On the asymptotic stability and numerical analysis of solutions to nonlinear stochastic differential equations with jumps

Wei Mao[1*], **Surong You**[2], **Xuerong Mao**[3]

[1] School of mathematics and information technology

Jiangsu Second Normal University, Nanjing 210013, China.

[2] Department of Applied Mathematics

Donghua Univerisity, Shanghai 201620, China.

[3] Department of Mathematics and Statistics

University of Strathclyde, Glasgow G1 1XH, U.K.

Abstract

This paper is concerned with the stability and numerical analysis of solution to highly nonlinear stochastic differential equations with jumps. By the It$\widehat{o}$ formula, stochastic inequality and semi-martingale convergence theorem, we study the asymptotic stability in the pth moment and almost sure exponential stability of solutions under the local Lipschitz condition and nonlinear growth condition. On the other hand, we also show the convergence in probability of numerical schemes under nonlinear growth condition. Finally, an example is provided to illustrate the theory.

Key words: Poisson random measure, Nonlinear stochastic differential equations, Asymptotic stability, Numerical analysis.

1 Introduction

During the past few decades, stochastic models that incorporate jumps have been proved successful at describing unexpected, abrupt changes of state and have been used with great success in a variety of application areas, including biology, epidemiology, mechanics, etc. In particular, they are used in mathematical finance in order to simulate asset prices, interest rates and volatilities [5,12].

Recently, qualitative theory about the existence and stability of SDEs with jumps have been studied intensively for many scholars. For example, Applebaum [1-3], Li [16], Rong [31], Yin [38], Yang [39] and Zhu [40]. Meantime, explicit solutions can hardly be obtained for SDEs with jumps. Thus appropriate numerical schemes such as the Euler (or Euler-Maruyama) are needed to apply them in practice or to study their properties. Here, we refer to Bruti-Liberati [6], Chalmers [7], Gardon [8], Higham [9,10], Liu [19], Platen [30] and references therein. For above mentioned papers, most of the existing convergence and stability theory require the coefficients of SDEs with jumps to satisfy linear growth condition. In fact, this condition is often not met by many systems in practice and the existing results of convergence and stability are somewhat restrictive for the purpose of practical applications. For example, consider the following SDEs with jumps:

$$dx(t) = f(x(t))dt + g(x(t))dw(t) + \int_0^1 h(x(t-), v)N(dt, dv), \tag{1.1}$$

where Poisson random measure $N(dt, dv)$ is generated by a Poisson point process $\bar{p}(t)$ (see Section 2 for more details). Here

$$f(x) = -2x - \frac{5}{2}x^3,\ g(x) = x^2 \text{ and } h(x, v) = 0.1v(x + x^2).$$

Obviously, the coefficients f, g, h satisfy the local Lipschitz condition but they do not satisfy the linear growth condition. Hence, we cannot apply the convergence and stability results ([2],[6],[8],[9],[10],[38],[39],[40]) to equation (1.1). Therefore, it is very important to establish the convergence and stability theory of SDEs with jumps under some weak conditions.

In the past few decades, many authors devoted themselves to find the other conditions to replace the linear growth condition. By using the Lyapunov-type functions, a lot of important results have been obtained by many scholars under the Khasminskhii-type conditions. For example, in terms of Lyapunov functions, Khasminskii [14] first investigated the existence and stability of solution for SDEs under the local Lipschitz condition. Next, Mao [23,24,26] extended the Khasminskii theorem ([14]) to the case of stochastic differential delay equations (SDDEs) and obtain the almost sure asymptotic stability of solution to SDDEs. Based on [24,26], Mao [28] also studied the almost surely asymptotic stability of the neutral SDDEs with Markovian switching. Meanwhile, under the Khasminskhii-type conditions, some authors studied the convergence of numerical solutions for stochastic differential systems. By using the Lyapunov functions, Marion [21] studied the convergence in probability of the Euler approximation to the exact solution for SDEs. Subsequently, Li [20] and Yuan [37] extended the results of Marion [21] to the case of SDDEs with markovian switching, respectively. Milosevic [22] studied the numerical solution of highly nonlinear neutral SDEs with time-dependent delay under the Khasminskhii-type conditions.

Although above mentioned works are very important and general, many SDEs obey these Khasminskhii-type conditions derived by the Lyapunov approach, but the greatest disadvantage of this approach is that no universal method has been constructed which enables us to find a Lyapunov function or determine that no such function exists. For example, Kolmanovskii [15] and Li [20]. In order to avoid constructing Lyapunov functions, the other general conditions are needed to satisfy highly nonlinear stochastic differential systems. For example, Boulanger [4] provided the polynomial growth condition and prove that there exists a control Lyapunov function such that single input nonlinear stochastic systems is asymptotically stable; Wu [34] assumed the coefficients of SDDEs to be polynomial and established the existence-and-uniqueness theorems of the global solution; In the meantime, Wu [35] and Liu [17] considered the suppression and stabilization of noise for stochastic differential systems. Under the polynomial growth condition, they showed that the noise perturbation may ensure the corresponding stochastic perturbed system is almost surely exponentially stable. Further, Wu [36] extended the results on stochastic suppression and stabilization of [35] to the case of SDDEs. Besides, many researchers also focused on the numerical solution of a class of highly nonlinear SDEs. In particular, they proved that numerical solutions converge to the true solutions in the strong sense under the polynomial growth condition. For detailed understanding on this, please refer to Hutzenthaler [11], Kumar [13], Mao [27], Szpruch [32], Sabanis [33].

Motivated by equation (1.1) and [11,27,33,35], we consider a class of nonlinear SDEs with jumps:

$$dx(t) \quad = \quad f(x(t))dt + g(x(t))dw(t) + \int_Z h(x(t-), v)N(dt, dv), \tag{1.2}$$

on $t \geq 0$. To the best of our knowledge, there are no literatures concerned with the related results on stability and numerical solution of SDEs with jumps under nonlinear growth conditions. By using the It$\widehat{o}$ formula, stochastic inequality and nonnegative semi-martingales convergence theorem, we prove that SDEs with jumps (1.2) is the asymptotically stable in the pth moment and almost surely exponentially stable under nonlinear growth conditions; Meantime, we show that the approximate solution converges in probability to the true solution of equation (1.2) under above mentioned conditions. Comparing with [11,27,33,35], the proof about stability and convergence of numerical solution for SDEs with jumps is not a straightforward generalization of that for SDEs without jumps. Unlike the Brown process $w(t)$ whose almost all sample paths are continuous, the Poisson random measure $N(dt, dv)$ is a jump process and has the sample paths which are right-continuous and have left limits. Therefore, there is a great difference between the stochastic integral with respect to the Brown process and the one with respect to the Poisson random measure. As a result, those results of [11,27,33,35] cannot be naturally extended to the jumps case.

In this paper, we prove that SDEs with jumps is asymptotically stable in the pth moment. However, most of the existing results on SDEs with jumps are about the exponential stability [1,2,38,39], while little is known on the asymptotic stability. Until recently, based on Mao's work [25], Zhu [40] studied the asymptotic stability in the p-th moment and almost sure stability for SDEs with Levy jump under the Lipschitz conditions and linear growth conditions. Unfortunately, the existing Theorem 3.2 for asymptotic stability in the p-th moment stability of SDEs with jumps require the operator LV (2.3) of [40] have the same order as some certain function at some instants. In fact, we will encounter a problem when we attempt to apply Theorem 3.2 of [40] to deduce the asymptotic stability in the p-th moment stability of the solution. Let us consider equation (1.1). If we choose $V(t,x)=x^2$, then LV operator becomes

$$\begin{aligned} LV(t,x) &= -4|x|^2-4|x|^4+\int_0^1\{[\frac{5}{2}(1+0.1v)^2-1]|x|^2+\frac{5}{3}0.01v^2|x|^4\}\pi(dv) \\ &\le -(2-\sqrt{e}-0.05e^2)|x|^2-(4-\frac{0.05}{3}e^2)|x|^4. \end{aligned}$$

Here, the terms $-(4-\frac{0.05}{3}e^2)|x|^4$ which has a higher degree than the degree of V, appear on the right-hand side and these prevent Theorem 3.2 of [40] from being used. It is due to this problem that we see the necessity to develop new stability crireria for SDEs with jumps under nonlinear growth conditions. In addition, although there exist a number of works concerned with exponential stability and numerical solutions for SDEs with jumps, those results in [2,3,6,9,10,30] cannot cover a wide range of highly nonlinear SDEs with jumps. In this case, we prove that SDEs with jumps (1.2) is almost surely exponentially stable and show that the approximate solutions converge in probability to the true solutions of equation (1.2) under nonlinear growth conditions.

