

Sets of Points Determining Only Acute Angles and Some Related Colouring Problems

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Abstract

We present both probabilistic and constructive lower bounds on the maximum size of a set of points $\mathcal{S} \subseteq \mathbb{R}^d$ such that every angle determined by three points in \mathcal{S} is acute, considering especially the case $\mathcal{S} \subseteq \{0, 1\}^d$. These results improve upon a probabilistic lower bound of Erdős and Füredi. We also present lower bounds for some generalisations of the acute angles problem, considering especially some problems concerning colourings of sets of integers.

1 Introduction

Let us say that a set of points $\mathcal{S} \subseteq \mathbb{R}^d$ is an **acute d -set** if every angle determined by a triple of \mathcal{S} is acute ($< \frac{\pi}{2}$). Let us also say that \mathcal{S} is a **cubic acute d -set** if \mathcal{S} is an acute d -set and is also a subset of the unit d -cube (i.e. $\mathcal{S} \subseteq \{0, 1\}^d$).

Let us further say that a triple $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^d$ is an **acute triple**, a **right triple**, or an **obtuse triple**, if the angle determined by the triple with apex \mathbf{v} is less than $\frac{\pi}{2}$, equal to $\frac{\pi}{2}$, or greater than $\frac{\pi}{2}$, respectively. Note that we consider the triples $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and $\mathbf{w}, \mathbf{v}, \mathbf{u}$ to be the same.

We will denote by $\alpha(d)$ the size of a largest possible acute d -set. Similarly, we will denote by $\kappa(d)$ the size of a largest possible cubic acute d -set. Clearly $\kappa(d) \leq \alpha(d)$, $\kappa(d) \leq \kappa(d+1)$ and $\alpha(d) \leq \alpha(d+1)$ for all d .

In [EF], Paul Erdős and Zoltán Füredi gave a probabilistic proof that $\kappa(d) \geq \left\lfloor \frac{1}{2} \left(\frac{2}{\sqrt{3}} \right)^d \right\rfloor$ (see also [AZ2]). This disproved an earlier conjecture of Ludwig Danzer and Branko Grünbaum [DG] that $\alpha(d) = 2d - 1$.

In the following two sections we give improved probabilistic lower bounds for $\kappa(d)$ and $\alpha(d)$. In section 4 we present a construction that gives further improved lower bounds for $\kappa(d)$ for small d . In section 5, we tabulate the best lower bounds known for $\kappa(d)$ and $\alpha(d)$ for small d . Finally, in sections 6–9, we give probabilistic and constructive lower bounds for some generalisations of $\kappa(d)$, considering especially some problems concerning colourings of sets of integers.

2 A probabilistic lower bound for $\kappa(d)$

Theorem 2.1

$$\kappa(d) \geq 2 \left\lfloor \frac{\sqrt{6}}{9} \left(\frac{2}{\sqrt{3}} \right)^d \right\rfloor \approx 0.544 \times 1.155^d.$$

For large d , this improves upon the result of Erdős and Füredi by a factor of $\frac{4\sqrt{6}}{9} \approx 1.089$. This is achieved by a slight improvement in the choice of parameters. This proof can also be found in [AZ3].

Proof: Let $m = \left\lfloor \frac{\sqrt{6}}{9} \left(\frac{2}{\sqrt{3}} \right)^d \right\rfloor$ and randomly pick a set \mathcal{S} of $3m$ point vectors from the vertices of the d -dimensional unit cube $\{0, 1\}^d$, choosing the coordinates independently with probability $\Pr[\mathbf{v}_i = 0] = \Pr[\mathbf{v}_i = 1] = \frac{1}{2}$, $1 \leq i \leq d$, for every $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_d) \in \mathcal{S}$.

Now every angle determined by a triple of points from \mathcal{S} is non-obtuse ($\leq \frac{\pi}{2}$), and a triple of vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ from \mathcal{S} is a right triple iff the scalar product $\langle \mathbf{u} - \mathbf{v}, \mathbf{w} - \mathbf{v} \rangle$ vanishes, i.e. iff either $\mathbf{u}_i - \mathbf{v}_i = 0$ or $\mathbf{w}_i - \mathbf{v}_i = 0$ for each i , $1 \leq i \leq d$.

Thus $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is a right triple iff $\mathbf{u}_i, \mathbf{v}_i, \mathbf{w}_i$ is neither $0, 1, 0$ nor $1, 0, 1$ for any i , $1 \leq i \leq d$. Since $\mathbf{u}_i, \mathbf{v}_i, \mathbf{w}_i$ can take eight different values, this occurs independently with probability $\frac{3}{4}$ for each i , so the probability that a triple of \mathcal{S} is a right triple is $\left(\frac{3}{4}\right)^d$.

Hence, the expected number of right triples in a set of $3m$ vectors is $3 \binom{3m}{3} \left(\frac{3}{4}\right)^d$. Thus there is *some* set \mathcal{S} of $3m$ vectors with no more than $3 \binom{3m}{3} \left(\frac{3}{4}\right)^d$ right triples, where

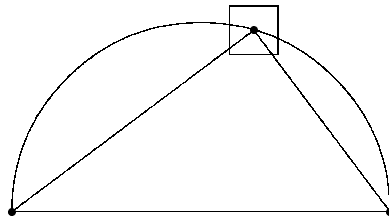
$$3 \binom{3m}{3} \left(\frac{3}{4}\right)^d < 3 \frac{(3m)^3}{6} \left(\frac{3}{4}\right)^d = m \left(\frac{9m}{\sqrt{6}}\right)^2 \left(\frac{3}{4}\right)^d \leq m$$

by the choice of m .

If we remove one point of each right triple from \mathcal{S} , the remaining set is a cubic acute d -set of cardinality at least $3m - m = 2m$. \square

3 A probabilistic lower bound for $\alpha(d)$

We can improve the lower bound in theorem 2.1 for non-cubic acute d -sets by a factor of $\sqrt{2}$ by slightly perturbing the points chosen away from the vertices of the unit cube. The intuition behind this is that a small random symmetrical perturbation of the points in a right triple is more likely than not to produce an acute triple, as the following diagram suggests.



Theorem 3.1

$$\alpha(d) \geq 2 \left\lfloor \frac{1}{3} \left(\frac{2}{\sqrt{3}} \right)^{d+1} \right\rfloor \approx 0.770 \times 1.155^d.$$

Before we can prove this theorem, we need some results concerning continuous random variables.

Definition 3.2 If $F(x) = \Pr[X \leq x]$ is the cumulative distribution function of a continuous random variable X , let $\bar{F}(x)$ denote $\Pr[X \geq x] = 1 - F(x)$.

Definition 3.3 Let us say that a continuous random variable X has **positive bias** if, for all t , $\Pr[X \geq t] \geq \Pr[X \leq -t]$, i.e. $\bar{F}(t) \geq F(-t)$.

Property 3.3.1 If a continuous random variable X has positive bias, it follows that $\Pr[X > 0] \geq \frac{1}{2}$.

Property 3.3.2 To show that a continuous random variable X has positive bias, it suffices to demonstrate that the condition $\bar{F}(t) \geq F(-t)$ holds for all **positive** t .

Lemma 3.4 *If X and Y are independent continuous random variables with positive bias, then $X + Y$ also has positive bias.*

Proof: Let f , g and h be the probability density functions, and F , G and H the cumulative distribution functions, for X , Y and $X + Y$ respectively. Then,

$$\begin{aligned}
 \overline{H}(t) - H(-t) &= \iint_{x+y \geq t} f(x)g(y) \, dy \, dx - \iint_{x+y \leq -t} f(x)g(y) \, dy \, dx \\
 &= \iint_{x+y \geq t} f(x)g(y) \, dy \, dx - \iint_{y-x \geq t} f(x)g(y) \, dy \, dx \\
 &\quad + \iint_{y-x \geq t} f(x)g(y) \, dy \, dx - \iint_{x+y \leq -t} f(x)g(y) \, dy \, dx \\
 &= \int_{-\infty}^{\infty} g(y) [\overline{F}(t-y) - F(y-t)] \, dy \\
 &\quad + \int_{-\infty}^{\infty} f(x) [\overline{G}(x+t) - G(-x-t)] \, dx
 \end{aligned}$$

which is non-negative because $f(t)$, $g(t)$, $\overline{F}(t) - F(-t)$ and $\overline{G}(t) - G(-t)$ are all non-negative for all t . □

Definition 3.5 *Let us say that a continuous random variable X is ϵ -uniformly distributed for some $\epsilon > 0$ if X is uniformly distributed between $-\epsilon$ and ϵ .*

Let us denote by j , the probability density function of an ϵ -uniformly distributed random variable:

$$j(x) = \begin{cases} \frac{1}{2\epsilon} & -\epsilon \leq x \leq \epsilon \\ 0 & \text{otherwise} \end{cases}$$

and by J , its cumulative distribution function:

$$J(x) = \begin{cases} 0 & x < -\epsilon \\ \frac{1}{2} + \frac{x}{2\epsilon} & -\epsilon \leq x \leq \epsilon \\ 1 & x > \epsilon \end{cases}$$

Property 3.5.1 *If X is an ϵ -uniformly distributed random variable, then so is $-X$.*

Lemma 3.6 *If X, Y and Z are independent ϵ -uniformly distributed random variables for some $\epsilon < \frac{1}{2}$, then $U = (Y - X)(1 + Z - X)$ has positive bias.*

Proof: Let G be the cumulative distribution function of U . By 3.3.2, it suffices to show that $\overline{G}(u) - G(-u) \geq 0$ for all positive u .

