

# The role of momentum and velocity for radiating electrons

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Radiation reaction remains one of the most fascinating open questions in electrodynamics. The development of multi-petawatt laser facilities capable of reaching extreme intensities has lent this topic a new urgency, and it is now more important than ever to properly understand it. Two models of radiation reaction, due to Landau and Lifshitz and to Sokolov, have gained prominence, but there has been little work exploring the relation between the two. We show that in the Sokolov theory electromagnetic fields induce a Lorentz transformation between momentum and velocity, which eliminates some of the counterintuitive results of Landau-Lifshitz. In particular, the Lorentz boost in a constant electric field causes the particle to lose electrostatic potential energy more rapidly than it otherwise would, explaining the long-standing mystery of how an electron can radiate while experience no radiation reaction force. These ideas are illustrated in examples of relevance to astrophysics and laser-particle interactions, where radiation reaction effects are particularly prominent.

## I. INTRODUCTION

Radiation reaction (RR)—how a charged particle interacts with the radiation it emits—is among the oldest and most controversial open questions in physics. In the century since its first formulation by Lorentz and Abraham [1, 2], there have been many theoretical investigations, but so far laboratory-based electromagnetic fields have not been sufficiently intense to produce an appreciable RR effect. A number of astrophysical scenarios do exist for which RR is important [3–5], but the impossibility of controlling the conditions of these events means a clear signature of RR has yet to be detected. A new generation of laser facilities, such as the Extreme Light Infrastructure (ELI), are anticipated to produce field intensities on the order of  $10^{23}$  W/cm<sup>2</sup>, where RR will not just be significant but will dominate the electron dynamics [6]. A full understanding of RR is vital to the success of these next generation laser facilities [7].

The first fully relativistic treatment of RR was given by Dirac [8] on the basis of energy-momentum conservation. This led to a third order differential equation, known as the Lorentz-Abraham-Dirac (LAD) equation, for the worldline of a radiating point charge. This equation has subsequently been rederived from a number of quite distinct physical principles [9–13]. Unfortunately, it suffers from some well-known anomalies, which render it unphysical; see [14] for a recent discussion. A number of alternatives have been proposed [15–20], among which two have received particular prominence in the literature: those of Landau and Lifshitz (LL) [15] and of Sokolov [20].

Landau and Lifshitz derived their equation by assuming that the RR force in LAD is a small correction to the Lorentz force of the applied fields. This allows the elimination of the third derivative terms, leading to the second order equation

$$\ddot{x}^a = \frac{e}{m} (F^a{}_b + \tau \dot{x}^c \partial_c F^a{}_b) \dot{x}^b + \tau \frac{e^2}{m^2} \Delta^a{}_b F^b{}_c F^c{}_d \dot{x}^d. \quad (1)$$

Here,  $e$  is the charge and  $m$  the mass of the particle,  $\tau = e^2/6\pi m \simeq 6.2 \times 10^{-24}$  s is the characteristic radiation time,  $F^a{}_b$  are components of the electromagnetic field, and  $\Delta^a{}_b = \delta^a_b - \dot{x}^a \dot{x}_b$  is the  $\dot{x}$ -orthogonal projection. Indices are raised and lowered with the metric tensor  $\eta_{ab} = \text{diag}(1, -1, -1, -1)$ , the Einstein summation convention is used throughout and  $c = \epsilon_0 = 1$ .

The RR force in (1) does not share the defects of LAD. In the decades since its introduction it has become the dominant description of radiation reaction, and has been applied to electron dynamics [21–28], ion acceleration [29–31], and high-energy synchrotron radiation [32]. However, its provenance as an approximation to an unphysical equation questions its validity. Furthermore, it does not conserve energy in both rapidly varying [33] and constant [34] fields. We show here that these anomalies can be removed by a simple and physically motivated redefinition of momentum.