The rest of the paper is organized as follows. In Section 2, we introduce some notations and hypotheses concerning equation (1.2); In Section 3, by applying the It$\widehat{o}$ formula, stochastic inequality and semi-martingale convergence theorem, we study the asymptotic stability in the pth moment and almost sure exponential stability of solutions to equation (1.2); While in Section 4 we investigate the convergence in probability of the numerical schemes (4.2) to equation (1.2) under above mentioned conditions; Finally, we give an example to illustrate the theory in Section 5.

2 Preliminaries and the global solution

Throughout this paper, unless otherwise specified, we use the following notation. Let $|x|$ be the Euclidean norm of a vector $x\in R^n$. If A is a matrix, its trace norm is denoted by $|A|=\sqrt{trace(A^\top A)}$. Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\ge0},P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets). Let $w(t)=(w_1(t),\cdots,w_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space $(\Omega,\mathcal{F},P)$.

Let $\{\bar{p} = \bar{p}(t), t \geq 0\}$ be a stationary $\mathcal{F}_t$-adapted and R^n-valued Poisson point process. Then, for $A \in \mathcal{B}(R^n - \{0\})$, here $0 \notin$ the closure of A, we define the Poisson counting measure N associated with $\bar{p}$ by

$$N((0,t] \times A) := \#\{0 < s \leq t, \bar{p}(s) \in A\} = \sum_{t_0 < s \leq t} I_A(\bar{p}(s)),$$

where $\#$ denotes the cardinality of set $\{.\}$. For simplicity, we denote $N(t, A) := N((0,t] \times A)$. It is known that there exists a σ- finite measure π such that

$$E[N(t,A)] = \pi(A)t, \quad P(N(t,A) = n) = \frac{exp(-t\pi(A))(\pi(A)t)^n}{n!}.$$

This measure π is called the Lévy measure. Moreover, by Doob-Meyer's decomposition theorem, there exists a unique $\{\mathcal{F}_t\}$-adapted martingale $\tilde{N}(t, A)$ and a unique $\{\mathcal{F}_t\}$-adapted natural increasing process $\hat{N}(t, A)$ such that

$$N(t,A) = \tilde{N}(t,A) + \hat{N}(t,A), \quad t > 0.$$

Here $\tilde{N}(t, A)$ is called the compensated Poisson random measure and $\hat{N}(t, A) = \pi(A)t$ is called the compensator. For more details on the Poisson point process and Lévy jumps, see [1,31].

In this paper, we assume that Poisson random measures N is independent of Brownian motion w. For $Z \in \mathcal{B}(R^n - \{0\})$, consider a nonlinear SDEs with Poisson random measures

$$dx(t) \;=\; f(x(t))dt + g(x(t))dw(t) + \int_Z h(x(t-), v)N(dt, dv), \quad t \geq 0, \tag{2.1}$$

with initial value $x(0) = x_0 \in R^n$, where

$$f : R^n \to R^n, \quad g : R^n \to R^{n\times m} \quad \text{and} \quad h : R^n \times Z \to R^n.$$

The well known conditions imposed for the existence and uniqueness of the global solution are the local Lipschitz condition and the linear growth condition (see e.g. Applebaum [1], Mao [25]). To be precise, let us state these conditions.

Assumption 2.1 *(The local Lipschitz condition) For each integer $d \geq 1$, there exist a positive constant k_d such that*

$$|f(x) - f(y)|^2 \vee |g(x) - g(y)|^2 \leq k_d(|x - y|^2),$$
$$\int_Z |h(x,v) - h(y,v)|^2 \pi(dv) \leq k_d(|x - y|^2),$$

for any $x, y \in R^n$ with $|x| \vee |y| \leq d$.

Assumption 2.2 *(The linear growth condition) There is a positive constant k such that*

$$|f(x)|^2 \vee |g(x)|^2 \vee \int_Z |h(x,v)|^2 \pi(dv) \le k(1+|x|^2),$$

for all $x \in R^n$.

In this paper we shall retain the local Lipschitz condition but replace the linear growth condition by a more general condition in order to guarantee the existence of a unique global solution. Let us now propose our more general conditions.

Assumption 2.3 *For any $x \in R^n$ and $v \in Z$, there exist positive constants $k_i, q_i, (i = 1,2,3)$ and a bounded function $\bar{h}(v)$ such that*

$$|f(x)|^2 \le k_1(1+|x|^{q_1+2}), \qquad |g(x)|^2 \le k_2(1+|x|^{q_2+2})$$

and

$$|h(x,v)|^2 \le k_3(1+|x|^{q_3+2})\bar{h}(v),$$

where $C_{\bar{h}} = \int_Z \bar{h}(v)\pi(dv) < \infty$.

Assumption 2.4 *For any $x \in R^n$ and $v \in Z$, there exist positive constants $\alpha_i, \beta_i, \gamma_i, (i = 1,2)$, such that*

$$x^\top f(x) + \frac{p-1}{2}|g(x)|^2 \le -\alpha_1|x|^2 - \alpha_2|x|^{\gamma_1+2}$$

and

$$|x+h(x,v)|^2 \le \bar{h}(v)(\beta_1|x|^2 + \beta_2|x|^{\gamma_2+2}).$$

We will denote by $C([0,\infty] \times R^n; R_+)$ the family of continuous functions from $[0,\infty) \times R^n$ to R_+. Also denote by $C^{2,1}(R_+ \times R^n; R_+)$ the family of all continuous non-negative functions $V(t,x)$ defined on $R_+ \times R^n$, they are continuously twice differentiable in x and once in t. Given $V \in C^{2,1}(R_+ \times R^n; R_+)$, we define an operator $LV : R_+ \times R^n \to R$ by

$$\begin{aligned} LV(t,x) = {} & V_t(t,x) + V_x(t,x)f(x) + \frac{1}{2}trace[g^\top(x)V_{xx}(t,x)g(x)] \\ & + \int_Z [V(t,x+h(x,v)) - V(t,x)]\pi(dv), \end{aligned} \tag{2.2}$$

where

$$V_t(t,x) = \frac{\partial V(t,x)}{\partial t}, \qquad V_x(t,x) = \Big(\frac{\partial V(t,x)}{\partial x_1}, \cdots, \frac{\partial V(t,x)}{\partial x_n}\Big),$$

$$V_{xx}(t,x) = \Big(\frac{\partial^2 V(t,x)}{\partial x_i \partial x_j}\Big)_{n\times n}.$$

Theorem 2.1 Let Assumptions 2.1 and 2.4 hold. If

$$2\alpha_1 - \beta_1 C_{\bar{h}} + \pi(Z) > 0, \quad \gamma_1 \geq \gamma_2 \ \text{ and } \ 2\alpha_2 > \beta_2 C_{\bar{h}}, \tag{2.3}$$

then for any given initial data x_0, there is a unique global solution $x(t)$ to SDEs with jumps (2.1) on $t \in [0,\infty)$. Moreover, the solution has the property that

$$\int_0^\infty E|x(t)|^2 dt \ < \ \infty, \tag{2.4}$$

for any $t \geq 0$.

The proof of this theorem is rather standard and we omit it here.

Remark 2.1 If $\gamma_1 = \gamma_2$, it is necessary for us to prove Theorem 2.1 by using $2\alpha_2 > \beta_2 C_{\bar{h}}$; If $\gamma_1 > \gamma_2$, we can relax this condition $2\alpha_2 > \beta_2 C_{\bar{h}}$ and only require $\alpha_2 > 0$. Meanwhile, we also obtain an important result (2.4) which is known as the H_∞-stability.

3 Asymptotic Stability of Solutions

In this section, we will discuss the asymptotic behavior of solution, including the asymptotic stability in the pth moment, almost sure exponential stability.

Theorem 3.1 Let Assumptions 2.1, 2.3 and 2.4 hold. If

$$\alpha_1 p - \beta_1^{\frac{p}{2}} C_{\bar{h},p} + \pi(Z) > 0, \quad \gamma_1 \geq \frac{p\gamma_2}{2} \ \text{ and } \ \alpha_2 p > \beta_2^{\frac{p}{2}} C_{\bar{h},p}, \tag{3.1}$$

where $C_{\bar{h},p} = \int_Z 2^{\frac{p}{2}-1}(\bar{h}(v))^{\frac{p}{2}}\pi(dv)$, then for any given initial data x_0, the unique global solution $x(t)$ has the property that

$$\lim_{t\to\infty} E|x(t)|^p = 0 \tag{3.2}$$

for any $p \geq 2$.

Proof. The proof is rather technical and we divide it into two steps.

