

New Periodic Orbits in the Solar Sail restricted three body problem

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In this paper we consider periodic orbits of a solar sail in the Earth-Sun restricted three-body problem. In particular, we consider orbits which are high above the ecliptic plane, in contrast to the classical Halo orbits about the collinear equilibria. We begin with the Circular Restricted Three-Body Problem (CRTBP) where periodic orbits about equilibria are naturally present at linear order. Using the method of Lindstedt-Poincaré, we construct n th order approximations to periodic solutions of the nonlinear equations of motion. In the second part of the paper we generalize to the Elliptic Restricted Three-Body Problem (ERTBP). A numerical continuation, with the eccentricity, e , as the varying parameter, is used to find periodic orbits above the ecliptic, starting from a known orbit at $e = 0$ and continuing to the required eccentricity of $e = 0.0167$. The stability of these periodic orbits is investigated.

Keywords: Periodic orbits, solar sail, elliptic three body problem

1. Introduction

While the concept of the solar sail has been with us for some time, it is only with recent advances in materials and structures that their use is being seriously considered. A solar sail consists essentially of a large mirror, which uses the momentum change due to photons reflecting off the sail for its impulse. A natural setting to consider the orbital dynamics of a solar sail is the restricted 3-body problem, with the Earth and Sun as the two primaries. We begin here with the analysis in the solar sail circular restricted three body problem (CRTBP) and later generalize to the elliptic restricted

three body problem (ERTBP).

There has been some work already carried out regarding solar sails in the 3-body problem. McInnes¹ first described the surfaces of equilibrium points. In Baoyin and McInnes,² the authors describe periodic orbits about equilibrium points in the solar sail three body problem, however they consider only equilibrium points on the axis joining the primary masses, corresponding to artificial Lagrange points (analogous to the classical ‘halo’ orbits^{3,4}).

In this paper we examine the solutions to the linearised equations of motion and discuss their stability in the CRTBP. We find that periodic orbits exist at linear order, and we use these linear solutions to find higher order approximations to periodic solutions of the non-linear system using the method of Lindstedt-Poincaré.^{5,6} These approximate orbits are then fine-tuned using a differential corrector to find initial conditions that yield periodic solutions to the full non-linear model.⁵

Following this we generalize the problem to the solar sail ERTBP⁷ in the Earth-Sun system. A numerical continuation, with the eccentricity e as the varying parameter, is used to find periodic orbits above the ecliptic, starting from a known orbit in the CRTBP ($e = 0$) and continuing to the required eccentricity $e = 0.0167$. The stability of some periodic orbits above the ecliptic are investigated and it is shown that they are unstable and that a bifurcation occurs at $e = 0$.

2. Equations of motion in the rotating frame

We follow the conventions set out in McInnes.¹ We consider a rotating coordinate system in which the primary masses are fixed on the x -axis with the origin at the centre of mass, the z -axis is the axis of rotation and the y -axis completes the triad. We choose our units to set the gravitational constant, the sum of the primary masses, the distance between the primaries, and the magnitude of the angular velocity of the rotating frame to be unity. We shall denote by $\mu = 3 \times 10^{-6}$ the dimensionless mass of the smaller body m_2 , the Earth, and therefore the mass of the larger body m_1 , the Sun, is given by $1 - \mu$ (see Figure 1).

Denoting by \mathbf{r} , \mathbf{r}_1 and \mathbf{r}_2 the position of the sail w.r.t. the origin, m_1 and m_2 respectively, the solar sail’s equations of motion in the rotating frame are

$$\frac{d^2\mathbf{r}}{dt^2} + 2\boldsymbol{\omega} \times \frac{d\mathbf{r}}{dt} = \mathbf{a} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - \nabla V \equiv \mathbf{F}, \quad (1)$$

with $\boldsymbol{\omega} = \hat{\mathbf{z}}$ and $V = -[(1 - \mu)/r_1 + \mu/r_2]$ where $r_i = |\mathbf{r}_i|$. These differ from

