than $k(\frac{d+1}{2})^k$. Their constructions also use the group $G$, with a different generating set. For large enough $d$, these graphs are bigger than those of Theorem 2. However, for $k \geq 27$, the graphs of Theorem 2 are larger than those of Macbeth, Šiagiová, Širáň & Vetřík for any $d - k \not\equiv 3 \pmod{4}$ satisfying

$$3k + 6 \ln k < d < 2\left(\frac{3}{2}\right)^k(1 - \frac{1}{k}) - k^2.$$ For specific values of the degree, we can do much better. If we set $d = k^2 - 2k$, then the graphs of Theorem 2 have orders exceeding those in [5] by a factor of at least $\frac{2}{e\pi^2}(\frac{3}{2})^k$, which exceeds 1 for $k \geq 14$.

### 3 The second construction

In our second construction, we conceive of the vectors of length $r$ as being partitioned into $k - 1$ long blocks, each of length $\ell$, and a single short block, of length $m$.

Again, the directed case is presented first, since it is simpler.

**Theorem 3.** For any $k, \ell, t \geq 2$ and positive $m < \ell$, there exist Cayley digraphs that have diameter $k$, outdegree $t^\ell + (r - 1)t^m - 1$, and order $rt^r$, where $r = (k - 1)\ell + m$.

**Proof.** As before, let $G = \mathbb{Z}_t^\ell \times \mathbb{Z}_r$, of order $rt^r$. As generators for the Cayley digraph, we use the $t^\ell$ long elements $(a_1, \ldots, a_{\ell}, 0, \ldots, 0; \ell)$, $a_i \in \mathbb{Z}_t$, together with the additional $(r - 1)t^m - 1$ nonzero short elements $(a_{1i}, \ldots, a_{mi}, 0, \ldots, 0; s)$, $a_i \in \mathbb{Z}_t$, $s \neq \ell$. Thus the digraph has outdegree $t^\ell + (r - 1)t^m - 1$. Long elements shift by $\ell$ and modify a long block; short elements shift arbitrarily and modify a short block.

The digraph has diameter $k$. Every element is the product of a single short element (establishing $m$ components of the vector and guaranteeing the final shift value) and $k - 1$ long elements (establishing the remaining $(k - 1)\ell = r - m$ components of the vector). On the other hand, $(1, \ldots, 1; 0)$ cannot be obtained as a product of fewer than $k$ generators. \hfill \Box

The Cayley digraph of Theorem 3 contains both of the butterfly networks $B_{t^{\ell}}(r)$ and $B_{t^m}(r)$ as subdigraphs. Its edges can be partitioned into $rt^{r-\ell}$ copies of the $t^\ell$-butterfly, from the long elements, $rt(r - 2)t^{r-m}$ copies of the $t^m$-butterfly, from the short elements that have nonzero shift, and a collection of directed cycles from the short elements with zero shift.

1The graphs in [5] are slightly larger than those of Macbeth, Šiagiová & Širáň [4], whose order is at most $k(\frac{d+1}{2})^k - k$. 

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Given $k$, $\ell$ and $t$, for judicious choice of $m$, these digraphs are larger than those of Abas & Vetrík [1]. For example, if we let $t = 2$, then for all $k \geq 31$ and sufficiently large $\ell$, the order of our digraphs is greater than that of those in [1] if

$$\ell - k - \log_2 \ell + 2 < m < \ell - \log_2 k\ell - \frac{2}{k}(\log_2 k + 2).$$

If $m$ is chosen optimally, we can do much better than that.

**Corollary 4.** *For any $k \geq 3$, there are arbitrarily large values of $d$ for which there exist Cayley digraphs that have diameter $k$, outdegree $d$, and order at least $\frac{1}{k} \left( \frac{k}{k+2} (d + 1) \right)^k$.***

**Proof.** We use the construction of Theorem 3. Let $t = 2$, and let $\ell$ be any sufficiently large positive integer such that $\log_2 k^2 \ell \leq \frac{3}{4} \ell$. Let $r = [k\ell - \log_2 k^2 \ell]$, and $m = r - (k - 1)\ell$, so $r = (k - 1)\ell + m$. Note that $0 < m < \ell$.

