Schwarz Methods for Second Order Maxwell Equations in 3D with Coefficient Jumps

Victorita Dolean, Martin Gander, Erwin Veneros

Schwarz Methods for Second Order Maxwell Equations in 3D with Coefficient Jumps

Victorita Dolean[1], Martin J. Gander[2], Erwin Veneros[3]

1 Introduction

Classical Schwarz methods need in general overlap to converge, but in the case of hyperbolic problems, they can also be convergent without overlap, see [7]. For the first order formulation of Maxwell equations, we have proved however in [18] that the classical Schwarz method without overlap does not converge in most cases in the presence of coefficient jumps aligned with interfaces.

Optimized Schwarz methods have been developed for Maxwell equations in first order form without conductivity in [8], and with conductivity in [5, 12]. These methods use modified transmission conditions, and often converge much faster than classical Schwarz methods. For DG discretizations of Maxwell equations, optimized Schwarz methods can be found in [9, 10, 6]. Optimized Schwarz methods were also developed for the second order formulation of Maxwell equations, see [1], and [16, 17] for scattering problems with applications.

While usually coefficient jumps hamper the convergence of domain decomposition methods, this is very different for optimized Schwarz methods. For diffusive problems, it was shown in [11] that jumps in the coefficients can actually lead to faster iterations, when they are taken into account correctly in the transmission conditions: optimized Schwarz methods benefit from jumps in the coefficients at interfaces. We had shown in [18] that this also holds for the special case of transverse magnetic modes (TMz) in the two dimensional first order Maxwell equations. We show in this short paper that these results for the TMz modes (and the corresponding ones for the transverse electric modes (TEz)) can be used to formulate optimized Schwarz methods for the 3D second order Maxwell equations which then in some cases converge faster, the bigger the coefficient jumps are.

Section de mathématiques, Université de Genève, 1211 Genève 4
victorita.dolean@unige.ch · martin.gander@unige.ch · erwin.veneros@unige.ch

2 Classical Schwarz for Second Order Maxwell Equations

The time dependent Maxwell equations in their second order formulation are

$$\varepsilon\partial_t^2\mathscr{E}+\nabla\times(\mu^{-1}\nabla\times\mathscr{E})=\partial_t\mathscr{J}, \tag{1}$$

where $\mathscr{E}=(\mathscr{E}_1,\mathscr{E}_2,\mathscr{E}_3)^T$ is the electric field, ε is the *electric permittivity*, μ is the *magnetic permeability*, and $\mathscr{J}$ is the applied current density. We assume that the applied current density is divergence free, $\operatorname{div}\mathscr{J}=0$. There is a similar system also for the magnetic field $\mathscr{H}=(\mathscr{H}_1,\mathscr{H}_2,\mathscr{H}_3)^T$,

$$\mu\partial_t^2\mathscr{H}+\nabla\times(\varepsilon^{-1}\nabla\times\mathscr{H})=\nabla\times\varepsilon^{-1}\mathscr{J}, \tag{2}$$

but we will only consider the equations (1) for the electric field in this short paper.

The time dependent Maxwell equations (1) form a system of hyperbolic partial differential equations [8]. Imposing incoming characteristics is equivalent to imposing the impedance condition

$$\mathscr{B}_{\mathbf{n}_j}(\mathscr{E}^{m,n})=\frac{1}{\mu_m}(\nabla\times\mathscr{E}^{m,n}\times\mathbf{n}_j)\times\mathbf{n}_j+\frac{i\omega}{Z_m}(\mathscr{E}^{m,n}\times\mathbf{n}_j)=\mathbf{s}, \tag{3}$$

where $Z_m=\sqrt{\frac{\mu_m}{\varepsilon_m}}$. We are interested here in the time-harmonic Maxwell equations, which are obtained by supposing that $\mathscr{E}(x,t)=e^{i\omega t}\mathbf{E}(x)$ for a fixed frequency ω. After some simplifications, we obtain from equation (1) the time harmonic second order Maxwell equation

$$\varepsilon\omega^2\mathbf{E}-\nabla\times(\mu^{-1}\nabla\times\mathbf{E})=-i\omega\mathbf{J}. \tag{4}$$