Let u be positive. Because $1 + Z - X$ is always positive, $U \geq u$ iff $Y > X$ and $Z \geq -1 + X + \frac{u}{Y-X}$. Similarly, $U \leq -u$ iff $X > Y$ and $Z \geq -1 + X + \frac{u}{X-Y}$. So,

$$\begin{aligned} \overline{G}(u) - G(-u) &= \iint_{y>x} j(x)j(y)\overline{J}\left(-1 + x + \frac{u}{y-x}\right) dy dx \\ &\quad - \iint_{x>y} j(x)j(y)\overline{J}\left(-1 + x + \frac{u}{x-y}\right) dy dx \\ &= \iint_{y>x} j(x)j(y) \left[J\left(1 - x - \frac{u}{y-x}\right) - J\left(1 - y - \frac{u}{y-x}\right) \right] dy dx \\ &\quad \text{(because } \overline{J}(x) = J(-x), \text{ and by variable renaming)} \end{aligned}$$

which is non-negative because j is non-negative and J is non-decreasing (so the expression in square brackets is non-negative over the domain of integration). \square

Corollary 3.6.1 *If X, Y and Z are independent ϵ -uniformly distributed random variables for some $\epsilon < \frac{1}{2}$, then $(Y - X)(Z - X - 1)$ has positive bias.*

Proof: $(Y - X)(Z - X - 1) = ((-Y) - (-X))(1 + (-Z) - (-X))$. The result follows from 3.5.1 and lemma 3.6. \square

Lemma 3.7 *If X, Y and Z are independent ϵ -uniformly distributed random variables, then $V = (Y - X)(Z - X)$ has positive bias.*

Proof: Let H be the cumulative distribution function of V . By 3.3.2, it suffices to show that $\overline{H}(v) - H(-v) \geq 0$ for all positive v .

Let v be positive. $V \geq v$ iff $Y > X$ and $Z \geq X + \frac{v}{Y-X}$ or $Y < X$ and $Z \leq X + \frac{v}{Y-X}$. Similarly, $V \leq -v$ iff $Y > X$ and $Z \leq X - \frac{v}{Y-X}$ or $Y < X$ and $Z \geq X - \frac{v}{Y-X}$. So,

$$\begin{aligned} \overline{H}(v) - H(-v) &= \iint_{y>x} j(x)j(y)\overline{J}\left(x + \frac{v}{y-x}\right) dy dx \\ &+ \iint_{y<x} j(x)j(y)J\left(x + \frac{v}{y-x}\right) dy dx \\ &- \iint_{y>x} j(x)j(y)J\left(x - \frac{v}{y-x}\right) dy dx \\ &- \iint_{y<x} j(x)j(y)\overline{J}\left(x - \frac{v}{y-x}\right) dy dx \\ &= \iint_{y>x} j(x)j(y) \left[J\left(-x - \frac{v}{y-x}\right) - J\left(-y - \frac{v}{y-x}\right) \right] dy dx \\ &+ \iint_{y<x} j(x)j(y) \left[J\left(x + \frac{v}{y-x}\right) - J\left(y + \frac{v}{y-x}\right) \right] dy dx \\ &\quad (\text{because } \overline{J}(x) = J(-x), \text{ and by variable renaming}) \end{aligned}$$

which is non-negative because j is non-negative and J is non-decreasing (so the expressions in square brackets are non-negative over the domains of integration). \square

We are now in a position to prove the theorem.

Proof of theorem 3.1

Let $m = \left\lceil \frac{1}{3} \left(\frac{2}{\sqrt{3}} \right)^{d+1} \right\rceil$, and randomly pick a set \mathcal{S} of $3m$ point vectors, $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{3m}$, from the vertices of the d -dimensional unit cube $\{0, 1\}^d$, choosing the coordinates independently with probability $\Pr[\mathbf{v}_{ki} = 0] = \Pr[\mathbf{v}_{ki} = 1] = \frac{1}{2}$ for every $\mathbf{v}_k = (\mathbf{v}_{k1}, \mathbf{v}_{k2}, \dots, \mathbf{v}_{kd})$, $1 \leq k \leq 3m$, $1 \leq i \leq d$.

Now for some ϵ , $0 < \epsilon < \frac{1}{2(d+1)}$, randomly pick $3m$ vectors, $\boldsymbol{\delta}_1, \boldsymbol{\delta}_2, \dots, \boldsymbol{\delta}_{3m}$, from the d -dimensional cube $[-\epsilon, \epsilon]^d$ of side 2ϵ centred on the origin, choosing the coordinates $\boldsymbol{\delta}_{ki}$, $1 \leq k \leq 3m$, $1 \leq i \leq d$, independently so that they are ϵ -uniformly distributed, and let $\mathcal{S}' = \{\mathbf{v}'_1, \mathbf{v}'_2, \dots, \mathbf{v}'_{3m}\}$ where $\mathbf{v}'_k = \mathbf{v}_k + \boldsymbol{\delta}_k$ for each k , $1 \leq k \leq 3m$.

Case 1: Acute triples in \mathcal{S}

Because $\epsilon < \frac{1}{2(d+1)}$, if $\mathbf{v}_j, \mathbf{v}_k, \mathbf{v}_l$ is an acute triple in \mathcal{S} , the scalar product $\langle \mathbf{v}'_j - \mathbf{v}'_k, \mathbf{v}'_l - \mathbf{v}'_k \rangle > \frac{1}{(d+1)^2}$, so $\mathbf{v}'_j, \mathbf{v}'_k, \mathbf{v}'_l$ is also an acute triple in \mathcal{S}' .

Case 2: Right triples in \mathcal{S}

If $\mathbf{v}_j, \mathbf{v}_k, \mathbf{v}_l$ is a right triple in \mathcal{S} then the scalar product $\langle \mathbf{v}_j - \mathbf{v}_k, \mathbf{v}_l - \mathbf{v}_k \rangle$ vanishes, i.e. either $\mathbf{v}_{ji} - \mathbf{v}_{ki} = 0$ or $\mathbf{v}_{li} - \mathbf{v}_{ki} = 0$ for each i , $1 \leq i \leq d$. There are six possibilities for each triple of coordinates:

$\mathbf{v}_{j_i}, \mathbf{v}_{k_i}, \mathbf{v}_{l_i}$	$(\mathbf{v}'_{j_i} - \mathbf{v}'_{k_i})(\mathbf{v}'_{l_i} - \mathbf{v}'_{k_i})$
0, 0, 0	$(\delta_{j_i} - \delta_{k_i})(\delta_{l_i} - \delta_{k_i})$
1, 1, 1	$(\delta_{j_i} - \delta_{k_i})(\delta_{l_i} - \delta_{k_i})$
0, 0, 1	$(\delta_{j_i} - \delta_{k_i})(1 + \delta_{l_i} - \delta_{k_i})$
1, 0, 0	$(\delta_{l_i} - \delta_{k_i})(1 + \delta_{j_i} - \delta_{k_i})$
0, 1, 1	$(\delta_{l_i} - \delta_{k_i})(\delta_{j_i} - \delta_{k_i} - 1)$
1, 1, 0	$(\delta_{j_i} - \delta_{k_i})(\delta_{l_i} - \delta_{k_i} - 1)$

Now, the values of the δ_{k_i} are independent and ϵ -uniformly distributed, so by lemmas 3.7 and 3.6 and corollary 3.6.1, the distribution of the $(\mathbf{v}'_{j_i} - \mathbf{v}'_{k_i})(\mathbf{v}'_{l_i} - \mathbf{v}'_{k_i})$ has positive bias, and by repeated application of lemma 3.4, the distribution of the scalar product $\langle \mathbf{v}'_j - \mathbf{v}'_k, \mathbf{v}'_l - \mathbf{v}'_k \rangle = \sum_{i=1}^d (\mathbf{v}'_{j_i} - \mathbf{v}'_{k_i})(\mathbf{v}'_{l_i} - \mathbf{v}'_{k_i})$ also has positive bias.

Thus, if $\mathbf{v}_j, \mathbf{v}_k, \mathbf{v}_l$ is a right triple in \mathcal{S} , then, by 3.3.1,

$$\Pr [\langle \mathbf{v}'_j - \mathbf{v}'_k, \mathbf{v}'_l - \mathbf{v}'_k \rangle > 0] \geq \frac{1}{2},$$

so the probability that the triple $\mathbf{v}'_j, \mathbf{v}'_k, \mathbf{v}'_l$ is an acute triple in \mathcal{S}' is at least $\frac{1}{2}$.

As in the proof of theorem 2.1, the expected number of right triples in \mathcal{S} is $3 \binom{3m}{3} \left(\frac{3}{4}\right)^d$, so the expected number of non-acute triples in \mathcal{S}' is no more than half this value. Thus there is *some* set \mathcal{S}' of $3m$ vectors with no more than $\frac{3}{2} \binom{3m}{3} \left(\frac{3}{4}\right)^d$ non-acute triples, where

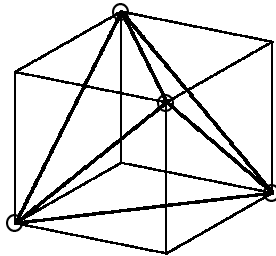
$$\frac{3}{2} \binom{3m}{3} \left(\frac{3}{4}\right)^d < \frac{3(3m)^3}{2 \cdot 6} \left(\frac{3}{4}\right)^d = m(3m)^2 \left(\frac{3}{4}\right)^{d+1} \leq m$$

by the choice of m .

