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COLLISION AND RE-ENTRY ANALYSIS UNDER ALEATORY AND EPISTEMIC UNCERTAINTY

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This paper presents an approach to the design of optimal collision avoidance and re-entry maneuvers considering different types of uncertainty in initial conditions and model parameters. The uncertainty is propagated through the dynamics, with a non-intrusive approach, based on multivariate Tchebycheff series, to form a polynomial representation of the final states. The collision probability, in the cases of precise and imprecise probability measures, is computed considering the intersection between the uncertainty region of the end states of the spacecraft and a reference sphere. The re-entry probability, instead, is computed considering the intersection between the uncertainty region of the end states of the spacecraft and the atmosphere.

INTRODUCTION

The analysis and design of optimal collision avoidance and re-entry maneuvers are of significant importance to mitigate the risk of a collision in orbit. It is known from previous studies[1,2] that some assumptions on the nature of the uncertainty region enclosing the predicted state of a spacecraft, or space debris, at the time of the collision are not generally applicable. Linearized models are not adequate, while techniques like the unscented transformation do not fully capture nonlinearities but simply improve the estimation of the first two statistical moments. Furthermore, the assumption of a Gaussian distribution of the uncertain quantities is in itself restrictive and requires a perfect knowledge of the model and of the source of uncertainty. In Jones et al.,[1] De Mars and Jah[2] and Morselli et al.[3] the nonlinearities in the propagation of uncertainty in the initial conditions have been captured with a high order representations of the uncertainty region of the terminal states. However, some assumptions were introduced on the nature of uncertainty.

This paper proposes a nonlinear approach to the design of optimal collision avoidance and re-entry maneuvers that accounts epistemic uncertainties in model parameters and probability distributions. The approach starts from a nonlinear representation of the uncertainty region of the terminal states with an expansion in Tchebycheff polynomials defined on sparse grids. Both uncertainties on initial conditions and model parameters are considered to partially account for unmodeled components (epistemic uncertainty in the definition of the model). We drop the assumption of perfectly known a priori probability distributions and we consider instead the case of sets of probability measures. The expectation of a collision is then computed considering all the pieces of evidence

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supporting the proposition that an intersection between the area defining all possible terminal positions of a spacecraft and a control volume is an empty set. An inverse proposition holds for the expectation of a re-entry event.

A worst case scenario maneuver maximizing the probability of success is then computed. The approach is base on a similar technique developed for the robust multiobjective design of asteroid deflection actions.[4] In both events an impulsive maneuver is applied. Since no assumptions on the nature of the uncertainty are introduced a priori, both collision avoidance and re-entry can be treated with the same approach, and an equivalent formulation. Besides, in the collision case we compared the results of the full numerical optimization with the analytical method which calculates the optimal deflection direction that maximizes the displacement of each virtual impactor on the impact plane[5]

The paper is structured as follows. First we present the dynamical model and the uncertainty space used in the numerical simulations. Then, we briefly introduce the Tchebycheff polynomial approximation theory. The encounter plane and re-entry geometry is then explained. After that, we describe the strategies used to quantify both aleatory and epistemic uncertainty. Finally, we present the results and the conclusions.

DYNAMICAL MODEL

We consider a simplified model with the Earth as main perturbator and a small body (e.g., an operational satellite) orbiting around it with negligible mass. We also take into account the solar radiation pressure (SRP) and the atmospheric drag. In an equatorial reference frame, the dynamical equations can be written as

$$\begin{aligned} \dot{\mathbf{r}} &= \mathbf{v}\,, \\ \dot{\mathbf{v}} &= \mathbf{F}_{earth} + \mathbf{F}_{srp} + \mathbf{F}_{drag}\,, \end{aligned} \tag{1}$$

where $\mathbf{r}, \mathbf{v}$ are the position and velocity vector, $\mathbf{r}_0 = \mathbf{r}(t_0), \mathbf{v}_0 = \mathbf{v}(t_0)$ are the initial condition at time t_0, and

$$\mathbf{F}_{earth} = -\frac{\mu}{r^3}\mathbf{r}\,, \tag{2}$$

$$\mathbf{F}_{srp} = \frac{\phi_\odot}{c}\,C_R\,\frac{A}{m}\,\hat{\mathbf{S}}\,, \tag{3}$$

$$\mathbf{F}_{drag} = -\frac{1}{2}\,C_D\,\frac{A}{m}\,\rho\,\mathbf{v}^2\hat{\mathbf{v}}\,, \tag{4}$$

where μ is the Earth gravitational parameter, R_e is the mean Earth's equatorial radius, (x, y, z) and r are, respectively, the components and the modulus of $\mathbf{r}$, $\phi_\odot$ is the solar radiation flux, c is the velocity of light, C_R is the reflectivity coefficient, A/m is the area-to-mass ration, $\hat{\mathbf{S}}$ is the direction of the Sun, C_D is the drag coefficient, and ρ is the density of the air atmosphere.

The initial uncertainty space is considered to be a hyperrectangle

$$\mathbf{r}_0 \in [\underline{\mathbf{r}}, \overline{\mathbf{r}}]\,, \quad \mathbf{v}_0 \in [\underline{\mathbf{v}}, \overline{\mathbf{v}}]\,, \quad \mathbf{p} \in [\underline{\mathbf{p}}, \overline{\mathbf{p}}]\,, \tag{5}$$

where $\mathbf{p}$ is the vector of model parameters $\mathbf{p} = [A/m, C_R, C_D, \rho]^T$.

A deflection maneuver can be reduced to an impulsive change of velocity $\Delta\mathbf{v}$, and included in the uncertainty model by increasing of the bounds of the velocity vector by a quantity $\pm\Delta\mathbf{v}$. We call

this space the complete uncertainty space. The components of the deflection vectors represent the control or decision variables that need to be optimized to obtain the best and worst case scenarios for the collision or the re-entry maneuver.

TCHEBYCHEFF POLYNOMIAL APPROXIMATION

The propagation of the initial uncertainty region through the dynamics (1) is usually performed using a Monte Carlo method over thousands of sample points. Recent works[1] have shown that it is possible to replace the propagation through the dynamics of the initial uncertainty region with a polynomial approximation of it. There are a number of possible choices for the polynomial basis. Among them polynomial chaos expansion (PCE) and mixture of Gaussian process are the most popular. In this work we use a Tchebycheff series on Smolyak sparse grids.[6] It is important, at this point, to remark that the use of high order polynomial expansions (including PCEs) to avoid running a full Monte Carlo simulation has been introduced in astrodynamics only recently but has been widely used in other fields for the past two decades. A lot of the existing machinery, therefore, has been imported and adapted to the case of space dynamics.