We are interested here in the heterogeneous case, where the domain Ω of interest consists of two non-overlapping subdomains Ω_1 and Ω_2 with interface Γ, and piecewise constant parameters ε_j and μ_j in Ω_j, $j=1,2$. We want to solve such problems using the Schwarz algorithm

$$\begin{aligned}&\begin{cases}\varepsilon_1\omega^2\mathbf{E}^{1,n}-\nabla\times(\mu_1^{-1}\nabla\times\mathbf{E}^{1,n}) = -i\omega\mathbf{J}, & \text{in } \Omega_1,\\ \mathscr{T}_{\mathbf{n}_1}(\mathbf{E}^{1,n}) = \mathscr{T}_{\mathbf{n}_1}(\mathbf{E}^{2,n-1}) & \text{on } \Gamma,\end{cases}\\ &\begin{cases}\varepsilon_2\omega^2\mathbf{E}^{2,n}-\nabla\times(\mu_2^{-1}\nabla\times\mathbf{E}^{2,n}) = -i\omega\mathbf{J}, & \text{in } \Omega_2,\\ \mathscr{T}_{\mathbf{n}_2}(\mathbf{E}^{2,n}) = \mathscr{T}_{\mathbf{n}_2}(\mathbf{E}^{1,n-1}) & \text{on } \Gamma,\end{cases}\end{aligned} \tag{5}$$

with the transmission condition

$$\mathscr{T}_{\mathbf{n}_j}(\mathbf{E}^{m,n})=(Id-A_j)(\frac{1}{\mu_m}\mathbf{n}_j\times\nabla\times\mathbf{E}^{m,n})-\frac{i\omega_j}{\mu_j}(Id+A_j)(\mathbf{n}_j\times(\mathbf{E}^{m,n}\times\mathbf{n}_j)). \tag{6}$$

with $\omega_j=\omega\sqrt{\varepsilon_j\mu_j}$, $j=1,2$. The classical Schwarz algorithm is obtained for the choice $A_j=0$, for $j=1,2$. We see that the classical Schwarz algorithm is exchanging characteristic information at the interfaces between subdomains, i.e. $\mathscr{T}_{\mathbf{n}_j}(\mathbf{E}^{m,n}) = \mathscr{B}_{\mathbf{n}_j}(\mathbf{E}^{m,n})$ where $\mathscr{B}$ is defined in (3).

In [18], we studied the classical Schwarz algorithm for the first order Maxwell equations on the domain $\Omega = \mathbb{R}^3$, with subdomains $\Omega_1 = (-\infty, 0] \times \mathbb{R}^2$ and $\Omega_2 = [0, \infty) \times \mathbb{R}^2$ and interface $\Gamma = \{0\} \times \mathbb{R}^2$ and the Silver-Müller radiation condition. We showed that the convergence factor of the classical Schwarz algorithm in 3D is $\rho_{\text{cla}} = \max\{\rho_{E\text{cla}}, \rho_{M\text{cla}}\}$, where $\rho_{E\text{cla}}$ and $\rho_{M\text{cla}}$ are the convergence factors of the TEz and TMz cases in 2D. We then proved that if there are coefficient jumps along the interface Γ, i.e. $\mu_1 \neq \mu_2$ and/or $\varepsilon_1 \neq \varepsilon_2$, the classical Schwarz algorithm is divergent in 3D if $\mu_1\varepsilon_2 \neq \mu_2\varepsilon_1$. If $\mu_1\varepsilon_2 = \mu_2\varepsilon_1$, we obtained $\rho_{E\text{cla}} = \rho_{M\text{cla}}$, and $\rho_{cla} < 1$ for the propagative modes, $|\mathbf{k}| < \omega_j$, $j = 1, 2$, but $\rho_{cla}(|\mathbf{k}|) = 1$ for the evanescent modes, $|\mathbf{k}| > \omega_j$, $j = 1, 2$, so the algorithm is stagnating for all evanescent modes. It is thus never convergent in 3D. We then investigated in [18] the 2D case of TMz modes in more detail, and found that the classical Schwarz algorithm in the presence of coefficient jumps is convergent in certain situations, depending on the jumps in ε and μ.