the classical equations of motion in the CR3BP by the radiation pressure acceleration term

$$\mathbf{a} = \beta \frac{(1 - \mu)}{r_1^2} (\hat{\mathbf{r}}_1 \cdot \mathbf{n})^2 \mathbf{n}, \quad (2)$$

where β is the sail lightness number, and is the ratio of the solar radiation pressure acceleration to the solar gravitational acceleration. Here \mathbf{n} is the unit normal of the sail and describes the sail's orientation. We define \mathbf{n} in terms of two angles γ and ϕ w.r.t. the rotating coordinate frame,

$$\mathbf{n} = (\cos(\gamma) \cos(\phi), \cos(\gamma) \sin(\phi), \sin(\gamma)), \quad (3)$$

where γ, ϕ are the angles the normal makes with the x - y and x - z plane respectively (see Figure 1).

Equilibria are given by the zeroes of \mathbf{F} in (1). We find a 3-parameter family of equilibria, as described in McInnes.¹ These are found by specifying the lightness number β and the sail angles γ and ϕ , and solving $\mathbf{F} = 0$. To simplify facilitate the search for orbits we initially assume $\phi = 0$ so the equilibrium (and sail normal) is in the x - z plane. In Figure 2 we show some of the equilibria near the Earth for low β values. Practically speaking, while a β value of about 0.3 – 0.4 is considered within the realm of possibility of current engineering, to put the analysis in this paper well within the near-term we will consider very modest β values of about 0.05.

3. Linearised system

We linearise about the equilibrium point (in the x - z plane) by making the transformation $\mathbf{r} \rightarrow \mathbf{r}_e + \delta\mathbf{r}$, Taylor expanding \mathbf{F} about \mathbf{r}_e , and neglecting

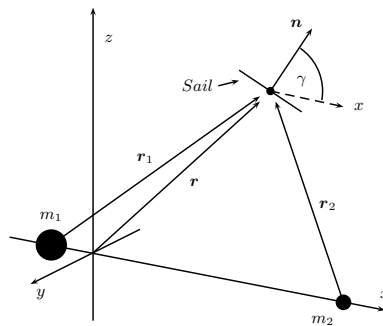


Fig. 1. The rotating coordinate frame and the sail position therein. The angles γ and ϕ which the sail normal makes with respect to the rotating frame are also shown.

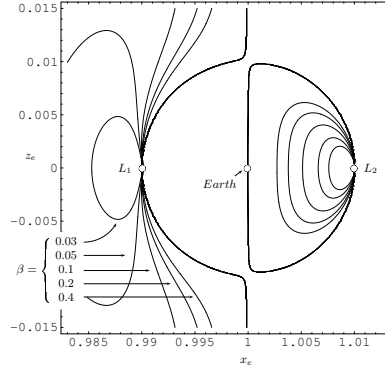


Fig. 2. Surfaces of equilibrium points in the x_e - z_e parameter space. Each curve is specified by a constant value of β , and the position of the equilibrium point along the curve is given by γ . The grey shaded regions denote areas where equilibrium is not possible.

the terms quadratic in $\delta\mathbf{r}$. We assume the orientation of the sail will remain fixed under perturbation of the sail position, in which case γ, ϕ and β are constants. Letting $\delta\mathbf{r} = (\delta x, \delta y, \delta z)^T$ and $\mathbf{X} = (\delta\mathbf{r}, \delta\dot{\mathbf{r}})^T$, our linear system is $\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}$ with

$$A = \begin{pmatrix} 0 & I \\ M & \Omega \end{pmatrix}, \quad M = \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (4)$$

where a dot denotes differentiation w.r.t. t ,

$$a = (\partial_x F^x)|_e, \quad b = (\partial_z F^x)|_e, \quad c = (\partial_y F^y)|_e, \quad d = (\partial_x F^z)|_e, \quad e = (\partial_z F^z)|_e,$$

and $b \neq d$. Here F^a denotes the a -th component of \mathbf{F} , and M is sparse due to $y_e = 0$.