Step 1. Let we show that $E|x(t)|^p \in L^1(R_+; R_+)$. By the Itô formula (see [1]) to $V(t, x(t)) = |x(t)|^p$, we have

$$
\begin{aligned}
|x(t)|^p &= |x(0)|^p + \int_0^t LV(s, x(s))ds + \int_0^t p|x(s)|^{p-2}x(s)^\top g(x(s))dw(s) \\
&\quad + \int_0^t \int_Z [|x(s-) + h(x(s-), v))|^p - |x(s-)|^p]\tilde{N}(ds, dv),
\end{aligned} \tag{3.3}
$$

where

$$
\begin{aligned}
LV(s, x) &= p|x|^{p-2}x^\top f(x) + \frac{p}{2}|x|^{p-2}|g(x)|^2 \\
&\quad + \frac{p(p-2)}{2}|x|^{p-4}|x^\top g(x)|^2 + \int_Z [|x + h(x, v))|^p - |x|^p]\pi(dv).
\end{aligned}
$$

Taking expectation on both side of (3.3), we get

$$
\begin{aligned}
E(|x(t)|^p) &= E|x_0|^p + E\int_0^t p|x(s)|^{p-2}[x(s)^\top f(x(s)) + \frac{p-1}{2}|g(x(s))|^2]ds \\
&\quad + E\int_0^t \int_Z [|x(s-) + h(x(s-), v))|^p - |x(s-)|^p]\pi(dv)ds.
\end{aligned} \tag{3.4}
$$

By the basic inequality $|a + b|^{\frac{p}{2}} \leq 2^{\frac{p}{2}-1}(|a|^{\frac{p}{2}} + |b|^{\frac{p}{2}})$ and assumption 2.4, it follows that

$$
\begin{aligned}
E(|x(t)|^p) &= E|x_0|^p + E\int_0^t [(-\alpha_1 p + \beta_1^{\frac{p}{2}} C_{\bar{h},p} - \pi(Z))|x(s)|^p \\
&\quad - \alpha_2 p|x(s)|^{p+\gamma_1} + \beta_2^{\frac{p}{2}} C_{\bar{h},p}|x(s)|^{\frac{p\gamma_2}{2}+p}]ds,
\end{aligned}
$$

where $C_{\bar{h},p} = \int_Z 2^{\frac{p}{2}-1}(\bar{h}(v))^{\frac{p}{2}}\pi(dv)$. Let $\tilde{C}_0 = \alpha_1 p - \beta_1^{\frac{p}{2}} C_{\bar{h},p} + \pi(Z)$, recalling that

$$
\gamma_1 \geq \frac{p\gamma_2}{2}, \ \alpha_1 p - \beta_1^{\frac{p}{2}} C_{\bar{h},p} + \pi(Z) > 0 \ \text{ and } \ \alpha_2 p > \beta_2^{\frac{p}{2}} C_{\bar{h},p},
$$

we see that there is a positive constant $\tilde{C} \in [0, \tilde{C}_0]$ such that

$$
-\alpha_2 p|x(s)|^{\gamma_1} + \beta_2^{\frac{p}{2}} C_{\bar{h},p}|x(s)|^{\frac{p\gamma_2}{2}} \leq \tilde{C}, \quad \forall x \in R^n. \tag{3.5}
$$

Hence

$$
E(|x(t)|^p) = E|x_0|^p - (\alpha_1 p - \beta_1^{\frac{p}{2}} C_{\bar{h},p} + \pi(Z) - \tilde{C})E\int_0^t |x(s)|^p ds. \tag{3.6}
$$

Note that $\alpha_1 p - \beta_1^{\frac{p}{2}} C_{\bar{h},p} + \pi(Z) - \tilde{C} > 0$, it follows from (3.6) that

$$
E\int_0^t |x(s)|^p ds \leq \frac{1}{\alpha_1 p - \beta_1^{\frac{p}{2}} C_{\bar{h},p} + \pi(Z) - \tilde{C}} E|x_0|^p.
$$

Using the Fubini theorem and letting $t \to \infty$, we obtain that

$$\int_0^\infty E|x(s)|^p ds < \infty.$$

Step 2. Now we prove that $E|x(t)|^p$ is uniformly continuous on $[0, \infty)$. By the Itô formula, for any $t > s$, we have

$$\begin{aligned} E|x(t)|^p &= E|x(s)|^p + pE\int_s^t |x(\sigma)|^{p-2}x(\sigma)^\top f(x(\sigma))d\sigma \\ &+\frac{p(p-1)}{2}E\int_s^t |x(\sigma)|^{p-2}|g(x(\sigma))|^2 d\sigma \\ &+E\int_s^t\int_Z [|x(\sigma-) + h(x(\sigma-), v))|^p - |x(\sigma-)|^p]\pi(dv)d\sigma. \end{aligned}$$

Then, by assumption 2.3, we have

$$\begin{aligned} |E|x(t)|^p - E|x(s)|^p| &\leq \frac{p}{2}E\int_s^t |x(\sigma)|^p d\sigma + \frac{p}{2}k_1 E\int_s^t (|x(\sigma)|^{p-2} + |x(\sigma)|^{p+\beta_1})d\sigma \\ &+\frac{p(p-1)}{2}k_2 E\int_s^t (|x(\sigma)|^{p-2} + |x(\sigma)|^{p+\beta_2})d\sigma \\ &+E\int_s^t\int_Z [|x(\sigma-) + h(x(\sigma-), v)|^p - |x(\sigma-)|^p]\pi(dv)d\sigma. \end{aligned} \tag{3.7}$$

Using the basic inequality $a^r b^{1-r} \leq ra + (1-r)b$ for any $r \in [0, 1]$, we derive that

$$\begin{aligned} k_1|x(\sigma)|^{p-2} &\leq k_1[1^{\frac{2}{p}}(|x(\sigma)|^p)^{1-\frac{2}{p}}] \\ &\leq \frac{2}{p}k_1 + (1-\frac{2}{p})k_1|x(\sigma)|^p \leq 2k_1 + k_1|x(\sigma)|^p. \end{aligned} \tag{3.8}$$

Similarly, we get

$$k_2|x(\sigma)|^{p-2} \leq 2k_2 + k_2|x(\sigma)|^p. \tag{3.9}$$

On the other hand, by the basic inequality (see [25]), there exists a $\varepsilon > 0$ such that

$$|x(\sigma-) + h(x(\sigma-), v)|^p \leq (1 + \varepsilon^{\frac{1}{p-1}})^{p-1}(\frac{1}{\varepsilon}|h(x(\sigma-), v)|^p + |x(\sigma-)|^p).$$

Then, assumption (2.3) implies that

$$|x(\sigma-) + h(x(\sigma-), v)|^p \leq (1 + \varepsilon^{\frac{1}{p-1}})^{p-1}\{\frac{1}{\varepsilon}[(2k_3\bar{h}(v))^{\frac{p}{2}}(1 + |x(\sigma-)|^{\frac{p}{2}(\beta_3+2)})] + |x(\sigma-)|^p\}.$$

Letting $\varepsilon = (\sqrt{2k_3\bar{h}(v)})^{p-1}$, we obtain

$$|x(\sigma-)+h(x(\sigma-),v)|^p \leq (1+\sqrt{2k_3\bar{h}(v)})^p(1+|x(\sigma-)|^p+|x(\sigma-)|^{\frac{p}{2}(\beta_3+2)}). \quad (3.10)$$

Inserting (3.8),(3.9) and (3.10) into (3.7), it follows that

$$\begin{aligned}
& |E|x(t)|^p - E|x(s)|^p| \\
\leq\ & C_1\int_s^t d\sigma + C_2\int_s^t E|x(\sigma)|^p d\sigma + \int_s^t [\frac{p}{2}k_1E|x(\sigma)|^{p+\beta_1} + \frac{p(p-1)}{2}k_2E|x(\sigma)|^{p+\beta_2} \\
& +\bar{C}_{\bar{h},p}E|x(\sigma)|^{\frac{p}{2}(\beta_3+2)}]d\sigma \\
\leq\ & [C_1 + C_2E|x_0|^p + \frac{p}{2}k_1E|x_0|^{p+\beta_1} + \frac{p(p-1)}{2}k_2E|x_0|^{p+\beta_2} + \bar{C}_{\bar{h},p}E|x_0|^{\frac{p}{2}(\beta_3+2)}](t-s),
\end{aligned}$$

where

$$\bar{C}_{\bar{h},p} = \int_Z (1+\sqrt{2k_3\bar{h}(v)})^p\pi(dv), \quad C_1 = pk_1 + p(p-1)k_2 + \bar{C}_{\bar{h},p},$$

$$C_2 = \frac{p+pk_1}{2} + \frac{p(p-1)}{2}k_2 + \bar{C}_{\bar{h},p} - \pi(Z).$$

This implies that $E|x(t)|^p$ is uniformly continuous on $[0,\infty)$. Finally, similar to the proof of [29], we can derive that

$$\lim_{t\to\infty} E|x(t)|^p = 0,$$

for any $p \geq 2$. Then the proof of Theorem 3.1 is completed.