An alternative description of RR using such a redefinition has recently been introduced by Sokolov [20]. This is derived according to principles arising from quantum electrodynamics (QED), and has the unusual feature that the momentum  $p$  is not parallel to the velocity  $\dot{x}$ . It is convenient to introduce the normalized momentum  $u = m^{-1}p$ , in terms of which Sokolov’s equations are

$$\dot{x}^a = \left( \delta^a_b + \tau \frac{e}{m} F^a{}_b \right) u^b, \quad (2)$$

$$\dot{u}^a = \frac{e}{m} F^a{}_b u^b + \tau \frac{e^2}{m^2} \mathcal{U}^a{}_b F^b{}_c F^c{}_d u^d. \quad (3)$$

Here, the tensor  $\mathcal{U}^a{}_b = \delta^a_b - u^a u_b$  projects out the components parallel to  $u$ , not  $\dot{x}$  (such a distinction is not required for LL). As such, (3) preserves the normalization of momentum  $u^2 = 1$ , equivalent to the Einstein relation  $E^2 = m^2 c^4 + \mathbf{p}^2 c^2$ . Though not yet as widespread as LL, the Sokolov model has attracted attention in recent years, with application to ion acceleration [35, 36] and generation of high energy synchrotron radiation [37–41]. Despite the emergence of two distinct theories, there has been surprisingly little discussion of the relation between

LL and Sokolov.

It is worth noting that classical physics is expected to emerge as the appropriate limit of an underlying quantum theory. Given the success of QED, some authors have explored whether its classical limit can shed light on the question of the correct theory of radiation reaction [42–45]. In [46], the QED predictions for an electron interacting with a plane electromagnetic wave have been compared with those of a range of classical theories, including LL and Sokolov. While agreement is found with the former, the latter is found to disagree with the QED result for momentum, but to agree on velocity, and hence on trajectory. However, there are subtleties involved in extracting the *kinetic particle momentum* from the total momentum operator, and it is possible that the lesson from this discrepancy is not that the Sokolov theory must be rejected, but rather that the definition of electron momentum in QED should be reconsidered.

The paper is organised as follows. Section II presents the relation between the particle momentum and its velocity for both the LL model and the Sokolov model. Section III considers the total energy of an electron experiencing a purely electrostatic field, where it is found that Sokolov, unlike LL, predicts behaviour consistent with an intuitive physical understanding. Section IV treats two examples of simple but physically relevant field-particle configurations modelling an electron in the magnetic field of a neutron star and in an intense laser pulse. The results of both models are compared and discussed.

## II. MOMENTUM AND VELOCITY

The LL and Sokolov theories agree to a very high precision in their predictions for the trajectory of an electron. Indeed, substituting (3) into the derivative of (2) yields precisely (1) with corrections of order  $O(\tau^2)$ . It follows that the distinction between the theories is less one of the motion of particles, and more of the evolution of their momenta.

As a particle moves through spacetime, it traces out a worldline, and its velocity  $\dot{x}$  is the tangent vector to this worldline. The direction of this vector is intrinsic to the worldline, while its normalization,  $|\dot{x}| = \sqrt{\dot{x}^2}$ , depends on the choice of parameterization. It is common to use proper time, defined by the condition  $|\dot{x}| = 1$ , though it is important to recognize that this is a choice. Momentum, on the other hand, describes the flow of energy through spacetime. For a noninteracting particle, its energy must flow along the worldline, but this case is of little interest. When the particle is allowed to interact, it can exchange energy and momentum with its environment, and a choice must then be made as to how to divide the energy-momentum between particle and environment. To an extent, this partition is arbitrary, and its ‘correctness’ should be judged by how the resultant

change in the particle’s momentum matches our expectations for particle-like behavior.

The primary difference between LL and Sokolov derives from this partition of electromagnetic momentum. The particle’s Coulomb field contains an infinite self-energy which in both theories is absorbed into a mass renormalization. But since the energy-momentum is quadratic in the fields, there is an additional contribution when the Coulomb field is superposed on a background field. Unlike LL, Sokolov interprets this also as contributing to the particle’s momentum, which is therefore not parallel to the particle’s velocity.