Sparse grids have been introduced by Sergey Smolyak (1962)[7] and allow to represent, integrate and interpolate functions on multidimensional hypercubes. A complete polynomial basis of maximum degree 4 in 10 unknown variables consists of $1\,001$ elements, while the corresponding sparse basis contains only 221 elements. We follow the construction of disjoint sparse grid presented in Judd et al.,[6] that use the extrema of Tchebycheff polynomials (also known as Tchebycheff-Gauss Lobatto points or Clenshaw-Curtis points).

Let n be the number of uncertain variables and $\mu \in \mathbb{N}^+$ be the level of approximation of the sparse grid. The the complete polynomial basis is given by

$$\mathcal{B} = \{\mathcal{T}_{\alpha_1}, \mathcal{T}_{\alpha_2} \ldots, \mathcal{T}_{\alpha_s}\}\,, \qquad s \in \mathbb{N}^+\,,$$

where $\alpha_i = (\alpha_{i_1}, \ldots, \alpha_{i_n})$ denotes the multi-index array corresponding to the i-th multidimensional Tchebycheff polynomial

$$\mathcal{T}_{\alpha_i} = \prod_{j=1}^{n} T_{\alpha_{i_j}}\,,$$

chosen in the space of all polynomial of degree at most 2^μ in n variables such that

$$\alpha_i \in \mathcal{H}^{n,\mu} = \{\alpha \in \mathbb{N}^n \,:\, \alpha \text{ satisfies the Smolyak rule at level } \mu\}\,,$$

and $T_{\alpha_{i_j}}$ is the univariate Tchebycheff polynomial corresponding to the variable of index j. For example, for $n = 2$ and $\mu = 1$ the Smolyak rule gives

$$\mathcal{H}^{2,1} = \{(0,0), (1,0), (0,1), (2,0), (0,2)\}\,,$$

and the corresponding Tchebycheff polynomial basis is

$$\mathcal{T}_{(0,0)} = 1\,,\ \mathcal{T}_{(1,0)} = x\,,\ \mathcal{T}_{(1,0)} = y\,,\ \mathcal{T}_{(2,0)} = 2x^2 - 1\,,\ \mathcal{T}_{(0,2)} = 2y^2 - 1\,.$$

Note the absence of the cross term $\mathcal{T}_{(1,1)} = xy$ from the basis. For details see Riccardi et al.[8]

The response function can be approximated with the finite series

$$\hat{Y}(X_0) = \sum_{\alpha \in \mathcal{H}^{n,\mu}} c_\alpha\, \mathcal{T}_\alpha(X_0)\,, \tag{6}$$

where each c_α is the unknown coefficient with respect to the element $\mathcal{T}_\alpha$, and X_0 are the initial uncertainty variables and belong to an hypercube.

The unknown coefficients can be computed by inverting the linear system

$$HC = Y\,, \tag{7}$$

with

$$H = \begin{bmatrix} T_{\alpha_1}(x_1) & \dots & T_{\alpha_s}(x_1) \\ \vdots & \ddots & \vdots \\ T_{\alpha_1}(x_s) & \dots & T_{\alpha_s}(x_s) \end{bmatrix}, \qquad C = \begin{bmatrix} c_{\alpha_1} \\ \vdots \\ c_{\alpha_s} \end{bmatrix}, \qquad Y = \begin{bmatrix} Y_1 \\ \vdots \\ Y_s \end{bmatrix}, \tag{8}$$

where $x_1, \dots, x_s$ are the Tchebycheff nodes in the sparse grid and the components of Y are the true values of the dynamical systems in these points. The system (7) cannot be inverted if the matrix H has not full rank. In most of the cases, this is guaranteed by choosing the Tchebycheff nodes. Figure 1 shows a two-dimensional sparse grid for different levels.

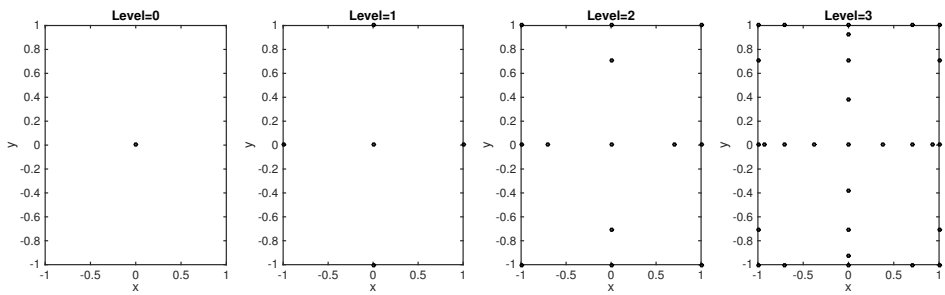

Figure 1. Sparse grid on Tchebycheff nodes for $\mu = 0, \dots, 3$.

COLLISION AND RE-ENTRY GEOMETRY

The probability of success of a collision avoidance maneuver is computed with respect to the miss distance on the target plane while the success of a re-entry maneuver is computed with respect to the distance from the center of the Earth. In this respect the underlying assumptions are that the target object is spherical and the Earth and its atmosphere are spherical and homogeneous. The methodology presented in the remainder of the paper, however, is not dependent on the model and is applicable as is to higher fidelity models.

The target plane or b-plane is the plane orthogonal to the relative velocities of the incoming objects with respect to the target and passing through the center of the target. In formulae, indicating with v_1 and v_2 the velocities of the spacecraft and of the target object at the assumed time of close approach, it is possible to define a frame centered in the target and with the following coordinate axis $(\hat{\xi}, \hat{\zeta}, \hat{\eta})$:

$$\hat{\eta} = \frac{v_1 - v_2}{|v_1 - v_2|}\,, \qquad \hat{\xi} = \frac{v_2 \times \hat{\eta}}{|v_2 \times \hat{\eta}|}\,, \qquad \hat{\zeta} = \hat{\eta} \times \hat{\xi}\,. \tag{9}$$

Hence, the b-plane is the subspace $(\hat{\xi}, \hat{\zeta})$. The miss distance, also called b-parameter, is the distance between the projection of the spacecraft on this plane and the target:

$$b = \sqrt{\xi^2 + \zeta^2}\,. \tag{10}$$

We say that a collision may occur if the proposition $b < b^*$ has a non-null probability of being true, with b^* a fixed threshold.