These results also hold for the second order Maxwell equations when the Schwarz algorithm (5,6) with classical transmission conditions is applied, and for the convergent cases from [18] in 2D, we have the following new contraction estimate:

Theorem 1 (Classical Schwarz in 2D). *If the classical Schwarz algorithm (5,6) in 2D converges, then we have the asymptotic convergence factor estimate*

$$\rho_{Mcla}(k, \omega_1, \omega_2, Z) = \rho_{Ecla}(k, \omega_1, \omega_2, Z) = 1 - O(h^2)$$

with $Z = \sqrt{\frac{\mu_1\varepsilon_2}{\mu_2\varepsilon_1}}$ *and h the uniform mesh size.*

Proof. As in [18], we can write the convergence factors for the TMz case as

$$\rho_{M\text{cla}}(k, \omega_1, \omega_2, Z) = \left| \frac{\left(\sqrt{k^2 - \omega_1^2} - i\omega_1 Z\right)\left(\sqrt{k^2 - \omega_2^2} - i\omega_2/Z\right)}{\left(\sqrt{k^2 - \omega_1^2} + i\omega_1\right)\left(\sqrt{k^2 - \omega_2^2} + i\omega_2\right)} \right|^{\frac{1}{2}}, \tag{7}$$

and for evanescent modes ($k > \omega_1, \omega_2$), equation (7) is equal to

$$\rho_{M\text{cla}}(k, \omega_1, \omega_2, Z) = 1 + \frac{(Z^2 - 1)\omega_1^2}{Z^2 k^2} \left(Z^2 - Y^2 - \frac{(Z^2 - 1)\omega_2^2}{k^2} \right), \tag{8}$$

with $Y = \frac{\omega_2}{\omega_1}$. From equation (8) we see that $\lim_{k\to\infty} \rho_{M\text{cla}} = 1$. If the classical Schwarz algorithm is convergent then $\rho_{M\text{cla}} < 1$, $\forall k$, the previous remark permits us to conclude that the maximum over all the frequencies must be at $k = k_{\max} = \frac{c_{\max}}{h}$, the largest frequency supported by the numerical grid, where h is the mesh size and $c_{\max}$ is a constant depending on the geometry. To conclude the proof, we just insert $k = c_{\max}/h$ into (8) and the result follows by expansion. The proof for the TEz case is similar.

3 Optimized Schwarz for Second Order Maxwell Equations

Since the classical Schwarz method is not an effective solver for Maxwell equations in the presence of coefficient jumps, we introduce now more effective transmission conditions which take the coefficient jumps into account. We consider algorithm (5,6) with the particular choice

$$A_j := \gamma_{jM} S_{TM} + \gamma_{jE} S_{TE}, \quad S_{TM} = \nabla_\tau \nabla_\tau \cdot, \quad S_{TE} = \nabla_\tau \times \nabla_\tau \times,$$

where τ is the tangential direction to the interface. We note that $S_{TM} - S_{TE} = \Delta_\tau I$, where Δ_τ is the Laplace-Beltrami operator in the tangential plane (for example , $\Delta_\tau = \partial_{yy} + \partial_{zz}$ when $\mathbf{n} = (1,0,0)$). The constants γ_{1E}, γ_{2E} and γ_{1M}, γ_{2M} can be chosen in order to optimize the algorithm.

Performing a Fourier transform in the yz plane, we find after a lengthy calculation the iteration matrix of the optimized Schwarz algorithm to be

$$IT = \begin{pmatrix} C_E & 0 \\ 0 & C_M \end{pmatrix} \tag{9}$$

with the coefficients

$$C_E = \frac{((\lambda_1 - i\omega_1/Z) - \gamma_{2M}|k|^2(\lambda_1 + i\omega_1/Z))((\lambda_2 - i\omega_2 Z) - \gamma_{1M}|k|^2(\lambda_2 + i\omega_2 Z))}{(2\omega_1 - i(\lambda_1 - i\omega_1)(1 - \gamma_{1M}|k|^2))(2\omega_2 - i(\lambda_2 - i\omega_2)(1 - \gamma_{2M}|k|^2))},$$

$$C_M = \frac{((\lambda_1 - i\omega_1 Z) - \gamma_{2E}|k|^2(\lambda_1 + i\omega_1 Z))((\lambda_2 - i\omega_2/Z) - \gamma_{1E}|k|^2(\lambda_2 + i\omega_2/Z))}{((1 - \gamma_{1E}|k|^2)(\lambda_1 - i\omega_1) + 2i\omega_1)((1 - \gamma_{2E}|k|^2)(\lambda_2 - i\omega_2) + 2i\omega_2)}, \tag{10}$$