Remark 3.1 From (3.2), we have that SDEs with jumps (2.1) is asymptotically stable in pth moment. In particular, when $p=2$, the condition (3.1) of Theorem 3.1 becomes the condition (2.3) of Theorem 2.1, then equation (2.1) has a unique solution $x(t)$ and the solution $x(t)$ is asymptotically stable in 2th moment.

Remark 3.2 It should also be mentioned that Mao [25] and Zhu [40] have studied the asymptotic stability of solutions for SDEs and SDEs with jumps, respectively. But the main results of [25] and [40] rely on the linear growth condition. Obviously, their results can not be used in this paper, the corresponding results of [25] and [40] on asymptotic stability are improved and generalized.

Next, we will study the almost sure exponential stablity of equation (2.1).

Lemma 3.1 (see [18],[25]) Let $A(t), U(t)$ be two $\mathcal{F}_t$-adapted increasing processes on $t \geq 0$ with $A(0) = U(0) = 0$ a.s. Let $M(t)$ be a real-valued local martingale with $M(0) = 0$ a.s. Let

ζ be a nonnegative $\mathcal{F}_0$-measurable random variable. Assume that $x(t)$ is nonnegative and

$$x(t) = \zeta + A(t) - U(t) + M(t) \quad \text{for} \quad t \geq 0.$$

If $\lim_{t\to\infty} A(t) < \infty$ a.s. then for almost all $\omega \in \Omega$,

$$\lim_{t\to\infty} x(t) < \infty \ \text{ and } \ \lim_{t\to\infty} U(t) < \infty,$$

that is, both $x(t)$ and $U(t)$ converge to finite random variables.

Theorem 3.2 Let Assumptions 2.1, 2.4 and condition (3.1) hold. Then for any given initial data x_0, the unique global solution $x(t)$ has the property that

$$\limsup_{t\to\infty} \frac{1}{t} log(|x(t)|) \leq -\frac{\varepsilon}{p} \tag{3.11}$$

for any $t \geq 0$.

Proof. For any $t \geq 0$ and $\varepsilon > 0$, applying the It$\hat{o}$ formula to $e^{\varepsilon t}V(t, x(t))$, we have

$$e^{\varepsilon t}V(t, x(t)) - V(0, x_0) \ = \ \int_0^t e^{\varepsilon s}[LV(s, x(s)) + \varepsilon V(s, x(s))]ds + M(t), \tag{3.12}$$

where

$$\begin{aligned} M(t) \ &= \ \int_0^t e^{\varepsilon s} V_x(s, x(s))g(s, x(s))dw(s) \\ &+ \int_0^t \int_Z e^{\varepsilon s}[V(s, x(s-) + h(x(s-), v)) - V(s, x(s-))]\tilde{N}(ds, dv) \end{aligned}$$

is a local martingale with the initial value $M(0) = 0$. By assumption 2.4, we compute

$$\begin{aligned} &\int_0^t e^{\varepsilon s}[LV(s, x(s)) + \varepsilon V(s, x(s))]ds \\ \leq \ &\int_0^t e^{\varepsilon s}[(\varepsilon - \alpha_1 p + \beta_1^{\frac{p}{2}} C_{\bar{h},p} - \pi(Z))|x(s)|^p - \alpha_2 p|x(s)|^{p+q} + \beta_2^{\frac{p}{2}} C_{\bar{h},p}|x(s)|^{\frac{p\gamma}{2}+p}]. \end{aligned}$$

Recalling (3.5), we have

$$\int_0^t e^{\varepsilon s}[LV(s, x(s)) + \varepsilon V(s, x(s))]ds \ \leq \int_0^t e^{\varepsilon s}(\varepsilon - \alpha_1 p + \beta_1^{\frac{p}{2}} C_{\bar{h},p} - \pi(Z) + \tilde{C})|x(s)|^p ds. \tag{3.13}$$

Inserting (3.13) into (3.12), we get

$$e^{\varepsilon t}|x(t)|^p \ = \ |x_0|^p - \int_0^t e^{\varepsilon s}(\alpha_1 p - \beta_1^{\frac{p}{2}} C_{\bar{h},p} + \pi(Z) - \tilde{C} - \varepsilon)|x(s)|^p ds + M(t).$$

Since $\alpha_1 p - \beta_1^{\frac{p}{2}} C_{\bar{h},p} + \pi(Z) - \tilde{C} > 0$, we choose sufficiently small $\varepsilon > 0$ such that $\alpha_1 p - \beta_1^{\frac{p}{2}} C_{\bar{h},p} + \pi(Z) - \tilde{C} - \varepsilon > 0$. By lemma 3.2, we obtain that

$$\limsup_{t\to\infty}(e^{\varepsilon t}|x(t)|^p) < \infty \quad a.s.$$

Hence, there exists a finite positive random variable η such that

$$\sup_{0\le t<\infty}(e^{\varepsilon t}|x(t)|^p) \le \eta \quad a.s.$$

This implies

$$\limsup_{t\to\infty}\frac{1}{t}log(|x(t)|) \le -\frac{\varepsilon}{p} \quad a.s.$$

Remark 3.3 Under the nonlinear growth conditions, we prove that SDEs with jumps (2.1) is almost sure exponentially stable which is studied by Applebaum [1,2], but they did not relax the linear growth conditions. In this way, the corresponding results of [1,2] are improved and generalized by Theorem 3.2.

4 Convergence in probability of numerical Solution

In this section, we study the convergence of numerical solutions for SDEs with jumps (2.1) under local Lipschitz condition and nonlinear growth condition.

First, we need to define the approximate solution of SDEs with jumps (2.1). For a given constant stepsize $h > 0$, we propose the Euler method for SDEs with jumps (2.1) as follows

$$y_{n+1} = y_n + f(y_n)h + g(y_n)\Delta w_n + \int_Z h(y_n, v)N(h, dv), \tag{4.1}$$

with initial value $y_0 = x_0$. For arbitrary stepsize $h > 0$, y_n denotes the approximation of $x(t)$ at time $t_n = nh$, $n = 0, 1, 2\cdots$. $\Delta w_n = w(t_{n+1}) - w(t_n)$ and $N(h, dv) = N(t_{n+1}, dv) - N(t_n, dv)$.

In the following convergence analysis, we find it convenient to use continuous-time approximation solution. To define the continuous extension, let us introduce one step processes

$$z(t) = y_n$$

for $t \in [t_n, t_{n+1})$. Hence we define the continuous version $y(t)$ as follows

$$y(t) = y(0) + \int_0^t f(z(s))ds + \int_0^t g(z(s))dw(s) + \int_0^t\int_Z h(z(s), v)N(ds, dv). \tag{4.2}$$

It is not hard to verify that $y(t_n) = y_n$, that is, $y(t)$ coincides with the discrete solutions at the grid-points.

Let we define three stopping times

$$\alpha_d = inf\{t \in [0,T] : |x(t)| \geq d\} \ \text{ and } \ \beta_d = inf\{t \in [0,T] : |y(t)| \geq d\},$$

$\gamma_d = \alpha_d \wedge \beta_d$, where as usual $inf\ \emptyset$ is set as ∞.

Lemma 4.1 If assumptions 2.1 and 2.3 hold, then there exists a positive constant C_d such that

$$E|y(t) - z(t)|^2 \leq C_d h, \qquad \forall \ t \in [0, \gamma_d \wedge T], \tag{4.3}$$

where C_d depend on d, but independent of h.

Proof. Similar to that of the SDEs, we can have the result.

Lemma 4.2 If assumptions 2.1 and 2.3 hold, then the Euler approximate solution $y(t)$ converges to the true solution $x(t)$ of SDEs with jumps (2.1); i.e.,

$$E[\sup_{0\leq t\leq T} |y(t\wedge \gamma_d - x(t\wedge \gamma_d))|^2] \leq \bar{C}_d h, \tag{4.4}$$

where $\bar{C}_d > 0$ depend on d, but independent of h.