In the re-entry case, we simply consider a reference sphere $R = R_E + h_A$ centered in the center of the Earth with R_E the mean radius of the Earth and h_A a reference altitude. The quantity of interest, in this case, is the modulus of the position vector $\mathbf{r}$. We say that a re-entry may occur if the proposition $r < r^*$ has a non-null probability of being true, with r^* a fixed threshold.

Optimal impulsive deflection maneuver

Recently, Vasile and Colombo[5] proposed an analytical method to increase the Minimum Orbit Intersection Distance (MOID) between two confocal Keplerian orbits under the effect of an impulsive deflection. Combining together the proximal motions equations and Gauss' planetary equations, they derived a simple analitycal formulation to compute for each deflection time t_d the transition matrix that links the velocity deviation δv at t_d to the position deviation r_{moid} at the time of close approach t_{moid}.

In formulae, let A_{moid} and G_d the matrices representing the proximity motion at t_{moid} and Gauss' equation at t_d, then in matrix form we have

$$\delta\mathbf{r}(t_{moid}) = A_{moid} G_d \delta\mathbf{v}(t_d) = T\delta\mathbf{v}(t_d) \; . \tag{11}$$

The optimal problem consists to maximize

$$||\delta\mathbf{r}(t_{moid})|| = \max(T\delta\mathbf{v}(t_d)) \quad \text{subject to } ||\delta\mathbf{v}|| \leq v_0 \; , \tag{12}$$

where v_0 is the maximum admissible deflection magnitude. The problem is equivalent to maximize the associated quadratic form T^tT and it can been solved by choosing an impulse vector $\delta\mathbf{v}(t_d)$ parallel to the eigenvector $\mathbf{s}_1$ of T^tT conjugated to the maximum eigenvalue λ_1. Then the optimal impulse in the tangential-normal-out-of-plane reference frame is $\delta v_{opt} = v_0 s_1$ and the maximum deflection is given by $\delta r_{max} = \sqrt{\lambda_1} v_0$.

If the impact parameter is the quantity of interest, then the problem becomes:

$$||\delta b(t_{moid})|| = \max(T_b\delta\mathbf{v}(t_d)) \quad \text{subject to } ||\delta\mathbf{v}|| \leq v_0 \; , \tag{13}$$

with the matrix T_b defined as:

$$T_b = Q\mathcal{R}T \; , \tag{14}$$

where $\mathcal{R}$ is the matrix defining the change of coordinates from the radial-transverse-out-of-plane reference frame to the $\hat{\xi}, \hat{\eta}, \hat{\zeta}$ reference frame, and Q is a square matrix that selects only the $\hat{\xi}$ and $\hat{\zeta}$ components.

UNCERTAINTY MODEL

We consider two types of uncertainty affecting the outcome of a deflection: aleatory and epistemic. The former type accounts for non reducible uncertainties that are generally well quantified with a known distribution, the latter instead accounts for reducible uncertainties that are due to our current knowledge of the system. Epistemic uncertainties include model uncertainties and uncertainties on the probability of an event, therefore, they cannot be fully captured with a single known distribution or by assuming a known correlation among variables. In this section we will investigate

the use of two different techniques for the treatment of epistemic uncertainties: one using Belief functions and p-boxes and the other using parametric distributions. In the latter case the assumption is that the probability density function of the uncertain quantities belongs to a family of known distributions, while in the former case the assumption is that the actual probability distribution is unknown but belongs to a set whose upper and lower probabilities are known. Note that the same approach used for parametric distributions is applicable to non-parametric distributions formed by a weighted combination of kernels.

In both the case of aleatory and epistemic uncertainty, the general quantity of interest is the intersection between two objects at a time t in the configuration space. An extension to finite times is possible by considering multiple snapshots or by introducing the time as further stochastic variable. Given $S_{B_1}(\mathbf{u}_{B_1}, X_{B_1,0}, t, t_0)$ and $S_{B_2}(\mathbf{u}_{B_2}, X_{B_2,0}, t, t_0)$ the sets of all possible positions occupied by object B_1 and B_2 at time t, we are interested in the probability associated to:

$$V = \int_{S_{B_1} \bigcap S_{B_2}} d\mathbf{r} \tag{15}$$

or

$$P(A) = \mathbb{E}(I_A) = \int_U I_A \, p(\mathbf{u}_{B_1}) p(\mathbf{u}_{B_2}) d\mathbf{u}_{B_1} d\mathbf{u}_{B_2} \,, \tag{16}$$

with

$$A = \{(\mathbf{u}_{B_1}, \mathbf{u}_{B_2}) \in U | V \leq V^*\} \,, \tag{17}$$

where $p(\mathbf{u}_{B_1})$ and $p(\mathbf{u}_{B_2})$ are the probability densities associated to the uncertain quantities of B_1 and B_2, I_A is the indicator function of this set and V^* a fixed quantity.

In the following we will analyze the simpler case in which one of the two objects is a given, known, sphere. In this case, we can replace the volume integral with the radial distance r from the center of the sphere and study the probability of $r < r^*$.

In the case of a collision at high velocity we can further reduce the problem from the distance to the center of the sphere in three dimensional space to the miss distance b measured on the impact plane of the sphere at time t. In the case of a re-entry we will use the distance from the center of the Earth.

Worst-Case Scenario

The miss distance is bound between an upper limit given by the best case scenario under uncertainty and a lower limit given by the worst case scenario under uncertainty. Conversely, the distance from the center of the Earth is bound between an upper limit given by the worst case scenario under uncertainty and a lower limit given by the best case scenario under uncertainty.

