AAS 15-544

AN INTRUSIVE APPROACH TO UNCERTAINTY PROPAGATION IN ORBITAL MECHANICS BASED ON TCHEBYCHEFF POLYNOMIAL ALGEBRA

Annalisa Riccardi[*], Chiara Tardioli[†], Massimiliano Vasile[‡]

The paper presents an intrusive approach to propagate uncertainty in orbital mechanics. The approach is based on an expansion of the uncertain quantities in Tchebycheff series and a propagation through the dynamics using a generalised polynomial algebra. Tchebycheff series expansions offer a fast uniform convergence with relaxed continuity and smoothness requirements. The paper details the proposed approach and illustrates its applicability through a set of test cases considering both parameter and model uncertainties. This novel intrusive technique is then compared against its non-intrusive counterpart in terms of approximation accuracy and computational complexity.

INTRODUCTION

Intrusive approaches for uncertainty quantification are methods that require a modification of either the system model or the algebra used to evaluate the quantities of interest. Existing intrusive approaches in orbital mechanics use Taylor series expansions of the uncertain quantities and differential algebra to estimate the quantities of interest. A wide range of applications of the technique to celestial mechanics problem can be found in literature, this includes as an example, the solution of the two-point boundary value problem,[1] nonlinear propagation of uncertainties[2] and optimal feedback control.[3] In the case of propagation of uncertainties, the variationals describe a hypercube in the uncertain space, with uncorrelated variables, that is mapped into a generally non-convex region through the system dynamics. The propagation is performed by introducing an algebra, on Taylor polynomials, that replaces the standard computer algebra. The so called Truncated Power Series Algebra (TPSA) was introduced by Berz in 1986 for the computation of transfer maps in particle optics[4,5] and extended to rigorous numerics in 1997 with the introduction of Taylor Models.[6]

In the case of Taylor series, the basis functions are monomials defining the d^{th} order variation with respect to a reference point. The residual error is proportional to the $(d+1)$ derivative and the $(d+1)$ order monomial. Numerical integration scheme as well as the evaluation of the dynamics are performed in the polynomial algebra therefore at each integration time step the full polynomial representation of the current state is available. The same idea can be generalized to a different set

[*]Research Fellow, Department of Mechanical & Aerospace Engineering, University of Strathclyde, 75 Montrose Street, Glasgow, UK.

[†]PhD Candidate, Department of Mechanical & Aerospace Engineering, University of Strathclyde, 75 Montrose Street, Glasgow, UK.

[‡]Professor, Department of Mechanical & Aerospace Engineering, University of Strathclyde, 75 Montrose Street, Glasgow, UK.

of basis functions providing that the corresponding algebraic rules between monomials can be defined. The idea of a more generic TPSA dates back at 1982[7] with the name of Ultra Arithmetic. However in 2003[8] the possibility of using an alternative TPSA than the one built on Taylor basis, as for example Tchebycheff basis, was discarded. This was because of several drawback related to polynomial multiplications and growth of the magnitude of the coefficients. It was with the work of Brisebarre and Joldes[9] in 2010 that a formal comparison of the TPSA with Taylor, Tchebycheff and Newton basis was given. The results proved, in the univariate case, that the interpolation polynomials constructed in the Tchebycheff and Newton basis algebra were able to achieve smaller reminders than Taylor models requiring nevertheless more computing time, for the same order of expansion. One of the main advantages of using Tchebycheff series expansions is the speed of uniform convergence over the interval of expansion, that guarantees near-minimax approximation. Given a generic function $f(x)$, Tchebycheff series converge to $f(x)$ if the function is piecewise smooth and continuous. In fact the series converges also when a finite number of discontinuities in $f(x)$ and its derivatives are present. On the contrary Taylor series may not converge to the function $f(x)$ even when convergence is achieved.

This paper is an extension of the work initiated in[9] to the multivariate case and its application to the propagation of uncertainties in astrodynamics. To the knowledge of the author, the only existing computational environment that implements a TPSA with multivariate Tchebycheff basis, has been developed under the name of CHEBINT.[10] However it is missing some of the functionality needed to integrate differential equations in astrodynamics. In this paper, a novel TSPA framework has been developed for the propagation of extended regions of the state space in astrodynamics. This paper presents only the algebra of Tchebycheff polynomials on Tchebycheff basis, however an equivalent algebra on Newton basis is also possible as shown for the univariate case in.[9] The advantage of using Newton basis on Tchebycheff nodes is duplex: reducing the computational complexity of the arithmetic operations in the algebra while maintaining the uniform approximation properties of Tchebycheff polynomials. However for this preliminary study, aiming at assessing the accuracy and the applicability of the methodology, only Tchebycheff basis are considered. The computational complexity and accuracy of the proposed Generalised Intrusive Polynomial Expansion (GIPE) and related Tchebycheff polynomial algebra are compared to an equivalent non-intrusive approach based on Tchebycheff polynomials on sparse grids.[11]

The next two sections are presenting the details of both the intrusive and non-intrusive techniques, followed by a discussion on the computational complexity of the two methods. The intrusive and non-intrusive methods are then applied to the propagation of uncertainties in the accelerated Kepler problem, considering uncertainties both on states and model parameters. As final test a multistage trajectory is analysed where at a predefined time step a manoeuvre is performed to deflect the uncertainty region. Some final remarks on the future development and applications of GIPE methodology are given in the conclusions.

INTRUSIVE METHOD: TCHEBYCHEFF POLYNOMIAL ALGEBRA

This section starts with a short introduction to multivariate Tchebycheff polynomials and then introduces the algebra used to propagate sets through the dynamics.

