# Robust Stabilization of Hybrid Uncertain Stochastic Systems by Discrete-time Feedback Control

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## SUMMARY

This paper aims to stabilize hybrid stochastic differential equations (SDEs) with norm bounded uncertainties by *feedback controls based on the discrete-time observations of both state and mode*. The control structure appears only in the drift part (the deterministic part) of an SDE and the controlled system will be robustly exponentially stable in mean-square. Our stabilization criteria are in terms of linear matrix inequalities (LMIs) whence the feedback controls can be designed more easily in practice. An example is given to illustrate the effectiveness of our results. Copyright © 2010 John Wiley & Sons, Ltd.

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## 1. INTRODUCTION

Stochastic systems have received a lot of attention as stochastic modeling has played a more and more important role in many branches of science and engineering in recent years (see e.g. [1–4]).

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In practice, some stochastic systems, such as electric power systems, manufacturing systems and financial systems, may experience abrupt changes in their structures and parameters because of environment changes, random failures of components, etc. Hybrid stochastic differential equations (SDEs) (also known as SDEs with Markovian switching) have been used to model such systems (see e.g. [5–8]). An area of particular interest in the study of hybrid SDEs is the automatic control, with subsequent emphasis being placed on the stability analysis. There is an intensive literature in the area (see e.g. [9–18]). Particularly, Mao [10,11] are two of the most cited papers while Mao and Yuan [13] is the first book in this area.

It is well known that parameter uncertainties exist inherently in many dynamic systems and are frequently a cause of instability and performance degradation. Therefore, a great deal of attention has been devoted to the study of linear or nonlinear uncertain systems. In particular, the problem of robust stability and stabilization for uncertain systems has become an important issue and attracted an increasing interest (see e.g. [19–23] and references therein). Moon et al. [24] investigated the robust stabilization of uncertain state-delayed systems. A delay-dependent stabilization condition using a memoryless controller was formulated in terms of matrix inequalities. Based on a reduction method, a delayed feedback controller was designed in [25] to stabilize uncertain systems with time-varying delay. As for the stochastic cases, Lu et al. [26] considered the robust state-feedback stabilization of uncertain stochastic delay systems by applying the linear matrix inequality method. Wang et al. in [27] designed a state feedback controller to stabilize bilinear uncertain time-delay stochastic systems with Markovian jumping parameters in mean square sense. A robust delayed-state-feedback controller that exponentially stabilized uncertain stochastic systems was proposed in [28]. It is worth noting that the state feedback controllers in these papers require continuous observations of the system state for all time, which is expensive and impractical. Recently, for the sake of saving costs and being more realistic, Mao [29] studied the stabilization problem of hybrid stochastic systems by feedback control based on discrete-time state observations, which develops the corresponding studies for deterministic systems (see e.g. [30–32]). Then Mao et al. in [33] provided a better bound on the duration $\tau$ between two consecutive state observations and

generalized the theory to a class of nonlinear hybrid stochastic systems, and You et al. [34] weakened the global Lipschitz assumption on coefficients and further investigated the asymptotic stabilization of nonlinear hybrid stochastic systems.

It is obserbed that the discrete-time feedback controls in [29, 33, 34] are based on the discrete-time observations of the state but they still depend on the continuous-time observations of the mode. This is perfectly fine if the mode of the system is fully observable at no cost. However, the mode is not obvious in many real-world situations and it costs to identify the current mode of the system in practice. For example, Geromel J and Gabriel G [35] pointed out the necessity to design the feedback control based on discrete-time observations of both state and mode from the numerical point of view when studying the state feedback sampled-data control design problem of Markov jump systems. Therefore, it is necessary and reasonable that we identify the mode at the discrete times when we make observations for the state. These motivate us to go a step further to design a feedback controller $u(x(\delta(t)), r(\delta(t)))$, where $\delta(t) = [t/\tau]\tau$ for $t \geq 0$ and $\tau > 0$, which is based on the discrete-time observations of both state and mode, to stabilize hybrid uncertain stochastic systems. It should be pointed out that our work is not a simple generalization of the existing results (see e.g. [29, 33, 34]). Mathematically speaking, the analysis in this paper becomes much more complicated due to the difficulties arisen from the discrete-time Markov chain $r([t/\tau]\tau)$. This is because that $r(t)$ is a jumping process (namely $r(t-)$ and $r(t+)$ may be at different states) and hence $r([t/\tau]\tau)$ dose NOT tend to $r(t)$ as $\tau$ tends to 0, although $x([t/\tau]\tau)$ tends to $x(t)$. Therefore, the effect caused by $x([t/\tau]\tau)$ can be controlled in terms of $x(t)$ and $x(t) - x([t/\tau]\tau)$ but it is inappropriate to expect the effect caused by $r([t/\tau]\tau)$ is similar to that caused by $r(t)$ no matter how small $\tau$ is. These difficulties will be presented in detail later when we perform the proofs of our main results.

The organization of the rest of this article is as follows. After giving some notations at the end of this section, the main problem is stated in Section 2 and related definition and some useful lemmas are also given for later use. The main results are presented in Section 3. Section 4 covers a numerical example to demonstrate the main results in Section 3. Finally, the article is concluded in Section 5.