If we remove one point of each non-acute triple from \mathcal{S}' , the remaining set is an acute d -set of cardinality at least $3m - m = 2m$. \square

4 Constructive lower bounds for $\kappa(d)$

In the following proofs, for clarity of exposition, we will represent point vectors in $\{0, 1\}^d$ as binary words of length d , e.g. $\mathcal{S}_3 = \{000, 011, 101, 110\}$ represents a cubic acute 3-set.



Concatenation of words (vectors) \mathbf{v} and \mathbf{v}' will be written \mathbf{vv}' .

We begin with a simple construction that enables us to extend a cubic acute d -set of cardinality n to a cubic acute $(d + 2)$ -set of cardinality $n + 1$.

Theorem 4.1

$$\kappa(d + 2) \geq \kappa(d) + 1$$

Proof: Let $\mathcal{S} = \{\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{n-1}\}$ be a cubic acute d -set of cardinality $n = \kappa(d)$. Now let $\mathcal{S}' = \{\mathbf{v}'_0, \mathbf{v}'_1, \dots, \mathbf{v}'_n\} \subseteq \{0, 1\}^{d+2}$ where $\mathbf{v}'_i = \mathbf{v}_i00$ for $0 \leq i \leq n - 2$, $\mathbf{v}'_{n-1} = \mathbf{v}_{n-1}10$ and $\mathbf{v}'_n = \mathbf{v}_{n-1}01$.

If $\mathbf{v}'_i, \mathbf{v}'_j, \mathbf{v}'_k$ is a triple of distinct points in \mathcal{S}' with no more than one of i, j and k greater than $n - 2$, then $\mathbf{v}'_i, \mathbf{v}'_j, \mathbf{v}'_k$ is an acute triple, because \mathcal{S} is an acute d -set. Also, any triple $\mathbf{v}'_k, \mathbf{v}'_{n-1}, \mathbf{v}'_n$ or $\mathbf{v}'_k, \mathbf{v}'_n, \mathbf{v}'_{n-1}$ is an acute triple, because its $(d + 1)$ th or $(d + 2)$ th coordinates (respectively) are $0, 1, 0$. Finally, for any triple $\mathbf{v}'_{n-1}, \mathbf{v}'_k, \mathbf{v}'_n$, if \mathbf{v}_k and \mathbf{v}_{n-1} differ in the r th coordinate, then the r th coordinates of $\mathbf{v}'_{n-1}, \mathbf{v}'_k, \mathbf{v}'_n$ are $0, 1, 0$ or $1, 0, 1$. Thus, \mathcal{S}' is a cubic acute $(d + 2)$ -set of cardinality $n + 1$. \square

Our second construction combines cubic acute d -sets of cardinality n to make a cubic acute $3d$ -set of cardinality n^2 .

Theorem 4.2

$$\kappa(3d) \geq \kappa(d)^2.$$

Proof: Let $\mathcal{S} = \{\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{n-1}\}$ be a cubic acute d -set of cardinality $n = \kappa(d)$, and let

$$\mathcal{T} = \{\mathbf{w}_{ij} = \mathbf{v}_i\mathbf{v}_j\mathbf{v}_{j-i \bmod n} : 0 \leq i, j \leq n - 1\},$$

each \mathbf{w}_{ij} being made by concatenating three of the \mathbf{v}_i .

Let $\mathbf{w}_{ps}, \mathbf{w}_{qt}, \mathbf{w}_{ru}$ be any triple of distinct points in \mathcal{T} . They constitute an acute triple iff the scalar product $\langle \mathbf{w}_{ps} - \mathbf{w}_{qt}, \mathbf{w}_{ru} - \mathbf{w}_{qt} \rangle$ does not vanish (is positive). Now,

$$\begin{aligned} \langle \mathbf{w}_{ps} - \mathbf{w}_{qt}, \mathbf{w}_{ru} - \mathbf{w}_{qt} \rangle &= \langle \mathbf{v}_p\mathbf{v}_s\mathbf{v}_{s-p} - \mathbf{v}_q\mathbf{v}_t\mathbf{v}_{t-q}, \mathbf{v}_r\mathbf{v}_u\mathbf{v}_{u-r} - \mathbf{v}_q\mathbf{v}_t\mathbf{v}_{t-q} \rangle \\ &= \langle \mathbf{v}_p - \mathbf{v}_q, \mathbf{v}_r - \mathbf{v}_q \rangle \\ &\quad + \langle \mathbf{v}_s - \mathbf{v}_t, \mathbf{v}_u - \mathbf{v}_t \rangle \\ &\quad + \langle \mathbf{v}_{s-p} - \mathbf{v}_{t-q}, \mathbf{v}_{u-r} - \mathbf{v}_{t-q} \rangle \end{aligned}$$

with all the index arithmetic modulo n .

If both $p \neq q$ and $q \neq r$, then the first component of this sum is positive, because \mathcal{S} is an acute d -set. Similarly, if both $s \neq t$ and $t \neq u$, then the second component is positive. Finally, if $p = q$ and $t = u$, then $q \neq r$ and $s \neq t$ or else the points would not be distinct, so the third component, $\langle \mathbf{v}_{s-p} - \mathbf{v}_{t-q}, \mathbf{v}_{u-r} - \mathbf{v}_{t-q} \rangle$ is positive. Similarly if $q = r$ and $s = t$.

Thus, all triples in \mathcal{T} are acute triples, so \mathcal{T} is a cubic acute $3d$ -set of cardinality n^2 . \square

Corollary 4.2.1 $\kappa(3^d) \geq 2^{2^d}$.

Proof: By repeated application of theorem 4.2 starting with \mathcal{S}_3 , a cubic acute 3-set of cardinality 4. \square

Corollary 4.2.2 If $d \geq 3$,

$$\kappa(d) \geq 10^{\frac{(d+1)^\mu}{4}} \approx 1.778^{(d+1)^{0.631}} \quad \text{where } \mu = \frac{\log 2}{\log 3}.$$

For small d , this is a tighter bound than theorem 2.1.

Proof: By induction on d . For $3 \leq d \leq 8$, we have the following cubic acute d -sets ($\mathcal{S}_3, \dots, \mathcal{S}_8$) that satisfy this lower bound for $\kappa(d)$ (with equality for $d = 8$):

$\mathcal{S}_3 : \kappa(3) \geq 4$ 000 011 101 110	$\mathcal{S}_4 : \kappa(4) \geq 5$ 0000 0011 0101 1001 1110	$\mathcal{S}_5 : \kappa(5) \geq 6$ 00000 00011 00101 01001 10001 11110
$\mathcal{S}_6 : \kappa(6) \geq 8$ 000000 000111 011001 011110 101010 101101 110011 110100	$\mathcal{S}_7 : \kappa(7) \geq 9$ 0000000 0000011 0001101 0110001 0111110 1010101 1011010 1100110 1101001	$\mathcal{S}_8 : \kappa(8) \geq 10$ 00000000 00000011 00000101 00011001 01100001 01111110 10101001 10110110 11001110 11010001

$$\begin{aligned}
 \text{If } \kappa(d) \geq 10^{\frac{(d+1)^\mu}{4}}, \text{ then } \kappa(3d) &\geq \kappa(d)^2 && \text{by theorem 4.2} \\
 &\geq 10^{\frac{2(d+1)^\mu}{4}} && \text{by the induction hypothesis} \\
 &= 10^{\frac{(3d+3)^\mu}{4}} && \text{because } 3^\mu = 2.
 \end{aligned}$$

So, since $\kappa(3d + 2) \geq \kappa(3d + 1) \geq \kappa(3d)$, if the lower bound is satisfied for d , it is also satisfied for $3d, 3d + 1$ and $3d + 2$. \square

Theorem 4.3 *If, for each r , $1 \leq r \leq m$, we have a cubic acute d_r -set of cardinality n_r , where n_1 is the least of the n_r , and if, for some dimension d_Z , we have a cubic acute d_Z -set of cardinality n_Z , where*

$$n_Z \geq \prod_{r=2}^m n_r,$$

then a cubic acute D -set of cardinality N can be constructed, where

$$D = \sum_{r=1}^m d_r + d_Z \quad \text{and} \quad N = \prod_{r=1}^m n_r.$$

This result generalises theorem 4.2, but before we can prove it, we first need some preliminary results.

Definition 4.4 *If $n_1 \leq n_2 \leq \dots \leq n_m$ and $0 \leq k_r < n_r$, for each r , $1 \leq r \leq m$, then let us denote by $\langle\langle k_1 k_2 \dots k_m \rangle\rangle_{n_1 n_2 \dots n_m}$, the number*

$$\langle\langle k_1 k_2 \dots k_m \rangle\rangle_{n_1 n_2 \dots n_m} = \sum_{r=2}^m \left((k_{r-1} - k_r \bmod n_r) \prod_{s=r+1}^m n_s \right).$$

Where the n_r can be inferred from the context, $\langle\langle k_1 k_2 \dots k_m \rangle\rangle$ may be used instead of $\langle\langle k_1 k_2 \dots k_m \rangle\rangle_{n_1 n_2 \dots n_m}$.

