# A note on $p$-Ascent Sequences

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Submitted: Date 1; Accepted: Date 2; Published: Date 3.
MR Subject Classifications: 05A15

**Abstract**

Ascent sequences were introduced by Bousquet-Mélou, Claesson, Dukes, and Kitaev in [1], who showed that ascent sequences of length $n$ are in 1-to-1 correspondence with $(\mathbf{2}+\mathbf{2})$-free posets of size $n$. In this paper, we introduce a generalization of ascent sequences, which we call *p-ascent sequences*, where $p \geq 1$. A sequence $(a_1, \ldots, a_n)$ of non-negative integers is a $p$-ascent sequence if $a_0 = 0$ and for all $i \geq 2$, $a_i$ is at most $p$ plus the number of ascents in $(a_1, \ldots, a_{i-1})$. Thus, in our terminology, ascent sequences are 1-ascent sequences. We generalize a result of the authors in [9] by enumerating $p$-ascent sequences with respect to the number of 0s. We also generalize a result of Dukes, Kitaev, Remmel, and Steingrímsson in [4] by finding the generating function for the number of $p$-ascent sequences which have no consecutive repeated elements. Finally, we initiate the study of pattern-avoiding $p$-ascent sequences.

## 1 Introduction

### 1.1 Ascent sequences

Ascent sequences were introduced by Bousquet-Mélou, Claesson, Dukes, and Kitaev in [1], who showed that ascent sequences of length $n$ are in 1-to-1 correspondence with $(\mathbf{2}+\mathbf{2})$-*free posets* of size $n$. Let $\mathbb{N} = \{0, 1, \ldots\}$ denote the natural numbers and $\mathbb{N}^*$ denote the set of all words over $\mathbb{N}$. A sequence $(a_1, \ldots, a_n) \in \mathbb{N}^n$ is an *ascent sequence of length n* if and only if it satisfies $a_1 = 0$ and $a_i \in [0, 1 + \mathrm{asc}(a_1, \ldots, a_{i-1})]$ for all $2 \leq i \leq n$, where

$$\mathrm{asc}(a_1, \ldots, a_i) = |\{j : a_j < a_{j+1}; 1 \leq j < i\}|$$

is the number of ascents in $(a_1, \ldots, a_n)$. For instance, (0, 1, 0, 2, 3, 1, 0, 0, 2) is an ascent sequence which has four ascents. We let *Asc* denote the set of all ascent sequences, where we assume that the empty word is also an ascent sequence. For any $n \geq 1$, we let $Asc_n$ denote the set of all ascent sequences of length $n$. If $a = (a_1, \ldots, a_n) \in Asc_n$, we let $|a| = n$ be the length of $a$, $\sum a = a_1 + \cdots + a_n$ equal the sum of the values of $a$, $|a|_0$ denote the number of occurrences of 0 in $a$, and $\mathrm{last}(a) = a_n$ denote the last letter of $a$. We say that $a = (a_1, \ldots, a_n) \in Asc_n$

*Partially supported by NSF grant DMS 0654060.

is an *up-down* ascent sequence if $a_1 < a_2 > a_3 < a_4 > \cdots$. That is, $a = (a_1, \ldots, a_n) \in Asc_n$ is an up-down ascent sequence if $a_i < a_{i+1}$ whenever $i$ is odd, and $a_i > a_{i+1}$ whenever $i$ is even. Throughout this paper, we will often identify a sequence $(a_1, \ldots, a_n)$ in $\mathbb{N}^n$ with the word $a_1 \ldots a_n$. Thus, instead of writing, say, $(0, 0, 0)$, we will simply write 000, or $0^3$.

There has been considerable work on ascent sequences in recent years, see, for example, [1, 4, 6, 9]. Ascent sequences are important because they are in bijection with several other interesting combinatorial objects. To be more precise, it follows from the work of [1, 3, 5] that there are natural bijections between $Asc_n$ and the following four classes of combinatorial objects:
**(1)** The set of $(\mathbf{2}+\mathbf{2})$-free posets of size $n$. Here we consider two posets to be equal if they are isomorphic, and an unlabeled poset is said to be $(\mathbf{2}+\mathbf{2})$-free if it does not contain an induced subposet that is isomorphic to $(\mathbf{2}+\mathbf{2})$, the union of two disjoint 2-element chains. $(\mathbf{2}+\mathbf{2})$-free posets are known to be in 1-to-1 correspondence with celebrated *interval orders*.
**(2)** The set $M_n$ of upper triangular matrices of non-negative integers such that no row or column contains all zero entries, and the sum of the entries is $n$.
**(3)** The set $R_n$ of permutations of $[n] = \{1, \ldots, n\}$, where in each occurrence of the pattern 231, either the letters corresponding to the 2 and the 3 are nonadjacent, or else the letters corresponding to the 2 and the 1 are nonadjacent in value. Here, a word contains an occurrence of the pattern 231 if it contains a subsequence of length 3 that is order-isomorphic to 231.
**(4)** The set $Mch_n$ of Stoimenow matchings on $[2n]$. A *matching* of the set $[2n] = \{1, 2, \ldots, 2n\}$ is a partition of $[2n]$ into subsets of size 2, each of which is called an *arc*. The smaller number in an arc is its *opener*, and the larger one is its *closer*. A matching is said to be *Stoimenow* if it has no pair of arcs $\{a < b\}$ and $\{c < d\}$ that satisfy one (or both) of the following conditions: (a) $a = c + 1$ and $b < d$ and (b) $a < c$ and $b = d + 1$. In other words, a Stoimenow matching has no pair of arcs such that one is nested within the other and either the openers or the closers of the two arcs differ by 1.

Remmel [11] showed that there is an interesting connection between the *Genocchi numbers* $G_{2n}$ and the *median Genocchi numbers* $H_{2n-1}$ and up-down ascent sequences. In particular, Remmel showed that $G_{2n}$ is the number of up-down ascent sequences of length $2n-1$, $H_{2n-1}$ is the number of up-down ascent sequences of length $2n-2$.

Let $p_n$ be the number of ascent sequences of length $n$. Bousquet-Mélou et al. [1] proved that

$$P(t) = \sum_{n \geq 0} p_n t^n = \sum_{n \geq 0} \prod_{i=1}^{n} \left(1 - (1-t)^i\right).$$

In fact, Bousquet-Mélou et al. [1] studied a more general generating function

$$F(t, u, v) = \sum_{w \in Asc} t^{|w|} u^{\mathrm{asc}(w)} v^{\mathrm{last}(w)}$$

and found an explicit form for such a generating function. Kitaev and Remmel [9] studied a refined version of this generating function. That is, they found an explicit formula for the generating function

$$G(t, u, v, z, x) := \sum_{w \in Asc} t^{|w|} u^{\mathrm{asc}(w)} v^{\mathrm{last}(w)} z^{|w|_0} x^{\mathrm{run}(w)},$$

where for any ascent sequence $w$, $\mathrm{run}(w) = 0$ if $w = 0^n$ for some $n$, and $\mathrm{run}(w) = r$ if $w = 0^r x v$, where $x$ is a positive integer and $v$ is a word. Thus $\mathrm{run}(w)$ keeps track of the initial sequences of 0s that start out $w$ if $w$ does not consist of all zeros. Kitaev and Remmel [9] were able to use their formula for $G(t,u,v,z,x)$ to prove that

$$A(t,z) := \sum_{w \in Asc} t^{|w|} z^{|w|_0} = 1 + \sum_{n \geq 0} \frac{zt}{(1-zt)^{n+1}} \prod_{i=1}^{n} (1-(1-t)^i). \tag{1}$$

### 1.2 $p$-ascent sequences

In this paper, we introduce a generalization of ascent sequences, which we call *p-ascent sequences*, where $p \geq 1$. A sequence $(a_1, \ldots, a_n)$ of non-negative integers is a *p-ascent sequence* if $a_0 = 0$ and for all $i \geq 2$, $a_i$ is at most $p$ plus the number of ascents in $(a_1, \ldots, a_{i-1})$. Thus, in our terminology, ascent sequences are 1-ascent sequences.