While it might appear unusual to have momentum and velocity aligned along different directions, this occurs in several other contexts. For example, spinning particles in gravitational [47] and electromagnetic fields [48] acquire a contribution to their momenta which is not parallel to velocity, while the canonical momentum of a charged particle generally does not even have a uniquely defined direction. Indeed, in some of the more ‘natural’ derivations of LAD [9, 49, 50], the troublesome Schott term (the derivative of acceleration) arises from taking the momentum to be  $p = m(\dot{x} - \tau\ddot{x})$ , which to a good approximation agrees with (2). In a recent derivation [51] of the classical radiation reaction by the integration of electromagnetic momentum, it was shown that the Schott term arises from the *bound field momentum*, in agreement with the calculation of Dirac [8]. Unlike the *emitted field momentum*, this bound momentum cannot escape to infinity, so it is reasonable to treat it as part of the particle’s momentum. This approach is consistent with the theory of Sokolov where it corresponds to the second term in the RHS of Eq. (2).

In theories for which  $p = m\dot{x}$ , the Einstein relation  $p^2 = m^2$  is a direct consequence of parameterizing the worldline by proper time. In the Sokolov theory, rather than being equivalent these relations are incompatible. Contraction of (3) with  $u$  indicates that its norm is preserved, and we can consistently set  $u^2 = 1$ . Squaring (2) then yields:

$$\dot{x}^2 = 1 - \left(\frac{2}{3}\alpha\chi\right)^2, \quad (4)$$

where  $\alpha \simeq 1/137$  is the fine structure constant and  $\chi = \frac{e\hbar}{m^2} \sqrt{F^a{}_b F^b{}_c u_a u^c}$  is the electric field in the zero (spatial) momentum frame in units of the Sauter-Schwinger field [52, 53].

The result (4) indicates that the worldline parameter is not strictly proper time. Rather, it is the time measured in the Lorentz frame in which the particle instantaneously has vanishing spatial momentum. However, the difference between this and true proper time is appreciable only when  $\chi \gtrsim 50$ , in which case quantum effects should be sufficiently important to invalidate the notion of a classical worldline. We therefore take (4) to imply  $\dot{x}^2 \simeq 1$  and interpret the time parameter as *effective proper time*.

To understand the relation between momentum and velocity in Sokolov's model, it is convenient to introduce the matrix

$$\Lambda^a{}_b = \delta_b^a + \tau \frac{e}{m} F^a{}_b \quad (5)$$

in terms of which (2) is  $\dot{x}^a = \Lambda^a{}_b u^b$ . The product  $\Lambda \Lambda^T$  yields

$$\Lambda^a{}_b \Lambda_c{}^b = \delta_c^a - \tau^2 \frac{e^2}{m^2} F^a{}_b F^b{}_c. \quad (6)$$

If the RHS of (6) were  $\delta_b^a$ ,  $\Lambda$  would be an element of the group  $SO(3,1)$ , implying that the emission of radiation induces a Lorentz transformation between velocity  $\dot{x}$  and normalized momentum  $u$ . While this does not hold exactly, even for the ultra-strong magnetic fields  $B \approx 10^{10}$  T surrounding some neutron stars, its violation is  $\sim 10^{-4}$ , while for lasers of intensity  $I \approx 10^{22}$  W/cm<sup>2</sup>, the strongest fields currently produced in the laboratory, it is  $\sim 10^{-12}$ . We thus interpret  $\Lambda$  as an *effective Lorentz transformation* between velocity and momentum.

The electromagnetic tensor  $F$  has electric fields for its spatio-temporal components and magnetic fields for its purely spatial components. It therefore follows from (2) that an electric field relates the particle's velocity to its momentum via a Lorentz boost, while a magnetic field does so via a spatial rotation. In particular, an electron with vanishing spatial momentum is unmoving in a pure magnetic field, while an electric field will imbue it with a nonzero velocity. This is illustrated in figure 1.