In order to find the worst and best case scenarios, we need to solve the following optimization problems. For the collision case it is

$$\overline{b} = \max_{\mathbf{d}} \max_{\mathbf{u}} b(\mathbf{u}, \mathbf{d}) \,, \qquad \underline{b} = \max_{\mathbf{d}} \min_{\mathbf{u}} b(\mathbf{u}, \mathbf{d}) \,, \tag{18}$$

while for the re-entry case it is

$$\overline{r} = \min_{\mathbf{d}} \max_{\mathbf{u}} r(\mathbf{u}, \mathbf{d}) \,, \qquad \underline{r} = \min_{\mathbf{d}} \min_{\mathbf{u}} r(\mathbf{u}, \mathbf{d}) \,, \tag{19}$$

where $\mathbf{u}$ represents the uncertainty state vector and $\mathbf{d}$ the control vector. We remind that $\mathbf{d}$ is the deflection maneuver, while $\mathbf{u}$ are the position and velocity of the objects at the initial time and/or some dynamical parameters. The solution corresponding to $\underline{b}$ represents the worst scenario for the collision, while for the re-entry the worst case scenario is given by the solution relative to $\overline{r}$.

Aleatory Uncertainty Model

When all the uncertainties are aleatory, the expectations $\mathbb{E}[r \leq r^*], \mathbb{E}[b \geq b^*]$ are given by the integrals

$$\int_{A_{r^*}} r(\mathbf{u})\, p(\mathbf{u})\, d\mathbf{u}\,, \qquad \int_{A_{b^*}} b(\mathbf{u})\, p(\mathbf{u})\, d\mathbf{u}\,, \tag{20}$$

where A_{r^*} and A_{b^*} are the sets

$$A_{r^*} = \{\mathbf{u} \mid r(\mathbf{u}) \leq r^*\}\,, \qquad A_{b^*} = \{\mathbf{u} \mid b(\mathbf{u}) \geq b^*\}\,, \tag{21}$$

for some $r^* \in [\underline{r}, \overline{r}]$ and $b^* \in [\underline{b}, \overline{b}]$, with $\underline{r}, \overline{r}$ given by eq. (18) and $\underline{b}, \overline{b}$ given by eq. (19), and p is the joint probability density function. Assuming that there is no correlation among variables, we can write p as

$$p = \prod_{j=1}^{n} p_j\,, \tag{22}$$

where p_j is the j-th marginal distribution mass relative to the variable with index j.

To compute each integral in eq. (20), we sample the initial uncertain space with a number M of points $\mathbf{u}_k$ and relative weights w_k. Then we evaluate the function of interest at each sample point and we construct the indicator function

$$I_{\bar{r}}^k = \begin{cases} 1 & \text{if } r(\mathbf{u}_k) \leq \bar{r} \\ 0 & \text{otherwise} \end{cases} \qquad k = 1, \ldots, M\,. \tag{23}$$

The integral can then be approximated with the expression

$$\sum_{k=1}^{M} \prod_{j=1}^{n} p_j(u_{j,k})\, I_{\bar{r}}^k\, w_k\,, \tag{24}$$

where $u_{j,k}$ is the j-th component of $\mathbf{u}_k$ and n is its dimension. Analogous formulae hold for b^*.

One possible choice for the samples is to build a grid using q Gauss-Lobatto points and weights per coordinate. The integral then becomes:

$$\sum_{k=1}^{M} I_{\bar{r}}^k \prod_{j=1}^{n} p_j(u_{j,k})\, w_{j,k}\,. \tag{25}$$

Such a grid, however, would require $M = q^n$ quadrature points and weights and an equal number of function evaluations, which is unfordable in high dimension even if one has only to evaluate the Tchebycheff polynomial.

An alternative approach is to use a low discrepancy sequence to generate M sample points in the domain U and then simply approximate the integral with:

$$\int_A f(\mathbf{u})\, p(\mathbf{u})\, d\mathbf{u} \approx \frac{1}{M} \sum_{k=1}^{M} I_{\bar{r}}^k\, p(\mathbf{u}_k) \tag{26}$$

where the samples $\mathbf{u}_k$ are taken from the low discrepancy sequence.[9] In the following we will use the Halton sequence.

Epistemic Uncertainty Model

In the following we will describe the analysis for the re-entry case. For the collision analogous formulae hold.

The construction of the upper and lower expectation curves in the range $[\underline{r}, \overline{r}]$ can be computed following two approaches. The first is based on the use of belief functions and basic probability assignments (bpa) and the second on sampling a parametric distribution function with unknown parameters.

Both approaches will lead to an upper and a lower expectation on the occurrence of an event. The event of interest is defined by the proposition:

$$I = r(\bar{\mathbf{d}}, \mathbf{u}) \leq r^* \tag{27}$$

where $\bar{\mathbf{d}}$ comes from the solution of the minmax problem. Since the probability of $r(\mathbf{d}, \mathbf{u}) \leq r^*$ is the expectation of I we will calculate the upper and lower expectations of I starting from two different assumptions on our knowledge of the probability associated to the uncertain quantities.

Belief Functions We first define a bpa structure starting from a p-box, for each uncertain variable, in which the upper and lower probabilities are known. The p-box is then partitioned in a number n_l of levels with probability mass m_{ij}.

Thus, each uncertain variable with index i is defined by n_l intervals u_{ij}, with $j = 1, ..., n_l$, each of which has probability mass m_{ij}. Then one can compute the Cartesian product of all the intervals to get a set of so called focal elements θ_k with associated probability mass $m(\theta_k) = \prod_i m_{ij}(u_{ij})$, with $i = 1, ..., n_u$ and n_u the number of uncertain variables.

Given the focal elements, the Belief in the realization of an event is defined as

$$Bel_{r^*} = \sum_{\theta \subseteq A_{r^*}} m(\theta)\,, \tag{28}$$

while the Plausibility in the realization of the same event is given by

$$Pl_{r^*} = \sum_{\theta \bigcap A_{r^*} \neq \emptyset} m(\theta)\,, \tag{29}$$

where the set A is defined as

$$A_{r^*} = \{\mathbf{u} | r(\bar{\mathbf{d}}, \mathbf{u}) \leq r^*\}\,. \tag{30}$$

In order to find the focal elements to be included in (28) we evaluate $\max_{\mathbf{u} \in \theta} r(\mathbf{d}, \mathbf{u})$ and check that this value is below r^*. Conversely, in order to find the focal elements to be included in (29) we compute $\min_{\mathbf{u} \in \theta} r(\mathbf{d}, \mathbf{u})$ and check that this value is below r^*. The summations (28) and (29) are then repeated for different $r^* \in [\underline{r}, \overline{r}]$.

