Polynomial Matrix Formulation-Based Capon Beamformer

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Abstract

This paper demonstrates the ease with which broadband array problems can be generalised from their well-known, straightforward narrowband equivalents when using polynomial matrix formulations. This is here exemplified for the Capon beamformer, which presents a solution to the minimum variance distortionless response problem. Based on the space-time covariance matrix of the array and the definition of a broadband steering vector, we formulate a polynomial MVDR problem. Results from its solution in the polynomial matrix domain are presented.

1. Introduction

Narrowband array processing methods are well established and rely in their computation on optimal tools such as the eigenvalue decomposition of a covariance matrix. When addressing broadband problems, the extension of these classical narrowband algorithms is generally not straightforward. The inclusion of tap-delay-line processors to manipulate explicit delays instead of narrowband gain factors to adjust the phase of signal can complicate matters. Often the covariance matrix has to be inflated to include a temporal dimension of a fixed, a-priori defined order Buckley 1987.

In this paper, we demonstrate how polynomial matrix formulations can be used to easily generalise and calculate broadband beamforming solutions from narrowband ones. We here extend work in Weiss *et al.* 2010 and address the definition and calculation of a Capon beamformer such as discussed in e.g. Lorenz & Boyd 2005.

2. Steering Vectors and Space-Time Covariance Matrix

For an M-element array receiving signals $x_m[n]$, $m = 1 \ldots M$, we define a data vector $\mathbf{x}[n] \in \mathbb{C}^M$. If a source illuminates the array, its signal will be received with different time delays $\tau_m \in \mathbb{R}$, and omitting any attenuation, we have

$$\mathbf{x}[n] = \begin{bmatrix} s[n-\tau_1] \\ \vdots \\ s[n-\tau_M] \end{bmatrix} = \begin{bmatrix} d[n-\tau_1] \\ \vdots \\ d[n-\tau_M] \end{bmatrix} * s[n] = \mathbf{a}[n] * s[n] \; . \tag{2.1}$$

The vector $\mathbf{a}[n] \in \mathbb{R}^M$ contains fractional delay filters (see e.g. Laakso *et al.* 1996), and characterises the direction of the source. Its z-transform $\boldsymbol{a}(z) = \sum_n \mathbf{a}[n] z^{-n}$, or short $\boldsymbol{a}(z) \bullet\!\!-\!\!\circ\, \mathbf{a}[n]$, is referred to as a broadband steering vector in Weiss *et al.* 2010.

If the array signal measures a superposition of K source signals $s_k[n]$, each characterised by a steering vector $\boldsymbol{a}_k(z)$, $k = 1 \dots K$, then

$$\mathbf{x}[n] = \sum_{k=1}^{K} \mathbf{a}_k[n] * s_k[n] + \mathbf{v}[n] \tag{2.2}$$

with $\mathbf{v}[n]$ containing independent and identically distributed Gaussian noise. The space-time covariance of the array vector is $\mathbf{R}[\tau] = \mathcal{E}\{\mathbf{x}[n]\mathbf{x}^{\mathrm{H}}[n-\tau]\} \in \mathbb{C}^{M\times M}$, with its z-transform the cross spectral density (CSD) matrix $\boldsymbol{R}(z) \bullet\!\!-\!\!\circ\, \mathbf{R}[\tau]$,

$$\boldsymbol{R}(z) = \sum_{k=1}^{K} \boldsymbol{a}_k(z)\boldsymbol{a}_k(z)P_k(z) + \mathbf{I}\sigma_v^2 \,, \tag{2.3}$$

where $P_k(z)$ is the power spectral density (PSD) of the kth source and σ_v^2 the noise power due to $\mathbf{v}[n]$ experienced at any array element.

3. Narrowband Capon Beamformer

In the narrowband case, the above definitions for steering vectors and space-time covariance matrix are evaluated on the unit circle, $z = e^{j\Omega}$, and for a particular normalised angular frequency Ω_0. With this evaluation, the steering vector $\mathbf{a}_k = \boldsymbol{a}_k(e^{j\Omega_0})$ collapses delays to scalar entries implementing phase shifts, and $\mathbf{R} = \boldsymbol{R}(e^{j\Omega_0})$ also contains only scalar entries rather than polynomials.