*Notation:* For $x \in R^n$, $|x|$ denotes its Euclidean norm. If $A$ is a vector or matrix, its transpose is denoted by $A^T$. For a matrix $A$, we let $|A| = \sqrt{\text{trace}(A^T A)}$ be its trace norm and $\|A\| = \max\{|Ax| : |x| = 1\}$ be the operator norm. If $A$ is a symmetric matrix ($A = A^T$), denote by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ its smallest and largest eigenvalue, respectively. For two symmetric matrices $A$ and $B$, $A > (<, \geq, \leq)B$ means that $A - B$ is positive definite (negative definite, positive semidefinite, negative semidefinite). The integer part of a real number $x$ will be denoted by $[x]$. Moreover, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ is a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous with $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets). Let $w(t) = (w_1(t), \cdots, w_m(t))^T$ be an $m$-dimensional Brownian motion defined on the probability space. Denote by $L^2_{\mathcal{F}_t}(\mathbb{R}^n)$ the family of all $\mathcal{F}_t$-measurable $\mathbb{R}^n$-valued random variables $\xi$ such that $\mathbb{E}|\xi|^2 < \infty$. Let $r(t), t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N \times N}$ given by

$$\mathbb{P}\{r(t + \Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$ and $\gamma_{ij} \geq 0$ is the transition rate from $i$ to $j$ if $i \neq j$, while $\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $w(\cdot)$. It is known that almost all sample paths of $r(t)$ are step functions with a finite number of simple jumps in any finite subinterval of $R_+ := [0, \infty)$. We stress that almost all sample paths of $r(t)$ are right continuous. Let $C(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$ denote the family of continuous functions from $\mathbb{R}^n \times \mathbb{R}_+$ to $\mathbb{R}_+$, and $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times S; \mathbb{R}_+)$ denote the family of continuous functions $V(x, t, i)$ from $\mathbb{R}^n \times \mathbb{R}_+ \times S$ to $\mathbb{R}_+$ such that for each $i \in S$, $V(x, t, i)$ is continuously twice differentiable in $x$ and once in $t$.

## 2. PRELIMINARIES AND PROBLEM STATEMENT

Given an unstable hybrid uncertain stochastic system

$$dx(t) = [A(r(t)) + \Delta A(t, r(t))]x(t)dt + \sum_{k=1}^{m}[B^k(r(t)) + \Delta B^k(t, r(t))]x(t)dw_k(t), \qquad (1)$$

on $t \geq 0$, with initial data

$$x(0) = x_0 \in L^2_{\mathcal{F}_0}(R^n), \quad r(0) = r_0 \in S, \tag{2}$$

where for any $i \in S, k = 1, 2, \cdots, m, A(i) = A_i$ and $B^k(i) = B_i^k$ are known constant matrices, while uncertainties $\Delta A(t, i), \Delta B^k(t, i)$ are assumed to be norm bounded, i.e.

$$\Delta A(t, i) = L_A F_A(t) N_i, \quad \Delta B^k(t, i) = L_B F_B(t) E_i^k, \tag{3}$$

with known constant matrices $L_A, N_i, L_B, E_i^k$ and unknown matrix functions $F_A(t)$ and $F_B(t)$ having Lebesgue-measurable elements and satisfying

$$F_A^T(t) F_A(t) \leq I, \quad F_B^T(t) F_B(t) \leq I, \quad \forall t \geq 0. \tag{4}$$

Now we aim to design a feedback control $u(x(\delta(t)), r(\delta(t)))$ based on the discrete-time observations of both state and mode in the drift part so that the controlled system

$$
\begin{aligned}
dx(t) =& [(A(r(t)) + \Delta A(t, r(t)))x(t) + C(r(t))u(x(\delta(t)), r(\delta(t)))]dt \\
&+ \sum_{k=1}^{m} [B^k(r(t)) + \Delta B^k(t, r(t))]x(t)dw_k(t)
\end{aligned}
\tag{5}
$$

will be mean-square exponentially stable, where $C(i) = C_i$ is a given matrix while $u$ is a mapping from $\mathbb{R}^n \times S$ to $\mathbb{R}^n$, $\tau > 0$ and

$$\delta(t) = [t/\tau]\tau \quad \text{for } t \geq 0, \tag{6}$$

in which $[t/\tau]$ is the integer part of $t/\tau$. We choose the structure control of the form $u(x, i) = K(i)x$ and then the controlled system (5) becomes

$$
\begin{aligned}
dx(t) =& [(A(r(t)) + \Delta A(t, r(t)))x(t) + C(r(t))K(r(\delta(t)))x(\delta(t))]dt \\
&+ \sum_{k=1}^{m} [B^k(r(t)) + \Delta B^k(t, r(t))]x(t)dw_k(t).
\end{aligned}
\tag{7}
$$

The equation (7) is in fact a special hybrid uncertain stochastic system with a bounded variable delay and the coefficients satisfy the Lipschitz condition and the linear growth condition. According to the existence-uniqueness theorem on SDDEs with Markovian switching (see e.g. [13]), equation (7) has a unique solution $x(t)$ under initial conditions (2) such that $\mathbb{E}|x(t)|^2 < \infty$ for all $t \geq 0$.

To close this section, we cite the following definition (see [13]) and two lemmas on matrix inequalities (see [36]) that will be in this paper.

*Definition 1*

The controlled system (7) with initial conditions (2) is said to be robustly exponentially stable in mean square, if there is a positive constant $\lambda > 0$ such that for any uncertainties satisfying (3) and (4), the solution $x(t)$ satisfies

$$\limsup_{t \to \infty} \frac{1}{t} \log(\mathbb{E}|x(t)|^2) \leq -\lambda.$$

*Lemma 1*

For any vectors $u \in \mathbb{R}^q, v \in \mathbb{R}^l$ and a matrix $M \in \mathbb{R}^{q \times l}$, the inequality

$$2u^T M v \leq r u^T M G M^T u + \frac{1}{r} v^T G^{-1} v$$

holds for any symmetric positive definite matrix $G \in \mathbb{R}^{l \times l}$ and number $r > 0$.