Parametric Distribution In order to avoid running two optimization problems per focal element for all the focal elements, an approximated approach is proposed. Consider the case in which one can reasonably assume that the uncertainty can be quantified with a family of beta distributions with unknown parameters α and β (any other parametric or non-parametric distribution would equally

work). The upper and lower probability distributions are the solutions of the two optimization problems

$$\min_{\alpha,\beta} \int_{A_{r^*}} p(\mathbf{u})\, d\mathbf{u}\,, \qquad \max_{\alpha,\beta} \int_{A_{r^*}} p(\mathbf{u})\, d\mathbf{u}\,, \tag{31}$$

where r^* is a fixed value in the interval $[\underline{r}, \overline{r}]$, p is the product of probability given by eq. (22) where each marginal density mass p_j is a beta distribution function with parameters α_j, β_j.

In the optimization problem, we replaced the calculation of the exact integral with the same procedure described for the computation of aleatory expectations. Using eq. (24) and the notation of the previous section, we re-write the optimization problem as

$$\min_{\alpha,\beta} \sum_k \prod_j p_j(u_{j,k})\, w_{j,k}\,, \qquad \max_{\alpha,\beta} \sum_k \prod_j p_j(u_{j,k})\, w_{j,k}\,, \tag{32}$$

where $k \in \{k | \mathbf{u}_k \in A_{r^*}\}$. Once the value of the two extreme beta distributions are found for r^*, the exact upper and lower expectation can be refined by solving the multidimensional quadrature problem with a higher number of sample points.

Sensitivity Analysis and Problem Decomposition

It is useful at this point to introduce the possibility to decompose the problem in subproblems of smaller dimension and obtain the sensitivity of the quantity of interest with respect to a selected group of variables. We use a technique similar to the one normally employed in anchored-ANOVA decomposition or cut-HDMR. Both decompositions have the goal to reduce the dimensionality of the problem and allow a more tractable calculation of the integrals even in high dimension.

We use the solution of the worst case scenario as an anchor to separate the effect of aleatory and epistemic variables. In other words we fix the value of the aleatory variables to that of the worst case solution when we study the sensitivity to the epistemic variables and vice versa.

We can now define two subsets of U, called U_a and U_e, respectively for aleatory and epistemic uncertainty, so that:

$$U = U_a \times U_e\,. \tag{33}$$

and calculate the quantities

$$Bel_{e,r^*} = \sum_{\theta \subseteq A_{e,r^*}} m(\theta)\,, \tag{34}$$

$$Pl_{e,r^*} = \sum_{\theta \bigcap A_{e,r^*} \neq \emptyset} m(\theta)\,, \tag{35}$$

with the set A_{e,r^*} defined as

$$A_{e,r^*} = \{\mathbf{u}_e \mid f(\mathbf{u}_e, \bar{\mathbf{u}}_a, \bar{\mathbf{d}}) \leq r^*\} \tag{36}$$

Problems (31) restricted to the epistemic variables then become:

$$\min_{\alpha,\beta} \int_{A_{e,r^*}} p(\mathbf{u}_e)\, d\mathbf{u}_e\,, \qquad \max_{\alpha,\beta} \int_{A_{e,r^*}} p(\mathbf{u}_e)\, d\mathbf{u}_e\,, \tag{37}$$

While the probability associated to the aleatory variables is:

$$P_a(r \leq r^*) = \int_{U_a} I(r(\bar{\mathbf{u}}_e, \mathbf{u}_a, \mathbf{d}) \leq r^*)\, p(\mathbf{u}_a)\, d\mathbf{u}_a \,. \tag{38}$$

It is interesting now to extend the use of the decomposition to get a cheap approximated representation of the upper and lower expectation. Let the polynomial representation be

$$\sum_{\alpha \in \mathcal{H}^{n,\mu}} c_\alpha\, \mathcal{T}_\alpha(X_0) = h(\mathbf{u}_e) g(\mathbf{u}_a) + e \,, \tag{39}$$

where h, g are polynomials and e is a small error. Then one can solve problem (37) to get a value for α_{min}, β_{min} and α_{max}, β_{max} and use these values to compute the full upper and lower expectations:

$$E_l = \int_{A_{r^*}} p_{\alpha_{min},\beta_{min}}(\mathbf{u}_e) p(\mathbf{u}_a)\, d\mathbf{u} \,, \qquad E_u = \int_{A_{r^*}} p_{\alpha_{max},\beta_{max}}(\mathbf{u}_e) p(\mathbf{u}_a)\, d\mathbf{u} \,, \tag{40}$$

with

$$A_{r^*} = \{\mathbf{u}_e \mid f(\mathbf{u}_e, \mathbf{u}_a, \bar{\mathbf{d}}) \leq r^*\} \,. \tag{41}$$

For the case of both the radial distance and the miss distance, that are positive defined quantities, this approximation has provided good results at a fraction of the computational cost.

Discussion

The method proposed in this paper requires the calculation of a multidimensional integral over the uncertain variables. If the probability density function of the uncertain variables is known a priori then one can exploit the surrogate model to build a further polynomial representation only of the probability associated to the quantity of interest. Since no Monte Carlo simulation is required once the Tchebycheff polynomial is available, computationally speaking the approach is equivalent to the approach of Coppola.[10]

Other authors tackled the problem with different approaches to the solution of the integral that gives the probability of a collision. Alfano[11] compared the computational cost of the main methods to compute collision probability for short-term encounters. In chronological order, they are: Foster (1992),[12] Chan (1997),[13] Patera (2001)[14] and Alfano (2005).[15] Each of them present a different approach to solve the integration of a Gaussian probability density function over a circular area in the encounter plane.

Although here we do not derive an analytical expression of the integral, if the probability distribution is known, the use of a polynomial representation on Tchebycheff bases allows for a quick calculation of the integral and, therefore, can be considered computationally equivalent to most of the methods analysed by Alfano.[11] Note, in fact, that the relation between uncertain variables and quantity of interest is given by the Tchebycheff polynomial expansion. That polynomial expansion also maps the uncertainty in the initial state to the one in the final state in the case in which an analytical mapping is not available.

If the distribution of the uncertain variables is not known a priori but can be represented as a family of parametric distributions (or non-parametric kernels), then the calculation of the upper and lower expectations requires the solution of a multidimensional integral for each possible value of the parameters characterizing the family.

If the distribution of the uncertain variables is not known a priori but cannot be represented as a family of parametric distributions (or non-parametric kernels), then the calculation of the upper and lower expectation requires the solution of n_{fe} optimization problems over all the focal elements to find the extremes of the distributions enveloping all possible distributions describing the quantity of interest.