with $\lambda_j = \sqrt{|\mathbf{k}|^2 - \omega_j^2}$, $j = 1,2$. If we choose for the parameters the values

$$\begin{aligned} \gamma_{1M} &= \frac{\lambda_2 - i\omega_2 Z}{|k|^2(\lambda_2 + i\omega_2 Z)}, \ \gamma_{1E} = \frac{\lambda_2 - i\omega_2/Z}{|k|^2(\lambda_2 + i\omega_2/Z)}, \\ \gamma_{2E} &= \frac{\lambda_1 - i\omega_1 Z}{|k|^2(\lambda_1 + i\omega_1 Z)}, \ \gamma_{2M} = \frac{\lambda_1 - i\omega_1/Z}{|k|^2(\lambda_1 + i\omega_1/Z)}, \end{aligned} \tag{11}$$

then the iteration matrix IT in (9) vanishes and we have convergence in two iterations. The corresponding transmission conditions are called transparent conditions, and are optimal, since they lead to a direct solver. But the operators corresponding to the symbols in (11) are non local and thus costly to use. We therefore propose to replace λ_1 and λ_2 in (11) by zeroth order approximations s_{1E}, s_{1M}, s_{2E} and s_{2E}. The convergence factor of the method is then the maximum of the spectral radius of (9) over all Fourier frequencies. We obtain

$$\rho_{\text{opt}} = \max\{\rho_{E\text{opt}}, \rho_{M\text{opt}}\}, \tag{12}$$

with

$$\begin{aligned}\rho_{E\text{opt}}(|\mathbf{k}|,\omega,\varepsilon_1,\varepsilon_2,\mu_1,\mu_2,s_{1M},s_{2M}) &= \left|\frac{(\lambda_2-s_{2M})(\lambda_1-s_{1M})}{(\lambda_2+s_{1M}\varepsilon_2/\varepsilon_1)(\lambda_1+s_{2M}\varepsilon_1/\varepsilon_2)}\right|^{1/2},\\ \rho_{M\text{opt}}(|\mathbf{k}|,\omega,\varepsilon_1,\varepsilon_2,\mu_1,\mu_2,s_{1E},s_{2E}) &= \left|\frac{(\lambda_1-s_{1E})(\lambda_2-s_{2E})}{(\lambda_2+s_{1E}\mu_2/\mu_1)(\lambda_1+s_{2E}\mu_1/\mu_2)}\right|^{1/2}.\end{aligned} \tag{13}$$

These factors can be optimized separately and they are once again the convergence factors of the TMz and TEz cases in 2D. In order to optimize we have to choose s_{jE}, s_{jM}, $j=1,2$ such that ρ_{opt} is as small as possible for all numerically relevant frequencies $k \in K := [k_{\min}, k_{\max}]$. Here $k_{\min}$ is the smallest frequency relevant to the subdomain, and $k_{\max} = \frac{c_{\max}}{h}$ is the largest frequency supported by the numerical grid, h being the mesh size, see for example [14]. We search for s_{jE} and s_{jM} of the form $s_{jE} = c_{jE}(1+i)$, $s_{jM} = c_{jM}(1+i)$ such that s_{jE}, s_{jM}, $j=1,2$ will be the solutions of the min-max problems

$$\min_{s_{1E},s_{2E}\in\mathbb{C}} \max_{k\in K} \rho_{M\text{opt}}(|\mathbf{k}|,\omega,\varepsilon_1,\varepsilon_2,\mu_1,\mu_2,s_{1E},s_{2E}), \tag{14}$$

$$\min_{s_{1M},s_{2M}\in\mathbb{C}} \max_{k\in K} \rho_{E\text{opt}}(|\mathbf{k}|,\omega,\varepsilon_1,\varepsilon_2,\mu_1,\mu_2,s_{1M},s_{2M}). \tag{15}$$

Since the optimization can be performed independently, we can use our results from [18] and obtain

