Expected utility theory for monitoring-based decision making

Carlo Cappello, Daniele Zonta, Branko Glišić

Abstract—**The main purpose of structural health monitoring (SHM) is to obtain information about the state of a structure, in order to guide bridge management decisions. Nevertheless, in practice, once a rigorous estimate of the structural state is available, decisions are usually made based on the decision maker's intuition or experience. In this paper, we present the implementation of expected utility theory (EUT) in those civil engineering decision problems in which decision makers have to act based on the output of SHM. EUT is an analytical quantitative framework that allows the identification of the financially most convenient decisions, based on the possible outcomes of each action and on the probabilities of each structural state occurring. The advantage of the presented implementation is the optimization of decision strategies in SHM. In the manuscript, we first formalize the solution of single-stage decision processes, in which the decision maker has to take only one action. Then, we formalize the solution of multi-stage decision processes, in which multiple actions may be taken over time. Finally, using an example based on a case study, we describe the variables involved in the analysis of SHM decision problems, discuss the possible results and address the issues that may arise in the application of EUT in real-life settings.**

Index Terms—**Structural health monitoring, smart structures, Bayesian analysis, decision-making, expected utility theory, bridge management, decision support systems.**

I. Introduction

There is a general agreement among researchers and practitioners on the claim that structural health monitoring (SHM), should help in making proper decisions about structural management; yet, at present, it is not fully clear how in practice a responsible manager is supposed to make decisions based on SHM data.

A typical approach to SHM-based decision making is the following: (1) sensors and data acquisition devices are installed on the structure, say a bridge; (2) the data coming from the monitoring system are analyzed; (3) an algorithm recognizes possible damage, identifies its location, and provides information on structural health and performance; (4) the responsible manager makes the decision on whether or not to repair the damage. In other words, decision-making is usually regarded as a process that occurs once the manager has understood the state of the structure based on SHM. The growing success in our community of keywords like *smart structure*, *intelligent monitoring* and *decision support systems*, indirectly supports the idea that decision-making is a mere output of SHM.

This optimistic view of SHM is in contrast with what appears to happen in real life: owners are typically very skeptical about the capacity of monitoring to support decisions and managers usually act based on their experience or on common sense, often disregarding the actions implicitly suggested by SHM [1].

In fact, SHM and decision making, although connected, are two separate processes, which should not be confused. Monitoring is about acquiring information, not about making decisions. The aim of monitoring is to understand the state of the structure based on the acquisition and interpretation of data, usually provided by sensors. A good SHM system correctly identifies the condition state of the structure. Conversely, decision-making is about choosing the optimal action to undertake. It entails the evaluation of the actual consequences of each action, consequences that depend on financial costs, social impact and management policies, which are concepts going beyond the mere structural engineering problem.

The bases of formal decision-making have been studied for a long time: axiomatic expected utility theory (EUT) was first introduced by von Neumann and Morgenstern in 1944 [2], and later developed in the form that we currently know by Raiffa and Schlaifer in 1961 [3]. According to EUT, each action is ranked based on its expected utility, which depends on both the consequences and the probabilities of each possible scenario [4]. Therefore, the rational decision-maker regards as optimal, and chooses, the action with the highest expected utility. In principle, EUT allows evaluation of any action, provided that the consequences and the probabilities involved can be quantified. This allows the modeling of a decision process even *a priori*, i.e. before the information on the current state of the structure is available.

Traditionally, optimal decision-making has been developed to guide decisions in the field of medicine and finance, in which the probabilities of different scenarios and consequences can be easily identified. There is nonetheless also a strong tradition in civil engineering. A recent overview is offered by Faber and Maes [5], where the authors point out a number of open issues arising when optimal decision-

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making has to be implemented in real-life settings.

One issue is the correct modelling of the problem uncertainties. These are traditionally distinguished between *aleatory*, when inherent to the nature of the phenomenon (e.g., a sensor noise), and *epistemic*, when related to our inability to investigate the problem completely (e.g., the incompleteness of a simplified structural model) [6].

Another problem stems from the observation that the concept of utility is inherently subjective: thus an objective approach to decision-making requires the identification of a 'stakeholder' (i.e. the beneficiary of the decision) and his/her decision principle. For civil engineering applications, Faber and Maes [5] identify the stakeholder as society, and propose models to reproduce its needs. In sustainable decision-making, the consequences for society, environment and economy are simultaneously accounted for through life or environmental quality indices [5]. Intergenerational effects of the consequences, or follow-up effects [7], can be also considered. Moreover, Maes and Faber [8] proposed a model for finding an optimal decision that applies when decisions are taken by groups, rather than individuals, as in the case of important structures and infrastructure.

Despite this lively interest in decision theory in civil engineering, the general impression is that the basic principles of EUT are still often disregarded by the SHM community, even in works that nominally focus on SHM-based decisions. Notable exceptions are the works by Flynn and Todd [9,10] and Flynn *et al.* [11]: in these publications, the *Bayes risk*, a metric strictly related to the EUT principles, is used to derive a global criterion for optimal sensor placement and the concept is demonstrated with laboratory experiments.