Proof. For simplicity, denote $e(t) = y(t) - x(t)$. From (2.1) and (4.2), we have

$$\begin{aligned} e(t) &= \int_0^t [f(z(s)) - f(x(s))]ds + \int_0^t [g(z(s)) - g(x(s))]dw(s) \\ &\quad + \int_0^t \int_Z [h(z(s), v) - h(x(s-), v)] N(ds, dv). \end{aligned}$$

Applying the It$\hat{o}$ formula to $|e(t)|^2$, we obtain

$$\begin{aligned} |e(t)|^2 &= 2\int_0^t (e(s), f(z(s)) - f(x(s)))ds + \int_0^t |g(z(s)) - g(x(s))|^2 ds \\ &\quad +2\int_0^t (e(s), g(z(s)) - g(x(s)))dw(s) \\ &\quad + \int_0^t \int_Z [|e(s) + h(z(s), v) - h(x(s-), v)|^2 - |e(s)|^2] N(ds, dv). \end{aligned} \tag{4.5}$$

The right side of (4.5) can be written as

$$
\begin{aligned}
|e(t)|^2 &= 2\int_0^t (e(s), f(z(s)) - f(x(s)))ds + \int_0^t |g(z(s)) - g(x(s))|^2 ds \\
&\quad +2\int_0^t (e(s), g(z(s)) - g(x(s)))dw(s) \\
&\quad +2\int_0^t \int_Z (e(s), h(z(s), v) - h(x(s-), v))\pi(dv)ds \\
&\quad +2\int_0^t \int_Z (e(s), h(z(s), v) - h(x(s-), v))\tilde{N}(ds, dv) \\
&\quad +\int_0^t \int_Z |h(z(s), v) - h(x(s-), v)|^2 N(ds, dv).
\end{aligned}
$$

Using the basic inequality $2ab \le a^2 + b^2$, we get

$$
\begin{aligned}
|e(t)|^2 &\le (1+\pi(Z))\int_0^t |e(s)|^2 ds + \int_0^t |f(z(s)) - f(x(s)|^2 ds \\
&\quad +\int_0^t |g(z(s)) - g(x(s))|^2 ds + 2\int_0^t (e(s), g(z(s)) - g(x(s)))dw(s) \\
&\quad +\int_0^t \int_Z |h(z(s), v) - h(x(s-), v)|^2 \pi(dv)ds \\
&\quad +2\int_0^t \int_Z (e(s), h(z(s), v) - h(x(s-), v))\tilde{N}(ds, dv) \\
&\quad +\int_0^t \int_Z |h(z(s), v) - h(x(s-), v)|^2 N(ds, dv). \qquad (4.6)
\end{aligned}
$$

Taking expectation on both sides of (4.6), it follows that

$$
E \sup_{0 \le t \le \gamma_d \wedge T} |e(t)|^2 \le (1+\pi(Z))E\int_0^{\gamma_d \wedge T} |e(t)|^2 dt + \sum_{i=1}^4 I_i, \qquad (4.7)
$$

where

$$
\begin{aligned}
I_1 &= E\int_0^{\gamma_d \wedge T} |f(z(t)) - f(x(t))|^2 dt + E\int_0^{\gamma_d \wedge T} |g(z(t)) - g(x(t))|^2 dt \\
&\quad +E\int_0^{\gamma_d \wedge T} \int_Z |h(z(t), v) - h(x(t-), v)|^2 \pi(du)dt, \\
I_2 &= 2E \sup_{0 \le t \le \gamma_d \wedge T} \int_0^t (e(s), g(z(s)) - g(x(s)))dw(s), \\
I_3 &= 2E \sup_{0 \le t \le \gamma_d \wedge T} \int_0^t \int_Z (e(s), h(z(s), v) - h(x(s-), v))\tilde{N}(ds, dv), \\
I_4 &= 2E \sup_{0 \le t \le \gamma_d \wedge T} \int_0^t \int_Z |h(z(s), v) - h(x(s-), v)|^2 N(ds, dv).
\end{aligned}
$$

By assumption 2.1 and lemma 4.1, we have

$$\begin{aligned} I_1 &\le 3k_d \int_0^{\gamma_d\wedge T} E|z(t)-x(t)|^2dt \\ &\le 3k_d \int_0^{\gamma_d\wedge T} (2E|z(t)-y(t)|^2+2E|y(t)-x(t)|^2)dt \\ &\le 6k_d \int_0^{\gamma_d\wedge T} E|y(t)-x(t)|^2dt+6k_dC_dhT. \end{aligned} \tag{4.8}$$

Let us estimate I_2. Using the Burkholder-Davis-Gundy inequality, we get

$$\begin{aligned} I_2 &\le 2E \sup_{0\le t\le \gamma_d\wedge T} \int_0^t |e(s)||g(z(s))-g(x(s))|dw(s) \\ &\le 6E[\sup_{0\le t\le \gamma_d\wedge T} |e(t)|(\int_0^{\gamma_d\wedge T} |g(z(t))-g(x(t))|^2dt)]^{\frac{1}{2}}. \end{aligned}$$

By the young inequality, there exists $\varepsilon>0$, such that

$$\begin{aligned} I_2 &\le 6[\varepsilon E \sup_{0\le t\le \gamma_d\wedge T} |e(t)|^2]^{\frac{1}{2}}[\frac{1}{\varepsilon}E(\int_0^{\tau_d\wedge T} |g(z(t))-g(x(t))|^2dt)]^{\frac{1}{2}} \\ &\le 6\varepsilon E \sup_{0\le t\le \gamma_d\wedge T} |e(t)|^2+\frac{6}{\varepsilon}E\int_0^{\gamma_d\wedge T} |g(z(t))-g(x(t))|^2dt \\ &\le \frac{1}{4}E \sup_{0\le t\le \gamma_d\wedge T} |e(t)|^2+288k_d\int_0^{\gamma_d\wedge T} E|y(t)-x(t)|^2dt+288k_dC_dhT. \end{aligned} \tag{4.9}$$

Next, we give the estimation of I_3. By the Burkholder-Davis-Gundy inequality, there exist a $C>0$ such that

$$\begin{aligned} I_3 &\le CE(\sum_{t\in D_p, t\le \gamma_d\wedge T} |e(t)|^2|h(z(t),p_t)-h(x(t-),p_t)|^2)^{\frac{1}{2}} \\ &\le CE\sup_{0\le t\le \gamma_d\wedge T} |e(t)|E(\sum_{t\in D_p, t\le \gamma_d\wedge T} |h(z(t),p_t)-h(x(t-),p_t)|^2)^{\frac{1}{2}}. \end{aligned}$$

From the Young inequality, we obtain that

$$\begin{aligned} I_3 &\le CE[\frac{1}{4C}\sup_{0\le t\le \gamma_d\wedge T} |e(t)|^2]^{\frac{1}{2}}E[4C(\sum_{t\in D_p, t\le \gamma_d\wedge T} |h(z(t),p_t)-h(x(t-)),p_t)|^2]^{\frac{1}{2}} \\ &\le \frac{1}{4}E\sup_{0\le t\le \gamma_d\wedge T} |e(t)|^2]+4C^2E(\sum_{t\in D_p, t\le \gamma_d\wedge T} |h(z(t),p_t)-h(x(t-),p_t)|^2] \\ &\le \frac{1}{4}E\sup_{0\le t\le \gamma_d\wedge T} |e(t)|^2+4C^2E\int_0^{\gamma_d\wedge T}\int_Z |h(z(t),v)-h(x(t-),v)|^2\pi(dv)dt \\ &\le \frac{1}{4}E\sup_{0\le t\le \gamma_d\wedge T} |e(t)|^2+8C^2k_d\int_0^{\gamma_d\wedge T} E|y(t)-x(t-)|^2dt+8C^2k_dC_dhT. \end{aligned} \tag{4.10}$$

Finally, let us estimate I_4. Since $N(dt,dv)=\tilde{N}(dt,dv)+\pi(dv)dt$ and $\tilde{N}(dt,dv)$ is a martingale, we get

$$\begin{aligned} I_4 &\leq 2E\int_0^{\gamma_d\wedge T}\int_Z |h(z(t),v)-h(x(t-),v)|^2 N(dt,dv) \\ &= 2E\int_0^{\gamma_d\wedge T}\int_Z |h(z(t),v)-h(x(t-),v)|^2 \pi(dv)dt. \end{aligned}$$

By assumption 2.1 and lemma 4.1, we have

$$\begin{aligned} I_4 &\leq 2k_d\int_0^{\gamma_d\wedge T} E|z(t)-x(t-)|^2 dt \\ &\leq 4k_d\int_0^{\gamma_d\wedge T} E|y(t)-x(t-)|^2 dt + 4k_d C_d hT. \end{aligned} \tag{4.11}$$

Substituting (4.8), (4.9), (4.10) and (4.11) into (4.7), we obtain that

$$E\sup_{0\leq t\leq T\wedge\gamma_d}|y(t)-x(t)|^2 \leq C_{1d}\int_0^T \sup_{0\leq s\leq t\wedge\gamma_d} E|y(s)-x(s)|^2 ds + C_{2d}h,$$

where $C_{1d}=2(1+\pi(Z)+298k_d+8C^2k_d)$, $C_{2d}=2(298+8C^2)k_dC_dT$. The Gronwall inequality implies that

$$E\sup_{0\leq t\leq T}|y(t\wedge\gamma_d)-x(t\wedge\gamma_d)|^2 \;\leq\; C_{2d}e^{C_{1d}T}h.$$

The proof is therefore completed.