Corollary 1 (2D asymptotically optimized contraction factor). *For TMz, the solution of (14) for $Y \neq 1$ gives the asymptotic convergence factor*

$$\rho^*_{Mopt} = \begin{cases} 1-\mathcal{O}(h^{1/4}) & \text{if } Z = Y,\\ \sqrt{\frac{\mu_{min}}{\mu_{max}}} + \mathcal{O}(h) & \text{if } Z \le Y < \sqrt{2}Z \text{ or } Y \le Z < \sqrt{2}Y,\\ \sqrt[4]{\frac{1}{2}} + \mathcal{O}(h) & \text{if } Z < \sqrt{2}Y \text{ or } Y > \sqrt{2}Z. \end{cases} \tag{16}$$

If $Z \neq 1$ and $Y = 1$, we obtain after excluding the resonance frequency [8]

$$\rho^*_{Mopt} = \sqrt{\frac{\mu_{min}}{\mu_{max}}} + \mathcal{O}(h).$$

For the TEz case, the same conclusion holds if we replace Y by Y^{-1} and μ by ε.

The results in 3D follow now by a systematic consideration of both cases together:

Theorem 2 (3D asymptotically optimized contraction factor, Case A). *If $Z \neq Y, Y^{-1}$ and $Y \neq 1$, the optimized convergence factor ρ^*_{opt} in (12) has the asymptotic behavior:*

1. *If $\min\{\max\{(ZY)^{-1}, ZY\}, \max\{Z/Y, Y/Z\}\} > \sqrt{2}$, then*

$$\rho^*_{opt} = \sqrt[4]{1/2} + \mathcal{O}(h). \tag{17}$$

2. *If $\min\{\max\{(ZY)^{-1}, ZY\}, \max\{Z/Y, Y/Z\}\} = \max\{Z/Y, Y/Z\} \le \sqrt{2}$, then*

$$\rho^*_{opt} = \sqrt{\frac{\mu_{min}}{\mu_{max}}} + \mathcal{O}(h). \tag{18}$$

3. If $\min\left\{\max\{(ZY)^{-1},ZY\},\max\{Z/Y,Y/Z\}\right\}=\max\{(YZ)^{-1},YZ\}\leq\sqrt{2}$*, then*

$$\rho^*_{opt}=\sqrt{\frac{\varepsilon_{min}}{\varepsilon_{max}}}+\mathscr{O}(h). \tag{19}$$

Proof. To prove 1. we use twice Corollary 1. If $\max\{Z/Y,Y/Z\}>\sqrt{2}$, we use the third result in (16) for the TMz case. Similarly if $\max\{ZY,(ZY)^{-1}\}>\sqrt{2}$ we use also the third result in (16) but for the TEz case. From equation (12) we know that ρ_{opt} is the maximum of $\rho_{E\text{opt}}$ and $\rho_{M\text{opt}}$, and if both of them have the asymptotic behaviour $\sqrt[4]{1/2}+\mathscr{O}(h)$, we get (17) as required.

For 2. we know that $\max\{Z/Y,Y/Z\}\leq\sqrt{2}$, which means that we can use the second result in (16), i.e. $\rho_{M\text{opt}}=\sqrt{\frac{\mu_{\min}}{\mu_{\max}}}+\mathscr{O}(h)$. We note that $Z/Y=\frac{\mu_1}{\mu_2}$ and $ZY=\frac{\varepsilon_2}{\varepsilon_1}$ which implies $1\geq\sqrt{\frac{\mu_{\min}}{\mu_{\max}}}\geq\sqrt[4]{\frac{1}{2}}$. If $\max\{(ZY)^{-1},ZY\}>\sqrt{2}$, by Corollary 1 we have $\rho_{E\text{opt}}=\sqrt[4]{1/2}+\mathscr{O}(h)$, and we clearly see $\rho_{M\text{opt}}>\rho_{E\text{opt}}$. If $\max\{(ZY)^{-1},ZY\}\leq\sqrt{2}$, we obtain by hypothesis the inequality $\max\{Z/Y,Y/Z\}\leq\max\{(ZY)^{-1},ZY\}\leq\sqrt{2}$, and this implies $\frac{\mu_{\min}}{\mu_{\max}}\geq\frac{\varepsilon_{\min}}{\varepsilon_{\max}}$. Then we obtain $\rho_{M\text{opt}}\geq\rho_{E\text{opt}}$ and thus (18).