It is also important to underline that the representation of the state space with Tchebycheff polynomials does not imply any probability distribution on either the uncertain inputs or the quantity of interest. On the other hand once the polynomial representation of the state space is available, it can be used to map the initial state and uncertain quantities onto the final states and quantity of interest. Likewise the solution of the *minmax* and *minmin* problems does not require any assumption on the distribution of the uncertain quantities. Furthermore, the use of a uniform distributions over the uncertain intervals to quantify epistemic uncertainties would be incorrect as it would need the strong assumption that the uncertain quantities are uniformly distributed.

NUMERICAL EXPERIMENTS

To illustrate the applicability of the proposed approach, we consider two hypothetical maneuvers to be conducted one before a conjunction event and the other before a re-entry event. For each example we study both the case of all aleatory uncertainties and the case of mixed aleatory and epistemic uncertainties.

We consider the dynamical model given by eq. (1). The uncertainty variables are the component of the position and velocity vectors in a equatorial reference frame, and the following model parameters: ratio-to-mass ratio A/m, reflectivity coefficient C_R, drag coefficient C_D, altitude H. A variation of H is considered to simulate the uncertainty on the density ρ of the Earth atmosphere.

The uncertainty space for the two cases is summarized in Table 1. For the collision we consider an orbit in LEO region with initial condition $r_a = 7\,200$ km and $v_a = \sqrt{-\mu/a + 2\mu/r_a}$, with semi-major axis $a = 6\,950$ km and μ the Earth's gravitational parameter. While for the re-entry we set $r_a = 42\,000$ km and $a = 24\,350$ km. Both orbits start at the apogee and we assume that the time of interest is the next time of passage at perigee. The uncertainties for re-entry have been increased with respect to the real case. The control variables are the component of the deflection maneuver and a variation of $1\% v_a$ in all their components is assumed for the re-entry, and of $5 v_a \cdot 10^{-5}$ for the collision.

The complete uncertainty regions, including the uncertainty in the controls, are shown in Figure 2 and Figure 3. Using an Intel i7 3.40GHz, a single polynomial evaluation takes about 4 millisecond and to compute the unknown coefficients for a level of approximation 2 it takes about 0.03 seconds. The Tchebycheff approximation gives a maximum absolute error of order 10^{-5} km for the collision and 1 km for the re-entry.

The worst case scenario is given by eq. (18), (19) for the collision and re-entry problem respectively. To find the solution of the minmax problem a numerical solver, called *ideaminmax*,[16] derived from IDEA,[17] is used. The solution of the maxmin problem associated to the collision case, instead, was solved in two ways. One using ideaminmax and the other solving problem (13) for the worst possible trajectory under uncertainty. The two approaches give equivalent results, although the solution of problem (13) is analytical. Figure 4 shows how the initial uncertainty region moves when the deflection corresponding to the worst case scenario is applied.

$\mathbf{u}$	Lower bound	Upper bound	$\mathbf{u}$	Lower bound	Upper bound
u_1	$-r_a(1+10^{-3})$	$r_a(1+10^{-3})$	u_1	$-r_a(1+10^{-5})$	$r_a(1+10^{-5})$
u_2	$-r_a \cdot 10^{-3}$	$r_a \cdot 10^{-3}$	u_2	$-r_a \cdot 10^{-5}$	$r_a \cdot 10^{-5}$
u_3	$-r_a \cdot 10^{-3}$	$r_a \cdot 10^{-3}$	u_3	$-r_a \cdot 10^{-5}$	$r_a \cdot 10^{-5}$
u_4	$-v_a \cdot 10^{-3}$	$v_a \cdot 10^{-3}$	u_4	$-v_a \cdot 10^{-5}$	$v_a \cdot 10^{-5}$
u_5	$-v_a(1+10^{-3})$	$v_a(1+10^{-3})$	u_5	$-v_a(1+10^{-5})$	$v_a(1+10^{-5})$
u_6	$-v_a \cdot 10^{-3}$	$v_a \cdot 10^{-3}$	u_6	$-v_a \cdot 10^{-5}$	$v_a \cdot 10^{-5}$
u_7	0.01	0.1	u_7	0.01	0.1
u_8	0.5	0.9	u_8	0.8	0.9
u_9	1	2	u_9	1	2
u_{10}	8	9	u_{10}	8	9

Table 1. Uncertainty space given as lower and upper bound for the re-entry (left) and collision (right). The variables $u_1, \ldots u_6$ are the components of the position and velocity vectors, and $u_7, \ldots, u_10$ represent the dynamical parameters $A/m, C_R, C_D$ and $F_{10.7}$, respectively.

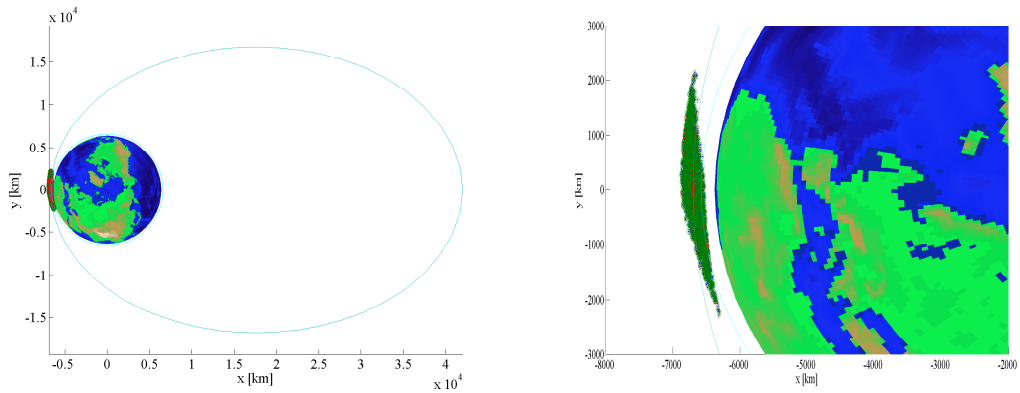

Figure 2. Complete uncertainty region of the re-entry maneuver.