The key difference between this analysis and the classical orbits about the collinear Lagrange points is the term $d \neq 0$, which appears precisely because we are linearising about an equilibrium point with $z_e \neq 0$. This means we cannot decouple the z -equation.

The characteristic equation of the Jacobian A is bi-cubic (whose corresponding cubic equation has real roots); this means the eigenvalues of A are either in pairs of pure imaginary conjugates or real and of opposite sign. Thus equilibria in the x - z plane will have the dynamical structure of centres and saddles, akin to the classical collinear Lagrange points.

If we label the eigenvectors associated with $\lambda_a i$ ($a = 1, 2$) as $\mathbf{u}_a + \mathbf{w}_a i$, and the eigenvectors associated with $\lambda_r, -\lambda_r$ as $\mathbf{v}_1, \mathbf{v}_2$, then the general

solution of the linear system (4) is

$$\begin{aligned} \mathbf{X}(t) = & \cos(\lambda_1 t)[A\mathbf{u}_1 + B\mathbf{w}_1] + \sin(\lambda_1 t)[B\mathbf{u}_1 - A\mathbf{w}_1] \\ & + \cos(\lambda_2 t)[C\mathbf{u}_2 + D\mathbf{w}_2] + \sin(\lambda_2 t)[D\mathbf{u}_2 - C\mathbf{w}_2] \\ & + Ee^{\lambda_r t}\mathbf{v}_1 + Fe^{-\lambda_r t}\mathbf{v}_2. \end{aligned} \quad (5)$$

We see that due to the coupling of the z -equation in the linear system, which in turn is due to our choice of $z_e \neq 0$, the linear order solution naturally contains periodic solutions in *both* linear frequencies. By setting $E = F = 0$ we may switch off the real modes, and by setting either $A = B = 0$ or $C = D = 0$ we have periodic solutions in the frequency of our choice.

4. High-order approximations to periodic orbits

The linear solutions given in the previous section will only closely approximate the motion of the sail given in (1) for small amplitudes. For larger amplitude periodic orbits, we compute high order approximations using the method of Linstedt-Poincaré.⁵ This procedure is well known and is described in the literature, for example.^{3,4}

We let ε be a perturbation parameter and expand each coordinate as $x \rightarrow x_e + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$ etc. We rescale the time coordinate $\tau = \omega t$ with $\omega = 1 + \varepsilon\omega_1 + \dots$, and group together the powers of ε in the high-order Taylor expansion of \mathbf{F} . We choose our linear solution to be

$$x_1 = kA_y \cos(\lambda\tau + \xi), \quad y_1 = A_y \sin(\lambda\tau + \xi), \quad z_1 = mA_y \cos(\lambda\tau + \xi), \quad (6)$$

where λ can be λ_1 or λ_2 , k, m are given in terms of components of the eigenvectors and A_y, ξ are free parameters. We use these linear solutions to build up non-linear approximations to periodic orbits one order at a time in the following way:

At each order of ε , the system to be solved will be

$$\begin{aligned} x_n'' - 2y_n' - ax_n - bz_n &= g_1(x_{n-1}, y_{n-1}, z_{n-1}, x_{n-2}, \dots) \\ y_n'' + 2x_n' - cy_n &= g_2(x_{n-1}, y_{n-1}, z_{n-1}, x_{n-2}, \dots) \\ z_n'' - dx_n - ez_n &= g_3(x_{n-1}, y_{n-1}, z_{n-1}, x_{n-2}, \dots), \end{aligned} \quad (7)$$

where prime denotes differentiation w.r.t. τ . The left hand side is the same form as the linear system (4), and on the right hand side the previous orders' solutions act as forcing terms. We use the freedom in ω_n to switch off the resonant or secular terms in the inhomogeneous part, that is those components on the right hand side of the form (6), and what remains is a series of trigonometric subharmonics up to order n .