If the array elements are followed by complex weight w_m organised in a vector $\mathbf{w} \in \mathbb{C}^M$, then the MVDR problem minimises the output power

$$\mathbf{w}_{\mathrm{opt}} = \arg\min_{\mathbf{w}} \mathbf{w}^{\mathrm{H}}\mathbf{R}\mathbf{w} \tag{3.1}$$

$$\text{s.t.} \quad \mathbf{C}^{\mathrm{H}}\mathbf{w} = \mathbf{f} \tag{3.2}$$

subject to constraints. W.l.o.g., if the beamformer is to receive source $k = 1$, then in the simplest case, $\mathbf{C} = \mathbf{a}_1$, and $\mathbf{f} = 1$ ensures unit (i.e. distortionless) gain. If the steering vectors $\mathbf{a}_k$, $k = 2 \dots L \leq K$ of L interfering sources are known, then this knowledge can be embedded as

$$\mathbf{C} = [\mathbf{a}_2 \ \dots \ \mathbf{a}_L] \quad , \qquad \mathbf{f} = \big[1 \ \underbrace{0 \ \dots \ 0}_{L}\big] \tag{3.3}$$

in the constraint equation (3.2).

The analytic solution to the problem in (3.1) and (3.2) is given as

$$\mathbf{w}_{\mathrm{opt}} = \mathbf{R}^{-1}\mathbf{C}\left(\mathbf{C}^{\mathrm{H}}\mathbf{R}^{-1}\mathbf{C}\right)^{-1}\mathbf{f} \,, \tag{3.4}$$

which is known as the Capon beamformer. Since the covariance $\mathbf{R}$ can be ill-conditioned, the inversion can be regularised in a number of fashions, see e.g. Lorenz & Boyd 2005. Unless the constraint only embraces the look direction $\mathbf{a}_1$, the term $\mathbf{C}^{\mathrm{H}}\mathbf{R}^{-1}\mathbf{C}$ may also be poorly conditioned.

4. Broadband Capon Beamformer

For the definition of the broadband problem, we follow the narrowband formulation in Sec. 3 but use broadband instead of narrowband quantities. The beamformer now has to implement a tap-delay-line with impulse responses $w_m[n]$ following every sensor, leading to a filter vector $\mathbf{w}[n] \circ\!\!-\!\!\bullet\, \boldsymbol{w}(z)$, whose aim is to minimise the output power. Obtain the latter by integrating over the output PSD, such that

$$\boldsymbol{w}_{\text{opt}} = \arg\min_{\mathbf{w}(z)} \oint_{|z|=1} R_e(z)\frac{dz}{z} \tag{4.1}$$

$$\text{s.t.} \quad \tilde{\boldsymbol{C}}(z)\boldsymbol{w}(z) = \boldsymbol{f}(z) \; . \tag{4.2}$$

The polynomial constraint matrix $\boldsymbol{C}(z)$ contains in its columns the broadband steering vectors $\boldsymbol{a}_k(z)$, $k = 1 \ldots (L+1)$,

$$\boldsymbol{C}(z) = [\boldsymbol{a}_1(z)\; \bar{\boldsymbol{a}}_2(z)\; \ldots\; \bar{\boldsymbol{a}}_{L+1}(z)] \; . \tag{4.3}$$

It is assumed that $\boldsymbol{a}_1(z)$ points in look direction, and that the remaining columns define interferers. In this case, the constraining vector takes the form

$$\boldsymbol{f}(z) = \big[F(z)\; \underbrace{0\; \ldots\; 0}_{L}\big]^{\text{T}} \; , \tag{4.4}$$

where $F(z)$ is the desired array response in look direction.

If no interferers are known and the desired response in look direction is unity, then the constraint in (4.2) simplifies to

$$\tilde{\boldsymbol{a}}_1(z)\boldsymbol{w}(z) = 1 \; . \tag{4.5}$$

Since $\boldsymbol{a}_k(z)$ contains — potentially fractional — delays, a $\boldsymbol{w}(z)$ satisfying (4.5) is likely non-causal, and the inclusion of a delay will be required to force a viable solution $\boldsymbol{w}_{\text{opt}}(z)$.

Different from a narrowband beamformer, the broadband formulation includes a component at DC. While this may be suppressed in an implementation, at DC the general constraint matrix $\boldsymbol{C}(z)$ becomes rank one, while $\mathbf{f}(z)$ imposes two inconsistent equations. One solution to this is to modify the broadband steering vectors of interferering sources, $\boldsymbol{a}_k(z)$, $k = 2 \ldots L$, to have a highpass characteristic $H(z)$, such that $\bar{\boldsymbol{a}}_k(z) = H(z)\boldsymbol{a}_k(z)$, $k = 2 \ldots L$ are included in (4.3).