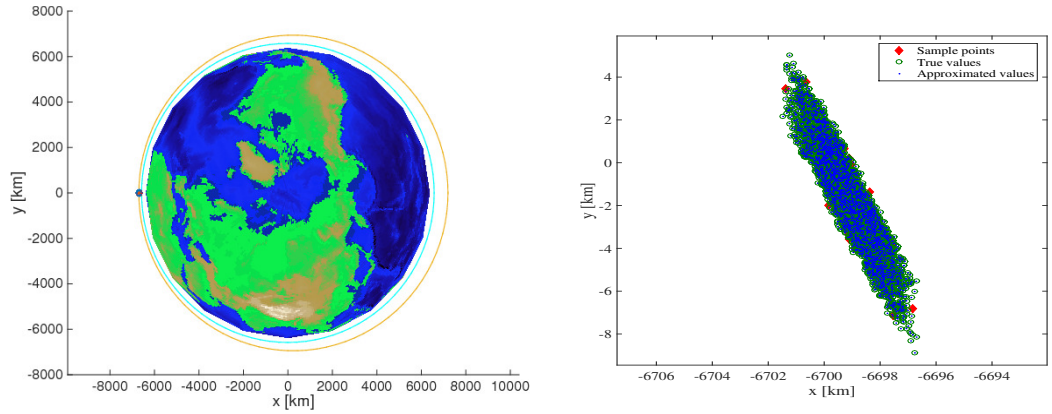

Figure 3. Complete uncertainty region of the collision maneuver.

Upper and Lower Expectations Once an optimal $\Delta\mathbf{v}$ maneuver is obtained from the solution of the minmax (maxmin respectively) problem, one can be interested in studying the expected proba-

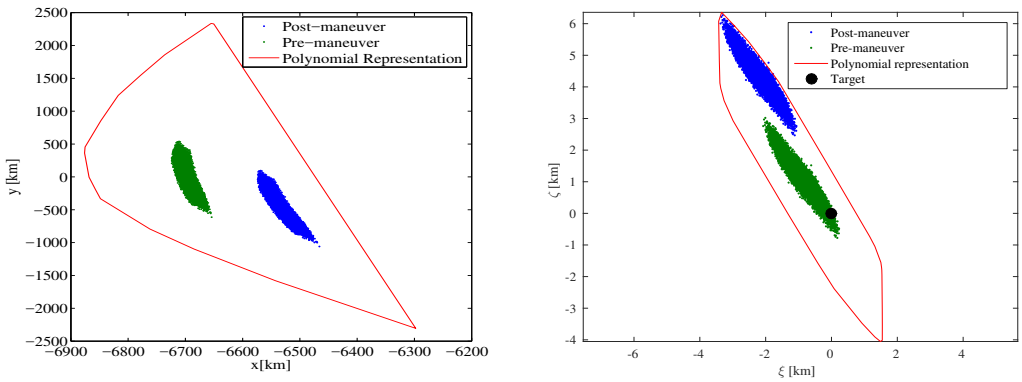

Figure 4. Post maneuver uncertainty region for the re-entry (left) and collision (right).

bility of re-entry or collision. We first assume that the uncertainty associated to all uncertain variables is epistemic. The marginal probabilities are assumed to belong to sets of probability measures bounded by two beta distributions with, respectively, $\alpha = 3$ and $\beta = 1$, and $\alpha = 1$ and $\beta = 3$. This p-box is then partitioned in two parts, resulting in two partially overlapping intervals with $m = 0.5$ for each coordinate.We then calculate the value of Bel and Pl for different radii $r^* \in [\underline{r}, \overline{r}]$ and b-parameters $b^* \in [\underline{b}, \overline{b}]$.

We then assume that the first six variables are aleatory with associated probability density taken from a beta distribution with parameters $\alpha = 1$ and $\beta = 1$. The uncertainty on model parameters is instead considered to be epistemic. Their probability is assumed to belong to a set of beta distributions with parameters $\alpha, \beta \in [1, 3]$. The parameters α and β are then numerically optimized by solving problems (32) with IDEA for different values of the threshold r^* (respectively b^*) to obtain the upper expectation E_u and the lower expectation E_l. The integrals in (31) are computed using $10\,000$ samples from the Halton low discrepancy sequence. Once the optimal α and β are found the upper and lower expectations are recomputed with $2 \cdot 10^5$ low discrepancy samples. A total of 6 values of r^* (respectively b^*) are taken to build an approximation to the whole curves. Figure 6 shows the results of the two approaches. As expected, the upper and lower expectations are inside the belief and plausibility curves since in the full epistemic case we have less information. The separation between the upper and lower expectation is the result of the epistemic uncertainty on model parameters. If this epistemic uncertainty was zero the two curves would coincide.

Sensitivity Analysis It is now interesting to decompose the problem and look at the sensitivity to aleatory and epistemic uncertainties separately. We use the decomposition technique explained in the previous section taking the worst case solution as anchor point. We first fix the value of all epistemic variables to the corresponding components of the worst case solution. We then compute the expectation only for the aleatory part by solving integral (20) over the space of the aleatory uncertain variables for different thresholds $\bar{r} \in [r_{min}, r_{max}]$ for the re-entry, $\bar{b} \in [b_{min}, b_{max}]$ for the collision. Since the dynamical parameters and control variables are fixed to value of the minmax (maxmin respectively) solution, the dimensionality of the problem is reduced to six. The integral is calculated numerically using formula (25) and low discrepancy sequences with $15\,625$ samples. It is assumed that aleatory uncertainties are quantified with a beta distribution with parameters $\alpha = 1$ and $\beta = 1$. The validity of this approximation has been compared to a fit of the distribution of the quantity of interest using the Matlab *fitdist* function (see Figure 5). The figure also shows the

approximation given by a full tensor product integral using an equal number of Gauss-Lobatto points and weights. For the collision case, the curves match well even with few sample points. For the re-entry case, there is a visible difference between the *fitdist* solution and the numerical integrals. One reason for this difference is that *fitdist* tries to find the best beta distribution fitting the data. It is however very possible that the data are not exactly distributed as a beta. Note that the same approach can be applied to any distribution, even on infinite support. In the latter case the minmax/maxmin solutions correspond to any $n\sigma$ confidence interval.

Finally, in Figure 7 we present the results of the sensitivity analysis with respect to epistemic uncertainty only. In this case, the state vector variables are kept fixed at the value obtained from the solution of the worst case scenario. The focal elements are constructed only taking the four uncertain model parameters and the intervals and probability masses per coordinate are built as before but this time partitioning the beta distribution in three parts with mass $m = 1/3$ each. Belief and Plausibility are then computed for different thresholds r^* (respectively b^*) according to (34) and (35).