*Lemma 2*

Let A, B, D, F, W be matrices with suitable dimensions. If $W > 0, F^T F \leq I$, then for any number $\varepsilon > 0$ such that $W^{-1} - \varepsilon DD^T > 0$, it holds that

$$(A + DFB)^T W(A + DFB) \leq A^T(W^{-1} - \varepsilon DD^T)^{-1} A + \varepsilon^{-1} B^T B.$$

## 3. MAIN RESULTS

Let us first present two lemmas in order to prove our main result.

*Lemma 3*

For any $t \geq 0, v > 0$ and $i \in S$,

$$P(r(s) \neq i \quad for \quad some \quad s \in [t, t+v]|r(t) = i) \leq 1 - e^{-\bar{\gamma}v},$$

where $\bar{\gamma} = \max_{i \in S}(-\gamma_{ii})$.

*Proof*

Given $r(t) = i$, define the stopping time

$$\rho_i = inf\{s \geq t : r(s) \neq i\},$$

where and throughout this paper we set inf $\emptyset = \infty$ (in which $\emptyset$ denotes the empty set as usual). It is well known (see [37]) that $\rho_i - t$ has the exponential distribution with parameter $-\gamma_{ii}$. Hence

$$P(r(s) \neq i \quad for \quad some \quad s \in [t, t+v]|r(t) = i)$$

$$= P(\rho_i - t \leq v|r(t) = i) = \int_0^v \frac{1}{-\gamma_{ii}} e^{\gamma_{ii}s} ds$$

$$= 1 - e^{\gamma_{ii}v} \leq 1 - e^{-\bar{\gamma}v}$$

as desired.

*Lemma 4*

Let $x(t)$ be the solution of the uncertain hybrid SDE (7). Set

$$M_A = 2\max_{i \in S}(\|A_i\|^2 + \|L_A\|^2 \cdot \|N_i\|^2), \quad M_{CK} = \max_{i \in S} \|C_i K_i\|^2,$$

$$M_B = 2\max_{i \in S} \sum_{k=1}^m (\|B_i^k\|^2 + \|L_B\|^2 \cdot \|E_i^k\|^2),$$

and define

$$K(\tau) = (6\tau^2 M_A + 6\tau M_B + 3\tau^2 M_{CK})e^{6\tau^2 M_A + 6\tau M_B} \tag{8}$$

for $\tau > 0$. If $\tau$ is small enough for $2K(\tau) < 1$, then

$$\mathbb{E}|x(t) - x(\delta(t))|^2 \leq \frac{2K(\tau)}{1 - 2K(\tau)} \mathbb{E}|x(t)|^2 \tag{9}$$

holds for all $t \geq 0$.

*Proof*

Fix any integer $l \geq 0$. For $t \in [l\tau, (l+1)\tau)$, we have $\delta(t) = l\tau$. Then it follows from (7) that

$$x(t) - x(\delta(t)) = x(t) - x(l\tau)$$

$$= \int_{l\tau}^t [(A(r(s)) + \Delta A(s, r(s)))x(s) + C(r(s))K(r(l\tau))x(l\tau)]ds$$

$$+ \sum_{k=1}^m \int_{l\tau}^t [B^k(r(s)) + \Delta B^k(s, r(s))]x(s)dw_k(s).$$

By the fundamental inequality $|a + b + c|^2 \leq 3|a|^2 + 3|b|^2 + 3|c|^2$, $H\ddot{o}lder's$ inequality and Doob's

martingale inequality , we can then derive

$$
\begin{aligned}
\mathbb{E}|x(t) - x(\delta(t))|^2 \leq & 3\mathbb{E}|\int_{l\tau}^{t}[A(r(s)) + \Delta A(s, r(s))]x(s)ds|^2 \\
& + 3\mathbb{E}|\int_{l\tau}^{t} C(r(s))K(r(l\tau))x(l\tau)ds|^2 \\
& + 3\mathbb{E}|\sum_{k=1}^{m}\int_{l\tau}^{t}[B^k(r(s)) + \Delta B^k(s, r(s))]x(s)dw_k(s)|^2 \\
\leq & 3\tau\int_{l\tau}^{t}\mathbb{E}(\|A(r(s)) + \Delta A(s, r(s))\|^2|x(s)|^2)ds \\
& + 3\tau\int_{l\tau}^{t}\mathbb{E}(\|C(r(s))K(r(l\tau))\|^2|x(l\tau)|^2)ds \\
& + 3\sum_{k=1}^{m}\int_{l\tau}^{t}\mathbb{E}(\|B^k(r(s)) + \Delta B^k(s, r(s))\|^2|x(s)|^2)ds.
\end{aligned} \tag{10}
$$

Note that $\|F_A(t)\| \leq 1$ for any $t \geq 0$ from (4). Consequently, for any $i \in S$,

$$
\|A_i + \Delta A(t, i)\|^2 \leq 2(\|A_i\|^2 + \|L_A F_A(t) N_i\|^2) \leq M_A. \tag{11}
$$