Tchebycheff Polynomials

Multivariate Tchebycheff polynomials, of d variables up to degree n, are defined on hyper-rectangular domain $\Omega = [-1, 1]^d$ as

$$T_{\mathbf{i}}(\mathbf{x}) = \prod_{r=1}^{d} T_{i_r}(x_r)\,,$$

where $\mathbf{x} = (x_1, \ldots, x_r, \ldots, x_d) \in \Omega$, $\mathbf{i} \in [0, n]^d \subset \mathbb{N}^d$ and

$$T_0(x_r) := 1, \quad T_{i_r}(x_r) := \cos(i_r \arccos(x_r))\,.$$

The definition of the polynomials can be extended to a generic hyper-rectangular $\overline{\Omega} = [\mathbf{a}, \mathbf{b}] \subset \mathbb{R}^d$. Being $\tau : \overline{\Omega} \rightarrow \Omega$ the linear mapping between the two hyper-rectangular then the Tchebycheff polynomials are defined over $\overline{\Omega}$ as

$$T_{\mathbf{i}}(\mathbf{x}) = T_{\mathbf{i}}(\tau(\mathbf{x}))\,,$$

where $\mathbf{x} \in \overline{\Omega}$. So without loss of generality the domain Ω is considered for further considerations. Tchebycheff polynomials are orthogonal with respect to the continuous scalar products and the weight function ω therefore

$$\int_{\Omega} \omega(\mathbf{x}) T_{\mathbf{i}}(\mathbf{x}) T_{\mathbf{j}}(\mathbf{x}) = 0 \text{ for } \mathbf{i} \neq \mathbf{j}, \qquad \omega(\mathbf{x}) = \frac{1}{\pi^{\mathrm{d}}} \prod_{\mathrm{r}=1}^{\mathrm{d}} \frac{1}{\sqrt{1 - \mathrm{x}_{\mathrm{r}}^2}}\,.$$

Hence Tchebycheff polynomials up to a certain degree n form an orthogonal basis on the function space $C_{n,d}$

$$< \mathcal{T}_0, \mathcal{T}_1, ..., \mathcal{T}_d >, \qquad \mathcal{T}_j := \{T_{\mathbf{i}} \,:\, |\mathbf{i}| = j\}\,,$$

where $|\mathbf{i}| = \sum_{r=1}^{d} i_r$. As shown below, the function space $C_{n,d}$ can be equipped with a set of operations, generating an algebra on the space of Tchebycheff polynomials.

Tchebycheff Approximation

Given a multivariate function $f(\mathbf{x})$ in d variables, its Tchebycheff approximation of order n, $C_{f(x)}$, is an element of the function space $C_{n,d}$

$$f(\mathbf{x}) \sim \sum_{\mathbf{i}, |\mathbf{i}| \leq n} c_{\mathbf{i}} T_{\mathbf{i}}(\mathbf{x})\,,$$

where the coefficients $c_{\mathbf{i}}$ can be determined by means of hyperinterpolation techniques or through algebraic manipulations of Tchebycheff polynomials as presented in the following section.

If $f(\mathbf{x})$ is an element of a normed function space $(\mathcal{F}(\mathbf{x}), \|\cdot\|)$, and $p(\mathbf{x})$ is an element of a sub-space $(\mathcal{P}(\mathbf{x}), \|\cdot\|)$, such as the space of polynomial approximations, $p(\mathbf{x})$ is near-best approximation of $f(\mathbf{x})$ within a relative acceptable small distance ρ if

$$\|f(\mathbf{x}) - p(\mathbf{x})\|(1 + \rho) \leq \|f(\mathbf{x}) - p^*(\mathbf{x})\|\,,$$

where $p^*(\mathbf{x})$ is the best approximation of f in $\mathcal{P}(\mathbf{x})$.[12,13] In Mason[14] it has been proved that the algebraic polynomials formed by either taking the partial sum of a multivariate Tchebycheff series of the first kind or by interpolating at a tensor product of Tchebycheff polynomial zeros are near best L_∞ approximations with

$$\rho = C \prod_{j=1}^{d} \log n_j \,,$$

where C is a constant independent from f and d, and $n_1, n_2, \ldots, n_d$ are the orders of the partial sum in $x_1, x_2, \ldots, x_d$ respectively. Hence if has been proved in Mason[14] that if f satisfy a Lipschitz condition of the form

$$\sum_{j=1}^{d} \omega_j(\delta_j) \cdot \sum_{j=1}^{d} \log \delta_j \to 0, \qquad \text{as } \{\delta_j\} \to 0$$

where

$$\omega_j(t) = \sup_{|x_j - x_j^*| \leq t} |f(x_1, \ldots, x_j, \ldots, x_d) - f(x_1, \ldots, x_j^*, \ldots, x_d)| \,,$$

then the multivariate Tchebycheff series of f, the multivariate polynomial interpolating f at a tensor product of Tchebycheff zeros, all converge in L_∞ to f as $\{n_j\} \to \infty$. This is a slightly weaker condition than the Dini-Lipschitz condition required in the univariate case.

Tchebycheff Polynomial Algebra

All elementary arithmetic operations as well as the elementary functions are defined on the function space $C_{n,d}$ such that given the approximation of any $f(\mathbf{x})$ and $g(\mathbf{x})$ multivariate functions, it stands that

$$C_{h(\mathbf{x})} = C_{f(\mathbf{x}) \oplus g(\mathbf{x})} = C_{f(\mathbf{x})} \otimes C_{g(\mathbf{x})} \,,$$

where $\oplus \in \{+, -, *, /\}$ and $\otimes$ is the corresponding operation over the space of Tchebycheff polynomials. This allows one to define a new algebra $(C_{n,d}, \otimes)$, of dimension

$$\mathcal{D} = \dim(C_{n,d}, \otimes) = \binom{n+d}{d} = \frac{(n+d)!}{n!d!} \,,$$

the elements of which are linear combination of the multivariate Tchebycheff basis in d variables up to degree n. Each element of the algebra $F(\mathbf{x})$ is uniquely identified by the set of its coefficients $\mathbf{c} = \{c_\mathbf{i} \,:\, |\mathbf{i}| \leq n\} \in \mathbb{R}^\mathcal{D}$ such that

$$F(\mathbf{x}) = \sum_{\mathbf{i}, |\mathbf{i}| \leq n} c_\mathbf{i} T_\mathbf{i}(\mathbf{x}) \,.$$

The coefficients have been ordered using the scheme presented in Giorgilli and Sansottera.[15]