Then we assumed that the epistemic uncertainty associated to the four model parameters could be quantified with families of beta distributions with unknown parameters $\alpha, \beta \in [1, 3]$. Using IDEA, we compute the optimal values for α and β associated to each of the four model parameters for different threshold. We took 10 values of r^* for the re-entry case and 6 values of b^* for the collision case. For the solution of problems (37) the integrals were approximated taking 1 000 low discrepancy samples. Once the values of α and β were available the upper and lower expectations were recomputed with 10 000 low discrepancy samples. As in the previous case, the upper and lower expectations are bounded from above and below by the Pl and Bel respectively. In this case, however, the enclosure is much tighter as there is no contribution of the aleatory variables and the worst case solution is used as an anchor point. Still Bel and Pl represent the lowest and upper-most limits as there is no assumption on the distribution bounded by the beta distributions. Conversely, the upper and lower expectations Eu and El were computed assuming that the uncertainty in the input was indeed quantified by a family of beta distributions.

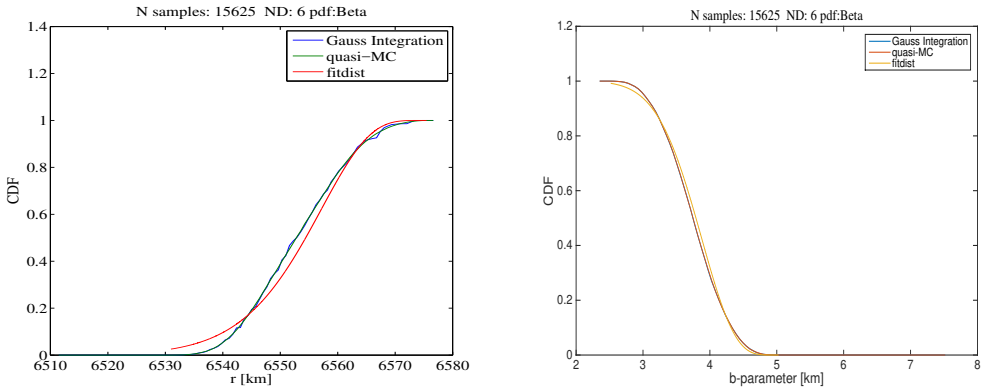

Figure 5. Aleatory decomposition for the re-entry (left) and collision (right).

FINAL REMARKS

We presented an approach to the design of optimal collision avoidance and re-entry maneuvers considering both aleatory and epistemic uncertainty in initial conditions and model parameters. To

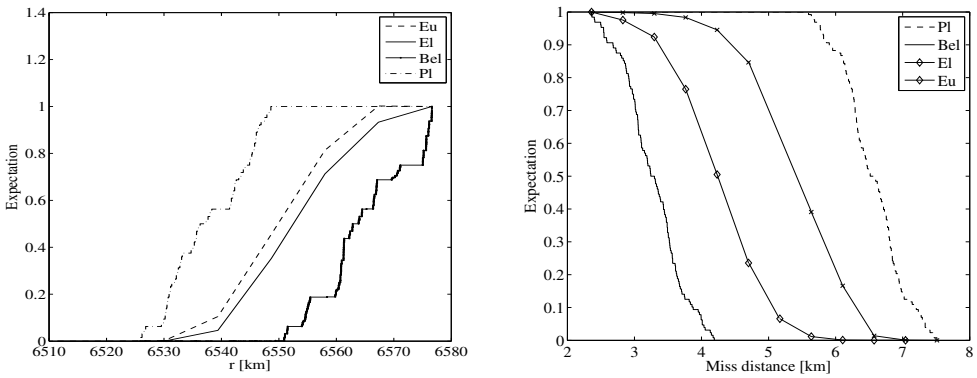

Figure 6. Upper and lower expectation for the re-entry (left) and collision (right).

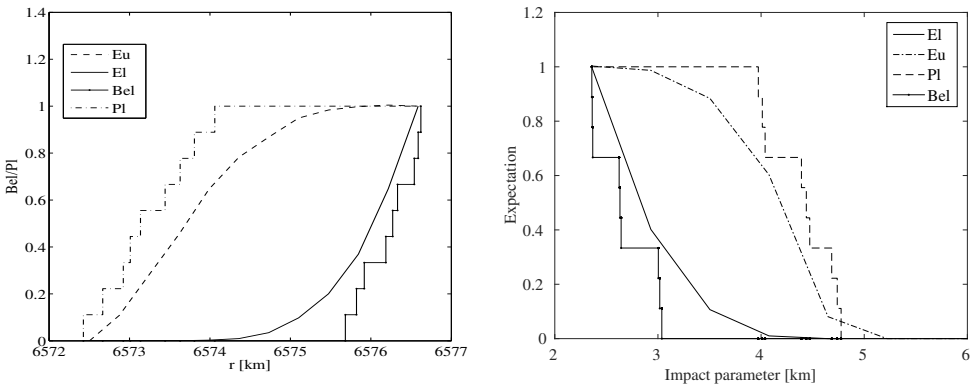

Figure 7. Upper and lower expectation for the re-entry (left) and collision (right) with the decomposition strategy.

reduce the computational cost of the optimization, we substitute the true dynamics with its approximation based on multivariate Tchebycheff polynomial expansion. The polynomial expansion represents a nonlinear mapping between the initial uncertainty space and the terminal conditions.

The re-entry probability, both in the case of precise and imprecise probability measures, is computed considering the radial distance from the Earth. The collision probability, instead, is computed considering the intersection between the uncertainty region of the end states of the spacecraft and a reference sphere. However, also for the collision case one could consider the 3D distance from the center of the sphere and this would not affect the uncertainty methods proposed in this paper.

Finally, we observe that the hypothesis of short-term encounter could be dropped by using an innovative intrusive approach to propagate the dynamics by overloading the operators with operators in the Tchebycheff polynomial algebra.[8] The Tchebycheff polynomial approximation computed at the time of interest becomes the initial state of the dynamics. The intrusive approach will return at each time step a Tchebycheff polynomial which can be then used in the optimization algorithm. Since we already start from a polynomial expansion and we need to propagate for time span less than the orbital period, the computational cost is reduce with respect to a Monte Carlo or sparse grid sampling. Besides, the intrusive approach does not require to set an a-priori time step. One could always restart the intrusive dynamics from the closest Tchebycheff polynomial.

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