Similarly, for any $i \in S$, $\|B_i^k + \Delta B^k(t, i)\|^2 \leq 2(\|B_i^k\|^2 + \|L_B\|^2\|E_i^k\|^2)$, and then

$$
\sum_{k=1}^{m}\|B_i^k + \Delta B^k(t, i)\|^2 \leq M_B. \tag{12}
$$

Substituting (11) and (12) together into (10), we have

$$
\begin{aligned}
\mathbb{E}|x(t) - x(\delta(t))|^2 \leq & 3(\tau M_A + M_B)\int_{l\tau}^{t}\mathbb{E}|x(s)|^2 ds + 3\tau^2 M_{CK}\mathbb{E}|x(l\tau)|^2 \\
\leq & 6(\tau M_A + M_B)\int_{l\tau}^{t}\mathbb{E}|x(s) - x(l\tau)|^2 ds \\
& + (6\tau^2 M_A + 6\tau M_B + 3\tau^2 M_{CK})\mathbb{E}|x(l\tau)|^2.
\end{aligned}
$$

The well-known Gronwall inequality shows

$$
\mathbb{E}|x(t) - x(\delta(t))|^2 \leq K(\tau)\mathbb{E}|x(l\tau)|^2 \leq 2K(\tau)(\mathbb{E}|x(t) - x(l\tau)|^2 + \mathbb{E}|x(t)|^2),
$$

which implies (9) holds for $t \in [l\tau, (l + 1)\tau)$. Then assertion (9) must hold for all $t \geq 0$ as $l \geq 0$ is

arbitrary.

*Theorem 1*

Assume that there exist positive definite matrices $Q_i$ and positive numbers $\rho_i, \varepsilon_i$ ($i \in S$) such that

$$\bar{Q}_i := Q_i(A_i + C_iK_i) + (A_i + C_iK_i)^TQ_i + \rho_iQ_iL_AL_A^TQ_i + \rho_i^{-1}N_i^TN_i + \sum_{j=1}^{N}\gamma_{ij}Q_j$$

$$+ \sum_{k=1}^{m}(B_i^k)^T(Q_i^{-1} - \varepsilon_iL_BL_B^T)^{-1}B_i^k + \varepsilon_i^{-1}\sum_{k=1}^{m}(E_i^k)^TE_i^k \qquad (13)$$

are all negative-definite matrices, and for any $i \in S$,

$$Q_i^{-1} - \varepsilon_iL_BL_B^T > 0. \qquad (14)$$

Let $K(\tau)$ be the same as defined in Lemma 4 and set

$$\lambda_M = \max_{i \in S}\lambda_{\max}(Q_i), \quad \lambda_m = \min_{i \in S}\lambda_{\min}(Q_i), \quad \lambda = \max_{i \in S}\lambda_{\max}(\bar{Q}_i),$$

$$M_{QCK} = \max_{i \in S}\|Q_iC_iK_i\|^2, \quad M_C = \max_{i \in S}\|C_i\|^2, \quad H_K = \max_{i,j \in S}\|K_j - K_i\|^2$$

(and of course $\lambda < 0$). If $\tau$ is sufficiently small for

$$\lambda_\tau := \lambda + 2\sqrt{\frac{2M_{QCK}K(\tau)}{1 - 2K(\tau)}} + 2\lambda_M\sqrt{\frac{2M_CH_K(1 - e^{-\bar{\gamma}\tau})}{1 - 2K(\tau)}} < 0, \qquad (15)$$

then the solution of (7) satisfies

$$\limsup_{t \to \infty}\frac{1}{t}\log(\mathbb{E}|x(t)|^2) \leq \frac{\lambda_\tau}{\lambda_M}.$$

In other words, the hybrid uncertain SDE (7) is exponentially stable in mean square.

*Proof*

Let $V(x(t), r(t)) = x^T(t)Q(r(t))x(t)$, where $Q(i) = Q_i$ as $r(t) = i$. Applying the generalized Itô formula to $V$, we get

$$dV(x(t), r(t)) = \mathcal{L}V(x(t), r(t))dt + dM_1(t),$$

where $M_1(t)$ is a martingale with $M_1(0) = 0$ and

$$\mathcal{L}V(x(t), i) = 2x^T(t)Q_i[(A_i + \Delta A(t, i))x(t) + C_iK(r(\delta(t)))x(\delta(t))] + \sum_{j=1}^{N}\gamma_{ij}x^T(t)Q_jx(t)$$

$$+ \sum_{k=1}^{m}x^T(t)(B_i^k + \Delta B^k(t, i))^TQ_i(B_i^k + \Delta B^k(t, i))x(t). \qquad (16)$$

According to (3), (4) and Lemma 1, the first term can be treated as

$$2x^T(t)Q_i(A_i + \Delta A(t,i))x(t) + 2x^T(t)Q_iC_iK(r(\delta(t)))x(\delta(t))$$

$$\leq 2x^T(t)Q_iA_ix(t) + \rho_ix^T(t)(Q_iL_AL_A^TQ_i)x(t) + \rho_i^{-1}x^T(t)(N_i^TN_i)x(t)$$

$$+2x^T(t)Q_iC_iK(r(\delta(t)))x(\delta(t))$$

$$\leq x^T(t)(Q_i(A_i + C_iK_i) + (A_i + C_iK_i)^TQ_i)x(t)$$

$$+\rho_ix^T(t)(Q_iL_AL_A^TQ_i)x(t) + \rho_i^{-1}x^T(t)(N_i^TN_i)x(t)$$

$$-2x^T(t)Q_iC_iK_i(x(t) - x(\delta(t))) - 2x^T(t)Q_iC_i(K(r(t)) - K(r(\delta(t))))x(\delta(t)). \quad (17)$$