The rule for adding or subtracting Tchebycheff polynomials is no different from any other polynomial algebra. Being $A(\mathbf{x})$ and $B(\mathbf{x})$ two elements of $(C_{n,d}, \otimes)$, identified by the set of coefficients $\mathbf{a}, \mathbf{b} \in \mathbb{R}^\mathcal{D}$ respectively, the result of the sum (and difference) is

$$C(\mathbf{x}) = A(\mathbf{x}) \pm B(\mathbf{x}) \,,$$

identified by the set of coefficients $\mathbf{c} \in \mathbb{R}^\mathcal{D}$ such that

$$\mathbf{c} = \mathbf{a} \pm \mathbf{b} \,.$$

The product between two elements of the algebra is a much more computational expensive operation. The multiplication of two polynomials of degree n is a polynomial of degree $2n$. Hence the multiplication of two elements of the algebra needs to be defined in such a way that the resulting is still an element of the algebra. The product of two elements of the Tchebycheff base follows the rule

$$T_{\mathbf{i}}(\mathbf{x})T_{\mathbf{j}}(\mathbf{x}) = \frac{1}{2^d} \sum_{\mathbf{t} \in \{-1,1\}^d} T_{I(\mathbf{i},\mathbf{j},\mathbf{t})}(\mathbf{x}) \,,$$

with

$$T_{I(\mathbf{i},\mathbf{j},\mathbf{t})}(\mathbf{x}) = \begin{cases} T_{\mathbf{i}+\mathbf{t}\cdot\mathbf{j}}(\mathbf{x}) & \text{if } |\mathbf{i}+\mathbf{t}\cdot\mathbf{j}| \leq \mathrm{n} \\ 0 & \text{otherwise} \end{cases} \quad ,$$

where $\mathbf{t} \cdot \mathbf{j}$ represents the element-wise multiplication and the $|\cdot|$ is the sum of the absolute value of the vector components. Applying this rule to the product between the Tchebycheff basis appearing simplifying the expression

$$C(\mathbf{x}) = A(\mathbf{x}) * B(\mathbf{x}) = \left(\sum_{\mathbf{i},|\mathbf{i}|\leq n} a_{\mathbf{i}} T_{\mathbf{i}}(\mathbf{x}) \right) \left(\sum_{\mathbf{i},|\mathbf{i}|\leq n} b_{\mathbf{i}} T_{\mathbf{i}}(\mathbf{x}) \right) ,$$

and collecting all the contributions it is possible to compute the coefficients $\mathbf{c}$ of the product approximation $C(\mathbf{x})$. Being this the most straight forward implementation of the product in the algebra it is not the most computational efficient. More efficient implementations based on Discrete Fourier Transformation (DFT) have already been studied by Giorgi[16] and will be considered by the authors for further studies.

In the same way as for arithmetic operation, it is possible to define a composition rule in the Tchebycheff algebra such that

$$C_{f(g(\mathbf{x}))} = C_{f(\mathbf{x})} \circ C_{g(\mathbf{x})} \,,$$

where $\circ$ is the composition function on $(C_{n,d}, \otimes)$ and $f(\mathbf{x})$ and $g(\mathbf{x})$ are two multivariate function in the real space. This is necessary to define the division operation and any elementary function on the algebra. Being $h(x)$ any of the function $\{1/x, \sin(x), \cos(x), \exp(x), \log(x), ...\}$, $H(x)$ its univariate Tchebycheff approximation and $A(\mathbf{x})$ an element of the algebra,

$$C(\mathbf{x}) = H(x) \circ A(\mathbf{x}) \,,$$

is the expansion of the composition of functions, where $\circ$ is the composition between polynomials.

Integration of dynamical systems

The procedure presented above allows one to create a new computational environment where each function, that can be defined by means of arithmetic operations and elementary functions, can be represented as an element of $(C_{n,d}, \otimes)$. It follows that expanding the flow of the system of autonomous ordinary differential equations of the form

$$\begin{cases} \dot{\mathbf{x}} = f(\mathbf{x}) \\ \mathbf{x}(t_0) = \mathbf{x_0} \end{cases}$$

requires declaring the uncertain initial condition $\mathbf{X}_0(\mathbf{x}_0) = (X_1(\mathbf{x}_0), \dots, X_d(\mathbf{x}_0)) \in (C_{n,d}, \otimes)^d$ as an element of the algebra:

$$
\begin{aligned}
X_1(\mathbf{x}_0) &= T_{(1,0,\dots,0)}(\mathbf{x}_0) \\
X_2(\mathbf{x}_0) &= T_{(0,1,\dots,0)}(\mathbf{x}_0) \\
&\dots \\
X_d(\mathbf{x}_0) &= T_{(0,0,\dots,1)}(\mathbf{x}_0)
\end{aligned}
$$

and apply one of the known integration scheme (forward Euler for example) with operations in the algebra to have at each integration step the full Tchebycheff expansion of the current state. For example:

$$\mathbf{X}_k = \mathbf{X}_{k-1} + h\, f(\mathbf{X}_{k-1}), \quad \mathbf{X}_k, \mathbf{X}_{k-1} \in (C_{n,d}, \otimes)^d\,,$$

where $\mathbf{X}_k$ is the polynomial representation in Tchebycheff base of the system flow at the k-th timestep.

NON-INTRUSIVE METHOD: TCHEBYCHEFF POLYNOMIAL INTERPOLATION

The most straightforward non-intrusive approach to perform propagation of uncertainties is the Monte Carlo method. It propagates the dynamics over thousands of sample points taken in the initial region of uncertainties. Recent works [17,18] have shown that it is possible to replace the propagation through the dynamics with a polynomial approximation of the final states as a function of the initial uncertain parameters. In this work the polynomial expansions in Tchebycheff series on Smolyak sparse grids is considered.

Smolyak sparse grid and polynomial basis

Sparse grids have been introduced by Sergey Smolyak (1962)[19] and allow to represent, integrate and interpolate functions on multidimensional hypercubes. Moreover they do not suffer the curse of dimensionality as the tensor product methods. For example, a complete polynomial basis of maximum degree 4 in 10 unknown variables consists of $1\,001$ elements, while the corresponding sparse basis contains only 221 elements.

The construction of disjoint sparse grid as presented in Judd et al.[20] has been followed. First unidimensional grid points are generated using the extrema of Tchebycheff polynomials (also known as Tchebycheff-Gauss Lobatto points or Clenshaw-Curtis points):

$$\zeta_j^d = -cos\left(\frac{\pi(j-1)}{n-1}\right)\,, \qquad j = 1, \dots, n$$

is the j-th extremum of a Tchebycheff unidimensional polynomial of degree $n-1$. Among the sets of consecutive extrema, it is chosen a sequence of sets $S_1, S_2, \dots$ satisfying two conditions:

(a) $|S_1| = 1$ and $|S_i| = 2^{i-1} + 1$ for $i \geq 2$, where $|\cdot|$ indicates the cardinality of a set.

(b) $S_i \subset S_{i+1}$, for $i \geq 1$ (nested condition).

The first three nested sets are $S_1 = \{0\}\,,\ S_2 = \{-1, 0, 1\}\,,\ S_3 = \{-1, -\sqrt{2}/2, 0, \sqrt{2}/2, 1\}$.