We note that $p$-ascent sequences of length $n$ can be encoded in terms of (usual) ascent sequences of length $n+2p-2$. Indeed, it is easy to see that $(a_1, a_2, \ldots, a_n)$ is a $p$-ascent sequence if and only if $(0,1,0,1,\ldots,0,1,a_1,a_2,\ldots,a_n)$ is an ascent sequence, where there are $p-1$ 0s and $p-1$ 1s preceding the $a_1 = 0$. Thus, $p$-ascent sequences can be thought of as a subset of ascent sequences of special type, namely, those ascents sequences that start out with $(01)^{p-1}0$.

The last observation allows to obtain a characterization of elements counted by $p$-ascent sequences in $(\mathbf{2}+\mathbf{2})$-free posets, the set of restricted permutations $R_n$, the set of upper triangular matrices $M_n$, and the set of Stoimenow matchings $Mch_n$ whenever we can characterize the images of ascent sequences whose corresponding words start with $(01)^{p-1}0$. We do not get into much detail here, but we provide two examples. We leave the other two cases to the interested reader to explore using [1, 3, 5]. The $(\mathbf{2}+\mathbf{2})$-free posets corresponding to $p$-ascent sequences are $(\mathbf{2}+\mathbf{2})$-free posets on $n+2p-2$ elements with the following property. Right before the last $2p-1$ steps in decomposition of such posets (the decomposition is described in [1]; we do not provide its details here due to space concerns), one obtains the poset with $p$ minimum elements and the other $p-1$ elements forming the pattern of the poset in Figure 1 corresponding to the case $p=5$. Of course, it would be interesting to give a direct characterization of such posets (e.g., in terms of forbidden sub-posets) but we were not able to succeed with that. On the other hand, permutations in $R_n$ corresponding to $p$-ascent sequences are easily seen via the bijection given in [1] (not to be provided here due to space concerns) to be the permutations that have consecutive blocks of elements $(2p+1)(2p-1)\ldots 1$ and $(2p)(2p-2)\ldots 2$ (the former block is to the left of the later block in all such permutations).

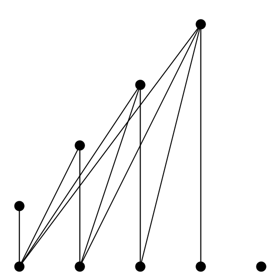

Figure 1: Type of poset obtained right before the last $2p-1$ steps in decomposition of the $(\mathbf{2}+\mathbf{2})$-free poset corresponding to a $p$-ascent sequence.

The main goal of this paper is to generalize the results of [9] to $p$-ascent sequences. That is, let $Asc(p)$ denote the set of $p$-ascent sequences, where, again, we consider the empty word to be a $p$-ascent sequence for any $p \geq 1$. Thus, the set of ascent sequences $Asc$ is $Asc(1)$ in our terminology. First, we shall study the generating functions

$$G^{(p)}(t,u,v,z,x) := \sum_{w \in Asc(p)} t^{|w|} u^{\mathrm{asc}(w)} v^{\mathrm{last}(w)} z^{|w|_0} x^{\mathrm{run}(w)}. \tag{2}$$

We shall find an explicit formula for $G^{(p)}(t,u,v,z,x)$ for any $p \geq 1$ (see Section 2) and then we shall use that formula to prove that

$$A^{(p)}(t,z) := \sum_{w \in Asc(p)} t^{|w|} z^{|w|_0} = 1 + \sum_{n \geq 0} \binom{p+n-1}{n} \frac{zt}{(1-zt)^{n+1}} \prod_{i=1}^{n} (1-(1-t)^i). \tag{3}$$

Duncan and Steingrímsson [6] introduced the study of pattern avoidance in ascent sequences. We initiate a similar study for $p$-ascent sequences. Given a word $w = w_1 \ldots w_n \in \mathbb{N}^*$, we let $\mathrm{red}(w)$ denote the word that is obtained from $w$ by replacing each copy of the $i$-th smallest element in $w$ by $i-1$. For example, $\mathrm{red}(238543623) = 015321401$. Then we say that a word $u = u_1 \ldots u_j$ *occurs* in $w$ if there exist $1 \leq i_1 < \cdots < i_j \leq n$ such that $\mathrm{red}(w_{i_1} w_{i_2} \ldots w_{i_j}) = u$. We say that $w$ *avoids* $u$ if $u$ does not occur in $w$.

For any word $u \in \mathbb{N}^*$ such that $\mathrm{red}(u) = u$, we let $a_{n,p,u}$ denote the number of $p$-ascent sequences $a$ of length $n$ avoiding $u$ and $r_{n,p,u}$ denote the number of sequences counted by $a_{n,p,u}$ with no equal consecutive letters. We prove a number of results about $a_{n,p,u}$ and $r_{n,p,u}$. For example, we will show that for all $p \geq 1$,

$$r_{n,p,10} = \binom{p+n-2}{n-1} \text{ and } a_{n,p,10} = \sum_{s=0}^{n-1} \binom{n-1}{s} \binom{p+s-1}{s}.$$

This paper is organized as follows. In Section 2, we shall find an explicit formula for $G^{(p)}(t,u,v,z,x)$. Unfortunately, we can not directly set $u = 1$ in that formula so that in Section 3, we shall find a formula for $G^{(p)}(t,1,1,1,x)$ via an alternative proof. This formula will also allow us to find an explicit formula for the generating function for the number of primitive $p$-ascent sequences. Finally, in Section 4, we shall study $a_{n,p,u}$ and $r_{n,p,u}$ for certain patterns $u$ of lengths 2 and 3.