For the last term, we can, using (3) (4) and Lemma 2, show that

$$\sum_{k=1}^m x^T(t)(B_i^k + L_BF_B(t)E_i^k)^TQ_i(B_i^k + L_BF_B(t)E_i^k)x(t)$$

$$\leq \sum_{k=1}^m x^T(t)[(B_i^k)^T(Q_i^{-1} - \varepsilon_iL_BL_B^T)^{-1}B_i^k + \varepsilon_i^{-1}(E_i^k)^TE_i^k]x(t). \quad (18)$$

Substituting (17) and (18) into (16), we have that for any $i \in S$,

$$\mathcal{L}V(x(t),i) \leq x^T(t)\bar{Q}_ix(t) - 2x^T(t)Q_iC_iK_i(x(t) - x(\delta(t)))$$

$$-2x^T(t)Q_iC_i(K(r(t)) - K(r(\delta(t))))x(\delta(t))$$

$$\leq \lambda|x(t)|^2 + 2\sqrt{M_{QCK}}|x(t)||x(t) - x(\delta(t))|$$

$$-2x^T(t)Q_iC_i(K(r(t)) - K(r(\delta(t))))x(\delta(t)). \quad (19)$$

Applying the generalized Itô formula now to $e^{\theta t}V(x(t), r(t))$, with $\theta = -\frac{\lambda_\tau}{\lambda_M} > 0$, we then obtain that, for any $t \geq 0$,

$$e^{\theta t}x^T(t)Q(r(t))x(t) = x^T(0)Q(r(0))x(0)$$

$$+ \int_0^t e^{\theta s}[\theta x^T(s)Q(r(s))x(s) + \mathcal{L}V(x(s), r(s))]ds + M_2(t),$$

where $M_2(t)$ is a continuous martingale with $M_2(0) = 0$. Taking the expectation on both sides, and then using (19) and Fubini's theorem, we have

$$
\begin{aligned}
e^{\theta t}\mathbb{E}(x^T(t)Q(r(t))x(t)) \leq & \lambda_M\mathbb{E}|x(0)|^2 + \int_0^t e^{\theta s}(\theta\lambda_M + \lambda)\mathbb{E}|x(s)|^2 ds \\
& + \int_0^t e^{\theta s}2\sqrt{M_{QCK}}\mathbb{E}(|x(s)||x(s) - x(\delta(s))|)ds \\
& - \int_0^t e^{\theta s}\mathbb{E}[2x^T(s)Q_iC_i(K(r(s)) - K(r(\delta(s))))x(\delta(s))]ds.
\end{aligned}
\tag{20}
$$

Setting $a = \sqrt{\frac{2M_C H_D(1 - e^{-\overline{\gamma}\tau})}{1 - 2K(\tau)}}$, $b = \sqrt{\frac{2M_{QCK}K(\tau)}{1 - 2K(\tau)}}$ and applying Lemma 3 and Lemma 4, we further derive that

$$
\begin{aligned}
& - 2\mathbb{E}[x^T(s)Q_iC_i(K(r(s)) - K(r(\delta(s))))x(\delta(s))] \\
\leq & a\lambda_M\mathbb{E}|x(s)|^2 + \frac{\lambda_M}{a}\mathbb{E}(\|C_i\|^2\|K(r(s)) - K(r(\delta(s)))\|^2|x(\delta(s))|^2) \\
\leq & a\lambda_M\mathbb{E}|x(s)|^2 + \frac{\lambda_M M_C}{a}\mathbb{E}\Big(\mathbb{E}(\|K(r(s)) - K(r(\delta(s)))\|^2|x(\delta(s))|^2|\mathcal{F}_{\delta(s)})\Big) \\
= & a\lambda_M\mathbb{E}|x(s)|^2 + \frac{\lambda_M M_C}{a}\mathbb{E}\Big(|x(\delta(s))|^2\mathbb{E}(\|K(r(s)) - K(r(\delta(s)))\|^2|\mathcal{F}_{\delta(s)})\Big) \\
= & a\lambda_M\mathbb{E}|x(s)|^2 + \frac{\lambda_M M_C}{a}\mathbb{E}\Big(|x(\delta(s))|^2\sum_{r(\delta(s))=i}\mathcal{I}_{\{r(\delta(s))=i\}}\mathbb{E}(\|K(r(s)) - K(r(\delta(s)))\|^2)\Big) \\
\leq & a\lambda_M\mathbb{E}|x(s)|^2 + \frac{\lambda_M M_C}{a}\mathbb{E}(|x(\delta(s))|^2\sum_{r(\delta(s))=i}\mathcal{I}_{\{r(\delta(s))=i\}}\max_{i,j\in S}\|K_j - K_i\|^2) \\
\leq & a\lambda_M\mathbb{E}|x(s)|^2 + \frac{\lambda_M M_C}{a}H_K(1 - e^{-\overline{\gamma}\tau})\mathbb{E}|x(\delta(s))|^2 \\
\leq & a\lambda_M\mathbb{E}|x(s)|^2 + \frac{\lambda_M M_C}{a}H_K(1 - e^{-\overline{\gamma}\tau})(2\mathbb{E}|x(s)|^2 + 2\mathbb{E}|x(s) - x(\delta(s))|^2) \\
\leq & a\lambda_M\mathbb{E}|x(s)|^2 + \frac{\lambda_M M_C}{a}\cdot\frac{2}{1 - 2K(\tau)}H_K(1 - e^{-\overline{\gamma}\tau})\mathbb{E}|x(s)|^2 \\
= & 2a\lambda_M\mathbb{E}|x(s)|^2
\end{aligned}
$$

and

$$
2\sqrt{M_{QCK}}\mathbb{E}(|x(s)||x(s) - x(\delta(s))|) \leq b\mathbb{E}|x(s)|^2 + \frac{M_{QCK}}{b}\mathbb{E}|x(s) - x(\delta(s))|^2 \leq 2b\mathbb{E}|x(s)|^2.
$$

Substituting these into (20) yields

$$
\lambda_m e^{\theta t}\mathbb{E}|x(t)|^2 \leq \lambda_M\mathbb{E}|x(0)|^2 + \int_0^t e^{\theta s}(\theta\lambda_M + \lambda_\tau)\mathbb{E}|x(s)|^2 ds = \lambda_M\mathbb{E}|x(0)|^2,
$$

which leads to

$$
\limsup_{t\to\infty}\frac{1}{t}\log(\mathbb{E}|x(t)|^2) \leq -\theta = \frac{\lambda_\tau}{\lambda_M}
$$

as required.