The analytical solution to the above MVDR problem is obtained by extending (3.4) to a polynomial notation,

$$\boldsymbol{w}_{\text{opt}}(z) = \boldsymbol{R}^{-1}(z)\boldsymbol{C}(z)\left(\boldsymbol{C}^{\text{H}}(z)\boldsymbol{R}^{-1}(z)\boldsymbol{C}(z)\right)^{-1}\boldsymbol{f}(z) \; , \tag{4.6}$$

such that a broadband Capon beamformer solution arises. The inversions in (4.6) are performed on parahermitian matrix terms $\boldsymbol{R}(z)$ and $\boldsymbol{C}^{\text{H}}(z)\boldsymbol{R}^{-1}(z)\boldsymbol{C}(z)$, which can be addressed as discussed in Weiss *et al.* 2015.

5. Numerical Examples

To demonstrate the polynomial broadband version of Capon beamformer, we test the approach for the scenario in Weiss *et al.* 2010, with 4 sources impinging on an $M = 8$ element linear equispaced sensor array. The signal of interest is located at 30° off broadside, with interferers at -30dB signal to interference noise ratio are located at angles $\vartheta = \{-40°, -10°, 80°\}$; these interferers have a highpass PSD with a lower passband edge at $\Omega = \frac{\pi}{10}$. Additionally, the array data is corrupted by additive white Gaussian noise at 20dB SNR.

The space-time covariance matrix is estimated over a sufficiently long window of data to yield a consistent estimate. The AWGN regularises the inverse $\boldsymbol{R}^{-1}(z)$, which is calculated via the SMD algorithm in Redif *et a.* 2015 following the procedure in Weiss *et al.* 2015. The broadband steering vectors have an order of 50. If the interfering sources are omitted from the problem formulation, the Capon beamformer only requires a correction

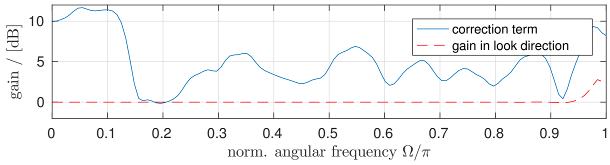

FIGURE 1. Gain of correction term $\tilde{\boldsymbol{a}}_1(z)\boldsymbol{R}(z)\boldsymbol{a}_1(z)$ and beamformer gain in look direction.

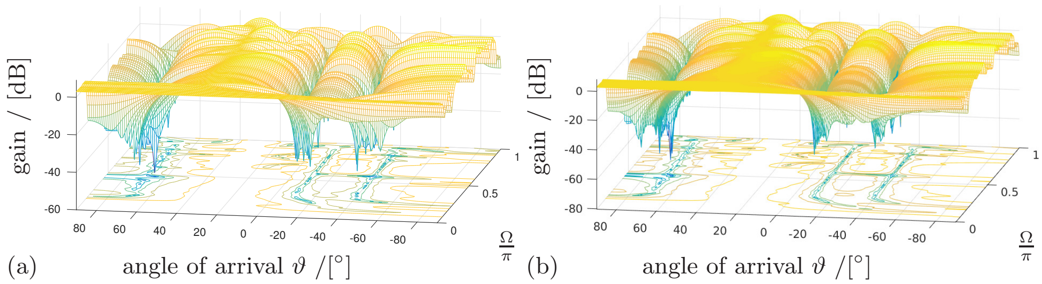

FIGURE 2. Directivity pattern of polynomial Capon beamformer (a) with look direction constraint only and (b) including constraints on interferers.

by the scalar term $\tilde{\boldsymbol{a}}_1(z)\boldsymbol{R}^{-1}(z)\boldsymbol{a}_1(z)$, which is shown in Fig. 1. The total beamformer has a polynomial order of 180, and a gain response as shown in Fig. 2(a), with the gain look direction shown in Fig. 1 indicating a distortionless response across most of the spectrum. The interferers are appropriately attenuated.

The same calcuation is applied for more extensive contraints with $L = 3$, which requires an additional regularisation to evaluate $(\tilde{\boldsymbol{C}}(z)\boldsymbol{R}^{-1}(z)\boldsymbol{C}(z) + \epsilon\mathbf{I})^{-1}$, with $\epsilon \ll 1$. The result is a Capon beamformer of polynomial order 260 with a gain response as shown in Fig. 2(b), demonstrating suitable suppression of the highpass interferers.

6. Conclusions

Based on polynomial matrix formulations, exemplarily a broadband version of the classical narrowband Capon beamformer has been derived. Only the inclusion of DC requires some extra considerations, and can be addressed by treating either the signal of interest or the interferers as highpass processes. Although not expanded here, careful regularisation is required when inverting polynomial matrices that may be poorly conditioned in space and frequency.

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