## 2 Main results

For $r \geq 1$, let $G_r^{(p)}(t,u,v,z)$ denote the coefficient of $x^r$ in $G^{(p)}(t,u,v,z,x)$. Thus $G_r^{(p)}(t,u,v,z)$ is the generating function of those $p$-ascent sequences that begin with $r \geq 1$ 0s followed by some element between 1 and $p$. We let $G_{a,\ell,m,n}^{(p,r)}$ denote the number of $p$-ascent sequences of length $n$, which begin with $r$ 0s followed by some element between 1 and $p$, have $a$ ascents, last letter $\ell$, and a total of $m$ zeros. We then let

$$G_r^{(p)}(t,u,v,z) = \sum_{a,\ell,m \geq 0,\ n \geq r+1} G_{a,\ell,m,n}^{(p,r)} t^n u^a v^\ell z^m. \tag{4}$$

The sequences of the form $0^n$ contribute a term $1 + tz + (tz)^2 + \cdots = \frac{1}{1-tz}$ to $G_r^{(p)}(t,u,v,z)$ since they have no ascents and no initial run of 0s (by definition). Hence

$$G^{(p)}(t,u,v,x,z) = \frac{1}{1-tz} + \sum_{r\geq 1} x^r G_r^{(p)}(t,u,v,z). \tag{5}$$

**Lemma 1.** *For $r \geq 1$, the generating function $G_r^{(p)}(t,u,v,z)$ satisfies*

$$(v-1-tv(1-u))G_r^{(p)}(t,u,v,z) = \\ t^{r+1}z^r uv(v^p-1) + t((v-1)z-v)G_r^{(p)}(t,u,1,z) + tuv^{p+1}G_r^{(p)}(t,uv,1,z). \tag{6}$$

*Proof.* Our proof follows the same steps as the proof of the $p=1$ case of the result that was provided in [9]. Fix $r \geq 1$. Let $x' = (x_1,\ldots,x_{n-1})$ be an ascent sequence beginning with $r$ 0s followed by a nonzero element, with $a$ ascents and $m$ zeros, where $x_{n-1} = \ell$. Then $x = (x_1,\ldots,x_{n-1},i)$ is an ascent sequence if and only if $i \in [0, a+p]$. Clearly, $x$ also begins with $r$ 0s followed by a nonzero element. Now, if $i = 0$, the sequence $x$ has $a$ ascents and $m+1$ zeros. If $1 \leq i \leq \ell$, $x$ has $a$ ascents and $m$ zeros. Finally if $i \in [\ell+1, a+p]$, then $x$ has $a+1$ ascents and $m$ zeros. Counting the sequences $0\ldots0q$ with $r$ 0s and $1 \leq q \leq p$ separately, we have

$$\begin{aligned}
G_r^{(p)}(t,u,v,z) &= t^{r+1}uvz^r\frac{v^p-1}{v-1} + \\
&\quad \sum_{\substack{a,\ell,m\geq 0\\ n\geq r+1}} G_{a,\ell,m,n}^{(p,r)} t^{n+1}\left(u^a v^0 z^{m+1} + \sum_{i=1}^{\ell} u^a v^i z^m + \sum_{i=\ell+1}^{a+p} u^{a+1}v^i z^m\right) \\
&= t^{r+1}uvz^r\frac{v^p-1}{v-1} + t\sum_{\substack{a,\ell,m\geq 0\\ n\geq r+1}} G_{a,\ell,m,n}^{(p,r)} t^n u^a z^m \left(z + \frac{v^{\ell+1}-v}{v-1} + u\frac{v^{a+p+1}-v^{\ell+1}}{v-1}\right) \\
&= t^{r+1}uvz^r\frac{v^p-1}{v-1} + tzG_r^{(p)}(t,u,1,z) + \\
&\quad tv\frac{G_r^{(p)}(t,u,v,z) - G_r^{(p)}(t,u,1,z)}{v-1} + tuv\frac{v^pG_r^{(p)}(t,uv,1,z) - G_r^{(p)}(t,u,v,z)}{v-1}.
\end{aligned}$$

The result follows. □

Next, just like in the proof of the $p=1$ case in [9], we use the kernel method to proceed. Setting $(v-1-tv(1-u)) = 0$ and solving for $v$, we obtain that the substitution $v = 1/(1+t(u-1))$ will eliminate the left-hand side of (6). We can then solve for $G_r^{(p)}(t,u,1,z)$ to obtain that

$$G_r^{(p)}(t,u,1,z) = \frac{t^r z^r u}{\gamma_1\delta_1^p}(1-\delta_1^p) + \frac{u}{\gamma_1\delta_1^p}G_r^{(p)}\left(t, \frac{u}{\delta_1}, 1, z\right) \tag{7}$$

where $\delta_1 = 1 + t(u-1)$ and $\gamma_1 = 1 + zt(u-1)$.

Next we let $\delta_k = u - (1-t)^k(u-1)$ and $\gamma_k = u - (1-zt)(1-t)^{k-1}(u-1)$ for $k \geq 1$. We also set $\delta_0 = \gamma_0 = 1$. Observe that $\delta_1 = u - (1-t)(u-1) = 1 + t(u-1)$ and $\gamma_1 = u - (1-zt)(u-1) = 1 + zt(u-1)$.

For any function of $f(u)$, we shall write $f(u)|_{u=\frac{u}{\delta_k}}$ for $f(u/\delta_k)$. It is then easy to check that

$$\delta_s|_{u=\frac{u}{\delta_k}} = \frac{\delta_{s+k}}{\delta_k}, \quad \gamma_s|_{u=\frac{u}{\delta_k}} = \frac{\gamma_{s+k}}{\delta_k}, \quad \frac{u}{\delta_s}|_{u=\frac{u}{\delta_k}} = \frac{u}{\delta_{s+k}}, \text{ and}$$
$$(u-1)|_{u=\frac{u}{\delta_k}} = \frac{(1-t)^k(u-1)}{\delta_k}.$$

Using these relations, one can iterate the recursion (7). For example,

$$\begin{aligned}\frac{u^k}{\gamma_1\cdots\gamma_k\delta_k^p}G_r^{(p)}(t,\frac{u}{\delta_k},1,z) &= \frac{u^k}{\gamma_1\ldots\gamma_k\delta_k^p}\left(\frac{t^rz^r\frac{u}{\delta_k}\left(1-\frac{\delta_{k+1}^p}{\delta_k^p}\right)}{\frac{\gamma_{k+1}}{\delta_k}\frac{\delta_{k+1}^p}{\delta_k^p}} + \frac{\frac{u}{\delta_k}}{\frac{\gamma_{k+1}}{\delta_k}\frac{\delta_{k+1}^p}{\delta_k^p}}G_r^{(p)}\left(t,\frac{u}{\delta_{k+1}},1,z\right)\right)\\ &= \frac{t^rz^ru^{k+1}\left(1-\frac{\delta_{k+1}^p}{\delta_k^p}\right)}{\gamma_1\cdots\gamma_{k+1}\delta_{k+1}^p} + \frac{u^{k+1}}{\gamma_1\cdots\gamma_{k+1}\delta_{k+1}^p}G_r^{(p)}\left(t,\frac{u}{\delta_{k+1}},1,z\right).\end{aligned}$$

Thus, by iterating recursion (7), we can derive that

$$G_r^{(p)}(t,u,1,z) = \frac{t^rz^ru(1-\delta_1^p)}{\gamma_1\delta_1^p} + \sum_{k=2}^{\infty}\frac{t^rz^ru^k\left(1-\frac{\delta_k^p}{\delta_{k-1}^p}\right)}{\gamma_1\cdots\gamma_k\delta_k^p}. \tag{8}$$

Note that since $\delta_0 = 1$, we can rewrite $\frac{t^{r+1}z^ru(1-\delta_1^p)}{\gamma_1\delta_1^p}$ as $\frac{t^rz^ru(\delta_0^p-\delta_1^p)}{\gamma_1\delta_0^p\delta_1^p}$ and we can rewrite $\frac{t^rz^ru^k\left(1-\frac{\delta_k^p}{\delta_{k-1}^p}\right)}{\gamma_1\cdots\gamma_k\delta_k^p}$ as $\frac{t^rz^ru(\delta_{k-1}^p-\delta_k^p)}{\gamma_1\cdots\gamma_k\delta_{k-1}^p\delta_k^p}$. Thus we have proved the following theorem.