In order to avoid repetitions of sparse grid points, a new sequence of sets $A_1, A_2, \dots$ has been defined such that $A_1 = S_1$ and $A_i = S_i \setminus S_{i-1}$ for $i \geq 2$. By construction these set are disjoint, their union satisfies $A_1 \cup \ldots \cup A_i = S_i$ and their cardinality is $|A_1| = 1, |A_2| = 2$ and $|A_i| = 2^{i-2}$ for $i \geq 3$.

Then the tensor product of the unidimensional sets of points A_i is constructed and the Smolyak rule is used to select multidimensional grid points. That is, the elements that belong to the set $A_{i_1}^{(1)} \times \ldots \times A_{i_d}^{(d)}$ are selected if the condition $d \leq i_1 + \ldots + i_d \leq d + l$ is satisfied, where d is the number of variables and l is the level of approximation of the sparse grid. For example, denoting by $\mathcal{H}^{d,l}$ a Smolyak grid in d dimension and level l, for $d = 2$ it is

- $\mathcal{H}^{2,0} = \{(0,0)\}$,
- $\mathcal{H}^{2,1} = \{(0,0), (-1,0), (1,0), (0,-1), (0,1)\}$,
- $\mathcal{H}^{2,1} = \{(0,0), (-1,0), (1,0), (0,-1), (0,1), (-\sqrt{2}/2, 0), (\sqrt{2}/2, 0), (0, -\sqrt{2}/2), (0, \sqrt{2}/2)\}$.

Figure 1 shows the two-dimensional sparse grid for different levels.

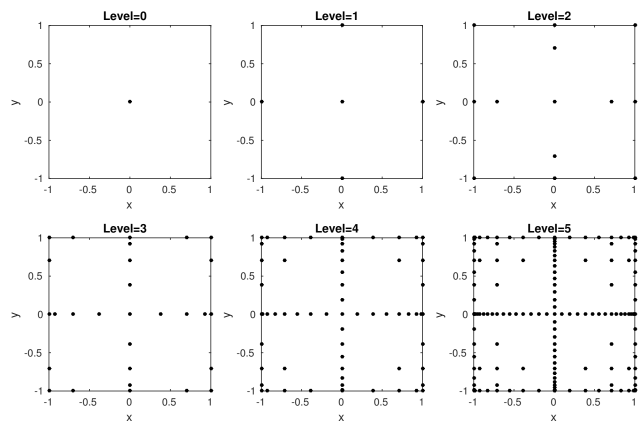


Figure 1: Sparse grid on Tchebycheff nodes for $l = 0, \ldots, 5$.

Analogously, multidimensional polynomial basis functions can be constructed using unidimensional disjoint sets. First disjoint set of unidimensional Tchebycheff functions are defined:

$$F_1 = \{1\}, \; F_2 = \{T_1, T_2\}, \; F_3 = \{T_3, T_4\}.$$

Then the multidimensional Tchebycheff basis functions are constructed using the tensor product of unidimensional basis functions. Finally, by applying the same Smolyak rule used to produce the grid points, a list of basis functions can be obtained. For example, denoting with $\mathcal{P}^{d,l}$ a Smolyak basis with dimension d and level of approximation l, for $d = 2$ it is:

- $\mathcal{P}^{2,0} = \{1\}$,
- $\mathcal{P}^{2,1} = \{1, T_{(1,0)}, T_{(0,1)}, T_{(2,0)}, T_{(0,2)}\}$
- $\mathcal{P}^{2,2} = \{1, T_{(1,0)}, T_{(0,1)}, T_{(2,0)}, T_{(0,2)}, T_{(1,1)}, T_{(3,0)}, T_{(2,1)}, T_{(1,2)}, T_{(0,3)}, T_{(4,0)}, T_{(2,2)}, T_{(0,4)}\}$.

Tchebycheff Polynomial Approximation

The polynomial interpolation on Tchebycheff extrema presented above is computed as

$$F(\mathbf{x}) = \sum_{\mathbf{i} \in \mathcal{H}^{d,l}} c_{\mathbf{i}} \, T_{\mathbf{i}}(\mathbf{x}) \,, \tag{1}$$

where $c_{\mathbf{i}}$ are the unknown coefficients with respect to the Tchebycheff basis element $T_{\mathbf{i}}$. They are computed by inverting the linear system

$$HC = Y \,,$$

with

$$H = \begin{bmatrix} T_{i_1}(x_1) & \dots & T_{i_s}(x_1) \\ \vdots & \ddots & \vdots \\ T_{i_1}(x_s) & \dots & T_{i_s}(x_s) \end{bmatrix} , \qquad C = \begin{bmatrix} c_{i_1} \\ \vdots \\ c_{i_s} \end{bmatrix} , \qquad Y = \begin{bmatrix} Y_1 \\ \vdots \\ Y_s \end{bmatrix} ,$$

where $s = |\mathcal{H}^{d,l}|$ is the cardinality of the set of grid points, $x_1, \dots, x_s$ are the grid nodes and the components of Y are the true values obtained integrating the dynamics in the initial grid points. The system cannot be inverted if the matrix H has not full rank. In most of the cases, this is guaranteed by choosing the Tchebycheff nodes.

COMPUTATIONAL COMPLEXITY

The two methods have not been compared in terms of computational cost because they have been implemented in two different programming languages on two different architectures. However a discussion on the computational complexity of each method is given below.

The computational complexity of the intrusive methods is proportional to the size of the Tchebycheff algebra and the computational complexity of the operations involved in the evaluation of the right hand side of the dynamical system. The complexity of the non-intrusive method instead is proportional to the number of sample points employed during the construction of the sparse grid and the inversion of a matrix of equal dimension for the approximation of the Tchebycheff coefficients.

In particular for the intrusive case, where N is the size of the algebra, adding two polynomials in Tchebycheff basis requires $O(N)$ operations. Multiplication is the most computational expensive operation, as it is implemented now, has a complexity of $O(N^2)$, otherwise faster implementations of polynomial multiplications exist that have an asymptotic complexity of $O(N \log N)$.[16] The dynamics is integrate forward only once with one of the available numerical scheme.

The non-intrusive approach here considered constructs the Tchebycheff approximation on Smolyak sparse grids. The number of points on a sparse grid grows polynomially with dimensionality d, meaning that sparse grids are not subject to the curse of dimensionality as it is the case for full tensor grids. For example, for the first two levels $l = 1$ and $l = 2$ the number of Smolyak points are $1 + 2d$ and $1 + 4d + (4d(d-1))/2$ respectively. This means that the number of points grows as $O(d^l)$ where l is the grid level and d the problem dimension. Grid level and polynomial degree are related by the equation $n = 2^l$. Hence the increase of polynomial degree is not linear. N forward integrations need to be performed, where N is the number of nodes on the sparse grid. Finally to perform the interpolation a matrix of dimension $N \times N$ needs to be inverted. Being a dense matrix the complexity of its inversion is $O(N^3)$.