Now, we will show the convergence in probability of the approximate solution $y(t)$ to the true solution $x(t)$ for SDEs with jumps (2.1).

Theorem 4.1. Let conditions of lemma 4.2 and assumption 2.4 hold. Then the approximate solution $y(t)$ converges to the true solution $x(t)$ of equation (2.1) in the sense of probability. That is

$$\lim_{h\to 0}\sup_{0\leq t\leq T}|x(t)-y(t)|^2 = 0, \quad \text{in} \quad \text{probability}. \tag{4.12}$$

Proof. Step 1. By Theorem 2.1, we have

$$E|x(\alpha_d\wedge T)|^2 \leq C. \tag{4.13}$$

Noting that $|x(\alpha_d)|\geq d$, as $\alpha_d<T$, we derive from (4.13) that

$$d^2P(\alpha_d\leq T)\leq E|x(\alpha_d\wedge T)|^2\leq C.$$

That is

$$P(\alpha_d \leq T) \leq \frac{C}{d^2}.$$

Letting $d \to \infty$, it follows that $\frac{C}{d^2} \to 0$. Let $\frac{\varepsilon}{3} = \frac{C}{d^2} \in (0,1)$. Thus, there exists a sufficiently large d^* such that

$$P(\alpha_d < T) \leq \frac{\varepsilon}{3}, \qquad \forall\ d \geq d^*. \tag{4.14}$$

Step 2. We will give the estimate of $P(\beta_d < T)$. By the $It\hat{o}'s$ formula to $V(t, y(t)) = |y(t)|^2$, it follows that

$$\begin{aligned} dV(t,y(t)) &= V_x(t,y(t))f(z(t))dt + V_x(t,y(t))g(z(t))dw(t) \\ &\quad + \frac{1}{2}trace[g^\top(z(t))V_{xx}(t,y(t))g(z(t))]dt \\ &\quad + \int_Z [V(t,y(t)+h(z(t),v)) - V(t,y(t))]\pi(dv)dt \\ &\quad + \int_Z [V(t,y(t)+h(z(t),u)) - V(t,y(t))]\tilde{N}(dt,dv). \end{aligned}$$

From the operator LV in (2.2),we have

$$\begin{aligned} dV(t,y(t)) &= LV(t,y(t))dt + [V_x(t,y(t))f(z(t)) - V_x(t,y(t))f(y(t))]dt \\ &\quad + \frac{1}{2}trace[g^\top(z(t))V_{xx}(t,y(t))g(z(t))]dt \\ &\quad - \frac{1}{2}trace[g^\top(y(t))V_{xx}(t,y(t))g(y(t))]\}dt \\ &\quad + \int_Z [V(t,y(t)+h(z(t),v)) - V(t,y(t)+h(y(t),v))]\pi(dv)dt \\ &\quad + V_x(t,y(t))g(z(t))dw_t + \int_Z [V(t,y(t)+h(z(t),v)) - V(t,y(t))]\tilde{N}(dt,dv). \end{aligned}$$

Integrating from 0 to $\beta_d \wedge t$ and taking expectations gives,

$$\begin{aligned} E|y(\beta_d \wedge t)|^2 &\leq E|y(0)|^2 + E\int_0^{\beta_d \wedge t} LV(s,y(s)))ds \\ &\quad + 2E\int_0^{\beta_d \wedge t} |y(s)||f(z(s)) - f(y(s))|ds \\ &\quad + E\int_0^{\beta_d \wedge t} [|g(z(s))|^2 - |g(y(s))|^2]ds \\ &\quad + E\int_0^{\beta_d \wedge t} \int_Z [|y(s)+h(z(s),v)|^2 - |y(s)+h(y(s),v)|^2]\pi(dv)ds \\ &\leq E|y(0)|^2 + E\int_0^{\beta_d \wedge t} LV(s,y(s))ds + \sum_{i=1}^{3} J_i. \end{aligned} \tag{4.15}$$

Let us estimate J_1. By assumption 2.1, the Jensen inequality and lemma 4.1, we have

$$
\begin{aligned}
J_1 &\le 2d\sqrt{k_d}\int_0^{\beta_d\wedge t}(E|z(s)-y(s)|^2)^{\frac{1}{2}}ds \\
&\le 2d\sqrt{k_d}\int_0^{\beta_d\wedge t}(E\sup_{0\le\sigma\le s}|z(\sigma)-y(\sigma)|^2)^{\frac{1}{2}}ds \\
&\le 2d\sqrt{k_d}\sqrt{C_d}h^{\frac{1}{2}}T.
\end{aligned}
\tag{4.16}
$$

Rearranging J_2 by plus-and minus technique, we obtain that

$$
\begin{aligned}
J_2 &\le E\int_0^{\beta_d\wedge t}[|g(z(s))|^2-|g(z(s))g(y(s))|+|g(z(s))g(y(s))|-|g(y(s))|^2]ds \\
&\le E\int_0^{\beta_d\wedge t}[|g(z(s))||g(z(s))-g(y(s))|+|g(y(s))||g(z(s))-g(y(s))|]ds.
\end{aligned}
$$

Using the $H\ddot{o}lder$ inequality, we get

$$
\begin{aligned}
J_2 \le \int_0^{\beta_d\wedge t} & [(E|g(z(s))|^2)^{\frac{1}{2}}(E|g(z(s))-g(y(s))|^2)^{\frac{1}{2}} \\
& +[(E|g(y(s))|^2)^{\frac{1}{2}}(E|g(z(s))-g(y(s))|^2)^{\frac{1}{2}}]ds.
\end{aligned}
$$

By assumption 2.3, we get

$$
\begin{aligned}
J_2 \le \int_0^{\beta_d\wedge t} & [(k_2(1+E|z(s)|^{q_2+2}))^{\frac{1}{2}}(E|g(z(s))-g(y(s))|^2)^{\frac{1}{2}} \\
& +[(k_2(1+E|y(s)|^{q_2+2}))^{\frac{1}{2}}(E|g(z(s))-g(y(s))|^2)^{\frac{1}{2}}]ds.
\end{aligned}
$$

Recalling the elementary inequality

$$
(a+b)^p\le a^p+b^p,\quad \forall a,b\ge 0,\ 0<p\le 1,
$$

it follows that

$$
\begin{aligned}
J_2 \le\ & \sqrt{k_2}\int_0^{\beta_d\wedge t}[(1+E|z(s)|^{\frac{1}{2}(q_2+2)})(E|g(z(s))-g(y(s))|^2)^{\frac{1}{2}}]ds \\
& +\sqrt{k_2}\int_0^{\beta_d\wedge t}[(1+E|y(s)|^{\frac{1}{2}(q_2+2)})(E|g(z(s))-g(y(s))|^2)^{\frac{1}{2}}]ds.
\end{aligned}
$$

By assumption 2.1 and lemma 4.1, we get

$$
\begin{aligned}
J_2 &\le 2\sqrt{k_2}(1+d^{\frac{1}{2}(q_2+2)})\int_0^{\beta_d\wedge t}[(E|g(z(s))-g(y(s))|^2)^{\frac{1}{2}}]ds \\
&\le 2\sqrt{k_2}(1+d^{\frac{1}{2}(q_2+2)})\sqrt{k_d}\int_0^{\beta_d\wedge t}(E|z(s)-y(s)|^2)^{\frac{1}{2}}ds \\
&\le 2\sqrt{k_2}(1+d^{\frac{1}{2}(q_2+2)})\sqrt{k_d}\sqrt{C_d}h^{\frac{1}{2}}T.
\end{aligned}
\tag{4.17}
$$

Finally, we estimate J_3. Rearranging J_3 by plus-and minus technique again, we obtain that

$$\begin{aligned}J_3 &\leq E\int_0^{\beta_d\wedge t}\int_Z[|y(s)||h(z(s),v)-h(y(s),v)|]\pi(dv)ds\\&+E\int_0^{\beta_d\wedge t}\int_Z[|h(z(s),v)||h(z(s),v)-h(y(s),v)|]\pi(dv)ds\\&+E\int_0^{\beta_d\wedge t}\int_Z[|h(y(s),v)||h(z(s),v)-h(y(s),v)|]\pi(dv)ds.\end{aligned}$$