**Theorem 2.**

$$G_r^{(p)}(t,u,1,z) = \sum_{k=1}^{\infty}\frac{t^rz^ru^k(\delta_{k-1}^p-\delta_k^p)}{\gamma_1\cdots\gamma_k\delta_{k-1}^p\delta_k^p}. \tag{9}$$

Note that we can rewrite (6) as

$$G_r^{(p)}(t,u,v,z) = \frac{t^{r+1}z^ruv(v^p-1)}{v\delta_1-1} + \frac{t(z(v-1)-v)}{v\delta_1-1}G_r^{(p)}(t,u,1,z) + \frac{uv^{p+1}t}{v\delta_1-1}G_r^{(p)}(t,uv,1,z). \tag{10}$$

For $s \geq 1$, we let

$$\bar{\delta}_s = \delta_s|_{u=uv} = uv-(1-t)^s(uv-1) \text{ and } \bar{\gamma}_s = \gamma_s|_{u=uv} = uv-(1-zt)(1-t)^{s-1}(uv-1)$$

and set $\bar{\delta}_0 = \bar{\gamma}_0 = 1$. Then using (10) and (9), we have the following theorem.

**Theorem 3.** *For all $r \geq 1$,*

$$G_r^{(p)}(t,u,v,z) =$$
$$t^rz^r\left(\frac{tuv(v^p-1)}{v\delta_1-1} + \frac{t(z(v-1)-v)}{v\delta_1-1}\sum_{k\geq1}\frac{(\delta_{k-1}^p-\delta_k^p)}{\gamma_1\cdots\gamma_k\delta_{k-1}^p\delta_k^p} + \frac{tuv^{p+1}}{v\delta_1-1}\sum_{k\geq1}\frac{(\bar{\delta}_{k-1}^p-\bar{\delta}_k^p)}{\bar{\gamma}_1\cdots\bar{\gamma}_k\bar{\delta}_{k-1}^p\bar{\delta}_k^p}\right). \tag{11}$$

It is easy to see from Theorem 3 that $G_r^{(p)}(t,u,v,z) = t^{r-1}z^{r-1}G_1^{(p)}(t,u,v,z)$. This is also easy to see combinatorially since every ascent sequence counted by $G_r^{(p)}(t,u,v,z)$ is of the form $0^{r-1}a$, where $a$ is a $p$-ascent sequence counted by $G_1^{(p)}(t,u,v,z)$. Hence

$$\begin{aligned} G^{(p)}(t,u,v,z,x) &= \frac{1}{1-tz} + \sum_{r\geq 1} G_r^{(p)}(t,u,v,z)x^r = \frac{1}{1-tz} + \sum_{r\geq 1} t^{r-1}z^{r-1}G_1^{(p)}(t,u,v,z)x^r \\ &= \frac{1}{1-tz} + \frac{x}{1-tzx}G_1^{(p)}(t,u,v,z). \end{aligned}$$

Thus we have the following theorem.

**Theorem 4.** $G^{(p)}(t,u,v,z,x) = \frac{1}{1-tz} + \frac{x}{1-tzx}G_1^{(p)}(t,u,v,z)$.

# 3 Specializations of our general results

In this section, we shall compute the generating function for $p$-ascent sequences by length and the number of zeros.

For $n \geq 1$, let $H_{a,b,\ell,n}^{(p)}$ denote the number of $p$-ascent sequences of length $n$ with $a$ ascents and $b$ zeros which have last letter $\ell$. Then we first wish to compute

$$H^{(p)}(t,u,v,z) = \sum_{n\geq 1,\ a,b,\ell\geq 0} H_{a,b,\ell,n}^{(p)} u^a z^b v^\ell t^n. \tag{12}$$

Using the same reasoning as in the previous section, we see that

$$\begin{aligned} H^{(p)}(t,u,v,z) &= tz + \sum_{\substack{a,b,\ell\geq 0\\ n\geq 1}} H_{a,b,\ell,n}^{(p)} t^{n+1}\left(u^a v^0 z^{b+1} + \sum_{i=1}^{\ell} u^a v^i z^b + \sum_{i=\ell+1}^{a+p} u^{a+1}v^i z^b\right) \\ &= tz + t\sum_{\substack{a,b,\ell\geq 0\\ n\geq r+1}} H_{a,b,\ell,n}^{(p)} t^n u^a z^b \left(z + \frac{v^{\ell+1}-v}{v-1} + u\frac{v^{a+p+1}-v^{\ell+1}}{v-1}\right) \\ &= tz + tzH^{(p)}(t,u,1,z) + \frac{tv}{v-1}\left(H^{(p)}(t,u,v,z) - H^{(p)}(t,u,1,z)\right) + \\ &\quad \frac{tuv}{v-1}\left(H^{(p)}(t,uv,1,z) - H^{(p)}(t,u,v,z)\right). \end{aligned}$$

Solving for $H^{(p)}(t,u,v,z)$, we see that we have the following lemma.

**Lemma 5.**

$$(v\delta_1 - 1)H^{(p)}(t,u,v,z) = \\ (v-1)tz + t(z(v-1)-v)H^{(p)}(t,u,1,z) + tuv^{p+1}H^{(p)}(t,uv,1,z). \tag{13}$$

Again, the substitution $v = \frac{1}{\delta_1}$ eliminates the left-hand side of (13). We can then solve for $H^{(p)}(u,1,z,t)$ to obtain the recursion

$$H^{(p)}(t,u,1,z) = \frac{(1-\delta_1)z}{\gamma_1} + \frac{u}{\gamma_1\delta_1^p}H^{(p)}\left(t,\frac{u}{\delta_1},1,z\right). \tag{14}$$

We can iterate the recursion (14) in the same manner as we iterated the recursion (7) in the previous section to prove that

$$H^{(p)}(t,u,1,z) = \sum_{n\geq 0} \frac{(\delta_n - \delta_{n+1})zu^n}{\gamma_1\cdots\gamma_{n+1}\delta_n^p}. \tag{15}$$

We can easily check that for all $n \geq 0$, $\delta_n - \delta_{n+1} = (1-u)t(1-t)^n$. Thus, as a power series in $u$, we can conclude the following.