The expression $\langle\langle k_1 k_2 \dots k_m \rangle\rangle_{n_1 n_2 \dots n_m}$ can be understood as representing a number in a number system where the radix for each digit is a different n_r — like the old British monetary system of pounds, shillings and pennies — and the digits are the difference of two adjacent $k_r \pmod{n_r}$. For example,

$$\langle\langle 2053 \rangle\rangle_{4668} = [2 - 0]_6 [0 - 5]_6 [5 - 3]_8 = 2 \times 6 \times 8 + 1 \times 8 + 2 = 106,$$

where $[a_2]_{n_2} \dots [a_m]_{n_m}$ is place notation with the n_r the radix for each place.

By construction, we have the following results:

Property 4.4.1

$$\langle\langle k_1 k_2 \dots k_m \rangle\rangle_{n_1 n_2 \dots n_m} < \prod_{r=2}^m n_r.$$

Property 4.4.2 *If $2 \leq t \leq m$ and $j_{t-1} - j_t \neq k_{t-1} - k_t \pmod{n_t}$, then*

$$\langle\langle j_1 j_2 \dots j_m \rangle\rangle_{n_1 n_2 \dots n_m} \neq \langle\langle k_1 k_2 \dots k_m \rangle\rangle_{n_1 n_2 \dots n_m}.$$

Lemma 4.5 *If $n_1 \leq n_2 \leq \dots \leq n_m$ and $0 \leq j_r, k_r < n_r$, for each r , $1 \leq r \leq m$, and the sequences of j_r and k_r are neither identical nor everywhere different (i.e. there exist both t and u such that $j_t = k_t$ and $j_u \neq k_u$), then*

$$\langle\langle j_1 j_2 \dots j_m \rangle\rangle_{n_1 n_2 \dots n_m} \neq \langle\langle k_1 k_2 \dots k_m \rangle\rangle_{n_1 n_2 \dots n_m}.$$

Proof: Let u be the greatest integer, $1 \leq u < m$, such that $j_u - j_{u+1} \neq k_u - k_{u+1} \pmod{n_{u+1}}$. (If $j_m = k_m$, then u is the greatest integer such that $j_u \neq k_u$. If $j_m \neq k_m$, then u is at least as great as the greatest integer t such that $j_t = k_t$.) The result now follows from 4.4.2. \square

We are now in a position to prove the theorem.

Proof of Theorem 4.3

Let $n_1 \leq n_2 \leq \dots \leq n_m$, and, for each r , $1 \leq r \leq m$, let $\mathcal{S}_r = \{\mathbf{v}_0^r, \mathbf{v}_1^r, \dots, \mathbf{v}_{n_r-1}^r\}$ be a cubic acute d_r -set of cardinality n_r . Let $\mathcal{Z} = \{\mathbf{z}_0, \mathbf{z}_1, \dots, \mathbf{z}_{n_Z-1}\}$ be a cubic acute d_Z -set of cardinality n_Z , where

$$n_Z \geq \prod_{r=2}^m n_r,$$

and let

$$D = \sum_{r=1}^m d_r + d_Z \quad \text{and} \quad N = \prod_{r=1}^m n_r.$$

Now let

$$\mathcal{T} = \{\mathbf{w}_{k_1 k_2 \dots k_m} = \mathbf{v}_{k_1}^1 \mathbf{v}_{k_2}^2 \dots \mathbf{v}_{k_m}^m \mathbf{z}_{k_Z} : 0 \leq k_r < n_r, 1 \leq r \leq m\},$$

where $k_Z = \langle\langle k_1 k_2 \dots k_m \rangle\rangle_{n_1 n_2 \dots n_m}$, be a point set of dimension D and cardinality N , each element of \mathcal{T} being made by concatenating one vector from each of the \mathcal{S}_r together with a vector from \mathcal{Z} . (In section 5, we will denote this construction by $d_1 \otimes \dots \otimes d_m \oplus d_Z$.)

By 4.4.1, we know that $k_Z < \prod_{r=2}^m n_r \leq n_Z$, so k_Z is a valid index into \mathcal{Z} .

Let $\mathbf{w}_{i_1 i_2 \dots i_m}, \mathbf{w}_{j_1 j_2 \dots j_m}, \mathbf{w}_{k_1 k_2 \dots k_m}$ be any triple of distinct points in \mathcal{T} . They constitute an acute triple iff the scalar product $q = \langle \mathbf{w}_{i_1 i_2 \dots i_m} - \mathbf{w}_{j_1 j_2 \dots j_m}, \mathbf{w}_{k_1 k_2 \dots k_m} - \mathbf{w}_{j_1 j_2 \dots j_m} \rangle$ does not vanish (is positive). Now,

$$\begin{aligned} q &= \langle \mathbf{v}_{i_1}^1 \mathbf{v}_{i_2}^2 \dots \mathbf{v}_{i_m}^m \mathbf{z}_{i_Z} - \mathbf{v}_{j_1}^1 \mathbf{v}_{j_2}^2 \dots \mathbf{v}_{j_m}^m \mathbf{z}_{j_Z}, \mathbf{v}_{k_1}^1 \mathbf{v}_{k_2}^2 \dots \mathbf{v}_{k_m}^m \mathbf{z}_{k_Z} - \mathbf{v}_{j_1}^1 \mathbf{v}_{j_2}^2 \dots \mathbf{v}_{j_m}^m \mathbf{z}_{j_Z} \rangle \\ &= \sum_{r=1}^m \langle \mathbf{v}_{i_r}^r - \mathbf{v}_{j_r}^r, \mathbf{v}_{k_r}^r - \mathbf{v}_{j_r}^r \rangle + \langle \mathbf{z}_{i_Z} - \mathbf{z}_{j_Z}, \mathbf{z}_{k_Z} - \mathbf{z}_{j_Z} \rangle. \end{aligned}$$

If, for some r , both $i_r \neq j_r$ and $j_r \neq k_r$, then the first component of this sum is positive, because \mathcal{S}_r is an acute set.

If, however, there is no r such that both $i_r \neq j_r$ and $j_r \neq k_r$, then there must be some t for which $i_t \neq j_t$ (or else $\mathbf{w}_{i_1 i_2 \dots i_m}$ and $\mathbf{w}_{j_1 j_2 \dots j_m}$ would not be distinct) and $j_t = k_t$, and

also some u for which $j_u \neq k_u$ (or else $\mathbf{w}_{j_1 j_2 \dots j_m}$ and $\mathbf{w}_{k_1 k_2 \dots k_m}$ would not be distinct) and $i_u = j_u$. So, by lemma 4.5, $i_Z \neq j_Z$ and $j_Z \neq k_Z$, so the second component of the sum for the scalar product is positive, because \mathcal{Z} is an acute set.

Thus, all triples in \mathcal{T} are acute triples, so \mathcal{T} is a cubic acute D -set of cardinality N . \square

Corollary 4.5.1

$$\text{If } d_1 \leq d_2 \leq \dots \leq d_m, \text{ then } \kappa\left(\sum_{r=1}^m r d_r\right) \geq \prod_{r=1}^m \kappa(d_r).$$

Proof: By induction on m . The bound is trivially true for $m = 1$.

Assume the bound holds for $m - 1$, and for each r , $1 \leq r \leq m$, let \mathcal{S}_r be a cubic acute d_r -set of cardinality $n_r = \kappa(d_r)$, with $d_1 \leq d_2 \leq \dots \leq d_m$ and thus $n_1 \leq n_2 \leq \dots \leq n_m$. By the induction hypothesis, there exists a cubic acute d_Z -set \mathcal{Z} of cardinality n_Z , where

$$d_Z = \sum_{r=2}^m (r-1)d_r \quad \text{and} \quad n_Z \geq \prod_{r=2}^m \kappa(d_r) = \prod_{r=2}^m n_r.$$

Thus, by theorem 4.3, there exists a cubic acute D -set of cardinality N , where

$$D = \sum_{r=1}^m d_r + d_Z = \sum_{r=1}^m d_r + \sum_{r=2}^m (r-1)d_r = \sum_{r=1}^m r d_r,$$

and

$$N = \prod_{r=1}^m n_r = \prod_{r=1}^m \kappa(d_r).$$

\square

5 Lower bounds for $\kappa(d)$ and $\alpha(d)$ for small d

The following table lists the best lower bounds known for $\kappa(d)$, $0 \leq d \leq 69$. For $3 \leq d \leq 9$, an exhaustive computer search shows that $\mathcal{S}_3, \dots, \mathcal{S}_8$ (corollary 4.2.2), are optimal and also that $\kappa(9) = 16$. For other small values of d , the construction used in theorem 4.3 provides the largest known cubic acute d -set. In the table, these constructions are denoted by $d_1 \otimes d_2 \otimes d_Z$ or $d_1 \otimes d_2 \otimes d_3 \otimes d_Z$. For $39 \leq d \leq 48$, the results of a computer program, based on the ‘probabilistic construction’ of theorem 2.1, provide the largest known cubic acute d -sets. Finally, for $d \geq 67$, theorem 2.1 provides the best (probabilistic) lower bound. $\kappa(d)$ is sequence A089676 in Sloane [S].