Using the *Hölder* inequality again, we get

$$\begin{aligned}J_3 &\leq d\sqrt{\pi(Z)}\int_0^{\beta_d\wedge t}[E\int_Z|h(z(s),v)-h(y(s),v)|^2\pi(dv)]^{\frac{1}{2}}ds\\&+\int_0^{\beta_d\wedge t}[(E\int_Z|h(z(s),v)|^2\pi(dv))^{\frac{1}{2}}(E\int_Z|h(z(s),v)-h(y(s),v)|^2\pi(dv))^{\frac{1}{2}}]ds\\&+\int_0^{\beta_d\wedge t}[(E\int_Z|h(y(s),v)|^2\pi(dv))^{\frac{1}{2}}(E\int_Z|h(z(s),v)-h(y(s),v)|^2\pi(dv))^{\frac{1}{2}}]ds.\end{aligned}$$

By assumption 2.1 and 2.3, it follows that

$$\begin{aligned}J_3 &\leq d\sqrt{\pi(Z)}\sqrt{k_d}\int_0^{\beta_d\wedge t}[E|z(s)-y(s)|^2]^{\frac{1}{2}}ds\\&+\sqrt{C_{\bar{h}}k_3}\sqrt{k_d}\int_0^{\beta_d\wedge t}[(1+E|z(s)|^{q_3+2})^{\frac{1}{2}}(E|z(s)-y(s)|^2)^{\frac{1}{2}}\\&+[(1+E|y(s)|^{q_3+2})^{\frac{1}{2}}(E|z(s)-y(s)|^2)^{\frac{1}{2}}]ds.\end{aligned}$$

By lemma 4.1, we have

$$\begin{aligned}J_3 &\leq d\sqrt{\pi(Z)}\sqrt{k_d}\int_0^{\beta_d\wedge t}[E|z(s)-y(s)|^2]^{\frac{1}{2}}ds\\&+\sqrt{C_{\bar{h}}k_3}\sqrt{k_d}\int_0^{\beta_d\wedge t}[(1+E|z(s)|^{\frac{1}{2}(q_3+2)})(E|z(s)-y(s)|^2)^{\frac{1}{2}}\\&+[(1+E|y(s)|^{\frac{1}{2}(q_3+2)})(E|z(s)-y(s)|^2)^{\frac{1}{2}}]ds\\&\leq [d\sqrt{\pi(Z)}\sqrt{k_d}+2\sqrt{C_{\bar{h}}k_3}\sqrt{k_d}(1+d^{\frac{1}{2}(q_3+2)})]\sqrt{C_d}h^{\frac{1}{2}}T.\end{aligned}\tag{4.18}$$

Inserting (4.16), (4.17) and (4.18) into (4.15), we obtain

$$E|y(\beta_d\wedge t)|^2 \leq E|y(0)|^2+C(d)Th^{\frac{1}{2}}+E\int_0^{\beta_d\wedge t}LV(s,y(s))ds.$$

Repeating the procedure from Theorem 2.1, we can prove that

$$E|y(\beta_d\wedge T)|^2\leq C+C(d)Th^{\frac{1}{2}}.\tag{4.19}$$

Since $|y(\beta_d)| \geq d$, as $\beta_d < T$, we derive from (4.19) that

$$\begin{aligned} C + C(d)Th^{\frac{1}{2}} &\geq E|y(\beta_d \wedge t)|^2 I_{\{\beta_d<T\}}(w)] \\ &\geq d^2 P(\beta_d \leq T). \end{aligned}$$

So we have

$$P(\beta_d \leq T) \leq \frac{C + C(d)Th^{\frac{1}{2}}}{d^2}. \tag{4.20}$$

Now, for any $\varepsilon \in (0,1)$, choose $d = d^*$ sufficiently large for $\frac{C}{d^{*2}} < \frac{\varepsilon}{6}$, and then choose h^* sufficiently small for$\frac{C(d)Th^{*\frac{1}{2}}}{d^{*2}} < \frac{\varepsilon}{6}$. It then follows from (4.20) that

$$P(\beta_d < T) \leq \frac{\varepsilon}{3}, \quad \forall \quad h \leq h^*, \tag{4.21}$$

as required.

Step 3. Let $\epsilon, \delta \in (0,1)$ be arbitrarily small, set

$$\bar{\Omega} = \{w : \sup_{0\leq t\leq T} |x(t) - y(t)|^2 \geq \delta\},$$

we have

$$\begin{aligned} P(\bar{\Omega}) &\leq P(\bar{\Omega} \cap \{\gamma_d > T\}) + P(\gamma_d < T) \\ &\leq P(\bar{\Omega} \cap \{\gamma_d > T\}) + P(\alpha_d < T) + P(\beta_d < T). \end{aligned}$$

By (4.14) and (4.21), we get

$$P(\bar{\Omega}) \leq P(\bar{\Omega} \cap \{\gamma_d > T\}) + \frac{2\varepsilon}{3}. \tag{4.22}$$

Using lemma 4.2, we have

$$\begin{aligned} \bar{C}_d h &\geq E[\sup_{0\leq t\leq T} |x(t) - y(t)|^2 I_{\gamma_d>T}(w)] \\ &\geq E[\sup_{0\leq t\leq T} |x(t) - y(t)|^2 I_{\gamma_d>T}(w) I_{\bar{\Omega}}(w)] \\ &\geq \delta P(\bar{\Omega} \cap \{\gamma_d > T\}). \end{aligned} \tag{4.23}$$

Inserting (4.23) into (4.22), we obtain that

$$P(\bar{\Omega}) \leq \frac{\bar{C}_d}{\delta} h + \frac{2\varepsilon}{3}.$$

Consequently, we can choose h sufficiently small for $\frac{\bar{C}_d h}{\delta} h < \frac{\varepsilon}{3}$ to obtain

$$P(\sup_{0 \le t \le T} |x(t) - y(t)|^2 \ge \delta) < \varepsilon.$$

The proof of Theorem 4.1 is now complete.

Remark 4.1 We note that the convergence results on numerical solution of [6],[9],[10],[30] are obtained under the Lipschitz and linear growth condition, while in this paper, we deal with the convergence of the approximate solution to the true solution of equation (2.1) under the local Lipschitz and nonlinear growth condition, so we generalize and improve the corresponding results of [6],[9],[10],[30].

5 An example

In this section, we construct one example to demonstrate the effectiveness of our theory.

Let $w(t)$ be a one-dimensional Brownian motion. $N(dt, dv)$ be a Poisson random measures and is given by $\pi(du)dt = \lambda f(v)dvdt$, where $\lambda = 2$ and

$$f(v) = \frac{1}{\sqrt{2\pi}v} e^{-\frac{(lnv)^2}{2}}, \qquad 0 \le v < \infty$$

is the density function of a lognormal random variable. Of course $w(t)$ and $N(dt, dv)$ are assumed to be independent.

Let us return to the nonlinear SDEs with jumps (1.1) with the coefficients f, g and h defined on page 2. Obviously, the coefficients f, g, h satisfy the local Lipschitz condition but they do not satisfy the linear growth condition. Through a straight computation, we have

$$x^\top f(x) + \frac{1}{2}|g(x)|^2 \le -2|x|^2 - 2|x|^4, \tag{5.1}$$

$$|x + h(x, v)|^2 \le \frac{5}{2}(1 + 0.1v)^2|x|^2 + \frac{5}{3}0.01v^2|x|^4, \tag{5.2}$$

where

$$\alpha_1 = 2,\ \alpha_1 = 2,\ \beta_1 = \frac{5}{2},\ \beta_2 = \frac{5}{3},\ \gamma_1 = 2,\ \gamma_2 = 2 \text{ and } \ \bar{h}(v) = (1 + 0.1v)^2.$$

So the inequalities (5.1) and (5.2) show that assumption 2.4 holds. Moreover, by the property of log-normal distribute $f(v)$, we can obtain that $\pi(Z) = 1$, and

$$\begin{aligned} C_{\bar{h}} = \int_Z \bar{h}(v)\pi(dv) &= 2\int_0^1 (1+0.1v)^2 \frac{1}{\sqrt{2\pi}v} e^{-\frac{(lnv)^2}{2}} dv \\ &\leq 1 + 0.4\sqrt{e} + 0.02e^2. \end{aligned}$$

Clearly, the above conditions imply that

$$\gamma_1 \geq \gamma_2,\ 2\alpha_2 - \beta_2 C_{\bar{h}} > 0 \text{ and } 2\alpha_1 - \beta_1 C_{\bar{h}} + \pi(Z) > 0.$$

Hence, by Theorem 3.1, 3.2, we have that the solution of equation (1.1) is asymptotically stable in the mean square sense and almost sure exponentially stable.