**Theorem 6.** $H^{(p)}(t,u,1,z) = \sum_{n=0}^{\infty} \frac{zt(1-u)u^n(1-t)^n}{\delta_n^p \prod_{i=1}^{n+1}\gamma_i}$.

We would like to set $u = 1$ in the power series $\sum_{s=0}^{\infty} \frac{zt(1-u)u^s(1-t)^s}{\delta_s \prod_{i=1}^{s+1}\gamma_i}$, but the factor $(1-u)$ in the series does not allow us to do that in this form. Thus our next step is to rewrite the series in a form where it is obvious that we can set $u = 1$ in the series. To that end, observe that for $k \geq 1$, $\delta_k = u-(1-t)^k(u-1) = 1+u-1-(1-t)^k(u-1) = 1-((1-t)^k-1)(u-1)$, so that by Newton's binomial theorem,

$$\begin{aligned}\frac{1}{\delta_k^p} &= \left(\frac{1}{1-(u-1)((1-t)^k-1)}\right)^p = \sum_{n=0}^{\infty}\binom{p-1+n}{n}((u-1)((1-t)^k-1))^n \\ &= \sum_{n=0}^{\infty}\binom{p-1+n}{n}(u-1)^n\left(\sum_{m=0}^{n}(-1)^{n-m}\binom{n}{m}(1-t)^{km}\right). \end{aligned} \tag{16}$$

Substituting (16) into Theorem 6, we see that

$$\begin{aligned}&H^{(p)}(t,u,1,z) = \\ &\frac{zt(1-u)}{\gamma_1} + \sum_{k\geq 1}\frac{zt(1-u)u^k(1-t)^k}{\prod_{i=1}^{k+1}\gamma_i}\sum_{n\geq 0}\binom{p-1+n}{n}(u-1)^n\sum_{m=0}^{n}(-1)^{n-m}\binom{n}{m}(1-t)^{km} = \\ &\frac{zt(1-u)}{\gamma_1} + \sum_{n\geq 0}\sum_{m=0}^{n}(-1)^{n-m-1}\binom{n}{m}(u-1)^{n-m}zt\sum_{k\geq 1}\frac{(u-1)^{m+1}u^k(1-t)^{k(m+1)}}{\prod_{i=1}^{k+1}\gamma_i} = \\ &\frac{zt(1-u)}{\gamma_1} + \sum_{n\geq 0}\binom{p-1+n}{n}\sum_{m=0}^{n}(-1)^{n-m-1}\binom{n}{m}(u-1)^{n-m}\frac{zt}{(1-zt)^{m+1}} \times \\ &\quad\sum_{k\geq 1}\frac{(u-1)^{m+1}(1-zt)^{m+1}u^k(1-t)^{k(m+1)}}{\prod_{i=1}^{k+1}\gamma_i}.\end{aligned}$$

In [9], we have proved the following lemma.

**Lemma 7.**

$$\sum_{k\geq 0}\frac{(u-1)^{m+1}(1-zt)^{m+1}u^k(1-t)^{k(m+1)}}{\prod_{i=1}^{k+1}\gamma_i} = -\sum_{j=0}^{m}(u-1)^j(1-zt)^ju^{m-j}\prod_{i=j+1}^{m}(1-((1-t)^i).$$

It thus follows that

$$H^{(p)}(t,u,1,z) = \frac{zt(1-u)}{\gamma_1} + \sum_{n\geq 0}\binom{p-1+n}{n}\sum_{m=0}^{n}(-1)^{n-m-1}\binom{n}{m}(u-1)^{n-m}\frac{zt}{(1-zt)^{m+1}} \times \left(-\frac{(u-1)^{m+1}(1-zt)^{m+1}}{\gamma_1} - \sum_{j=0}^{m}(u-1)^j(1-zt)^j u^{m-j}\prod_{i=j+1}^{m}(1-(1-t)^i)\right).$$

There is no problem in setting $u = 1$ in this expression to obtain that

$$H^{(p)}(t,1,1,z) = \sum_{n\geq 0}\binom{p-1+n}{n}\frac{zt}{(1-zt)^{n+1}}\prod_{i=1}^{n}(1-(1-t)^i). \tag{17}$$

Clearly, our definitions ensure that $1 + H(t,1,1,z) = A^{(p)}(t,z)$ as defined in the introduction so that we have the following theorem.

**Theorem 8.** *For all $p \geq 1$,*

$$A^{(p)}(t,z) = \sum_{w\in Asc(p)} t^{|w|}z^{|w|_0} = 1 + \sum_{n\geq 0}\binom{p-1+n}{n}\frac{zt}{(1-zt)^{n+1}}\prod_{i=1}^{n}(1-(1-t)^i). \tag{18}$$

The case $p = 1$ in Theorem 8 gives exactly the same formula for $A^{(1)}(t,z)$ as that derived in [9], which should be the case. We also note that the authors conjectured in [9] that

$$1 + \sum_{k=0}^{\infty}\frac{zt}{(1-zt)^{k+1}}\prod_{i=1}^{k}(1-((1-t)^i) = 1 + \sum_{m=1}^{\infty}\prod_{i=1}^{m}(1-(1-t)^{i-1}(1-zt)). \tag{19}$$

This was proved independently by Jelínek [7], Levande [10], and Yan [13]. It would be interesting to find an analogue of this relation for $p > 1$.

Next we can use the same techniques as in [4] to find the generating function for the number of *primitive p-ascent sequences*. That is, let $r_{n,p}$ denote the number of $p$-ascent sequences $a$ of length $n$ such that $a$ has no consecutive repeated letters and $a_{n,p}$ denote the number of $p$-ascent sequences $a$ of length $n$.

If $R^{(p)}(t) = 1 + \sum_{n\geq 1} r_{n,p}t^n$ and $A^{(p)}(t) = 1 + \sum_{n\geq 1} a_{n,p}t^n$, then it is easy to see that

$$A^{(p)}(t) = A^{(p)}(t,1) = R^{(p)}\left(\frac{t}{1-t}\right) = R^{(p)}(t + t^2 + \cdots), \tag{20}$$

since each element in a primitive $p$-ascent sequence can be repeated any specified number of times. Setting $x = \frac{t}{1-t}$ so that $t = \frac{x}{1+x}$, we see that (20) implies that

$$R^{(p)}(x) = A^{(p)}\left(\frac{x}{1+x}\right). \tag{21}$$

Using our formula (18) for $A^{(p)}(t)$ and simplifying will yeild the following theorem.

**Theorem 9.** *For all $p \geq 1$,* $R^{(p)}(t) = 1 + t\sum_{n=0}^{\infty}\binom{p-1+n}{n}(1+t)^n\prod_{i=1}^{n}\left(1-\left(\frac{1}{1+t}\right)^i\right).$

Finally if we replace $t$ by $t + t^2 + \cdots + t^k = t\frac{(t^k-1)}{t-1}$ in Theorem 9, then we can obtain the generating function for the number of $p$-ascent sequences $a$ such that the maximum length of a consecutive sequence of repeated letters is less than or equal to $k$:

$$1 + t\frac{t^k - 1}{t-1} \sum_{n=0}^{\infty} \binom{p-1+n}{n} \left(\frac{t^{k+1}-1}{t-1}\right)^n \prod_{i=1}^{n} \left(1 - \left(\frac{t-1}{t^{k+1}-1}\right)^i\right). \tag{22}$$

# 4 Pattern avoidance in $p$-ascent sequences

In this section, we shall prove some simple results about pattern avoidance in $p$-ascent sequences thus extending the studies initiated in [6] for ascent sequences.

We begin by considering patterns of length 2. There are three such patterns, 00, 01, and 10. Recall that $a_{n,p,u}$ (resp., $r_{n,p,u}$) is the number of (resp., primitive) $p$-ascent sequences of length $n$ that avoid a pattern $u$. The only $p$-ascent sequences that avoid 01 are the sequences that consist of all zeros so that $a_{n,p,01} = 1$ for all $n, p \geq 1$ and $r_{n,p,01}$ equals 1 if $n = 1$ and 0 otherwise.