Best Lower Bounds Known for $\kappa(d)$

d	$\kappa(d)$
0	= 1
1	= 2
2	= 2
3	= 4 <i>computer, \mathcal{S}_3</i>
4	= 5 <i>computer, \mathcal{S}_4</i>
5	= 6 <i>computer, \mathcal{S}_5</i>
6	= 8 <i>computer, \mathcal{S}_6</i>
7	= 9 <i>computer, \mathcal{S}_7</i>
8	= 10 <i>computer, \mathcal{S}_8</i>
9	= 16 <i>computer, $3\otimes 3\otimes 3$</i>
10	≥ 16
11	≥ 20 $3\otimes 4\otimes 4$
12	≥ 25 $4\otimes 4\otimes 4$
13	≥ 25
14	≥ 30 $4\otimes 5\otimes 5$
15	≥ 36 $5\otimes 5\otimes 5$
16	≥ 40 $4\otimes 6\otimes 6$
17	≥ 48 $5\otimes 6\otimes 6$
18	≥ 64 $6\otimes 6\otimes 6$ or $3\otimes 3\otimes 3\otimes 9$
19	≥ 64
20	≥ 72 $6\otimes 7\otimes 7$
21	≥ 81 $7\otimes 7\otimes 7$
22	≥ 81
23	≥ 100 $3\otimes 4\otimes 4\otimes 12$
24	≥ 125 $4\otimes 4\otimes 4\otimes 12$
25	≥ 144 $7\otimes 9\otimes 9$

d	$\kappa(d)$
26	≥ 160 $8\otimes 9\otimes 9$
27	≥ 256 $9\otimes 9\otimes 9$
28	≥ 256
29	≥ 257 <i>theorem 4.1</i>
30	≥ 257
31	≥ 320 $9\otimes 11\otimes 11$
32	≥ 320
33	≥ 400 $11\otimes 11\otimes 11$
34	≥ 400
35	≥ 500 $11\otimes 12\otimes 12$
36	≥ 625 $12\otimes 12\otimes 12$
37	≥ 625
38	≥ 626 <i>theorem 4.1</i>
39	≥ 678 <i>computer</i>
40	≥ 762 <i>computer</i>
41	≥ 871 <i>computer</i>
42	≥ 976 <i>computer</i>
43	≥ 1086 <i>computer</i>
44	≥ 1246 <i>computer</i>
45	≥ 1420 <i>computer</i>
46	≥ 1630 <i>computer</i>
47	≥ 1808 <i>computer</i>
48	≥ 2036 <i>computer</i>
49	≥ 2036
50	≥ 2037 <i>theorem 4.1</i>
51	≥ 2304 $17\otimes 17\otimes 17$

d	$\kappa(d)$
52	≥ 2560 $16\otimes 18\otimes 18$
53	≥ 3072 $17\otimes 18\otimes 18$
54	≥ 4096 $18\otimes 18\otimes 18$ or $9\otimes 9\otimes 9\otimes 27$
55	≥ 4096
56	≥ 4097 <i>theorem 4.1</i>
57	≥ 4097
58	≥ 4608 $18\otimes 20\otimes 20$
59	≥ 4608
60	≥ 5184 $20\otimes 20\otimes 20$

d	$\kappa(d)$
61	≥ 5184
62	≥ 5832 $20\otimes 21\otimes 21$
63	≥ 6561 $21\otimes 21\otimes 21$
64	≥ 6561
65	≥ 6562 <i>theorem 4.1</i>
66	≥ 8000 $11\otimes 11\otimes 11\otimes 33$
67	≥ 8342 <i>theorem 2.1</i>
68	≥ 9632 <i>theorem 2.1</i>
69	≥ 11122 <i>theorem 2.1</i>

The following tables summarise the best lower bounds known for $\alpha(d)$. For $3 \leq d \leq 6$, the best lower bound is Danzer and Grünbaum's $2d - 1$ [DG]. For $7 \leq d \leq 26$, the results of a computer program, based on the 'probabilistic construction' but using sets of points close to the surface of the d -sphere, provide the largest known acute d -sets. An acute 7-set of cardinality 14 and an acute 8-set of cardinality 16 are displayed. For $27 \leq d \leq 62$, the largest known acute d -set is cubic. Finally, for $d \geq 63$, theorem 3.1 provides the best (probabilistic) lower bound.

Best Lower Bounds Known for $\alpha(d)$

d	$\alpha(d)$
0	= 1
1	= 2
2	= 3
3	= 5 [DG]
4–6	$\geq 2d - 1$ [DG]
7	≥ 14 <i>computer</i>
8	≥ 16 <i>computer</i>
9	≥ 19 <i>computer</i>
10	≥ 23 <i>computer</i>
11	≥ 26 <i>computer</i>
12	≥ 30 <i>computer</i>
13	≥ 36 <i>computer</i>
14	≥ 42 <i>computer</i>
15	≥ 47 <i>computer</i>

d	$\alpha(d)$
16	≥ 54 <i>computer</i>
17	≥ 63 <i>computer</i>
18	≥ 71 <i>computer</i>
19	≥ 76 <i>computer</i>
20	≥ 90 <i>computer</i>
21	≥ 103 <i>computer</i>
22	≥ 118 <i>computer</i>
23	≥ 121 <i>computer</i>
24	≥ 144 <i>computer</i>
25	≥ 155 <i>computer</i>
26	≥ 184 <i>computer</i>
27–62	$\geq \kappa(d)$
63	≥ 6636 <i>theorem 3.1</i>

$\alpha(7) \geq 14$
(62, 1, 9, 10, 17, 38, 46)
(38, 54, 0, 19, 38, 14, 25)
(60, 33, 42, 9, 48, 3, 12)
(62, 35, 41, 44, 16, 39, 44)
(62, 34, 7, 45, 48, 37, 12)
(28, 33, 42, 8, 49, 39, 45)
(40, 16, 22, 12, 0, 0, 25)
(45, 17, 26, 67, 25, 20, 29)
(38, 6, 35, 0, 32, 18, 0)
(62, 0, 42, 45, 49, 3, 48)
(30, 0, 9, 44, 49, 37, 48)
(0, 20, 31, 27, 34, 21, 28)
(48, 19, 24, 22, 33, 20, 73)
(43, 17, 25, 27, 32, 64, 19)

$\alpha(8) \geq 16$
(34, 49, 14, 51, 0, 36, 46, 0)
(31, 17, 14, 51, 1, 5, 44, 31)
(33, 50, 48, 20, 34, 35, 15, 0)
(0, 16, 16, 52, 32, 36, 45, 0)
(37, 31, 46, 52, 13, 0, 0, 22)
(2, 50, 13, 52, 3, 3, 46, 0)
(1, 50, 48, 51, 1, 5, 46, 31)
(24, 0, 43, 2, 17, 20, 32, 16)
(11, 49, 0, 11, 19, 8, 32, 19)
(0, 48, 48, 52, 1, 34, 12, 2)
(0, 48, 47, 51, 34, 37, 47, 32)
(34, 49, 14, 51, 34, 36, 13, 34)
(0, 46, 31, 0, 0, 23, 29, 29)
(16, 40, 29, 23, 54, 3, 17, 16)
(2, 15, 14, 50, 2, 36, 15, 33)
(12, 36, 28, 30, 3, 45, 48, 45)

6 Generalising $\kappa(d)$

We can understand $\kappa(d)$ to be the size of the largest possible set \mathcal{S} of binary words such that, for any ordered triple of words $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ in \mathcal{S} , there exists an index i for which $(\mathbf{u}_i, \mathbf{v}_i, \mathbf{w}_i) = (0, 1, 0)$ or $(\mathbf{u}_i, \mathbf{v}_i, \mathbf{w}_i) = (1, 0, 1)$. We can generalise this in the following way:

Definition 6.1 *If T_1, \dots, T_m are ordered k -tuples from $\{0, \dots, r-1\}^k$ (which we will refer to as the matching k -tuples), then let us define $\kappa[r, k, T_1, \dots, T_m](d)$ to be the size of the largest possible set \mathcal{S} of r -ary words of length d such that, for any ordered k -tuple of words $(\mathbf{w}_1, \dots, \mathbf{w}_k)$ in \mathcal{S} , there exist i and j , $1 \leq i \leq d$, $1 \leq j \leq m$, for which $(\mathbf{w}_{1i}, \dots, \mathbf{w}_{ki}) = T_j$.*

Thus we have $\kappa(d) = \kappa[2, 3, (0, 1, 0), (1, 0, 1)](d)$. If the set of matching k -tuples is closed under permutation, we will abbreviate by writing a list of matching *multisets* of cardinality k , rather than ordered tuples. For example, instead of $\kappa[2, 3, (0, 0, 1), (0, 1, 0), (1, 0, 0)](d)$, we write $\kappa[2, 3, \{0, 0, 1\}](d)$.

We can find probabilistic and, in some cases, constructive lower bounds for general $\kappa[r, k, T_1, \dots, T_m](d)$ using the approaches we used for cubic acute d -sets. To illustrate this, in the remainder of this paper, we will consider the set of problems in which it is simply required that at some index the k -tuple of words be all different (pairwise distinct). First, we express this in a slightly different form.

Let us say that an **r -ary d -colouring** is some colouring of the integers $1, \dots, d$ using r colours. Let us also say that a set \mathcal{R} of r -ary d -colourings is a **k -rainbow set**, for some $k \leq r$ if for any set $\{c_1, \dots, c_k\}$ of k colourings in \mathcal{R} , there exists some integer t , $1 \leq t \leq d$, for which the colours $c_1(t), \dots, c_k(t)$ are all different, i.e. $c_i(t) \neq c_j(t)$ for any i and j , $1 \leq i, j \leq k$, $i \neq j$. For conciseness, we will denote “a k -rainbow set of r -ary d -colourings” by “a $\mathcal{RSC}[k, r, d]$ ”.

Let us further say that a set $\{c_1, \dots, c_k\}$ of k d -colourings is a **good k -set** if there exists some integer t , $1 \leq t \leq d$, for which the colours $c_1(t), \dots, c_k(t)$ are all different, and a **bad k -set** if there exists no such t .