The digraph has diameter $k$ and order $r2^\ell$, which (rounding $r$ down) is at least

$$n_0 = (k\ell - \log_2 k^2 \ell)2^{k\ell - \log_2 k^2 \ell} = \left( \frac{1}{k} - \frac{\log_2 k^2 \ell}{k^2 \ell} \right)^{2^k}.$$

Its degree is $d = 2^\ell + (r - 1)2^m - 1$, which (substituting for $m$ and rounding $r$ up) is less than

$$d^+ = 2^\ell + (k\ell - \log_2 k^2 \ell)2^{k\ell - \log_2 k^2 \ell + 1 - (k - 1)\ell} - 1 = \left( 1 + \frac{2}{k} - \frac{2\log_2 k^2 \ell}{k^2 \ell} \right)2^\ell - 1.$$

Let $\theta = \frac{\log_2 k^2 \ell}{k\ell}$. Note that the condition on $\ell$ implies that $0 \leq \frac{2}{k} \leq \frac{1}{4}$, since $k \geq 3$.

Now,

$$kn_0 \left( \frac{k}{k+2} (d^+ + 1) \right)^{-k} = (1 - \theta)^k \left( 1 + \frac{2\theta}{k + 2 - 2\theta} \right)^k > (1 - \theta)^k \left( 1 + \frac{2k\theta}{k + 2 - 2\theta} \right),$$

which is at least 1 if $k \geq 2$ and $0 \leq \theta \leq \frac{k - 2}{2k - 2}$. Since $k \geq 3$ and $\theta \leq \frac{1}{4}$, the result follows.

These digraphs have asymptotic order exceeding $\frac{d^k}{e^{\ell^k}}$, a factor of $\frac{2^{k-1}}{e^{\ell^k}}$ larger than those of Abas & Vetrík, and within a linear factor in $k$ of the Moore bound.

It is worth briefly explaining the choice of values for $t$ and $r$ in the proof of Corollary 4. Suppose we fix $t$ and $r$ (and hence the order $rt^t$), and also fix the diameter $k$. What is the optimal choice for $\ell$, that minimises the degree $t^\ell + (r - 1)t^{r - (k - 1)\ell} - 1$? Differentiating with respect to $\ell$ and equating to zero yields $\ell = \frac{1}{k}(r + \log_t(k - 1)(r - 1))$. Solving for $r$ then gives

$$r = \frac{1}{\ln t} W\left( \frac{t^{k\ell - 1} \ln t}{k - 1} \right) + 1,$$

where $W$ is the *Lambert W function*, defined implicitly by $W(z)e^{W(z)} = z$. Asymptotically, $W(z) = \ln z - \ln \ln z + o(1)$. Applying this approximation for $W$ then yields $r \approx k\ell - \log_t k^2 \ell$. 

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Using this value for \( r \) results in a digraph whose order is asymptotically at least \( \frac{1}{\ell} \left( \frac{k}{k + \ell} (d + 1) \right)^k \). Setting \( t = 2 \) makes this maximal.

The results in the undirected case are similar. As before, we just add elements to the generating set to make it symmetric.

**Theorem 5.** For any \( k, \ell, t \geq 2 \) and positive \( m < \ell \), there exist Cayley graphs that have diameter \( k \), degree \( 2t^\ell + (2r - 3)t^m - r \), and order \( rt^r \), where \( r = (k - 1)\ell + m \).

**Proof.** Let \( G = \mathbb{Z}_t^r \times \mathbb{Z}_r \). As generators for the Cayley graph of \( G \) with these parameters, we use:

- the \( t^\ell \) long elements \((a_1, \ldots, a_\ell, 0, \ldots, 0; t), a_i \in \mathbb{Z}_t\)
- their \( t^\ell \) inverses \((0, \ldots, 0, a_1, \ldots, a_\ell; -t)\)
- the \((r - 2)(t^m - 1)\) short elements \((a_1, \ldots, a_m, 0, \ldots, 0; s), a_i \in \mathbb{Z}_t\) not all zero, \( s \notin \{0, \ell\}\)
- their \((r - 2)(t^m - 1)\) inverses \((0, \ldots, 0, a_1, \ldots, a_m, 0, \ldots, 0; -s)\)
- the \( t^m - 1 \) nonzero short elements \((a_1, \ldots, a_m, 0, \ldots, 0; 0)\); this set is symmetric
- the \( r - 3 \) short elements \((0, \ldots, 0; s), s \notin \{0, \pm \ell\}; \) this set is also symmetric

Thus the graph has degree \( 2t^\ell + (2r - 3)t^m - r \). As in the directed case, it has order \( rt^r \) and diameter \( k \).