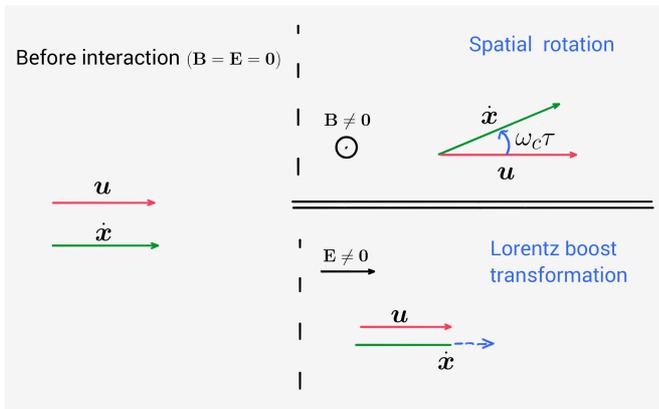


Figure 1: (Color online) Schematic showing how the electromagnetic tensor components change the relation between the particle's momentum and its velocity.

### III. ENERGY CONSIDERATIONS

So far, we have been considering only the kinetic 4-momentum. However, in a purely electrostatic field,

$$F_{ab} = \partial_a \varphi \eta_b - \partial_b \varphi \eta_a \quad \text{with} \quad \eta^a \partial_a \varphi = 0, \quad (7)$$

it is of interest also to consider the *total energy*  $\mathcal{E} = m(\eta \cdot u) + e\varphi$ , where  $\eta$  is the 4-velocity of the laboratory frame, which picks out the time-component of the *canonical 4-momentum*  $mu + eA$  in that frame while annihilating the spatial components. For the electrostatic field (7) we choose the 4-potential  $A = \varphi\eta$ .

In the absence of radiation reaction (the limit  $\tau \rightarrow 0$ ), the particle's total energy is conserved,  $\dot{\mathcal{E}} = m(\eta \cdot \dot{u}) + e\dot{x} \cdot \partial\varphi = 0$ . (In more general field configurations this is not the case, hence the restriction in the present Section to electrostatic fields.) Intuitively, since radiation carries away energy, RR should cause the total energy of the particle to decrease,  $\dot{\mathcal{E}} < 0$ . The radiation emitted is greatest when the particle's acceleration is orthogonal to its 3-velocity, so we expect  $-\dot{\mathcal{E}}$  to be maximized when  $u \cdot \partial\varphi \approx 0$ . And since the radiation emitted is proportional to  $e^2$ , we do not expect  $\dot{\mathcal{E}}$  to depend on the sign of the charge. Let us see if these properties are respected by the theoretical models.

Substituting (7) into (1) and contracting with  $\eta$  yields for the rate of change of total energy according to LL,

$$\dot{\mathcal{E}} = \underbrace{-\tau e(u \cdot \partial)^2 \varphi}_{(a)} + \tau \frac{e^2}{m} \left[ \underbrace{(u \cdot \partial\varphi)^2}_{(b)} + \underbrace{\{(\eta \cdot u)^2 - 1\}}_{(c)} \partial\varphi^2 \right] \eta \cdot u. \quad (8)$$

Two differences occur in Sokolov's theory: the term involving the derivative of the fields is not present in (3), eliminating (a), and the field-dependent term in (2) contributes to  $\dot{\varphi}$ , canceling with the  $-\partial\varphi^2$  contribution to (c). Hence the rate of change of energy according to Sokolov becomes

$$\dot{\mathcal{E}} = \tau \frac{e^2}{m} \left[ \underbrace{(u \cdot \partial\varphi)^2}_{(b)} + \underbrace{(\eta \cdot u)^2}_{(c')} \partial\varphi^2 \right] \eta \cdot u. \quad (9)$$

Consider each term in turn:

- (a) appears in LL only and can contribute either positively or negatively, depending on the direction of the particle's momentum relative to both the electric field and its derivative. Moreover, it is linear in  $e$ , so if it is positive for an electron it will be negative for a positron, and vice versa.
- (b) has the same form in both LL and Sokolov, and leads to an *increase* in the particle's energy. It is maximized when the particle's momentum is directed along the polarization of the field, and vanishes when it is perpendicular.
- (c) appears in LL and in a slightly modified form in Sokolov. Since  $\partial\varphi$  is *spacelike*, this term leads to a *decrease* in the particle's energy, and it is insensitive to the direction of the particle's motion. In LL, this term is always large enough to compensate for the

gain of energy from (b), but not necessarily more than that. In Sokolov it is enhanced, so that  $\dot{\mathcal{E}} < 0$  provided only that  $\partial\varphi \neq 0$ .