**10-avoiding $p$-ascent sequences**
Let us consider $r_{n,p,10}$. In this case, we are looking for $p$-ascent sequences which avoid 10 and have no repeated letters. It is clear that any such a sequence $a$ must be of the form $a = a_1 \ldots a_n$, where $0 = a_1 < a_2 < \cdots < a_n$. For each $1 \leq i \leq n$, the word $a_1 \ldots a_i$ has $i-1$ ascents so that $a_{i+1} \leq i - 1 + p$. It follows that $r_{n,p,10}$ counts all words $a_1 a_2 \ldots a_n$, where $0 = a_1 < a_2 < \cdots < a_n \leq p + n - 2$ so that $r_{n,p,10} = \binom{p+n-2}{n-1}$. Hence by Newton's Binomial Theorem,

$$R_{10}^{(p)}(t) = 1 + \sum_{n \geq 1} \binom{p-1+n-1}{n-1} t^n = 1 + \frac{t}{(1-t)^p}. \tag{23}$$

It is easy to see that the $p$-ascent sequences counted by $a_{n,p,10}$ arise by taking a sequence $d_1 \ldots d_s$ counted by $r_{s,p,10}$ for some $s \leq n$ and replacing each letter $d_i$ by one or more copies so that the resulting word is of length $n$. The number of ways to do this for a given $d_1 \ldots d_s$ is the number of solutions to $b_1 + \cdots + b_s = n$, where $b_i \geq 1$, which is $\binom{n-1}{s-1}$. Thus

$$a_{n,p,10} = \sum_{s=1}^{n} \binom{n-1}{s} r_{s,p,10} = \sum_{s=1}^{n} \binom{n-1}{s-1} \binom{p+s-2}{s-1} = \sum_{s=0}^{n-1} \binom{n-1}{s} \binom{p+s-1}{s}. \tag{24}$$

It also follows that $A_{10}^{(p)}(t) = R_{10}^{(p)}\left(\frac{t}{1-t}\right) = 1 + \frac{t(1-t)^{p-1}}{(1-2t)^p}$.

We note that the sequence $(a_{n,2,10})_{n \geq 1}$ starts out $1, 3, 8, 20, 48, 112, 256, \ldots$ and this is the sequence A001792 in the OEIS [12] which has many combinatorial interpretations.

**00-avoiding $p$-ascent sequences**
If a $p$-ascent sequence $a = a_1 \ldots a_n$ avoids 00, then all its elements must be distinct. Note that for each $2 \leq i \leq n$, $a_1 \ldots a_{i-1}$ can have at most $i-2$ ascents so that $a_i \leq p + i - 2$. Let $\max(a)$ denote the maximum of $\{a_1, \ldots, a_n\}$. If $a$ avoids 00, then by the pigeon hole principle, it must be the case that $\max(a) \geq n - 1$. Thus, if $a$ avoids 00, then $n - 1 \leq \max(a) \leq n + p - 2$.

Now consider 2-ascent sequences that avoid 00. Suppose that $a = a_1 \dots a_n$ is a 2-ascent sequence which avoids 00. Then we know that $\max(a) \in \{n-1, n\}$. If $\max(a) = n$, $a$ must be strictly increasing and there must be some smallest $k \geq 1$ such that $a_k = k$, In such a situation, it is easy to see that $a$ must be of the form $0, 1, \dots, k-2, k, k+1, \dots n$. Thus there are $n-1$ 2-ascent sequences $a$ of length $n$ such that $a$ avoids 00 and $\max(a) = n$.

Next, suppose that $a = a_1 \dots a_n$ is a 2-ascent sequence that avoids 00 and $\max(a) = n-1$. Then there are two cases. Namely, it could be that there is no $k \leq n$ such that $a_k = k$. In that case, $a$ is the increasing sequence $a = 012\dots(n-1)$. Otherwise, let $j$ equal the smallest $i$ such that $a_i = i$. Then $a$ must be strictly increasing up to $a_j$ so that $a$ starts out $012\dots(j-2)j$. Since $\max(a) = n-1$, it follows that $\{a_1, \dots, a_n\} = \{0, 1, \dots, n-1\}$ so that there must be some $j < k \leq n$ such that $a_k = j-1$. In that case, $a_{k-1} > a_k$ so that $a$ has at least one descent. However, if $\max(a) = n-1$, $a$ can have at most one descent. Thus, once we have placed $j-1$, the remaining elements must be placed in increasing order. It is then easy to check that no matter where we place $j-1$ after position $j$, the resulting sequence will be a 2-ascent sequence. It follows that the number of 2-ascent sequences which avoid 00 and have one descent is $\sum_{j=1}^{n-1}(n-j) = \binom{n-1}{2}$.

Thus, we have the following theorem.

**Theorem 10.** *For all* $n \geq 1$, $a_{n,2,00} = n - 1 + 1 + \binom{n-1}{2} = 1 + \binom{n}{2}$.

The sequence $(a_{n,3,00})_{n\geq 1}$ starts out $1, 3, 9, 24, 57, 122, 239, 435, 745, 1213, 1893, 2850, \dots$, which is the sequence A089830 in the OEIS [12], whose generating function is $\frac{1-3x+6x^2-5x^3+3x^4-x^5}{(1-x)^6}$.

In this case, if $a = a_1 \dots a_n$ is a 3-ascent sequence which avoids 00, then we know that $n - 1 \leq \max(a) \leq n + 1$. We shall prove that

$$\sum_{n\geq 1} a_{n,3,00} x^n = \frac{x(1-3x+6x^2-5x^3+3x^4-x^5)}{(1-x)^6}$$

by classifying the 3-ascent sequences $a$ which avoid 00 by the $\max(a)$ and $\text{des}(a)$, where $\text{des}(a)$ is the number of *descents* in $a$, that is, the number of elements followed by smaller elements.

**Case 1**. $\text{des}(a) = 0$.
Suppose that $a = a_1 \dots a_n$ is an increasing 3-ascent sequence that avoids 00. Now, if $\max(a) = n-1$, then $a = 012\dots(n-1)$. If $\max(a) = n$, then exactly one element from $[n] = \{1, \dots, n-1\}$ does not appear in $a$. If $i$ does not appear in $a$, then $a = 01\dots(i-1)(i+1)(i+2)\dots n$, which is a 3-ascent sequence. Thus, there are $n-1$ increasing 3-ascent sequences whose maximum is $n$. Finally, if $\max(a) = n+1$, then two elements from $[n]$ do not appear in $a$. Again, it is easy to check that no matter which two elements from $[n]$ we leave out, the resulting increasing sequence will be a 3-ascent sequence. Thus, there are $\binom{n}{2}$ increasing 3-ascent sequences whose maximum is $n+1$. Therefore, the total number of increasing 3-ascents sequences of length $n$ is $1 + (n-1) + \binom{n}{2} = \binom{n+1}{2}$.

**Case 2.** $\text{des}(a) = 1$.
In this case, if $a = a_1 \dots a_n$ is a 3-ascent sequence such that $\text{des}(a) = 1$ and $a$ avoids 00, then $\max(a) \in \{n-1, n\}$. Suppose that $a_j > a_{j-1}$. Then we have two subcases depending on whether $a_j = j$ or $a_j = j+1$.