In calculating the solution at n th order, we find two sets of solutions depending on whether n is even or odd. When n is even, the n th order solutions have the form (letting $T = \lambda\tau + \xi$)

$$\begin{aligned} x_n &= p_{n0} + p_{n2} \cos(2T) + \dots + p_{nn} \cos(nT), \\ y_n &= q_{n2} \sin(2T) + \dots + q_{nn} \sin(nT), \\ z_n &= s_{n0} + s_{n2} \cos(2T) + \dots + s_{nn} \cos(nT), \end{aligned} \quad (8)$$

with $\omega_{n-1} = 0$. When n is odd, the solutions at n th order have the form

$$\begin{aligned} x_n &= p_{n3} \cos(3T) + \dots + p_{nn} \cos(nT), \\ y_n &= q_{n1} \sin(T) + q_{n3} \sin(3T) + \dots + q_{nn} \sin(nT), \\ z_n &= s_{n1} \cos(T) + s_{n3} \cos(3T) + \dots + s_{nn} \cos(nT), \end{aligned} \quad (9)$$

and ω_{n-1} solves

$$\frac{2\lambda\beta_{n1}}{(c + \lambda^2)} + \frac{b\gamma_{n1}}{(e + \lambda^2)} - \alpha_{n1} = 0. \quad (10)$$

Here α_{nj}, β_{nj} and γ_{nj} are the coefficients of the \cos, \sin and \cos terms in the functions g_1, g_2 and g_3 respectively at order n given in (7), and the coefficients p_{nj}, q_{nj} and s_{nj} are given by

$$\begin{aligned} -(a + j^2\lambda^2)p_{nj} - 2j\lambda q_{nj} - bs_{nj} - \alpha_{nj} &= 0, \\ q_{nj} = \frac{-2j\lambda p_{nj} - \beta_{nj}}{(c + j^2\lambda^2)}, \quad s_{nj} = \frac{-dp_{nj} - \gamma_{nj}}{(e + j^2\lambda^2)}, \end{aligned} \quad (11)$$

with the exception of $q_{n0} = 0$ and $p_{n1} = 0$.

With these high order approximations, we may find approximate initial data from which to integrate the system of equations (1). However these will not evolve to exactly periodic trajectories, as they are only approximations to periodic solutions. Thus we must define a differential corrector with which to adjust the initial data so as to close the orbit.

With the high order approximations and the differential corrector,⁵ we may integrate the full nonlinear system of equations (1) to find large amplitude periodic orbits. An example of this for one family of periodic orbits is shown in Figure 3.

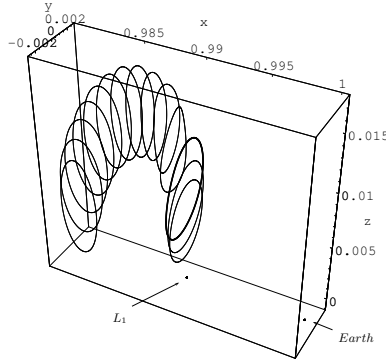


Fig. 3. A family of orbits with $\beta = 0.05$. Each orbit has the same amplitude and is about a different equilibrium point along the β level curve shown in Figure 2, each equilibrium point being defined by a different γ value. For reference the Earth (to scale) and L_1 are shown.

5. Equations of motion for the Solar Sail ERTBP

We will consider the solar sail ERTBP⁷ where the equations of motion are expressed in the rotating-pulsating frame:⁸

$$\begin{aligned} x'' - 2y' &= \frac{1}{1+e \cos f} \left(\frac{\partial \Omega}{\partial x} + a_x \right) \\ y'' + 2x' &= \frac{1}{1+e \cos f} \left(\frac{\partial \Omega}{\partial y} + a_y \right) \\ z'' + z &= \frac{1}{1+e \cos f} \left(\frac{\partial \Omega}{\partial z} + a_z \right) \end{aligned} \quad (12)$$

