A Theoretical Study of Two-Period Relaxations for Lot-Sizing Problems with Big-Bucket Capacities

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Abstract

In this paper, we study two-period subproblems proposed by [1] for lot-sizing problems with big-bucket capacities and nonzero setup times, complementing our previous work [3] investigating the special case of zero setup times. In particular, we study the polyhedral structure of the mixed integer sets related to various two-period relaxations. We derive several families of valid inequalities and investigate their facet-defining conditions. We also discuss the separation problems associated with these valid inequalities.

1 Introduction

In this study, we investigate multi-item production planning problems with big bucket capacities, i.e., each resource is shared by multiple items, which can be produced in a specific time period. These real-world problems are very interesting, as they remain challenging to solve to optimality and also to achieve strong bounds. The uncapacitated and single-item relaxations of the problem have been previously studied by [7]. The work of [6] introduced and studied the single-period relaxation with "preceding inventory", where a number of cover and reverse cover inequalities are defined for this relaxation. Finally, we also note the relevant study of [5], who studied a single-period relaxation and compared with other relaxations.

We present a formulation for this problem following the notation of [2]. Let NT, NI and NK indicate the number of *periods*, *items*, and *machine types*, respectively. We represent the production, setup, and inventory variables for item i in period t by x_t^i, y_t^i, and s_t^i, respectively. We note that our model can be generalized to involve multiple levels as in [1], however, we omit this for the sake of simplicity.

$$\min \sum_{t=1}^{NT} \sum_{i=1}^{NI} f_t^i y_t^i + \sum_{t=1}^{NT} \sum_{i=1}^{NI} h_t^i s_t^i \tag{1}$$

$$\text{s.t. } x_t^i + s_{t-1}^i - s_t^i = d_t^i \qquad t \in \{1, \ldots, NT\}, i \in \{1, \ldots, NI\} \tag{2}$$

$$\sum_{i=1}^{NI} (a_k^i x_t^i + ST_k^i y_t^i) \le C_t^k \qquad t \in \{1, \ldots, NT\}, k \in \{1, \ldots, NK\} \tag{3}$$

$$x_t^i \le M_t^i y_t^i \qquad t \in \{1, \ldots, NT\}, i \in \{1, \ldots, NI\} \tag{4}$$

$$y \in \{0,1\}^{NT\text{x}NI}; x, s \ge 0 \tag{5}$$

The objective function (1) minimizes total cost, where f_t^i and h_t^i indicate the setup and inventory cost coefficients, respectively. The flow balance constraints (2) ensure that the demand for each item i in period t, denoted by d_t^i, is satisfied. The big bucket capacity constraints (3) ensure that the capacity C_t^k of machine k is not exceeded in time period t, where a_k^i and ST_k^i represent the per unit production time and setup time for item i, respectively. The constraints (4) guarantee that the setup variable is equal to 1 if production occurs, where M_t^i represents the maximum number of item i that can be produced in period t, based on the minimum of remaining cumulative demand and capacity available. Finally, the integrality and non-negativity constraints are given by (5).

2 Two-Period Relaxation

Let $I = \{1, \ldots, NI\}$. We present the feasible region of a two-period, single-machine relaxation of the multi-item production planning problem, denoted by X^{2PL} (see [1] for details).

$$x_{t'}^i \le \widetilde{M}_{t'}^i y_{t'}^i \qquad i \in I, t' = 1, 2 \tag{6}$$

$$x_{t'}^i \le \widetilde{d}_{t'}^i y_{t'}^i + s^i \qquad i \in I, t' = 1, 2 \tag{7}$$

$$x_1^i + x_2^i \le \widetilde{d}_1^i y_1^i + \widetilde{d}_2^i y_2^i + s^i \qquad i \in I \tag{8}$$

$$x_1^i + x_2^i \le \widetilde{d}_1^i + s^i \qquad i \in I \tag{9}$$

$$\sum_{i \in I} (a^i x_{t'}^i + ST^i y_{t'}^i) \le \widetilde{C}_{t'} \qquad t' = 1, 2 \tag{10}$$

$$x, s \ge 0, y \in \{0,1\}^{2 \times NI} \tag{11}$$

Since we consider a single machine, we dropped the k index from this formulation, however, all parameters are defined in the same lines as before. The obvious choice for the horizon would be $t+1$, in which case the definition of the parameter $\widetilde{M}^i_{t'}$ is the same as of the basic definition of $M^i_{t+t'-1}$, for all i and $t' = 1, 2$. Similarly, capacity parameter $\widetilde{C}_{t'}$ is the same as $C_{t+t'-1}$, for all $t' = 1, 2$. Cumulative demand parameter $\widetilde{d}^i_{t'}$ represents simply $d^i_{t+t'-1,\, t+1}$, for all i and $t' = 1, 2$, i.e., $\widetilde{d}^i_1 = d^i_{1,2}$ and $\widetilde{d}^i_2 = d^i_2$. We note the following polyhedral result for X^{2PL} from [1].