On the other hand, let us define the approximate solution of equation (1.1). Similar to (4.1), we get

$$y_{n+1} = y_n + (-2y_n - \frac{5}{2}y_n^3)h + y_n^2 \Delta w_n + 0.1\int_0^1 v(y_n + y_n^2)N(h, dv), \tag{5.3}$$

with initial value $y_0 = x_0$. Then, by the step function $z(s) = \sum_{n=1}^{\infty} y_n I_{[t_n, t_{n+1})}(s)$, we have the continuous Euler approximate solution $y(t)$ of equation (1.1)

$$\begin{aligned} y(t) &= y(0) + \int_0^t (-2z(s) - \frac{5}{2}z^3(s))ds + \int_0^t z^2(s)dw(s) \\ &\quad +0.1\int_0^t \int_0^1 v(z(s) + z^2(s))N(ds, dv). \end{aligned} \tag{5.4}$$

Note that conditions of Theorem 4.1 are satisfied, then Theorem 4.1 implies that the convergence in probability of numerical solution $y(t)$ and the true solution $x(t)$ to equation (1.1).

Conclusion

In this paper, we generalize and extend the existing theory of stability and convergence of numerical solution to the case of highly nonlinear SDEs with jumps. Under nonlinear growth conditions, we investigate the asymptotic stability in the pth moment and almost sure exponential stability of solutions to SDEs with jumps; Meantime, we obtain that the approximate solution converges in probability to the true solution of SDEs with jumps under the above mentioned conditions. Finally, an example is provided to demonstrate the effectiveness of the main results in this paper.

Acknowledgements

The authors would like to thank the referees for their valuable comments and suggestions. The authors would also like to thank the Royal Society of Edinburgh, the National Natural Science Foundation of China under NSFC grant (No.11401261), Qing Lan Project of Jiangsu Province (2012), the NSF of Higher Education Institutions of Jiangsu Province (13KJB110005), the grant of Jiangsu Second Normal University (JSNU-ZY-02) and the Jiangsu Government Overseas Study Scholarship for their financial support.

References

[1] D. Applebaum, Lévy Processes and Stochastic Calculus, Cambridge University Press, 2004.

[2] D. Applebaum, M. Siakalli, Asymptotic stability of stochastic differential equations driven by Lévy noise, J. Appl. Probab, 46 (2009), 1116-1129.

[3] D. Applebaum, M. Siakalli, Stochastic stabilization of dynamical systems using Lévy noise, Stoch. Dyn, 10 (2010), 509-527.

[4] C. Boulanger, Stabilization of a class of nonlinear stochastic systems, Nonlinear Anal. 41 (2000), 277-286.

[5] D. Bates, Jumps and stochastic volatility: exchange rate processes implicit in deutsche mark options, Rev. Financ. Studies, 9 (1996), 96-107.

[6] N. Bruti-Liberati, E. Platen, Strong approximations of stochastic differential equations with jumps, J. Comput. Appl. Math, 205 (2007), 982-1001.

[7] G. Chalmers, D. Higham Asymptotic stability of a jump-diffusion equation and its numerical approximation, SIAM J. Sci. Comp, 31 (2008), 1141-1155.

[8] A. Gardon, The order of approximations for solutions of Ito-type stochastic differential equations with jumps, Stoch. Anal. Appl, 22 (2004), 679-699.

[9] D. Higham, P. E. Kloeden, Numerical methods for nonlinear stochastic differential equations with jumps, Numer. Math, 110 (2005), 101-119.

[10] D. Higham, P. E. Kloeden, Convergence and stability of implicit methods for jump-diffusion systems, Internat. J. Numer. Anal. Modeling, 3 (2006), 125-140.

[11] M. Hutzenthaler, A. Jentzen and P. E. Kloeden, Strong and weak divergence in finite time of Euler's method for SDEs with non-globally Lipschitz coefficients, Proceedings of the Royal Society A, 467 (2011), 1563-1576.

[12] P. Jorion, On jump processes in the foreign exchange and stock markets, Rev. Financ. Studies, 1 (1988), 427-445.

[13] C. Kumar and S. Sabanis, Strong Convergence of Euler Approximations of Stochastic Differential Equations with Delay Under Local Lipschitz Condition, Stoch. Anal. Appl, 32 (2014), 207-228.

[14] R. Z. Khasminskii, Stochastic Stability of Differential Equations, Sijthoff and Noordhoof, 1980.

[15] V. Kolmanovskii and L. Shaikhet, Construction of Lyapunov functionals for stochastic hereditary systems: A survey of some recent results, Math. Computer Model, 36 (2002), 691-716.

[16] C. Li, Z. Dong, S. Rong, Almost sure stability of linear stochastic differential equations with jumps, Probability Theory and Related Fields, 123 (2002), 121-155.

[17] L. Liu, Y. Shen, Noise suppresses explosive solutions of differential systems with coefficients satisfying the polynomial growth condition, Automatica, 48 (2012), 619-624.

[18] R. Liptser, A. N. Shiryaev, Theory of Martingale, Kluwer Academic Publishers, Dordrecht, 1989.

[19] X. Liu, C. Li, Weak approximations and extrapolations of stochastic differential equations with jumps, SIAM J. Numer. Anal, 37 (2004), 1747-1767.

[20] X. Li, X. Mao, Y. Shen, Approximate solutions of stochastic differential delay equations with Markovian switching, J. Difference Equ. Appl, 16 (2010), 195-207.

[21] G. Marion, X. Mao and E. Renshaw, Convergence of the Euler shceme for a class of stochastic differential equations, International Mathematical Journal, 1 (2002), 9-22.

[22] M. Milosevic, Highly nonlinear neutral stochastic differential equations with time-dependent delay and the Euler-Maruyama method, Math. Comput. Modelling, 54 (2011), 2235-2251.

[23] X. Mao, LaSalle-type theorems for stochastic differential delay equations, J. Math. Anal. Appl, 236 (1999), 350-369.

[24] X. Mao, A note on the LaSalle-type theorems for stochastic differential delay equations, J. Math. Anal. Appl, 268 (2002), 125-142.

[25] X. Mao, Stochastic Differential Equations and their Applications, Horwood, Chichester, 2008.

[26] X. Mao and M. Rassias, Khasminskii-type theorems for stochastic differential delay equations, Stoch. Anal. Appl, 23 (2005), 1045-1069.

[27] X. Mao, L. Szpruch, Strong convergence rates for backward Euler-Maruyama method for non-linear dissipative-type stochastic differential equations with super-linear diffusion coefficients, Stochastics An International Journal of Probability and Stochastic Processes, 85 (2013), 144-171.

[28] X. Mao, Y. Shen and C. Yuan, Almost surely asymptotic stability of neutral stochastic differential delay equations with Markovian switching, Stoch. Proc. Appl, 118 (2008), 1385-1406.

[29] X. Mao, C. Yuan, Stochastic Differential Equations with Markovian Switching, Imperial College Press, London, UK, 2006.

[30] E. Platen and N. Bruti-Liberati, Numerical Solution of Stochastic Differential Equations with Jumps in Finance, Springer, Berlin, 2010.

[31] S. T. Rong, Theory of Stochastic Differential Equations with Jumps and Applications, Springer, Berlin, 2005.

[32] L. Szpruch, X.Mao, D. Higham and J. Pan, Numerical simulation of a strongly nonlinear Ait-Sahalia type interest rate model, BIT Numerical Mathematics, 51 (2011), 405-425.

[33] S. Sabanis, Euler approximations with varying coefficients: the case of super-linearly growing diffusion coefficients, arXiv:1308.1796 [math.PR], (2013).

[34] F. Wu and S. Hu, A study of a class of nonlinear stochastic delay differential equations, Stochastics and Dynamics, 10 (2010), 97-118.

[35] F. Wu, S. Hu, Suppression and stabilisation of noise, Internat. J. Control, 82 (2009), 2150-2157.

[36] F. Wu and S. Hu, Stochastic suppression and stabilization of delay differential systems, Int. J. Robust. Nonlinear Control, 21 (2011), 488-500.

[37] C. Yuan, W. Glover, Approximate solutions of stochastic differential delay equations with Markovian switching, J. Comput. Appl. Math, 194 (2006), 207-226.

[38] G. Yin, F. Xi, Stability of regime-switching jump diffusions, SIAM J. Control and Optimization, 48 (2010), 4525-4549.

[39] Z. Yang, G. Yin, Stability of nonlinear regime-switching jump diffusion, Nonlinear Anal. Theory Methods Appl, 75 (2012), 3854-3873.

[40] Q. Zhu, Asymptotic stability in the pth moment for stochastic differential equations with Lévy noise, J. Math. Anal. Appl, 416 (2014), 126-142.