Given \( k, \ell \) and \( t \), for appropriate choice of \( m \), these graphs are larger than those of Macbeth, Šiagiová, Širáň & Vetrík [5]. For example, if we let \( t = 2 \), then for all \( k \geq 69 \) and sufficiently large \( \ell \), the order of our graphs is greater than that of those in [5] if

\[
\ell + k - \log_2 3^k \ell + 1 < m < \ell - \log_2 k\ell - \frac{\ell}{k} (\log_2 k + 2) - 1.
\]

If \( m \) is chosen optimally, we have the following.

**Corollary 6.** For any \( k \geq 3 \), there are arbitrarily large values of \( d \) for which there exist Cayley graphs that have diameter \( k \), degree \( d \), and order at least

\[
\frac{1}{k} \left( \frac{k}{2k + d} \left( d + k \log_2 \frac{d}{2} - \log_2 \log_2 d - \log_2 8k^2 \right) \right)^k.
\]

**Proof.** We use the construction of Theorem 5. As in the proof of Corollary 4, let \( t = 2 \), and let \( \ell \) be any sufficiently large positive integer such that \( \log_2 k^2 \ell \leq \frac{3}{4} \ell \). Let \( r = \lceil k\ell - \log_2 k^2 \ell \rceil \), and \( m = r - (k - 1)\ell \), so \( r = (k - 1)\ell + m \).

The graph has diameter \( k \) and order \( r2^r \), which is at least

\[
n_0 = \left( k\ell - \log_2 k^2 \ell \right) 2^{k\ell - \log_2 k^2 \ell} = \left( \frac{1}{k} - \frac{\log_2 k^2 \ell}{k^2 \ell} \right) 2^{k\ell}.
\]
Its degree is \( d = 2^\ell + 1 + (2r - 3)2^m - r \), which (substituting for \( m \) and rounding \( r \) up in the second term) is less than

\[
2^{\ell + 1} + (2k\ell - 2\log_2 k^2\ell - 1)2^{k\ell - \log_2 k^2\ell + 1 - (k - 1)\ell} - r = \left( 2 + \frac{4}{k} - \frac{1 + 4\log_2 k^2\ell}{k^2\ell} \right)2^\ell - r.
\]

Thus, \( \frac{1}{2}(d + r) \) is less than \( q = \left( 1 + \frac{2}{k} - \frac{2\log_2 k^2\ell}{k^2\ell} \right)2^\ell \), and by the argument in the proof of Corollary 4 (with \( q = d' + 1 \)), we know that \( kn_0 > \left( \frac{kq}{k+2} \right)^k > \left( \frac{k}{2k+4} (d + r) \right)^k \).

It remains to establish the appropriate lower bound for \( r \).

Now, \( kn_0 < 2^{k\ell} \) and \( q > \frac{d'}{2} \), so \( 2^\ell > \frac{kd}{2k+4} \) and thus \( \ell > \log_2 \frac{kd}{2k+4} = \log_2 \frac{d}{2} - \log_2 \left( 1 + \frac{2}{k} \right) \).

Since \( (1 + \frac{2}{k})^k < e^2 < 2^3 \), we have \( \log_2 \left( 1 + \frac{2}{k} \right) < \frac{3}{k} \) and thus \( \ell > \log_2 \frac{d}{2} - \frac{3}{k} \).

Now, \( r \geq k\ell - \log_2 k^2\ell \), so

\[
r > k\log_2 \frac{d}{2} - 3 - \log_2 k^2 - \log_2 \left( \log_2 \frac{d}{2} - \frac{3}{k} \right),
\]

which is greater than \( k\log_2 \frac{d}{2} - \log_2 d - \log_2 8k^2 \), as required.

These graphs have asymptotic order exceeding \( \frac{1}{e^2} \left( \frac{d}{2} \right)^k \), a factor of \( \frac{1}{e^2} \left( \frac{d}{2} \right)^k \) larger than those of Macbeth, Šiagiová, Širáň & Vetrík.

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