Proposition 2.1 *Assume that* $\widetilde{M}^i_t > 0, \forall t \in \{1, \ldots, NT\}, \forall i \in \{1, \ldots, NI\}$ *and* $ST^i < \widetilde{C}_t, \forall t \in \{1, \ldots, NT\}, \forall i \in \{1, \ldots, NI\}$. *Then conv(*$X^{2PL}$*) is full-dimensional.*

For the sake of simplicity, we will drop subscript t and symbol $\tilde{}$ in the following notations. In this paper, we investigate the case of $a^i = 1, \forall i \in \{1, \ldots, NI\}$ with nonzero setups. We establish two relaxations of X^{2PL} and study their polyhedral structures. For a given t, we define the first relaxation of X^{2PL}, denoted by $LR1$, as set of $(x, y) \in \mathbb{R}^{NI} \times \mathbb{Z}^{NI}$ satisfying

$$\begin{aligned}
& x^i \le M^i y^i, i \in I \\
& \sum_{i=1}^{NI} (x^i + ST^i y^i) \le C \\
& x^i \ge 0, y^i \in \{0, 1\}, i \in I
\end{aligned}$$

Next, we present a result from the literature [4] concerning this relaxation.

Definition 2.1 *Let* $S_1 \subseteq I$ *and* $S_2 \subseteq I$ *such that* $S_1 \cap S_2 = \emptyset$. *We say that* (S_1, S_2) *is a* generalized cover *of* I *if* $\sum_{i \in S_1} (M^i + ST^i) + \sum_{i \in S_2} ST^i - C = \delta > 0$.

Proposition 2.2 *(see [4]) Let* (S_1, S_2) *be a generalized cover of* I, *and let* $L_1 \subseteq I \backslash (S_1 \cup S_2)$ *and* $L_2 \subseteq I \backslash (S_1 \cup S_2)$ *such that* $L_1 \cap L_2 = \emptyset$. *Then,*

$$\begin{aligned}
& \sum_{i \in S_1 \cup L_1} x^i + \sum_{i \in S_1 \cup S_2 \cup L_1 \cup L_2} ST^i y^i - \sum_{i \in S_1} (M^i + ST^i - \delta)^+ y^i - \sum_{i \in S_2} (ST^i - \delta)^+ y^i \\
& - \sum_{i \in L_1} (\overline{q}^i - \delta) y^i - \sum_{i \in L_2} (\overline{ST}^i - \delta) y^i \le C - \sum_{i \in S_1} (M^i + ST^i - \delta)^+ - \sum_{i \in S_2} (ST^i - \delta)^+
\end{aligned}$$

is valid for $LR1$, *where* $A \ge \max(\max_{i \in S_1} (M^i + ST^i), \max_{i \in S_2} ST^i, \delta)$, $\overline{q}^i = \max(A, M^i + ST^i)$, *and* $\overline{ST}^i = \max(A, ST^i)$.

For a given t, second relaxation of X^{2PL}, denoted by $LR2$, can be defined as the set of $(x, y, s) \in \mathbb{R}^{NI} \times \mathbb{Z}^{NI} \times \mathbb{R}^{NI}$ satisfying

$$\begin{aligned}
& x^i \le M^i y^i, i \in I \\
& x^i \le d^i y^i + s^i, i \in I \\
& \sum_{i=1}^{NI} (x^i + ST^i y^i) \le C \\
& x^i \ge 0, y^i \in \{0, 1\}, s^i \ge 0, i \in I
\end{aligned}$$

In this talk, we will present the trivial facet-defining inequalities for $LR2$, and then derive several classes of valid inequalities such as *cover* and *partition* inequalities. We will also present item- and period-extended versions of some of these families of inequalities, and we will establish facet-defining conditions for all families of inequalities. We will also discuss the separation problems associated with these valid inequalities.

References

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