PROPAGATION OF UNCERTAINTIES IN SPACE DYNAMICS

The application of the proposed intrusive method in astrodynamics, and the comparison with its non-intrusive counterpart, is illustrated by a simple example of two-body dynamics in the plane. Uncertainties in the states and model parameters are considered in the first two examples. Finally the propagation of state uncertainties is analysed in the case of a multi-phase trajectory where at each time interval a manoeuvre occurs.

Accelerated Kepler problem

The accelerated Kepler problem (AKP)[21] is obtaining by adding a constant force in magnitude and direction to the classical two-body Kepler problem. It can described common problems in orbital mechanics. For instance, it can model a low-thrust trajectory discretised in short arcs of constant thrust, or it can model the solar radiation pressure of a satellite orbit over a short period of time and outside of the shadow of the Earth.

The AKP has the interesting property to be integrable, according to the Liouville-Arnold theorem, that is there exist three integrals, independent and in involutions of the AKP.[22] These are the total energy, the angular momentum and a generalized Runge-Lenz vector.[23] The integrability can be also proved using the Hamilton-Jacobi approach in parabolic coordinates: the Hamiltonian of the AKP separates in these variables in two unidimensional Hamiltonians.[24,25] Recently, a closed-form solution in terms of elementary function has been shown, together with an analytical expressions .[26]

The motion of an arbitrary point P with negligible mass in the gravitational field induced by a larger body at the origin, and subject to an additional constant force is considered. In an inertial reference frame the dynamical equations are

$$\ddot{\mathbf{x}} = -\frac{k}{|\mathbf{x}|^3}\mathbf{x} + \mathbf{A},$$

where $\mathbf{x} = (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{\mathbf{2}}$ is the position vector, k is the gravitational parameter and $\mathbf{A}$ is the constant force vector. Without loss of generality, the constant force can be assumed to be in the y-direction, otherwise a coordinate rotation must be performed first. In this hypothesis, the equations of motion are

$$\begin{aligned} \ddot{x} &= -\frac{k}{r^3}x \\ \ddot{y} &= -\frac{k}{r^3}y + A \end{aligned}$$

where $r = \sqrt{x^2 + y^2} \in \mathbb{R}$ and A is the magnitude of the constant force. The limit $A \to 0$ can represent a low-thrust propulsion or other smaller perturbations, however the value of A is arbitrary and not necessary small.

Experiments set up

The AKP is made dimensionless using $k = 1$. It is integrated with a fixed stepsize Runge-Kutta 4th order scheme, with initial conditions

$$\begin{aligned} x(0) &= 1, \qquad & v_x(0) &= 0, \\ y(0) &= 0, \qquad & v_y(0) &= \sqrt{1+e}, \end{aligned}$$

Table 1: Uncertainty on states parameters

	$e = 0$	$e = 0.5$
x	0.01	0.005
y	0.01	0.005
v_x	0.005	0.0001
v_y	0.005	0.0001

where e is the eccentricity of the orbit considered for the test cases. Intrusive and non-intrusive methods are compared against a Monte Carlo sampling of 10000 points. The Root Mean Square Error (RSME) measure is used for the comparison defined as

$$RMSE = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{x}_i - x_i)^2},$$

where N is the number of samples, x_i is the true value of the state (obtained by forward integration in the sampling points) and $\hat{x}_i$ is the approximated value computed evaluating the polynomial approximation obtained with the two methods.

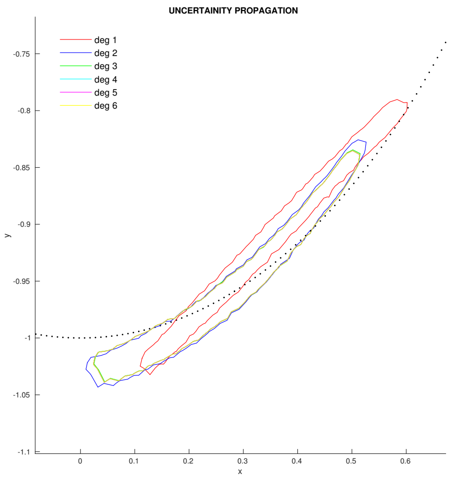
Uncertainties on initial state

The first study is the propagation of uncertainties on position and velocity for the case of a circular ($e = 0$) and elliptical orbit ($e = 0.5$). The acceleration is considered zero so the problem is the simple Kepler problem. The uncertain parameters in Table 1 have been considered on the initial states. Therefore the intervals of definition of the states are $(x, y) \in [1-0.01, 1+0.01]\times[-0.01, 0.01]$ and $(v_x, v_y) \in [-0.005, 0.005] \times [1 - 0.005, 1 + 0.005]$ for the circular case; $(x, y) \in [1 - 0.005, 1 + 0.005] \times [-0.005, 0.005]$ and $(v_x, v_y) \in [-0.0001, 0.0001] \times [\sqrt{1.5} - 0.0001, \sqrt{1.5} + 0.0001]$ for the elliptic case. Being the problem adimensionalized the uncertainties corresponds to roughly $\mathrm{DU} \cdot \mathrm{u_{pos}}$ and $\mathrm{DU/TU} \cdot \mathrm{u_{vel}}$, where u_{pos} and u_{vel} are the uncertainties on the position and velocity respectively and DU and TU are the planetary canonical units that for the case of Earth are

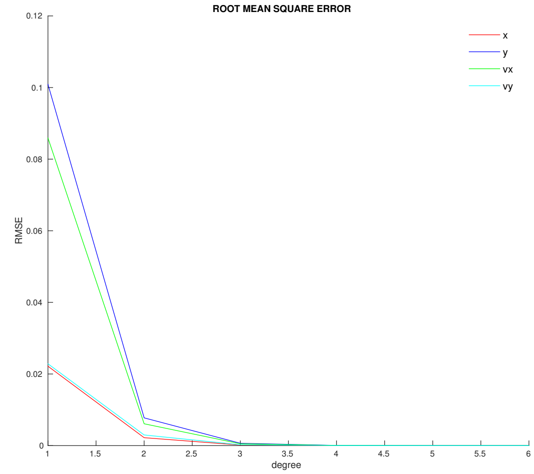
$$\mathrm{DU} = 6378.136 \text{ [Km]}, \quad \mathrm{TU} = 806.78 \text{ [s]}.$$

The uncertainties for the elliptic case have been chosen smaller because the orbit is propagated for longer time (one revolution around the central body). A test has been performed initially to assess the accuracy of the Tchebycheff approximation. The dynamic of the Kepler problem, only for the case of a circular orbit, has been integrated in the Tchebycheff algebra for different values of the degree of the polynomials. In Figure 2 the polynomial approximations of the position state at $t = 5$, for polynomials with degree up to 6, are compared. The metric used to evaluate the accuracy of the approximation is the RMSE with respect to the single point integration of the Monte Carlo sampling. A polynomial approximation of degree 4, that corresponds to an error on the final states of order 10^{-4} has been selected for the whole set of experiments hereafter presented.