Let us now begin to consider the design of a feedback controller based on the discrete-time observations of both state and mode that robustly stabilizes hybrid uncertain stochastic system (1) in mean-square sense. According to Theorem 1, we need to design $K(\cdot)$, namely $K_i$ for $i \in S$, and further find $Q_i$ and positive numbers $\rho_i, \varepsilon_i$ in order for (13) and (14) to hold for any $i \in S$. Fortunately, we can transfer conditions (13) and (14) into LMIs, which it is easy to check in practice by using MATLAB Toolbox. The following theorem describes the details.

*Theorem 2*

Assume that there are solutions $P_i = P_i^T > 0, \rho_i > 0, \varepsilon_i > 0$ and $Y_i$ $(i \in S)$ to the following LMIs

$$\Pi_i = \begin{pmatrix} \Pi_{11i} & P_i N_i^T & \Pi_{31i}^T & \Pi_{41i}^T & \Pi_{51i}^T \\ N_i P_i & -\rho_i I & 0 & 0 & 0 \\ \Pi_{31i} & 0 & \Pi_{33i} & 0 & 0 \\ \Pi_{41i} & 0 & 0 & \Pi_{44i} & 0 \\ \Pi_{51i} & 0 & 0 & 0 & \Pi_{55i} \end{pmatrix} < 0, \qquad (21)$$

where

$$\Pi_{11i} = A_i P_i + P_i A_i^T + C_i Y_i + Y_i^T C_i^T + \gamma_{ii} P_i + \rho_i L_A L_A^T,$$

$$\Pi_{33i} = diag(-P_1, \cdots, -P_{i-1}, -P_{i+1}, \cdots, -P_N),$$

$$\Pi_{31i} = (\sqrt{\gamma_{i1}} P_i, \cdots, \sqrt{\gamma_{i,i-1}} P_i, \sqrt{\gamma_{i,i+1}} P_i, \cdots, \sqrt{\gamma_{iN}} P_i)^T,$$

$$\Pi_{44i} = diag(\varepsilon_i L_B L_B^T - P_i, \varepsilon_i L_B L_B^T - P_i, \cdots, \varepsilon_i L_B L_B^T - P_i),$$

$$\Pi_{41i} = (P_i (B_i^1)^T, P_i (B_i^2)^T, \cdots, P_i (B_i^m)^T)^T,$$

$$\Pi_{55i} = diag(-\varepsilon_i I, -\varepsilon_i I, \cdots, -\varepsilon_i I),$$

$$\Pi_{51i} = (P_i (E_i^1)^T, P_i (E_i^2)^T, \cdots, P_i (E_i^m)^T)^T.$$

Then by setting $Q_i = P_i^{-1}$ and $K_i = Y_i P_i^{-1}$, the controlled system (7) will be robustly exponentially stable in mean square if $\tau > 0$ is small enough such that (15) holds.

*Proof*

We first observe that by the well-known Schur complements (see e.g. [13]), the LMIs (21) are

equivalent to the following matrix inequalities

$$A_i P_i + P_i A_i^T + C_i Y_i + Y_i^T C_i^T + \gamma_{ii} P_i + \rho_i L_A L_A^T$$

$$+\rho_i^{-1} P_i N_i^T N_i P_i + \sum_{j \neq i} (\sqrt{\gamma_{ij}} P_i) P_j^{-1} (\sqrt{\gamma_{ij}} P_i)$$

$$+\sum_{k=1}^m P_i (B_i^k)^T (P_i - \varepsilon_i L_B L_B^T)^{-1} B_i^k P_i + \varepsilon_i^{-1} \sum_{k=1}^m P_i (E_i^k)^T E_i^k P_i < 0 \qquad (22)$$

and

$$\varepsilon_i L_B L_B^T - P_i < 0. \qquad (23)$$

Recalling that $K_i = Y_i P_i^{-1}$ and $P_i = P_i^T$, we can rewrite (22) as

$$A_i P_i + P_i A_i^T + C_i K_i P_i + P_i K_i^T C_i^T + \gamma_{ii} P_i + \rho_i L_A L_A^T$$

$$+\rho_i^{-1} P_i N_i^T N_i P_i + \sum_{j \neq i} (\sqrt{\gamma_{ij}} P_i) P_j^{-1} (\sqrt{\gamma_{ij}} P_i)$$

$$+\sum_{k=1}^m P_i (B_i^k)^T (P_i - \varepsilon_i L_B L_B^T)^{-1} B_i^k P_i + \varepsilon_i^{-1} \sum_{k=1}^m P_i (E_i^k)^T E_i^k P_i < 0. \qquad (24)$$

Multiplying $P_i^{-1}$ from left and then from right, and noting $Q_i = P_i^{-1}$, we see that the matrix

inequalities (24) are equivalent to the following matrix inequalities

$$Q_i A_i + A_i^T Q_i + Q_i C_i K_i + K_i^T C_i^T Q_i + \gamma_{ii} Q_i + \rho_i Q_i L_A L_A^T Q_i + \rho_i^{-1} N_i^T N_i + \sum_{j \neq i} \gamma_{ij} Q_j$$

$$+\sum_{k=1}^m (B_i^k)^T (Q_i^{-1} - \varepsilon_i L_B L_B^T)^{-1} B_i^k + \varepsilon_i^{-1} \sum_{k=1}^m (E_i^k)^T E_i^k < 0, \qquad (25)$$

which means the matrices in (13) are all negative-definite. Moreover, from (23) we have

$$Q_i^{-1} - \varepsilon_i L_B L_B^T > 0.$$

Then the required assertion follows directly from Theorem 1.