The rate of change of energy according to Sokolov is fully in keeping with our expectations: it is always negative; it is highest when the particle's momentum is perpendicular to the field; and it does not depend on the sign of the particle's charge. None of these properties is shared by the LL analogue. However, under the conditions to maximize RR ( $u \cdot \partial\varphi = 0$ ,  $\eta \cdot u \gg 1$ ), the two predictions for  $\dot{\mathcal{E}}$  converge. Concerns over the interpretation of LL are therefore very much ones of principle rather than practical difficulties.

The benefits of the Sokolov model are clearly demonstrated in the long-standing problem of the hyperbolic motion of an electron accelerating in a constant electric field,  $\varphi = -E_0 z$ . The LAD equation for this case gives zero RR force,  $\dot{u}^a = \frac{e}{m} F^a_b u^b$ , and this result is inherited by both LL and Sokolov. This has caused significant confusion, leading some researchers to argue that a charge with constant proper acceleration should not radiate [54, 55]. While it has now been established that this is not the case, explanations of how this is compatible with energy conservation have not been compelling [34], relying on the electron's behaviour as it enters and leaves the constant field, rather than providing a local energy balance.

According to the LL result (8),  $\dot{\mathcal{E}} = 0$ , while (9) gives  $\dot{\mathcal{E}} = -\tau \frac{e^2}{m} E_0^2 \eta \cdot u$  for Sokolov, consistent with loss of energy to radiation. Although there is no radiation reaction force, *per se*, the effective Lorentz boost means the electron moves through the potential more rapidly than it would if it did not radiate, and thus converts potential energy into radiation. This situation was considered in [20], but there lacked the detail of the Lorentz boost which provides the physical mechanism for the energy exchange.

#### IV. EXAMPLES

In this section, we consider the motion of an electron in two cases which are simple enough to solve, yet capture the key physics in situations in which radiation reaction is most important.

##### A. Constant magnetic field

The strongest known magnetic fields are found around magnetars, and can exceed  $10^{10}$  T. In such fields, motion across the field lines is strongly suppressed, so to a good approximation the field can be taken as constant along an electron's orbit,

$$F^a_b = B(\epsilon^a \lambda_b - \lambda^a \epsilon_b). \quad (10)$$

$B$  is the constant strength of the magnetic field, directed along the vector  $\kappa$ , which together with  $\eta$ ,  $\epsilon$  and  $\lambda$  forms an orthonormal frame ( $\eta^2 = -\epsilon^2 = -\lambda^2 = -\kappa^2 = 1$ , with all other scalar products vanishing).

The LL equation has been studied in the field (10), yielding simple expressions for the momentum [19]. Defining the contractions  $u_\ell = (\epsilon + iu_\parallel) = (\eta - i\kappa) \cdot u$  and  $u_\perp = u_\epsilon + iu_\lambda = -(\epsilon + i\lambda) \cdot u$ , simple requirements of Lorentz invariance lead to

$$u_\ell = \sqrt{1 + |u_\perp|^2} \frac{u_{\ell 0}}{\sqrt{\epsilon_0^2 - u_{\parallel 0}^2}}, \quad (11)$$

where the subscript '0' denotes the value at time  $s = 0$ . LL gives the transverse momentum as

$$u_\perp = \frac{e^{i(\omega_c s + \theta)}}{\sqrt{\mathcal{A} e^{2\tau\omega_c^2 s} - 1}} \quad (12)$$

where  $\omega_c = eB/m$  is the cyclotron frequency,  $\mathcal{A} = 1 + |u_{\perp 0}|^{-2}$  and  $\theta$  is the angle between the initial transverse momentum and the  $\epsilon$  direction.