We will denote by $\rho_{r,k}(d)$ the size of the largest possible $\mathcal{RSC}[k, r, d]$, abbreviating $\rho_{k,k}(d)$ by $\rho_k(d)$. Now, $\rho_k(d) = \kappa[k, k, \{0, 1, \dots, k-1\}](d)$ and

$$\rho_{r,k}(d) = \kappa[r, k, \{0, \dots, k-1\}, \dots, \{r-k, \dots, r-1\}](d),$$

where the matching multisets are those of cardinality k with k distinct members.

Clearly, $\rho_{r,k}(d) \leq \rho_{r,k}(d+1)$, $\rho_{r,k}(d) \leq \rho_{r+1,k}(d)$ and $\rho_{r,k}(d) \geq \rho_{r,k+1}(d)$. Also, $\rho_{r,1}(d)$ is undefined because any set of colourings is a 1-rainbow, $\rho_{r,k}(1) = r$ if $k > 1$, and $\rho_{r,2}(d) = r^d$ because any two distinct r -ary d -colourings (or r -ary words of length d) differ somewhere.

In the next two sections we will give a number of probabilistic and constructive lower bounds for $\rho_{r,k}(d)$, for various r and k .

7 A probabilistic lower bound for $\rho_{r,k}(d)$

Theorem 7.1

$$\rho_{r,k}(d) \geq (k-1)m \quad \text{where} \quad m = \left\lfloor \sqrt[k-1]{\frac{k!}{k^k}} \left(\sqrt[k-1]{\frac{(r-k)! r^k}{(r-k)! r^k - r!}} \right)^d \right\rfloor.$$

Proof: This proof is similar that of theorem 2.1.

Randomly pick a set \mathcal{R} of km r -ary d -colourings, choosing the colours from $\{\chi_0, \dots, \chi_{r-1}\}$ independently with probability $\Pr[c(i) = \chi_j] = 1/r$, $1 \leq i \leq d$, $0 \leq j < r$ for every $c \in \mathcal{R}$.

Now the probability that a set of k colourings from \mathcal{R} is a bad k -set is

$$(1-p)^d \quad \text{where} \quad p = \frac{r!/(r-k)!}{r^k}.$$

Hence, the expected number of bad k -sets in a set of km d -colourings is $\binom{km}{k}(1-p)^d$. Thus there is *some* set \mathcal{R} of km d -colourings with no more than $\binom{km}{k}(1-p)^d$ bad k -sets, where

$$\binom{km}{k}(1-p)^d < \frac{(km)^k}{k!}(1-p)^d = m \frac{k^k}{k!} m^{k-1}(1-p)^d \leq m$$

by the choice of m .

If we remove one colouring of each bad k -set from \mathcal{R} , the remaining set is a $\mathcal{RSC}[k, r, d]$ of cardinality at least $km - m = (k-1)m$. \square

The following results follow directly:

$$\rho_3(d) \geq 2 \left\lfloor \frac{\sqrt{2}}{3} \left(\frac{3}{\sqrt{7}} \right)^d \right\rfloor \approx 0.943 \times 1.134^d.$$

$$\rho_{4,3}(d) \geq 2 \left\lfloor \frac{\sqrt{2}}{3} \left(\frac{4}{\sqrt{10}} \right)^d \right\rfloor \approx 0.943 \times 1.265^d.$$

$$\rho_4(d) \geq 3 \left\lfloor \sqrt[3]{\frac{3}{32}} \sqrt[3]{\frac{32}{29}}^d \right\rfloor \approx 1.363 \times 1.033^d.$$

8 Constructive lower bounds for $\rho_{r,k}(d)$

In the following proofs, for clarity of exposition, we will represent r -ary d -colourings as r -ary words of length d , e.g. $\mathcal{R}_{3,3,3} = \{000, 011, 102, 121, 212, 220\}$ represents a 3-rainbow set of ternary 3-colourings (using the colours χ_0, χ_1 and χ_2). Concatenation of words (colourings) c and c' will be written $c.c'$.

We begin with a construction that enables us to extend a $\mathcal{RSC}[k, r, d]$ of cardinality n to one of cardinality $n + 1$ or greater.

Theorem 8.1 *If for some $r \geq k \geq 3$, and some d , we have a $\mathcal{RSC}[k, r, d]$ of cardinality n , and for some $r', k - 2 \leq r' \leq r - 2$, and d' , we have a $\mathcal{RSC}[k - 2, r', d']$ of cardinality at least $n - 1$, then we can construct a $\mathcal{RSC}[k, r, d + d']$ of cardinality $N = n - 1 + r - r'$.*

Proof: Let $\mathcal{R} = \{c_0, c_1, \dots, c_{n-1}\}$ be a $\mathcal{RSC}[k, r, d]$ of cardinality n (using colours $\chi_0, \dots, \chi_{r-1}$) and $\mathcal{R}' = \{c'_0, c'_1, \dots, c'_{n'-1}\}$ be a $\mathcal{RSC}[k - 2, r', d']$ of cardinality $n' \geq n - 1$ (using colours $\chi_0, \dots, \chi_{r'-1}$).

Now let $\mathcal{Q} = \{q_0, q_1, \dots, q_{N-1}\}$ be a set of r -ary $(d + d')$ -colourings where $q_i = c_i.c'_i$ for $0 \leq i \leq n - 2$, and $q_{n-1+j} = c_{n-1}.(r' + j)^{d'}$ for $0 \leq j < r - r'$, each element of \mathcal{Q} being made by concatenating two component colourings, the first from \mathcal{R} and the second being either from \mathcal{R}' or a monochrome colouring.

If $\{q_{i_1}, \dots, q_{i_k}\}$ is a set of colourings in \mathcal{Q} with no more than one of the i_m greater than $n - 2$, then it is a good k -set because of the first components, since \mathcal{R} is a k -rainbow set.

On the other hand, if $\{q_{i_1}, \dots, q_{i_k}\}$ is a set of colourings in \mathcal{Q} with no more than $k - 2$ of the i_m less than $n - 1$, then it too is a good k -set because of the second components, since \mathcal{R}' is a $(k - 2)$ -rainbow set using colours $\chi_0, \dots, \chi_{r'-1}$ and the second components of the colourings with indices greater than $n - 2$ are each monochrome of a different colour, drawn from $\chi_{r'}, \dots, \chi_{r-1}$.

Thus \mathcal{Q} is a $\mathcal{RSC}[k, r, d + d']$ of cardinality N . □

Corollary 8.1.1 $\rho_{r,3}(d + 1) \geq \rho_{r,3}(d) + r - 2$.

Proof: This follows from the theorem due to the fact that there is a 1-rainbow set of 1-ary 1-colourings of any cardinality. □

Corollary 8.1.2 $\rho_{r,4}(d + \lceil \log_2(\rho_{r,4}(d) - 1) \rceil) \geq \rho_{r,4}(d) + r - 3$.

Proof: Since $\rho_{r,2}(d) = r^d$, we have $\rho_{2,2}(d') \geq \rho_{r,4}(d) - 1$ iff $d' \geq \log_2(\rho_{r,4}(d) - 1)$. □

Theorem 8.2 *If, for each s , $1 \leq s \leq m$, we have a $\mathcal{RSC}[3, r, d_s]$ of cardinality n_s , where n_1 is the least of the n_s , and if, for some d_Z , we have a $\mathcal{RSC}[3, r, d_Z]$ of cardinality n_Z , where*

$$n_Z \geq \prod_{s=2}^m (1 + 2 \lfloor \frac{n_s}{2} \rfloor),$$

then a $\mathcal{RSC}[3, r, D]$ of cardinality N can be constructed, where

$$D = \sum_{s=1}^m d_s + 2d_Z \quad \text{and} \quad N = \prod_{s=1}^m n_s.$$

This result for 3-rainbow sets corresponds to theorem 4.3 for cubic acute d -sets. Before we can prove it, we need some further preliminary results.

Definition 8.3 *If $n_1 \leq n_2 \leq \dots \leq n_m$ and $0 \leq k_r < n_r$, for each r , $1 \leq r \leq m$, then let us denote by $\langle\langle k_1 k_2 \dots k_m \rangle\rangle_{n_1 n_2 \dots n_m}^+$, the number*

$$\langle\langle k_1 k_2 \dots k_m \rangle\rangle_{n_1 n_2 \dots n_m}^+ = \sum_{r=2}^m \left((k_{r-1} + k_r \bmod n_r) \prod_{s=r+1}^m n_s \right).$$

The definition of $\langle\langle k_1 k_2 \dots k_m \rangle\rangle_{n_1 n_2 \dots n_m}^+$ is the same as that for $\langle\langle k_1 k_2 \dots k_m \rangle\rangle_{n_1 n_2 \dots n_m}$ (see 4.4), but with addition replacing subtraction. By construction, we have

$$\langle\langle k_1 k_2 \dots k_m \rangle\rangle_{n_1 n_2 \dots n_m}^+ < \prod_{r=2}^m n_r,$$

and, if $2 \leq t \leq m$ and $j_{t-1} + j_t \neq k_{t-1} + k_t \pmod{n_t}$, then

$$\langle\langle j_1 j_2 \dots j_m \rangle\rangle_{n_1 n_2 \dots n_m}^+ \neq \langle\langle k_1 k_2 \dots k_m \rangle\rangle_{n_1 n_2 \dots n_m}^+.$$