From the above theorem we can see that, to get the robust controller, we should first find solutions

for (21) and then obtain small $\tau$ according to condition (15) after calculating all related quantities.

*Remark*. As we can see, the controller in this paper is based on the discrete-time observations of both

state and mode. It is more practical and costs less compared with most existing results on the same

topic and hence our work is a useful improvement. Our controller is mode-dependent and this may be unrealistic in some real-world situations where the system mode is inaccessible. So the problem of mode-independent control for hybrid systems has attracted a lot of attention and some excellent results have appeared (see e.g. [38, 39]). However, there are also some practical systems which are mode available or part available. In this case, the mode-independent results may be conservative due to the full negligence of mode information. In fact, some papers designed both mode-independent controllers and mode-dependent controllers, corresponding to the cases of mode inaccessible and mode accessible or part accessible, respectively (see e.g. [40]).

Note that on one hand, the most existing mode-independent results totally neglect system mode no matter whether it is available or not, and hence the controller $u(t) = Kx(t)$ will be conservative the case where the system mode is available or part available; on the other hand, the mode-dependent controller $u(t) = K_{r(t)}x(t)$ is not applicable to the case where the system mode is inaccessible. We hence feel that we may design a new controller, like $u(t) = (K_{r(t)} + \rho(t)\Delta K_{r(t)})x(t)(\Delta K_{r(t)} = K - K_{r(t)}$ and $\rho(t) = 1$ or $0$), in order to combine both mode-dependent and mode-independent methods together to establish better results. This will be our further research topic.

## 4. NUMERICAL EXAMPLE

Linear hybrid stochastic differential equations have been extensively employed to model many real-world systems (see e.g. [5, 7, 8]). Due to the practical constraints and different limitations for different practical problems, most real-world models are very special, with constant coefficients, diagonal matrix coefficients or some other specific matrix coefficients. To illustrate the theory developed in this paper better, we will use more general system matrices instead of diagonal matrices etc.

Let us consider a two-dimensional linear hybrid uncertain SDE

$$dx(t) = [A(r(t)) + \Delta A(t, r(t))]x(t)dt + [B(r(t)) + \Delta B(t, r(t))]x(t)dw(t) \qquad (26)$$

on $t \geq 0$. Here $w(t)$ is a scalar Brownian motion and $r(t)$ is a Markov chain on the state space $S = \{1, 2\}$ with the generator

$$\Gamma = \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix},$$

$\Delta A(t, r(t))$ and $\Delta B(t, r(t))$ are defined as in (3) and (4) and all the other matrices are given by

$$A_1 = \begin{bmatrix} -2.1 & 0.1 \\ 0.1 & 1.1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1.9 & -0.1 \\ -0.1 & 0.9 \end{bmatrix}, \quad L_A = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix},$$

$$B_1 = \begin{bmatrix} 0.2 & 1 \\ 0 & 0.2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad L_B = \begin{pmatrix} 0.2 \\ 0.2 \end{pmatrix},$$

$$N_1 = \begin{pmatrix} 0.2 & 0.1 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0.13 & 0.21 \end{pmatrix},$$

$$E_1 = \begin{pmatrix} 0.4 & 0.3 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0.24 & 0.31 \end{pmatrix}.$$

The numerical simulation (Figure 5.1) shows that this hybrid uncertain SDE is not almost surely exponentially stable (and then certainly not mean-square exponentially stable as the latter implies the former).

Now we aim to design a feedback control based on discrete-time observations of both state and mode to stabilize the system. Assume that the controlled linear hybrid uncertain SDE has the form

$$dx(t) = [(A(r(t)) + \Delta A(t, r(t)))x(t) + C(r(t))K(r(\delta(t)))x(\delta(t))]dt$$

$$+ [B(r(t)) + \Delta B(t, r(t))]x(t)dw(t), \tag{27}$$

where

$$C_1 = \begin{pmatrix} 0.1 \\ 0.7 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0.2 \\ 0.5 \end{pmatrix}.$$