Since the field is constant, the derivative terms in (1) do not contribute, so the solutions (11)–(12) are equally valid in the Sokolov theory. Defining the analogous contractions  $\dot{x}_\ell = \gamma + i\dot{x}_\parallel = (\eta - i\kappa) \cdot \dot{x}$ ,  $\dot{x}_\perp = \dot{x}_\epsilon + i\dot{x}_\lambda = -(\epsilon + i\lambda) \cdot \dot{x}$ , it follows from (2)

$$\dot{x}_\ell = u_\ell, \quad \dot{x}_\perp = (1 + i\tau\omega_c)u_\perp \simeq e^{i\tau\omega_c}u_\perp. \quad (13)$$

As anticipated, velocity is related to momentum by a rotation around the direction of the magnetic field. Because the field is homogeneous, this discrepancy between the directions of momentum and velocity does not affect the rate at which the particle spirals inwards. Essentially, as the particle rotates in the magnetic field, its transverse momentum simply lags slightly behind its velocity.

##### B. Electromagnetic plane wave

The strongest fields available in the laboratory are those produced by high power lasers. By tightly focusing short laser pulses, present laser facilities can produce intensities  $\sim 10^{22}$  W/cm<sup>2</sup>, and it is anticipated that forthcoming facilities could exceed  $10^{23}$  W/cm<sup>2</sup>.

To simplify the analysis, we ignore the focusing and treat the laser pulse as a plane wave. However, by allowing an arbitrary longitudinal profile we can model the short duration. We therefore take the field as

$$\frac{e}{m} F^a_b = a_\epsilon(\phi) (\epsilon^a k_b - k^a \epsilon_b) + a_\lambda(\phi) (\lambda^a k_b - k^a \lambda_b), \quad (14)$$

where  $a_\epsilon$  ( $a_\lambda$ ) is a dimensionless measure of the electric field strength in the  $\epsilon$  ( $\lambda$ ) direction and  $k = \omega(\eta + \kappa)$  is the null wave 4-vector, with  $\omega$  the frequency of the pulse, and  $\phi = k \cdot x$ .

The Landau-Lifshitz equation in the field (14) has been studied extensively [56–60]. However, a number of conceptual and technical differences arise in the Sokolov theory.

Assume  $a_\epsilon = a$  and  $a_\lambda = 0$ , so the electric and magnetic fields are oriented in the  $\epsilon$  and  $\lambda$  directions, respectively (this can always be achieved at a given  $\phi$  by rotating  $\epsilon$  and  $\lambda$ ). Then (2) yields

$$\begin{aligned} \gamma &= \varepsilon + \tau\omega a u_\epsilon, & \dot{x}_\epsilon &= u_\epsilon + \tau\omega a \varepsilon - \tau\omega a u_\parallel, \\ \dot{x}_\parallel &= u_\parallel + \tau\omega a u_\epsilon, & \dot{x}_\lambda &= u_\lambda. \end{aligned} \quad (15)$$

Again, the magnetic field induces a rotation between momentum and velocity, while the electric field causes an effective Lorentz boost, the plane wave assumption (14) ensuring these are of equal magnitude  $\tau\omega a$ . These results hold regardless of the polarization in the  $\epsilon$ – $\lambda$  plane.

Since the field components  $a_\epsilon$  and  $a_\lambda$  depend on the coordinate  $\phi$ , the Sokolov equation (3) for momentum differs through the field derivative terms from LL, so solutions to the latter cannot be imported from the literature as they were for the constant magnetic field. Nevertheless, we can follow the approach in [56], changing the independent variable from proper time  $s$  to phase  $\phi$ . This is possible since the field drops out of the phase derivative:

$$\dot{\phi} = k \cdot \dot{x} = \omega(\gamma - \dot{x}_\parallel) = \omega(\varepsilon - u_\parallel). \quad (16)$$