where

$$\Omega = \frac{1}{2}(x^2 + y^2 + z^2) + \frac{(1-\mu)}{|\mathbf{r}_1|} + \frac{\mu}{|\mathbf{r}_2|}$$

and where a_x, a_y, a_z are the components of the solar sail acceleration $\mathbf{a} = (a_x, a_y, a_z)^T$ and where $(\cdot)'$ denotes differentiating with respect to the true anomaly f . The pulsating-rotating frame is convenient as the true anomaly appears in the equations of motion as the independent variable and therefore we do not need to integrate Kepler's equations. We note that when $e = 0$ in (12) the equations are the equations of motion for the solar sail in the CRTBP. Therefore, we can treat the eccentricity, e , as a continuation parameter from a known periodic orbit in the CRTBP ($e=0$ in (12)). In the solar sail ERTBP the time appears explicitly in the equations of motion through the true anomaly f . Therefore, the differential equations (12) are non-autonomous. As the true anomaly f is periodic of period $2\pi k$ where k is an integer, any periodic orbit will also have to be of period $2\pi k$.

We can therefore continue from the 1 year periodic orbit highlighted in Figure 3 at $e = 0$ to the required $e = 0.0167$.

6. Periodic Orbits in the solar sail ERTBP

The continuation algorithm used to find periodic orbits above the ecliptic in the ERTBP is based on a monodromy variant of Newton's method.⁹ The initial orbit which will serve as a starter in the numerical continuation is given in the solar sail CRTBP⁵ ($e = 0$). If e is incremented by a suitably small value, the trajectory remains close enough for Newton's method to converge to a periodic orbit. This process is repeated until a closed orbit is found with the required $e = 0.0167$. In this section we apply the continuation to a 1 year periodic orbit above the ecliptic in the solar sail CRTBP.

The Newton method starts with an orbit $\mathbf{X}(t)$ initialized at $t = 0$ on a surface of section. In our case we require that the orbit be exactly 1 year so the return map in the rotating-pulsating frame is defined by a T-map of period $f = 2\pi$. It is assumed that the orbit is close to a natural periodic orbit $\mathbf{\Gamma}(t)$. The Newton method provides an iterative improvement to the choice of initial conditions for a periodic orbit:⁹

$$\mathbf{X}^*(0) = \mathbf{X}(0) + (I - M)^{-1}[\mathbf{X}(T) - \mathbf{X}(0)] \quad (13)$$

where $\mathbf{X}^*(0)$ is the improved initial condition and M is the monodromy matrix. One of the problems encountered with this Newton method is that the determinant of $(I - M)$ maybe zero and therefore the inverse is not well defined. In addition the determinant of $(I - M)$ may be very small and in these cases convergence of iterations may become poor. However, this problem is resolved by using the Moore-Penrose pseudo inverse. The implementation of Newton's method relies on the computation of the monodromy matrix as follows:

Let $\mathbf{\Gamma}(t)$ denote a periodic with period $T = 2\pi$ which satisfies the condition $\mathbf{\Gamma}(T) = \mathbf{\Gamma}(0)$, by letting $\mathbf{x} = \mathbf{X}(t) - \mathbf{\Gamma}(t)$, we may linearize the nonlinear system about this periodic orbit, resulting in the variational equations

$$\dot{\mathbf{x}} = A(t)\mathbf{x}$$

where

$$A(t) = A(t + T) = \left. \frac{\partial f}{\partial \mathbf{X}} \right|_{\mathbf{X}(t) = \mathbf{\Gamma}(t)}$$

Recasting the variational equations in terms of the state transition matrix (or principle fundamental matrix) $\Phi = \partial \mathbf{X}(t) / \partial \mathbf{X}(0)$, we have

$$\dot{\Phi} = A(t)\Phi, \quad \Phi(0) = I$$

where Φ is a 6×6 matrix. Two periodic orbits are illustrated in Figure 4 for $e = 0$ and $e = 0.0167$