If $a_j = j+1$, then there must be two elements $1 \leq u < v \leq j$, which do not appear in $a_1 \ldots a_j$. Clearly, we have $\binom{j}{2}$ ways to pick $u$ and $v$. We then have three subcases depending on whether $u$ and $v$ appear in $a$. If both $u$ and $v$ appear in $a$, then $a$ must start out $a_1 \ldots a_j uv$ so that $a_{j+3} \ldots a_n$ must be an increasing sequence from $[n]-[j+1]$ of length $n-j-2$. Clearly, there are $n-j-1$ such subsequences and it is easy to check that we can attach any such subsequence at the end of the sequence $a_1 \ldots a_j uv$ to obtain a 3-ascent sequence avoiding 00. If $u$ appears in $a$, but $v$ does not appear in $a$, then $a$ must be of the form $a_1 \ldots a_j u\gamma$, where $\gamma$ is the increasing sequence $(j+2)(j+3)\ldots n$. Similarly if $v$ appears in $a$, but $u$ does not appear in $a$. then $a$ must be of the form $a_1 \ldots a_j v\gamma$, where $\gamma$ is the increasing sequence $(j+2)(j+3)\ldots n$. It follows that the number of 3-ascent sequences is $\sum_{j=2}^{n-1} \binom{j}{2}(n-j+1)$. One can verify by Mathematica that $\sum_{j=2}^{n-1} \binom{j}{2}(n-j+1) = \binom{n}{3} + \binom{n+1}{4}$.

If $a_j = j$, there is one element $u$ in $[j]$ which does not appear in $a_1 \ldots a_j$, so that the sequence must start out $a_1 \ldots a_j u$. The rest of the sequence must be the increasing rearrangement of $\{j+1, \ldots, n\} - \{v\}$ for some $v \in \{j+1, \ldots, n\}$. Thus, we have $j-1$ choices for $u$ and $n-j$ choices for $v$. Hence the number of 3-ascent sequences $a$ where $\text{des}(a) = 1$ and for some $j$, $a_j > a_{j+1}$ and $a_j = j$ is $\sum_{j=2}^{n-1}(j-1)(n-j)$. One can check by Mathematica that $\sum_{j=2}^{n-1}(j-1)(n-j) = \binom{n}{3}$.

Thus, the number of 3-ascent sequences with one descent, which avoid 00 is $2\binom{n}{3} + \binom{n+1}{4}$.

**Case 3** $\text{des}(a) = 2$.
In this case, it must be that $\max(a) = n-1$, so that $a$ must contain all the elements in the sequence $0, 1, \ldots, n-1$. Now, suppose that the first descent of $a$ occurs at position $j$. Then we have two cases depending on whether $a_j = j$ or $a_j = j+1$.

If $a_j = j$, there must be some $u$, where $1 \leq u \leq j-1$, which does not appear in $a_1 \ldots a_j$ and $a_{j+1} = u$. We have $j-1$ choices for $u$. The sequence $a_{j+2} \ldots a_n$ must be a rearrangement of $(j+1)(j+2)\ldots(n-1)$, which has one descent. The bottom element of the descent pair that occurs in $a_{j+2} \ldots a_n$ must equal $s$ for some $j+1 \leq s \leq n-2$ and the top element of the descent must equal $t$, where $s+1 \leq t \leq n-1$. It is easy to check that any choice of $s$ and $t$ will yield a 3-ascent sequence, so that the number of choices for the sequence $a_{j+2} \ldots a_n$ is $\sum_{s=(j+1)}^{n-2} n-1-s = \binom{n-1-j}{2}$. It follows that the number of 3-ascent sequences in this case is $\sum_{j=2}^{n-2}(j-1)\binom{n-1-j}{2}$, which can be shown by Mathematica to be equal to $\binom{n-1}{4}$.

If $a_j = j+1$, then there must be two elements $1 \leq u \leq v \leq j$ that do not appear in $a_1 \ldots a_j$. We have $\binom{j}{2}$ ways to choose $u$ and $v$. We then have two further subcases depending on whether $a_{j+1} = v$ or $a_{j+1} = v$.

If $a_{j+1} = v$, then our sequences start out $a_1 \ldots a_j = (j+1)v$ and where every $u$ occurs in the sequence $a_{j+2} \ldots a_n$, it will cause a second descent so that there are $n-j-1$ choices in this case. If $a_{j+1} = u$, then the sequence $a_{j+2} \ldots a_n$ must be a rearrangemetn of the sequence $v(j+2)(j+3)\ldots(n-1)$ with one descent and we can argue as we did in the case where $a_j = j$ that there are $\binom{n-j-1}{2}$ choices for the sequence $a_{j+2} \ldots a_n$. Thus the total number of choices in the case where $a_j = j+1$ is $\sum_{j=1}^{n-2} \binom{j}{2}\binom{n-j}{2} = \binom{n+1}{5}$ where the last equality can be checked by Mathematica.

Putting all the cases together, we see that the number of 3-ascent sequences of length $n$, which avoid 00 is equal to

$$\binom{n+1}{2} + 2\binom{n}{3} + \binom{n+1}{4} + \binom{n-1}{4} + \binom{n+1}{5} = \binom{n+1}{2} + 2\binom{n}{3} + \binom{n-1}{4} + \binom{n+2}{5}.$$

Thus we have the following theorem.

**Theorem 11.** *For all $n \geq 1$,* $a_{n,3,00} = \binom{n+1}{2} + 2\binom{n}{3} + \binom{n-1}{4} + \binom{n+2}{5}$,

Note that it follows from Newton's binomial theorem that

$$\sum_{n\geq 1} \binom{n+1}{2} x^n = \frac{x}{(1-x)^3}, \quad \sum_{n\geq 1} 2\binom{n}{3} x^n = \frac{2x^3}{(1-x)^4},$$
$$\sum_{n\geq 1} \binom{n-1}{4} x^n = \frac{x^5}{(1-x)^5}, \text{ and} \sum_{n\geq 1} \binom{n+2}{5} x^n = \frac{x^3}{(1-x)^6}.$$

Adding these series together and simplifying, we have the following theorem.

**Theorem 12.** $\sum_{n\geq 1} a_{n,3,00} x^n = \frac{x(1-3x+6x^2-5x^3+3x^4-x^5)}{(1-x)^6}.$

We note that Burstein and Mansour [2] gave a combinatorial interpretation to the $n$-th element in sequence A089830 as the number of words $w = w_1 \ldots w_{n-1} \in \{1,2,3\}^*$, which avoid the vincular pattern 21-2 (also denoted in the literature $\underline{21}2$; see [8]). That is, there are no subsequences of the form $w_i w_{i+1} w_j$ in $w$ such that $i+1 < j$ and $w_i = w_j > w_{i+1}$. We ask the question whether one can construct a simple bijection between such words and the set of 3-ascent sequences of length $n$, which avoid 00.

We note that the sequence $(a_{n,4,00})_{n\geq 1}$ starts out $1, 4, 16, 58, 190, 564, 1526, 3794 \ldots$. This is the sequence A263851 in the OEIS [12].

**012-avoiding $p$-ascent sequences** Now suppose that $a = a_1 \ldots a_n$ is a $p$-ascent sequence such that $a$ avoids 012. The first thing to observe is that if $a_i = 1$ for some $i$, then since $a_1 = 0$, it must be the case that $a_j \in \{0,1\}$ for all $j \geq i$. The second thing to observe is that $a_i \leq p$ for all $i$. That is, the only way that $a$ can have an element $a_k > p$ is if $a_1 \ldots a_{k-1}$ has at least $a_k - p$ ascents. Since the first ascent in a $p$-ascent sequence must be of one of the forms $01, 02, \ldots, 0p$, such an ascent sequence would not avoid 012.