Finally, for 3., one can proceed as for 2 to obtain (19).

Theorem 3 (3D asymptotically optimized contraction factor, Case B). *If* $Z=Y$ *or* $Z=Y^{-1}$*, then the optimized convergence factor* ρ^*_{opt} *in (12) satisfies*

$$\rho^*_{opt}=1-\mathscr{O}(h^{1/4}). \tag{20}$$

Proof. We use the first result in (16) of Corollary 1 and proceed as in Theorem 2.

Theorem 4 (3D asymptotically optimized contraction factor, Case C). *If* $Y=1$ *and* $Z\neq Y$*, then the optimized convergence factor* ρ^*_{opt} *in (12) satisfies*

$$\rho^*_{opt}=\sqrt{\frac{\mu_{min}}{\mu_{max}}}+\mathscr{O}(\sqrt{h}). \tag{21}$$

Proof. After excluding the resonance frequency, we apply the second part of Corollary 1. Note that in this case $\rho_{M\text{opt}}=\rho_{E\text{opt}}$.

Theorem 2 and 4 contain the important result that in the presence of jumps in the coefficients, the convergence of the optimized Schwarz method for Maxwell equations gets faster when the jump increases, the method benefits from the jumps! In the first part of Theorem 2, the convergence is independent of the jump in the coefficients, and in all these cases the nonoverlapping method converges independently of the mesh parameter, also unusual for optimized Schwarz methods without jumps in the coefficients. In the case of $Z=Y$ or $Z=Y^{-1}$ ($\mu_1=\mu_2$ or $\varepsilon_1=\varepsilon_2$) in Theorem 3 however, the convergence factor depends on h and deteriorates as h goes to zero, as in the case without jumps presented in [8].

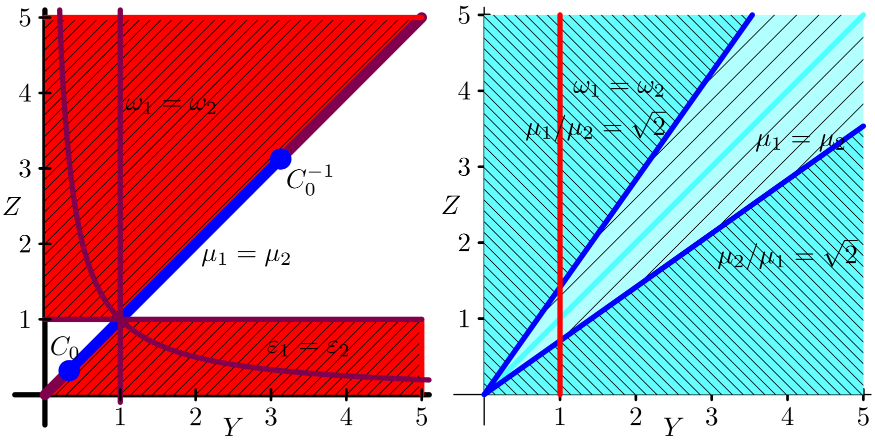

Fig. 1 Convergence regions in blue and divergence regions in red for classical Schwarz (left) and optimized Schwarz (right)

We now illustrate graphically the improvement of the optimized Schwarz method over the classical one in 2D. We show in Figure 1 in red the divergence regions and in blue the convergence regions for different values of Z and Y. In the left graphic the white part is still an open problem. In the right the light blue line have convergence dependant of the mesh size h, the light blue region have convergence dependent on the coefficients $\mu's$ and the dark blue region have convergence independent of the mesh size h and the coefficients $\mu's$, the red line is the zone of resonance corrected with theorem 1. We clearly see that the optimization of the transmission conditions transforms an algorithm that fails for a large range of problems into one that works in all cases.

4 Conclusions

Classical Schwarz methods applied to 3D Maxwell equations with jumps in the coefficients aligned with the interfaces do not converge, and this is also the case for the second order formulation of Maxwell equations. Using however optimized transmission conditions, we showed that one can obtain Schwarz methods for the 3D Maxwell equations that converge independently of the mesh parameter in some cases, and even become faster as the jumps get larger at the interfaces. These methods directly benefit from the jumps in the coefficients. We presented precise asymptotic convergence factor estimates for the many different cases of coefficient jumps, and are currently working on the numerical implementation of these methods in the full 3D setting.