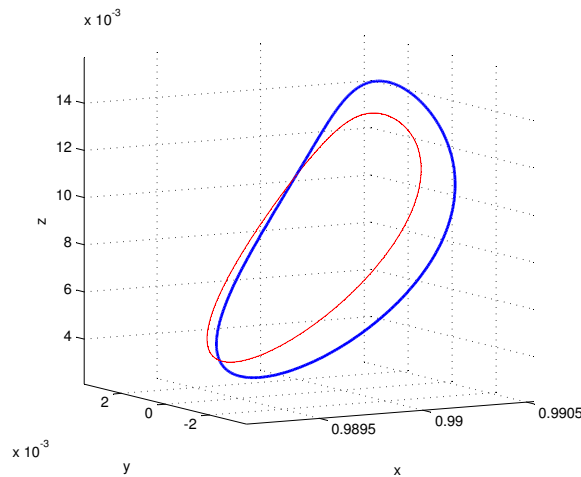


Fig. 4. Periodic Orbits in the rotating-pulsating frame: the thin line orbit is for $e = 0$ and the thick lined orbit is for $e = 0.0167$

7. Stability of periodic orbits in the Solar Sail RTBP

Additionally, we consider the linear stability of these periodic orbits in the nonlinear system. The stability of periodic orbits is determined using Floquet theory¹⁰ and depends on the behavior of the eigenvalues of the monodromy matrix M . The eigenvalues of M will be denoted by

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$$

To these eigenvalues λ_i correspond the characteristic (Floquet) exponents α_i defined by

$$\lambda_i = e^{\alpha_i T}$$

The orbit is stable at linear order if and only if the real parts of all the characteristic exponents are less than or equal to zero. In the circular case $e = 0$ the differential equations are autonomous and it is well known that one of the characteristic exponents of every periodic solution will be zero.¹¹

The eigenvalues of the periodic orbits above the ecliptic⁶ indicate that the orbits are unstable and of the form:

$$\{1, 1, \lambda_i, \bar{\lambda}_i, \lambda_r, 1/\lambda_r\}$$

where the bar denotes complex conjugacy. The unit eigenvalues appear as the solar sail CRTBP is an autonomous system¹¹ therefore the characteristic exponents are of the form

$$\{0, 0, \alpha_i, \bar{\alpha}_i, \pm\alpha_r\}$$

However, in the elliptic case, t is contained explicitly in the equations of motion, through the true anomaly f . The form of the eigenvalues in the elliptic case are:

$$\{\lambda_j, \bar{\lambda}_j, \lambda_i, \bar{\lambda}_i, \lambda_r, 1/\lambda_r\}$$

and the characteristic exponents are of the form

$$\{\alpha_j, \bar{\alpha}_j, \alpha_i, \bar{\alpha}_i, \pm\alpha_r\}$$

which is consistent with periodic orbits in the the classical ERTBP.¹² This implies that in the solar sail ERTBP there is a bifurcation at $e = 0$, in the sense that the eigenvalues change form. However, in each case the periodic orbit is unstable and requires active control to maintain the solar sail on the orbit.

8. Conclusion

In this paper we have considered periodic orbits above the ecliptic of a solar sail in the Earth-Sun restricted three-body problem. We begin with the Circular Restricted Three-Body Problem (CRTBP) where periodic orbits about equilibria are naturally present at linear order. Using the method of Lindstedt-Poincaré, we construct n th order approximations to periodic solutions of the nonlinear equations of motion and use these to compute high amplitude orbits above the ecliptic plane. In the second part of the paper we generalize to the Elliptic Restricted Three-Body Problem (ERTBP). A numerical continuation, with the eccentricity, e , as the varying parameter, is used to find periodic orbits of 1 year above the ecliptic, starting from a known orbit at $e = 0$ and continuing to the required eccentricity of $e = 0.0167$. The stability of these periodic orbits is investigated and they are shown to be unstable. Additionally it is shown that a bifurcation occurs at $e = 0$ in that the eigenvalues of the monodromy matrix change form, but the orbits remain unstable.

Acknowledgments

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