**2-ascent sequences.** Now, suppose that $a = a_1 \ldots a_n$ is a 2-ascent sequence such that $a$ avoids 012. If $a$ has no 1s, then $a_i \in \{0,2\}$ for all $i \geq 2$, so that there are $2^{n-1}$ such 2-ascent sequences. If $a$ contains a 1, then let $k$ be the smallest $j$ such that $a_j$ equals 1. It then follows that $a_i \in \{0,2\}$ for $2 \leq i < k$ and $a_j \in \{0,1\}$ for $k < j \leq n$. Thus, there are $2^{n-2}$ such 2-ascent sequences, so that the number of 2-ascent sequences that avoid 012 and contain a 1 is $(n-1)2^{n-2}$. Hence, for $n \geq 1$,

$$a_{n,2,012} = 2^{n-1} + (n-1)2^{n-2} = (n+1)2^{n-2}. \tag{25}$$

We note that the sequence $(a_{n,2,012})_{n\geq 1}$ starts out $1, 3, 8, 20, 48, 112, 256, \ldots$, and this is, again, as in the case of $(a_{n,2,10})_{n\geq 1}$, the sequence A001792 in the OEIS [12]. Next, we will explain this fact combinatorially.

It is easy to see that each sequence counted by $(a_{n,2,012})_{n\geq 1}$ can be obtained by taking a number of 2s (maybe none) followed by a number of 1s, and placing any number of 0s (maybe none) between these 1s and 2s making sure that the total length of the sequence is $n$, and this sequence begins with a 0. On the other hand, it is also straightforward to see that sequences counted by $(a_{n,2,10})_{n\geq 1}$ are of two types: they are either of the form

$$\underbrace{0\ldots0}_{i_0\geq 1}\underbrace{1\ldots1}_{i_1\geq 1}\underbrace{2\ldots2}_{i_2\geq 1}\cdots\underbrace{a\ldots a}_{i_a\geq 1}, \tag{26}$$

where $0, 1, \ldots, a$ all appear or of the form

$$\underbrace{0\ldots0}_{i_0\geq1}\underbrace{1\ldots1}_{i_1\geq1}\underbrace{2\ldots2}_{i_2\geq1}\cdots\underbrace{a\ldots a}_{i_a\geq1}\underbrace{(a+2)\ldots(a+2)}_{i_{a+2}\geq1}\underbrace{(a+3)\ldots(a+3)}_{i_{a+3}\geq1}\underbrace{(a+4)\ldots(a+4)}_{i_{a+4}\geq1}\ldots, \quad (27)$$

where $a \geq 0$ exists. A bijection between the classes of sequences is given by turning sequences of the form (26) into

$$\underbrace{0\ldots0}_{i_0}2\underbrace{0\ldots0}_{i_1-1}2\underbrace{0\ldots0}_{i_2-1}\ldots2\underbrace{0\ldots0}_{i_a-1},$$

and the sequences of the form (27) into

$$\underbrace{0\ldots0}_{i_0}2\underbrace{0\ldots0}_{i_1-1}2\underbrace{0\ldots0}_{i_2-1}\ldots2\underbrace{0\ldots0}_{i_a-1}1\underbrace{0\ldots0}_{i_{a+2}-1}1\underbrace{0\ldots0}_{i_{a+3}-1}1\underbrace{0\ldots0}_{i_{a+4}-1}\ldots.$$

**3-ascent sequences.** Now, suppose that $a = a_1 \ldots a_n$ is a 3-ascent sequence such that $a$ avoids 012. If $a$ has no 1s, then $a_i \in \{0, 2, 3\}$ for all $i \geq 2$. It is then easy to see that if $b_1 \ldots b_n$ is the sequence that arises from $a_1 \ldots a_n$ by replacing each 2 by a 1 and each 3 by a 2, then $b$ is a 2-ascent sequence that avoids 012. Thus, there are $(n+1)2^{n-2}$ such sequences. Now, suppose that $a$ contains a 1. Then let $k$ be the smallest $j$ such that $a_j$ equals 1. It then follows that $a_i \in \{0, 2, 3\}$ for $2 \leq i < k$ and $a_j \in \{0, 1\}$ for $k < j \leq n$. It is then easy to see that if $b_1 \ldots b_{k-1}$ is the sequence that arises from $a_1 \ldots a_{k-1}$ by replacing each 2 by a 1 and each 3 by a 2, then $b_1 \ldots b_{k-1}$ is a 2-ascent sequence that avoids 012. Thus, from our argument above, it follows that there are $k2^{k-3}$ choices for $a_1 \ldots a_{k-1}$ and $2^{n-k}$ choices for $a_{k+1} \ldots a_n$. Therefore, given $k$, we have $k2^{n-3}$ choices for $a$. Thus,

$$a_{n,3,012} \quad = \quad (n+1)2^{n-2} + \sum_{k=2}^{n} k2^{n-3} = 2^{n-4}(n^2 + 5n + 2) \quad (28)$$

where the last equality can be checked by Mathematica. We note that the sequence $(a_{n,3,012})_{n\geq1}$ starts out $1, 4, 13, 38, 104, 272, 688, \ldots$ and this is the sequence A049611 in the OEIS [12] having several combinatorial interpretations.

**$p$-ascent sequences for an arbitrary $p$.** In general, we can obtain a simple recursion for $a_{n,p,012}$. That is, suppose that $a = (a_1, \ldots, a_n)$ is a $p$-ascent sequence such that $a$ avoids 012. Now, if $a$ has no 1s, then $a_i \in \{0, 2, 3, \ldots, p\}$ for all $i \geq 2$. It is then easy to see that if $b = (b_1, \ldots, b_n)$ is the sequence that arises from $a$ by replacing each $i \geq 2$, by an $i-1$, then $b$ is a $(p-1)$-ascent sequences that avoids 012. Thus, there are $a_{n,p-1,012}$ such sequences. Now suppose that $a$ contains a 1. Then let $k$ be the smallest $j$ such that $a_j$ equals 1. It then follows that $a_i \in \{0, 2, 3, \ldots, p\}$ for $2 \leq i < k$ and $a_j \in \{0, 1\}$ for $k < j \leq n$. It is then easy to see that if $b_1 \ldots b_{k-1}$ is the sequence that arises from $a_1 \ldots a_{k-1}$ by replacing each $i \geq 2$ by an $i-1$, then $b_1 \ldots b_{k-1}$ is a 2-ascent sequences that avoids 012. It follows that there are $a_{k-1,p-1,012}$ choices for $a_1 \ldots a_{k-1}$ and $2^{n-k}$ choices for $a_{k+1} \ldots a_n$. Thus, given $k$, we have $2^{n-k}a_{k-1,p-1,012}$ choices for $a$. It follows that

$$a_{n,p,012} = a_{n,p-1,012} + \sum_{k=2}^{n} a_{k-1,p-1,012}2^{n-k}. \quad (29)$$

For example, using our formula for $a_{n,3,012}$, one can compute that $a_{n,4,012} = \frac{2^{n-5}}{3}(n^3 + 12n^2 + 29n + 6)$. The sequence $(a_{n,4,012})_{n\geq1}$ starts out $1, 5, 19, 63, 192, 552, 1520, 4048, 10496, 26264, \ldots$ and this is the sequence A049612 in the OEIS [12].

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