Substituting (14) in (3) and contracting with  $k$  then yields

$$\dot{\phi} = \Omega(\phi) = \frac{\Omega_0}{1 + \tau\Omega_0 \int_{\phi_0}^{\phi} [a_\epsilon^2(\phi') + a_\lambda^2(\phi')] d\phi'}, \quad (17)$$

where  $\Omega(\phi)$  is the instantaneous frequency as measured by the particle (i.e.  $\Omega/\omega$  is the Doppler factor). Note that (17) is valid for LL as well as Sokolov, and moreover  $\dot{\Omega} < 0$ , so the frequency observed by the particle *decreases* as it traverses the pulse.

Using (17) and defining the reduced momentum  $\tilde{u} = u/\Omega$ , we can now rewrite (2)–(3) as derivatives with respect to  $\phi$ :

$$\frac{dx^a}{d\phi} = \Lambda^a_b \tilde{u}^b, \quad \frac{d\tilde{u}^a}{d\phi} = \Omega^{-1} \Lambda^a_b \frac{e}{m} F^b_c \tilde{u}^c. \quad (18)$$

Not only does RR Lorentz transform the velocity relative to the reduced momentum, but the same Lorentz transformation relates the effective total force to the Lorentz force. While the former is a universal effect, the latter is a consequence of the specific field configuration (14).

Since (18) is linear in  $\tilde{u}$ , its solution is simply obtained by exponentiating the integral of the matrix multiplying it on the RHS. Moreover, since  $k$  is null, the exponentiation terminates at second order, allowing us to write the

solution in the compact form

$$\tilde{u}^a = \left[ \delta_b^a + (I^a k_b - k^a I_b) + \frac{1}{2} \left( \frac{1}{\Omega^2} - \frac{1}{\Omega_0^2} - I^2 \right) k^a k_b \right] \tilde{u}_0^b, \quad (19)$$

where we have introduced the vector

$$I^a(\phi) = \int_{\phi_0}^{\phi} \frac{a_\epsilon(\phi') \epsilon^a + a_\lambda(\phi') \lambda^a}{\Omega(\phi')} d\phi'. \quad (20)$$

The solution (19) differs from that found for LL [56] only by the terms in  $I^a$  arising from the field derivatives. From (19) and (17) we readily obtain the particle's momentum,  $u^a = \Omega \tilde{u}^a$ , and position,  $x^a = x_0^a + \int \Lambda^a_b \tilde{u}^b d\phi$ .

## V. DISCUSSION AND CONCLUSION

In conclusion, recent rapid advances in laser technology have promoted the long-standing issue of radiation reaction from an intellectual curiosity to a problem that urgently needs clarification. Of the many models proposed to describe radiation reaction, those of Landau and Lifshitz and of Sokolov have attained particular prominence. While the predictions they make for the motion of a radiating charge are consistent with each other, they differ on the evolution of its energy-momentum. This suggests the distinction is not a question of one theory being right and the other wrong, but rather how to interpret the different momenta of the two theories.

In the Sokolov theory, normalized momentum is related to velocity by a Lorentz transformation, with electric fields generating boosts and magnetic fields inducing spatial rotations. With this notion of momentum, a particle in an electrostatic field necessarily loses energy as a consequence of radiation emission, which is not the case for the Landau-Lifshitz momentum, which is directed along the velocity.

It is worth noting that, in addition to a more satisfactory interpretation of momentum, the Sokolov theory has a distinct numerical advantage, as there is no need to calculate derivatives of the electromagnetic field. In [61], a number of classical radiation reaction theories were considered, with Sokolov among the most computationally efficient, with less than half the overhead required for Landau-Lifshitz.

As a final remark, we note that there is currently considerable activity in the recoil of a massive body to the emission of gravitational radiation [62], stimulated by the prospect of detecting gravitational waves. While the focus of the present paper has been purely on electromagnetic radiation reaction, there has been substantial cross-fertilization of ideas across the two fields [63]. The perspective offered here may therefore also be of relevance to gravitational radiation reaction.

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