In Figure 3 and Table 2 the uncertainties regions obtained with the two methods, on the states uncertainty problem, are compared.

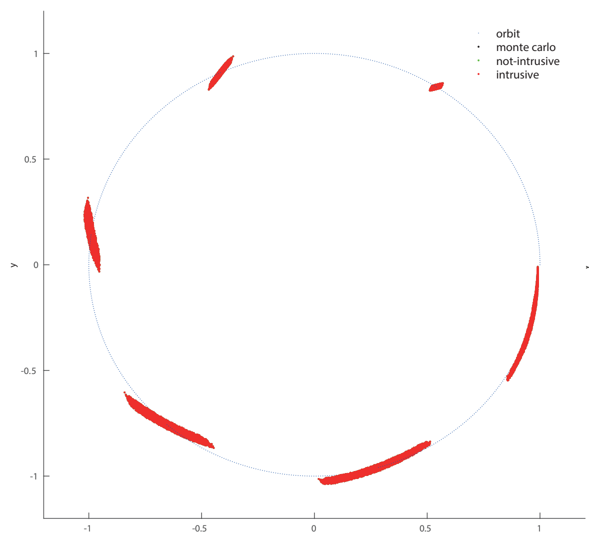


(a) position state uncertainty, t=5

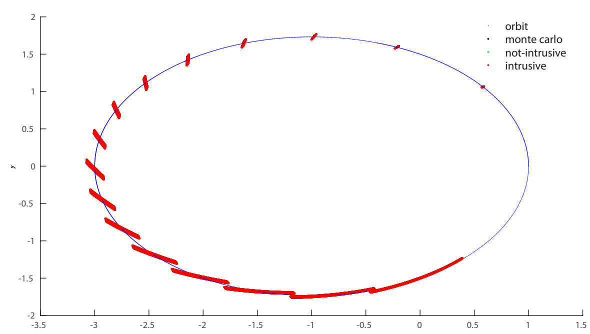


(b) degree versus accuracy

Figure 2: Uncertainty regions on the position state for different value of polynomial degree and Root Mean Square Error



(a) Circular $e = 0$



(b) Elliptic $e = 0.5$

Figure 3: Propagation of uncertainties on the initial state in the Kepler problem dynamics

As seen in the Table 2 the two methods are comparable in term of accuracy. The loss of accuracy in the position and velocity along the y-axis observed by the intrusive approach in the case of an elliptic orbit can be due to the propagation of the truncation error of the Tchebycheff expansions that adds up to the error of the integration scheme. This requires a further rigorous statistical test to be proved. However it is general true, for any intrusive method, that if the propagation of the truncation error is greater than the approximation error at the final state, the solution obtained with the intrusive

Table 2: Comparison of RMSE at final time (uncertainty on states)

	e=0 (t=6)		e=0.5 (t=16)	
	intrusive	non-intrusive	intrusive	non-intrusive
x	0.132821 e-04	0.385858 e-04	0.326001 e-04	0.200703 e-04
y	0.578751 e-04	0.276079 e-04	0.107113 e-03	0.133100 e-04
v_x	0.450878 e-04	0.888662 e-04	0.433208 e-04	0.315385 e-04
v_y	0.188817 e-04	0.763898 e-04	0.257869 e-03	0.304058 e-04

method loses in accuracy with respect to its non-intrusive counterpart. In this case the intervals of definition of the uncertain variables need to be sectioned to gain accuracy on smaller intervals. Therefore two separate propagation are carried forward to the next integration steps resulting in an overall increase of the computational cost of the intrusive method.

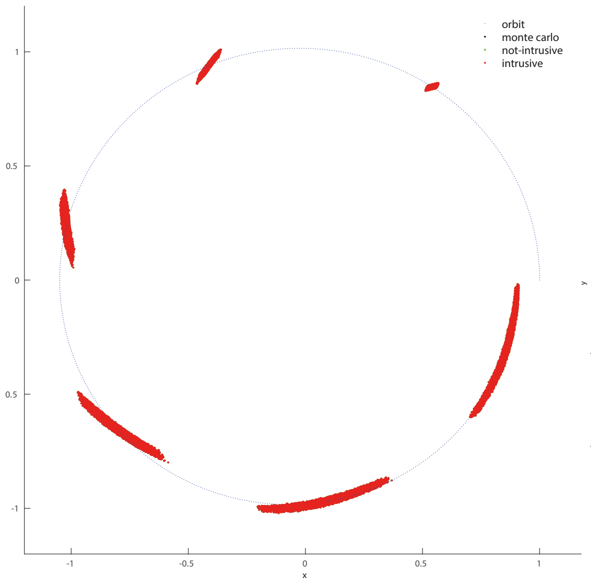
Uncertainties on initial state and model parameters

The same analysis has been repeated adding to the list of uncertain variables a model parameter. The acceleration along the y-axis has been selected as uncertain parameter. To the reference value $A = 0.01$ has been added an uncertainty of 10%. The dynamics has been propagated as above for both the circular an elliptic case. In Figure 4 and Table 3 the uncertainty regions, in the space of the position parameters, for the AKP obtained with the intrusive and non-intrusive method are compared. The integration of the elliptic case has been stopped at $t = 15$ because the system gets close to a singularity. Being the Tchebycheff polynomial defined on an interval it is possible, in the case of singularities, to continue with the integration on subintervals where the system is continuous. This is a technique that has not been implemented yet in the proposed methodology, hence the system has been integrated till the time $t = 15$. As for the previous case the two techniques are comparable. The intrusive method gains better accuracy in the circular case. The worsening of the results for the non-intrusive case can be attributed to the use of an incomplete base of Tchebycheff polynomials. The polynomial obtained with the non-intrusive approach is missing some monomials because it makes use of a reduced set of basis functions. It is possible that, in this particular test case, some information are lost by the use of such a reduced basis. Further rigorous statistical tests need to be performed to prove this hypothesis.