Our aim is to find $K_1$ and $K_2$ in $R^{1 \times 2}$ and then make sure $\tau$ is sufficiently small for this controlled SDE to be exponentially stable in mean square. To apply Theorem 2, we first find that LMIs (21) have the feasible solutions

$$P_1 = \begin{bmatrix} 8.8284 & 0.0897 \\ 0.0897 & 8.5196 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 9.0255 & 0.1382 \\ 0.1382 & 9.4558 \end{bmatrix},$$
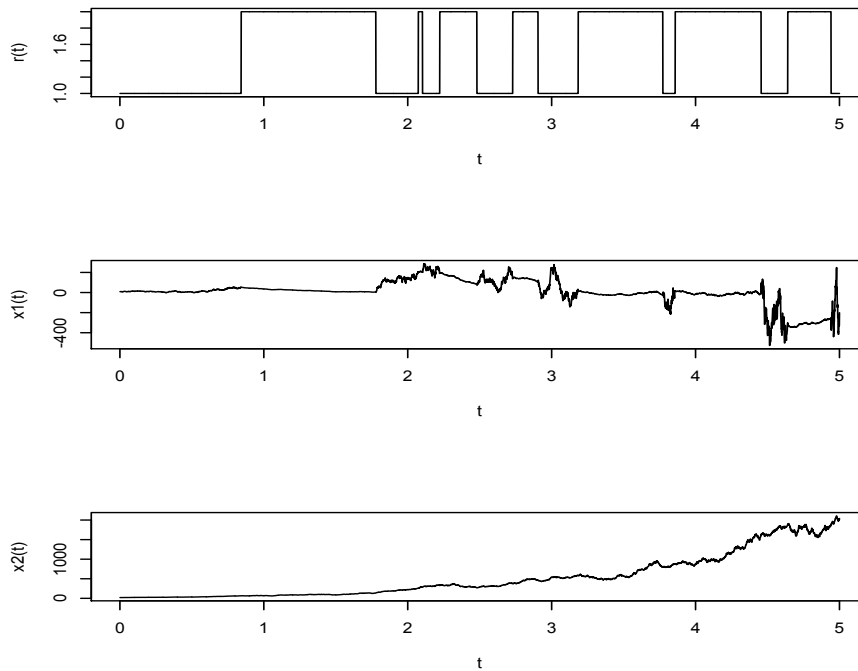
Fig. 5.1. Computer simulation of the paths of $r(t), x_1(t)$ and $x_2(t)$ for the hybrid uncertain SDE (26) using the Euler–Maruyama method with step size $10^{-6}$ and initial values $r(0) = 1, x_1(0) = 10$ and $x_2(0) = 20$.

$$Y_1 = \begin{pmatrix} 3.9907 & -30.7673 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 23.5315 & -40.1093 \end{pmatrix},$$

$$\rho_1 = 21.8561, \quad \rho_2 = 21.6545, \quad \varepsilon_1 = 18.0638, \quad \varepsilon_2 = 18.8368.$$

A further calculation shows that

$$K_1 = \begin{pmatrix} 0.4888 & -3.6165 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 2.6728 & -4.2808 \end{pmatrix},$$

and

$$\lambda_M = 0.1177, \quad \lambda_m = 0.1053, \quad \lambda = -0.2384,$$

$$M_{QCK} = 0.0911, \quad M_C = 0.5, \quad H_K = 5.2112,$$

$$M_A = 8.8482, \quad M_{CK} = 7.3861, \quad M_B = 2.1970.$$

It is easy to show that (15) holds whenever $\tau < 0.0031$. So by Theorem 2, if we set $K_i(i = 1, 2)$ as above and make sure $\tau < 0.0031$, then the controlled system (27) is mean-square exponentially stable. The numerical simulation (Figure 5.2) supports this result clearly.
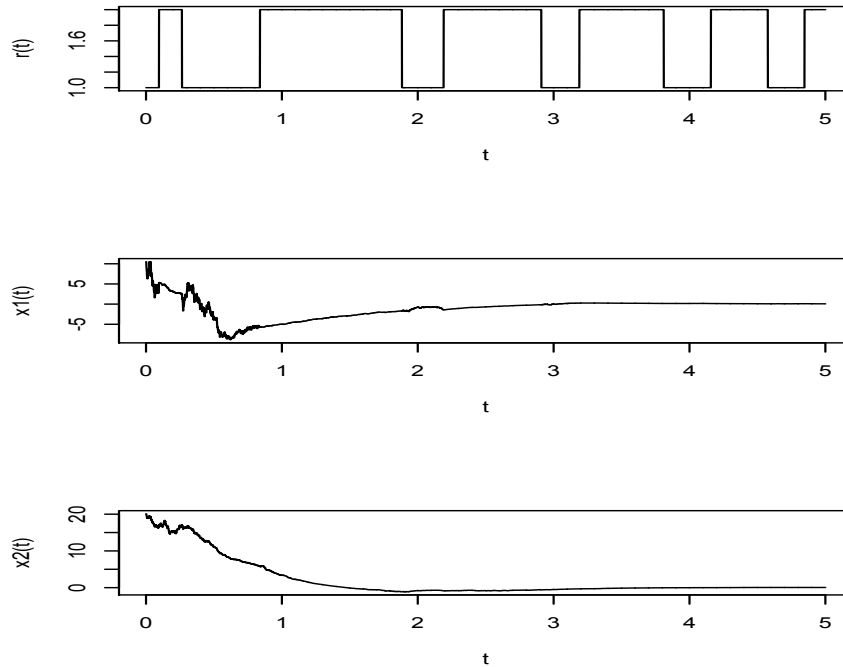


Fig. 5.2. Computer simulation of the paths of $r(t), x_1(t)$ and $x_2(t)$ for the controlled hybrid uncertain SDE (27) with $\tau = 0.002$ using the Euler–Maruyama method with step size $10^{-6}$ and initial values $r(0) = 1, x_1(0) = 10$ and $x_2(0) = 20$.

## 5. CONCLUSION

In this paper, we have showed that unstable hybrid uncertain stochastic systems can be stabilized by linear state feedback controls based on the discrete-time observations of both state and mode. Such discrete-time controls are more realistic and cost less compared with continuous-time feedback controls. A method in terms of LMIs for designing the robust controller has also been developed. An numerical example is provided to demonstrate the effectiveness of our theory.

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