References

1. Alonso-Rodriguez, A., Gerardo-Giorda, L.: New nonoverlapping domain decomposition methods for the harmonic Maxwell system. SIAM J. Sci. Comput. **28**(1), 102–122 (2006)
2. Chevalier, P., Nataf, F.: An OO2 (Optimized Order 2) method for the Helmholtz and Maxwell equations. In: 10th International Conference on Domain Decomposition Methods in Science and in Engineering, pp. 400–407. AMS (1997)
3. Després, B.: Décomposition de domaine et problème de Helmholtz. C.R. Acad. Sci. Paris **1**(6), 313–316 (1990)
4. Després, B., Joly, P., Roberts, J.: A domain decomposition method for the harmonic Maxwell equations. In: Iterative methods in linear algebra, pp. 475–484. North-Holland, Amsterdam (1992)
5. Dolean, V., El Bouajaji, M., Gander, M.J., Lanteri, S.: Optimized Schwarz methods for Maxwell's equations with non-zero electric conductivity. In: Domain decomposition methods in science and engineering XIX, *Lect. Notes Comput. Sci. Eng.*, vol. 78, pp. 269–276. Springer, Heidelberg (2011).
6. Dolean, V., El Bouajaji, M., Gander, M.J., Lanteri, S., Perrussel, R.: Domain decomposition methods for electromagnetic wave propagation problems in heterogeneous media and complex domains. In: Domain decomposition methods in science and engineering XIX, *Lect. Notes Comput. Sci. Eng.*, vol. 78, pp. 15–26. Springer, Heidelberg (2011).
7. Dolean, V., Gander, M.J., : Why Classical Schwarz Methods Applied to Hyperbolic Systems Can Converge even Without Overlap, In: Domain decomposition methods in science and engineering XVII, *Lect. Notes Comput. Sci. Eng.*, vol. 60, pp. 467–475. Springer, Heidelberg (2007).
8. Dolean, V., Gerardo-Giorda, L., Gander, M.: Optimized Schwarz methods for Maxwell equations. SIAM J. Scient. Comp. **31**(3), 2193–?2213 (2009)
9. Dolean, V., Lanteri, S., Perrussel, R.: A domain decomposition method for solving the three-dimensional time-harmonic Maxwell equations discretized by discontinuous Galerkin methods. J. Comput. Phys. **227**(3), 2044–2072 (2008)
10. Dolean, V., Lanteri, S., Perrussel, R.: Optimized Schwarz algorithms for solving time-harmonic Maxwell's equations discretized by a discontinuous Galerkin method. IEEE. Trans. Magn. **44**(6), 954–957 (2008)
11. Dubois, O.: Optimized Schwarz methods for the advection-diffusion equation and for problems with discontinuous coefficients. Ph.D. thesis, McGill University (2007)
12. El Bouajaji, M., Dolean, V., Gander, M.J., Lanteri, S.: Optimized Schwarz methods for the time-harmonic Maxwell equations with damping. SIAM Journal on Scientific Computing **34**(4), A2048–A2071 (2012).
13. Gander, M., Magoulès, F., Nataf, F.: Optimized Schwarz methods without overlap for the Helmholtz equation. SIAM J. Sci. Comput. **24**(1), 38–60 (2002)
14. Gander, M.J.: Optimized Schwarz methods. SIAM J. Numer. Anal. **44**(2), 699–731 (2006)
15. Gander, M.J., Halpern, L., Magoulès, F.: An optimized Schwarz method with two-sided robin transmission conditions for the Helmholtz equation. Int. J. for Num. Meth. in Fluids **55**(2), 163–175 (2007)
16. Peng, Z., Lee, J.F.: Non-conformal domain decomposition method with second-order transmission conditions for time-harmonic electromagnetics. J. Comput. Physics **229**(16), 5615–5629 (2010).
17. Peng, Z., Rawat, V., Lee, J.F.: One way domain decomposition method with second order transmission conditions for solving electromagnetic wave problems. J. Comput. Physics **229**(4), 1181–1197 (2010).
18. Dolean, V., Gander, M.J., Veneros, E.: Optimized Schwarz Methods for Maxwell Equations with Discontinuous Coefficients