Lemma 8.4 *If $n_1 \leq n_2 \leq \dots \leq n_m$, with all the n_r odd except perhaps n_1 , and $0 \leq j_r, k_r, l_r < n_r$, for each r , $1 \leq r \leq m$, and the sequences of j_r , k_r and l_r are neither pairwise identical nor anywhere pairwise distinct, i.e. there is some u, v and w such that $j_u \neq k_u$, $k_v \neq l_v$ and $l_w \neq j_w$ but no t such that $j_t \neq k_t$, $k_t \neq l_t$ and $l_t \neq j_t$, then either*

$$\langle\langle j_1 \dots j_m \rangle\rangle_{n_1 \dots n_m}, \langle\langle k_1 \dots k_m \rangle\rangle_{n_1 \dots n_m}, \langle\langle l_1 \dots l_m \rangle\rangle_{n_1 \dots n_m} \text{ are pairwise distinct}$$

or

$$\langle\langle j_1 \dots j_m \rangle\rangle_{n_1 \dots n_m}^+, \langle\langle k_1 \dots k_m \rangle\rangle_{n_1 \dots n_m}^+, \langle\langle l_1 \dots l_m \rangle\rangle_{n_1 \dots n_m}^+ \text{ are pairwise distinct.}$$

Proof: Without loss of generality, we can assume that we have $j_1 = k_1$, that $t > 1$ is the least integer for which $j_t \neq k_t$, and that $k_t = l_t$. We will consider two cases:

Case 1: $k_{t-1} \neq l_{t-1}$

Since $j_{t-1} = k_{t-1} \neq l_{t-1}$ and $j_t \neq k_t = l_t$, we have $j_{t-1} - j_t \neq k_{t-1} - k_t$ and $k_{t-1} - k_t \neq l_{t-1} - l_t$, and so $\langle\langle j_1 \dots j_m \rangle\rangle \neq \langle\langle k_1 \dots k_m \rangle\rangle$ and $\langle\langle k_1 \dots k_m \rangle\rangle \neq \langle\langle l_1 \dots l_m \rangle\rangle$. Similarly, $j_{t-1} + j_t \neq k_{t-1} + k_t$ and $k_{t-1} + k_t \neq l_{t-1} + l_t$, and so $\langle\langle j_1 \dots j_m \rangle\rangle^+ \neq \langle\langle k_1 \dots k_m \rangle\rangle^+$ and $\langle\langle k_1 \dots k_m \rangle\rangle^+ \neq \langle\langle l_1 \dots l_m \rangle\rangle^+$.

If $j_{t-1} - j_t \neq l_{t-1} - l_t$, then $\langle\langle j_1 \dots j_m \rangle\rangle \neq \langle\langle l_1 \dots l_m \rangle\rangle$. If $j_{t-1} - j_t = l_{t-1} - l_t$ then $(j_{t-1} + j_t) - (l_{t-1} + l_t) = (j_{t-1} - j_t + 2j_t) - (l_{t-1} - l_t + 2l_t) = 2(j_t - l_t) \not\equiv 0 \pmod{n_t}$ because $j_t \neq l_t$ and n_t is odd, so $j_{t-1} + j_t \neq l_{t-1} + l_t$ and $\langle\langle j_1 \dots j_m \rangle\rangle^+ \neq \langle\langle l_1 \dots l_m \rangle\rangle^+$.

Case 2: $k_{t-1} = l_{t-1}$

Since $j_{t-1} = k_{t-1} = l_{t-1}$ and $j_t \neq k_t = l_t$, we have $j_{t-1} - j_t \neq k_{t-1} - k_t$ and $j_{t-1} - j_t \neq l_{t-1} - l_t$, and so $\langle\langle j_1 \dots j_m \rangle\rangle \neq \langle\langle k_1 \dots k_m \rangle\rangle$ and $\langle\langle j_1 \dots j_m \rangle\rangle \neq \langle\langle l_1 \dots l_m \rangle\rangle$.

If $k_1 = l_1$, let u be the least integer such that $k_u \neq l_u$. Since $k_{u-1} = l_{u-1}$, we have $k_{u-1} - k_u \neq l_{u-1} - l_u$. If $k_1 \neq l_1$, let u be the least integer such that $k_u = l_u$. Since $k_{u-1} \neq l_{u-1}$, we still have $k_{u-1} - k_u \neq l_{u-1} - l_u$. Thus, $\langle\langle k_1 \dots k_m \rangle\rangle \neq \langle\langle l_1 \dots l_m \rangle\rangle$. \square

Proof of Theorem 8.2

Let $n_1 \leq n_2 \leq \dots \leq n_m$, and, for each s , $1 \leq s \leq m$, let $\mathcal{R}_s = \{c_0^s, c_1^s, \dots, c_{n_s-1}^s\}$ be a $\mathcal{RSC}[3, r, d_s]$ of cardinality n_s , and let $n'_s = 1 + 2 \lfloor n_s/2 \rfloor$ be the least odd integer not less than n_s . Let $\mathcal{Z} = \{z_0, z_1, \dots, z_{n_Z-1}\}$ be a $\mathcal{RSC}[3, r, d_Z]$ of cardinality n_Z , where

$$n_Z \geq \prod_{s=2}^m n'_s,$$

and let

$$D = \sum_{s=1}^m d_s + 2d_Z \quad \text{and} \quad N = \prod_{s=1}^m n_s.$$

Now let

$$\mathcal{Q} = \{c_{k_1}^1 \cdot c_{k_2}^2 \cdot \dots \cdot c_{k_m}^m \cdot z_{k_Z} \cdot z_{k_Z}^+ : 0 \leq k_s < n_s, 1 \leq s \leq m\},$$

where $k_Z = \langle\langle k_1 k_2 \dots k_m \rangle\rangle_{n'_1 n'_2 \dots n'_m}$ and $k_Z^+ = \langle\langle k_1 k_2 \dots k_m \rangle\rangle_{n'_1 n'_2 \dots n'_m}^+$ be a set of D -colourings of cardinality N , each element of \mathcal{Q} being made by concatenating one colouring from each of the \mathcal{R}_s together with two colourings from \mathcal{Z} . (Below, we will denote this construction by $d_1 \otimes \dots \otimes d_m \oplus d_Z \oplus d_Z$.)

Let $c_{i_1}^1 \cdot c_{i_2}^2 \cdot \dots \cdot c_{i_m}^m \cdot z_{i_Z} \cdot z_{i_Z}^+$, $c_{j_1}^1 \cdot c_{j_2}^2 \cdot \dots \cdot c_{j_m}^m \cdot z_{j_Z} \cdot z_{j_Z}^+$ and $c_{k_1}^1 \cdot c_{k_2}^2 \cdot \dots \cdot c_{k_m}^m \cdot z_{k_Z} \cdot z_{k_Z}^+$ be any three distinct colourings in \mathcal{Q} . If, for some s , $i_s \neq j_s$, $j_s \neq k_s$ and $k_s \neq i_s$, then these three colourings comprise a good 3-set because \mathcal{R}_s is a 3-rainbow set.

If, however, there is no s such that i_s, j_s and k_s are all different, then the condition of lemma 8.4 holds, and so either i_Z, j_Z and k_Z are all different, or i_Z^+, j_Z^+ and k_Z^+ are all different, and the three colourings comprise a good 3-set because \mathcal{Z} is a 3-rainbow set.

Thus, any three colourings in \mathcal{Q} comprise a good 3-set, so \mathcal{Q} is a $\mathcal{RSC}[3, r, D]$ of cardinality N . □

Corollary 8.4.1 *If $\rho_{r,3}(d)$ is odd, then $\rho_{r,3}(4d) \geq \rho_{r,3}(d)^2$.*

Proof: By theorem 8.2 using the construction $d \otimes d \oplus d \oplus d$. □

Corollary 8.4.2 $\rho_{r,3}(4d + 2) \geq \rho_{r,3}(d)^2$.

Proof: By 8.1.1, if $n = \rho_{r,3}(d)$, we can construct a $\mathcal{RSC}[3, r, d + 1]$ of cardinality $n + 1 \geq 1 + 2 \lfloor n/2 \rfloor$. By theorem 8.2, we can then construct a $\mathcal{RSC}[3, r, 4d + 2]$ of cardinality n^2 using the construction $d \otimes d \oplus (d + 1) \oplus (d + 1)$. □

Corollary 8.4.3 $\rho_3(4^d) \geq 3^{2^d}$.

Proof: By repeated application of 8.4.1 starting with $\rho_{3,3}(1) = 3$. □

Our final construction enables us to combine k -rainbow sets of r -ary d -colourings for arbitrary k .

Theorem 8.5 *If we have a $\mathcal{RSC}[k, r, d_1]$ of cardinality n_1 , a $\mathcal{RSC}[k, r, d_2]$ of cardinality $n_2 \geq n_1$, and a $\mathcal{RSC}[k, r, d_Z]$ of cardinality $n_Z \geq n_2$, with n_Z coprime to each integer in the range $[2, \dots, h]$ where $h = \binom{k}{2} - 1$, then a $\mathcal{RSC}[k, r, D]$ of cardinality N can be constructed, where $D = d_1 + d_2 + h d_Z$ and $N = n_1 n_2$.*

As before, we first need a preliminary result:

Lemma 8.6 *Given distinct pairs of integers (a, b) and (c, d) with $0 \leq a, b, c, d < n$ for some n , and given a positive integer h such that n is coprime to each integer in the range $[2, \dots, h]$, then if we let $b_{-1} = a$ and $d_{-1} = c$, and $b_r = b + ra \pmod{n}$ and $d_r = d + rc \pmod{n}$ for $0 \leq r \leq h$, then if $b_i = d_i$ for some i , $-1 \leq i \leq h$, we have $b_j \neq d_j$ for all $j \neq i$.*

Proof: We consider two cases:

Case 1: $i = -1$

Since $a = c$, $(b + ja) - (d + jc) = b - d \not\equiv 0 \pmod{n}$ since (a, b) and (c, d) are distinct, and b and d both less than n .