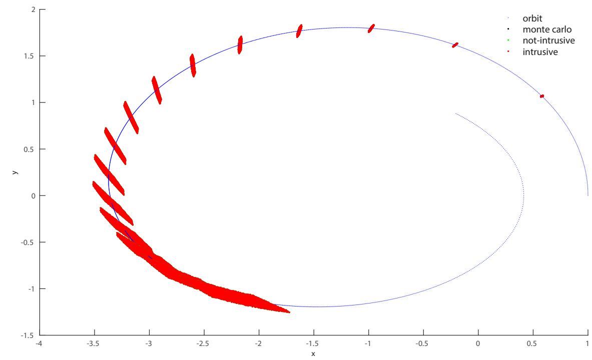
Multi-phase uncertainty propagation

The last example here presented is a multi-phase integration of a trajectory. The simple Kepler problem of a circular orbit has been considered. At each time $t = i$ for $i = 1, .., 6$ the velocity vector is rotated of an angle $\alpha = 0.09$ [rad] and its magnitude multiplied by a constant $c = 1.1$. This can represents for example a manoeuvre aiming at deflecting the current orbit.
An uncertainty of 1%, with respect to the reference value, on the angle and on the multiplicative constant has been considered. The same uncertainties on the state of Table 1 has been used.
In Figure 5 and Table 4 the uncertainty regions obtained with the two methods and the RMSE, computed with respect to the Monte Carlo sampling, are reported for comparison.

The two methods are comparable in terms of accuracy. However the two methods treat the manoeuvre in a different way. For the intrusive case the manoeuvre is the results of an algebraic ma-



(a) Circular $e = 0$



(b) Elliptic $e = 0.5$

Figure 4: Propagation of uncertainties on the initial state and force parameter in the Accelerated Kepler problem dynamics

Table 3: Comparison of RMSE at final time (uncertainty on states and model parameter)

	e=0 (t=6)		e=0.5 (t=15)	
	intrusive	non-intrusive	intrusive	non-intrusive
x	0.268012 e-04	0.122555 e-03	0.608590 e-04	0.379348 e-04
y	0.137101 e-03	0.605071 e-04	0.208880 e-04	0.362687 e-04
v_x	0.119142 e-03	0.145818 e-03	0.109825 e-03	0.254841 e-04
v_y	0.760445 e-04	0.255403 e-03	0.487156 e-04	0.339847 e-04

nipulation of the polynomial. The state at the instant that the manoeuvre occurs is composed with the polynomial representation of the manoeuvre itself. The advantage of the intrusive approach relies indeed in the possibility of representing non linear regions of uncertainties, in a polynomial form, at each instant of time. Moreover the ability of using the same polynomial as initial guess of the next integration step. The non intrusive method instead needs to always define the variable of the polynomial on a hyper-rectangular region of uncertainty. In case the region of uncertainty is not an hyper-rectangular a Principal Component Analysis (PCA) needs to be used to define the best hyper-rectangular enclosure. In the presented test case the PCA was not necessary because the manoeuvre applied is the same at each time step. In case a different manoeuvre is applied at each step the intrusive method can simply enlarge the dimension of the algebra with one variable, perform the composition and continuing with the integration. On the other hand the interpolation method needs to add one dimension for each manoeuvre parameter on the hyper-rectangular defining the box of the initial state uncertainty or perform a PCA analysis on the current states.

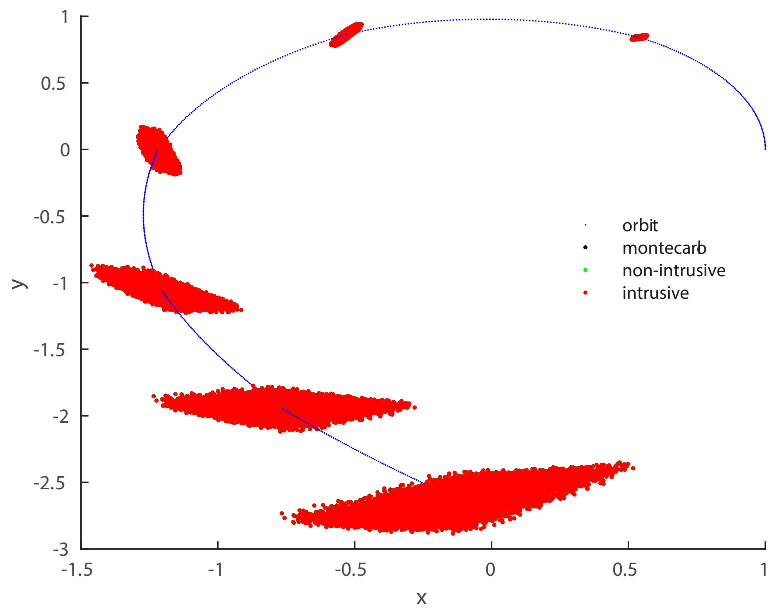


Figure 5: Propagation of uncertainties on the initial state and manoeuvre parameter in the Kepler problem dynamics

Table 4: Comparison of RMSE at final time (uncertainty on states and manoeuvre parameters)

	e=0 (t=6)	
	intrusive	non-intrusive
x	0.166485 e-03	0.294136 e-03
y	0.136589 e-03	0.939322 e-04
v_x	0.662739 e-04	0.148801 e-03
v_y	0.116262 e-03	0.107878 e-03

CONCLUSIONS

The work is a first step towards the definition of a Generalised Intrusive Polynomial Expansion (GIPE) technique for uncertainty propagation. Only the Tchebycheff basis has been considered in this first development. The definition of a multivariate computational polynomial algebra on the space of Tchebycheff polynomials is, to the knowledge of the authors, one of the novelty of the work, together with its application to space dynamics problems. The intrusive method has been compared to its non intrusive counterpart: the Tchebycheff polynomial interpolation on sparse grid. The two techniques are compared on the propagation of uncertainties, both on states and model parameters, in the Accelerated Keplerian Problem (AKP) dynamics. They show comparable results in term of accuracy. While only a discussion of the computational complexity of the two techniques is given in this paper, further work need to be performed to assess the scalability of the two methods when increasing the number of uncertain variables. The potentiality of the intrusive method, that is not built on a sampling of the initial space, lies in its flexibility of manipulating the non linear uncertainty region through algebraic operations.

ACKNOWLEDGMENT

The work in this paper was partially supported by the Marie Curie FP7-PEOPLE-2012-ITN Stardust, grant agreement 317185.

REFERENCES

[1] P. D. Lizia, R. Armellin, and M. Lavagna, "Application of high order expansions of two-point boundary value problems to astrodynamics," *Celestial Mechanics and Dynamical Astronomy*, Vol. 102, No. 4, 2008, pp. 355–375, 10.1007/s10569-008-9170-5.

[2] R. Armellin, P. Di Lizia, F. Bernelli-Zazzera, and M. Berz, "Asteroid close encounters characterization using differential algebra: the case of Apophis," *Celestial Mechanics and Dynamical Astronomy*, Vol. 107, No. 4, 2010, pp. 451–470, 10.1007/s10569-010-9283-5.

[3] P. D. Lizia, R. Armellin, A. Morselli, and F. Bernelli-Zazzera, "High order optimal feedback control of space trajectories with bounded control," *Acta Astronautica*, Vol. 94, No. 1, 2014, pp. 383–394.

[4] M. Berz, "The New Method of TPSA Algebra for the Description of Beam Dynamics to High Orders," Tech. Report AT–6: TN–86–16, Los Alamos National Laboratory, Los Alamos, N.M., 1986.

[5] M. Berz, "The method of power series tracking for the mathematical description of beam dynamics," *Nuclear Instruments and Methods*, Vol. A258, 1987, pp. 431 +.

[6] M. Berz, "From Taylor Series to Taylor Models," *in AIP Conference Proceedings 405*, 1997, pp. 1–23.

[7] C. Epstein, W. Miranker, and T. Rivlin, "Ultra-arithmetic I: Function data types," *Mathematics and Computers in Simulation*, Vol. 24, No. 1, 1982, pp. 1 – 18, http://dx.doi.org/10.1016/0378-4754(82)90045-3.

[8] K. Makino and M. Berz, "Taylor Models and Other Validated Functional Inclusion Methods," *International Journal of Pure and Applied Mathematics*, Vol. 4, No. 4, 2003, pp. 379–456.

[9] N. Brisebarre and M. Joldeş, "Chebyshev Interpolation Polynomial-based Tools for Rigorous Computing," *Proceedings of the 2010 International Symposium on Symbolic and Algebraic Computation*, ISSAC '10, New York, NY, USA, ACM, 2010, pp. 147–154, 10.1145/1837934.1837966.

[10] K. Poppe and R. Cools, "CHEBINT: A MATLAB/Octave Toolbox for Fast Multivariate Integration and Interpolation Based on Chebyshev Approximations over Hypercubes," *ACM Trans. Math. Softw.*, Vol. 40, Oct. 2013, pp. 2:1–2:13, 10.1145/2513109.2513111.

[11] K. L. Judd, L. Maliar, S. Maliar, and R. Valero, "Smolyak method for solving dynamic economic models: Lagrange interpolation, anisotropic grid and adaptive domain," *Journal of Economic Dynamics and Control*, Vol. 44, No. 0, 2014, pp. 92 – 123, http://dx.doi.org/10.1016/j.jedc.2014.03.003.

[12] J. Mason, *Orthogonal polynomial approximation methods in numerical analysis, Approximation Theory*. A. Talbot (Ed.), New York, Academic Press, 1970.

[13] K. O. Geddes and J. C. Mason, "Polynomial Approximation by Projections on the Unit Circle," *SIAM Journal on Numerical Analysis*, Vol. 12, No. 1, 1975, pp. pp. 111–120.

[14] J. Mason, "Near-best multivariate approximation by Fourier series, Chebyshev series and Chebyshev interpolation," *Journal of Approximation Theory*, Vol. 28, No. 4, 1980, pp. 349 – 358, http://dx.doi.org/10.1016/0021-9045(80)90069-6.

[15] A. Giorgilli and M. Sansottera, "Methods of algebraic manipulation in perturbation theory," *Proceedings of the 3rd La Plata International School on Astronomy and Geophysics*, La Plata, Argentina, Universidad Nacional de La Plata and Asociacin Argentina de Astronoma Publishers, 2012.

[16] P. Giorgi, "On Polynomial Multiplication in Chebyshev Basis," *Computers, IEEE Transactions on*, Vol. 61, June 2012, pp. 780–789, 10.1109/TC.2011.110.

[17] B. A. Jones, A. Doostan, and G. H. Born, "Nonlinear Propagation of Orbit Uncertainty Using Non-Intrusive Polynomial Chaos," *Journal of Guidance Control Dynamics*, Vol. 36, mar 2013, pp. 430–444, 10.2514/1.57599.

[18] C. Tardioli, M. Kubicek, M. Vasile, E. Minisci, and A. Riccardi, "Comparison of non-intrusive approaches to uncertainty propagation in orbital mechanics," *AAS Astrodynamics Specialists Conference, Vail, Colorado, USA*, Vol. AAS 15-545, August 9–13 2015.

[19] S. Smolyak, "Quadrature and interpolation formulas for tensor products of certain classes of functions," *Dofl. Akad. Nauk*, Vol. 158, 1963, pp. 1042–1045.

[20] K. L. Judd, L. Maliar, S. Maliar, and R. Valero, "Smolyak method for solving dynamic economic models: Lagrange interpolation, anisotropic grid and adaptive domain," *Journal of Economic Dynamics and Control*, Vol. 44, 2014, pp. 92–123.

[21] F. Namouni and M. Guzzo, "The accelerated Kepler problem," *Celestial Mechanics and Dynamical Astronomy*, Vol. 99, No. 1, 2007, pp. 31–44, 10.1007/s10569-007-9087-4.

[22] J. Liouville, "Mémoire sur lintégration des équations différentielles du mouvement d'un nombre quelconque de points materiels," *J. de Mathematiques Pures et Appliquees*, Vol. 14, 1849, pp. 257–299.

[23] P. J. Redmond, "Generalization of the Runge-Lenz Vector in the Presence of an Electric Field," *Physical Review*, Vol. 133, Mar. 1964, pp. 1352–1353, 10.1103/PhysRev.133.B1352.

[24] P. S. Epstein, "Zur Theorie des Starkeffektes," *Annalen der Physik*, Vol. 355, 1916, pp. 489–520, 10.1002/andp.19163551302.

[25] L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics*. Physics. Pergamon Press, Oxford, 1969.

[26] G. Lantoine and R. P. Russell, "Complete closed-form solutions of the Stark problem," *Celestial Mechanics and Dynamical Astronomy*, Vol. 109, April 2011, pp. 333–366.