Case 2: $i \neq -1$

By the reversing the argument in case 1, $a \neq c$, i.e. $b_{-1} \neq d_{-1}$. For $j \geq 0$, since $b + ia = d + ic$, we have $(b + ja) - (d + jc) = (j - i)a - (j - i)c = (j - i)(a - c) \not\equiv 0 \pmod{n}$ since $a \neq c$ and $|j - i| \leq h$ so $j - i$ is coprime to n . \square

Proof of Theorem 8.5

Let $\mathcal{R}_1 = \{c_0^1, \dots, c_{n_1-1}^1\}$, $\mathcal{R}_2 = \{c_0^2, \dots, c_{n_2-1}^2\}$ and $\mathcal{Z} = \{z_0, \dots, z_{n_Z-1}\}$ be k -rainbow sets of r -ary d_1 -, d_2 - and d_Z -colourings of cardinality n_1 , n_2 and n_Z , respectively.

Now let

$$\mathcal{Q} = \{c_i^1 \cdot c_j^2 \cdot z_{j+i} \cdot z_{j+2i} \dots z_{j+hi} : 0 \leq i < n_1, 0 \leq j < n_2\},$$

where $h = \binom{k}{2} - 1$ and the subscript arithmetic is modulo n_Z , be a set of D -colourings of cardinality N , each element of \mathcal{Q} being made by concatenating $h+2$ component colourings: one from \mathcal{R}_1 , one from \mathcal{R}_2 , and h from \mathcal{Z} .

Let

$$\mathcal{S} = \{c_{i_1}^1 \cdot c_{j_1}^2 \cdot z_{j_1+i_1} \dots z_{j_1+hi_1}, c_{i_2}^1 \cdot c_{j_2}^2 \cdot z_{j_2+i_2} \dots z_{j_2+hi_2}, \dots, c_{i_k}^1 \cdot c_{j_k}^2 \cdot z_{j_k+i_k} \dots z_{j_k+hi_k}\}$$

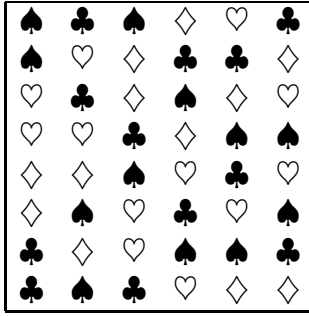
be any set of k distinct colourings in \mathcal{Q} , and let $b_{s,-1} = i_s$ and $b_{s,t} = j_s + ti_s \pmod{n_Z}$, for each s and t , $1 \leq s \leq k$, $0 \leq t \leq h$, so the s^{th} colouring in \mathcal{S} is $c_{b_{s,-1}}^1 \cdot c_{b_{s,0}}^2 \cdot z_{b_{s,1}} \dots z_{b_{s,h}}$.

Now, for any s, s' and t , $1 \leq s, s' \leq k$, $-1 \leq t \leq h$, if $b_{s,t} = b_{s',t}$, then by lemma 8.6 we know that for all $u \neq t$, $b_{s,u} \neq b_{s',u}$. So for each pair $\{s, s'\}$, $b_{s,t} = b_{s',t}$ for no more than one value of t . Now there are $h+2$ possible values of t , but only $\binom{k}{2} = h+1$ different pairs $\{s, s'\}$, so there is *some* t for which $b_{s,t} \neq b_{s',t}$ for all pairs $\{s, s'\}$ and the $(t+2)^{\text{th}}$ component colourings of the elements in \mathcal{S} are all different. Since \mathcal{R}_1 , \mathcal{R}_2 and \mathcal{Z} are all k -rainbow sets, we know that \mathcal{S} is a good k -set.

Thus, any k colourings from \mathcal{Q} comprise a good k -set, so \mathcal{Q} is a $\mathcal{RSC}[k, r, D]$ of cardinality N . \square

Corollary 8.6.1 $\rho_4(6.7^d) \geq 7^{2^d}$.

Proof: The following 4-rainbow set of 4-ary 6-colourings of cardinality 8 — a version of $\mathcal{R}_{4,4,6}$ (see below) displayed with different symbols for each colour — shows that $\rho_4(6) \geq 7$.



The result follows by repeated application of theorem 8.5, noting that 7 is coprime to 2, 3, 4 and $5 = \binom{4}{2} - 1$. □

9 Lower bounds for $\rho_{r,k}(d)$ for small r , k and d

We conclude with tables of the best lower bounds known for $\rho_3(d)$, $\rho_{4,3}(d)$ and $\rho_4(d)$ for small d . For very small d , exhaustive computer searches have determined the values of $\rho_{r,k}(d)$. For other small values of d , the constructions used in theorems 8.2 and 8.5 provide the largest known rainbow sets. In the tables, these constructions are denoted $d_1 \otimes d_2 \oplus d_Z \oplus d_Z$, etc., with superscript minus signs (d^-) to denote the removal of a single colouring from a largest rainbow set of d -colourings (to satisfy the requirement that the cardinality be odd). For $\rho_3(d)$, the probabilistic lower bound of theorem 7.1 is better than the constructions for $d \geq 71$; for $\rho_{4,3}(d)$, this is the case for $d \geq 26$.

Some k -rainbow sets of r -ary d -colourings, for small k , r and d

$\mathcal{R}_{3,3,3}$ $\rho_3(3) \geq 6$	$\mathcal{R}_{3,3,6}$ $\rho_3(6) \geq 13$	$\mathcal{R}_{4,3,3}$ $\rho_{4,3}(3) \geq 9$	$\mathcal{R}_{4,3,4}$ $\rho_{4,3}(4) \geq 16$	$\mathcal{R}_{4,4,6}$ $\rho_4(6) \geq 8$
000	000000	000	0000	000000
011	000111	011	0011	011111
102	000222	022	0102	101222
121	011012	103	0220	112033
212	022120	131	1013	220312
220	101120	213	1212	233103
	112021	232	1230	323230
	112102	323	1302	332321
	112210	330	2031	
	120012		2103	
	202012		2121	
	210120		2320	
	221201		3113	
			3231	
			3322	
			3333	

Best Lower Bounds Known for $\rho_3(d)$ and $\rho_{4,3}(d)$

d	$\rho_3(d)$	d	$\rho_{4,3}(d)$
1	= 3	1	= 4
2	= 4 <i>computer, 8.1.1</i>	2	= 6 <i>computer, 8.1.1</i>
3	= 6 <i>computer, $\mathcal{R}_{3,3,3}$</i>	3	= 9 <i>computer, $\mathcal{R}_{4,3,3}$</i>
4	= 9 <i>computer, 1⊗1⊗1⊗1</i>	4	= 16 <i>computer, $\mathcal{R}_{4,3,4}$</i>
5	= 10 <i>computer, 8.1.1</i>	5	≥ 18 8.1.1
6	= 13 <i>computer, $\mathcal{R}_{3,3,6}$</i>	6	≥ 20 8.1.1
7	≥ 14 8.1.1	7	≥ 22 8.1.1
8	≥ 15 8.1.1	8	≥ 25 $2^- \otimes 2^- \otimes 2 \otimes 2$
9	≥ 16 8.1.1	9	≥ 27 8.1.1
10	≥ 17 8.1.1	10	≥ 36 $1 \otimes 3 \otimes 3 \otimes 3$ or $2 \otimes 2 \otimes 3 \otimes 3$
11	≥ 27 $1 \otimes 1 \otimes 1 \otimes 4 \otimes 4$	11	≥ 54 $2 \otimes 3 \otimes 3 \otimes 3$
12	≥ 28 8.1.1	12	≥ 81 $3 \otimes 3 \otimes 3 \otimes 3$
13	≥ 29 8.1.1	13	≥ 83 8.1.1
14	≥ 36 $2 \otimes 4 \otimes 4 \otimes 4$	14	≥ 90 $2 \otimes 4^- \otimes 4 \otimes 4$
15	≥ 54 $3 \otimes 4 \otimes 4 \otimes 4$	15	≥ 135 $3 \otimes 4^- \otimes 4 \otimes 4$
16	≥ 81 $4 \otimes 4 \otimes 4 \otimes 4$	16	≥ 225 $4^- \otimes 4^- \otimes 4 \otimes 4$
...
70	≥ 6723 $16 \otimes 18 \otimes 18 \otimes 18$	25	≥ 363 8.1.1
71	≥ 7064 <i>theorem 7.1</i>	26	≥ 424 <i>theorem 7.1</i>

Best Lower Bounds Known for $\rho_4(d)$

d	$\rho_4(d)$
1	= 4
2	= 4 <i>computer</i>
3	= 5 <i>computer, 8.1.2</i>
4	= 5 <i>computer</i>
5	= 6 <i>computer, 8.1.2</i>
6	= 8 <i>computer, $\mathcal{R}_{4,4,6}$</i>
...	...
42	≥ 49 $6^- \oplus 6^- \oplus 6^- \oplus 6^- \oplus 6^- \oplus 6^-$

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