Even more curiously, the recent focus of the SHM community on EUT does not address the use of monitoring data, but rather the problem of whether or not to install a monitoring system. Since 2010, a number of papers have investigated the nature of the benefit of SHM, [1,12–15], recognizing that SHM finds its practical justification in its capacity of influencing the actions of a decision maker. In these works, the benefit of monitoring is formally quantified through the concept of value of information (VoI), which has its background in the seminal work by Lindley [16], later formalized by Raiffa and Schlaifer [3] and finally by DeGroot [17] (see the literature review by [13]). The VoI of SHM is defined as the difference between the expected utility when the owner is not informed by the SHM and the expected utility upon receipt of the monitoring information. It represents the money saved by consulting SHM and can likewise be regarded as the maximum cost the owner is willing to pay for information. The VoI is useful to rank different SHM strategies, to compare the benefit of a given SHM system with its cost, or to compare different SHM solutions with alternatives such as repairing and rehabilitation [13].

In this paper, we wish to fill this apparent gap between SHM practice and formal decision theory by introducing a unified process, consistent with the EUT principles, whereby monitoring information is directly used to select the optimal action to undertake. Unlike previous works in which decision is based on heuristics, or EUT is only implicitly assumed, here we aim to establish a rigorous formal procedure based on monitoring observations that generally applies to any decision instance. In the next section, we formally state the problem of logical inference in SHM and we introduce the basic principles of EUT, including the elementary instruments used solve the problem of decision-making (decision problem). In Section III, we formalize single-stage decision problems, and we illustrate their solution with the aid of a real-life case study. In Section IV, we extend these findings to multi-stage decision problems, again with the aid of a real-life case study. Concluding remarks are presented at the end of the paper.

II. Problem Statement

We have already observed that SHM aims to infer the state of the structure based on sensor observations, whereas decision-making aims at making the optimal choice based on the probability of each structural state. Stated in very simple terms, the *SHM-based decision problem* can be summarized as follows:

given a set of observations from the monitoring system, what is the financially optimal action to undertake?

Before attacking this problem, in this section we introduce the basics that are necessary to formalize the solution, including Bayesian inference theory, as applied to SHM, and von Neumann and Morgenstern's decision theory.

A. Glossary

In this section, we formally define the concepts of *sensor*, *observation*, *state*, *model*, *action* and *outcome*, which we will use in the rest of the manuscript.

Sensor. In this paper, we refer to a sensor in logical terms as suggested by Hall and McMullen [18]: 'any device which functions as a source of information'. We assume that sensors are the source of each *observation*.

Observation. We assume that each piece of information provided directly by sensors is an observation. When we describe a process of data analysis that occurs at time t, we assume that all the observations collected at that moment and before are available and contained in a vector $\mathbf{y}(t)$. The space of the observations is indicated by $\Omega_{\mathbf{y}}$.

State. The state S is a parameter that represents the condition of the structure. Depending on the case, it can be specified by: a discrete variable S defined in the domain $\Omega_S = \{S_1,\ldots,S_i,\ldots,S_M\}$ (e.g., the structure can be S_1 = 'severely damaged, S_2 = 'moderately damaged' or S_3 = 'not damaged'); by a vector of continuous parameters $\boldsymbol{\theta}$ defined in the domain $\Omega_{\boldsymbol{\theta}}$ (e.g., a damage index, crack size); or by a combination of the two.

Model. This is the way we assume that the observations $\mathbf{y}$ are correlated to the state S. In structural engineering problems, the model typically has a physical background (e.g., a finite element model or an analytical model). Sometimes the model can be heuristic. In mathematical terms, it is a function $f(\dots)$ that provides a distribution of the observations $\mathbf{y} = f(S,\boldsymbol{\lambda})$, given the state S, other parameters $\boldsymbol{\lambda}$ (e.g., geometrical properties, parameters we estimated beforehand, constants), and the uncertainty that may affect the observations and the model itself. Depending on the problem, the uncertainty is a parameter included in either S or $\boldsymbol{\lambda}$.

Action. An action a indicates any of the possible options the decision maker can choose at a decision point, formally called the *decision node*. The set of possible actions is indicated by Ω_a, and may be in general either discrete (e.g., a_1 = 'do nothing', a_2 = 'repair', a_3 = 'retrofit', a_4 = 'replace') or continuous, the former being of interest in SHM-based decision making. Decision-making can be performed in several stages. The first stage happens after SHM has identified damage to the structure for the first time. Based on the action resulting from the decision, the state of the structure may be changed, and further decisions that have to be made based on future SHM results might include new actions that were not available in the first stage. Hence, the set of possible actions Ω_a can be divided into subsets $\Omega_a^{(i)}$, where i = 1, 2, 3, … denotes the stage in the whole decision-making process. The actions corresponding to the i-th stage are denoted by $a^{(i)}$.

Outcome. An outcome z (also known as *consequence* or *reward*) is the quantification of the consequences of an action. The nature of outcomes can be various and, in the case of a civil structure, may include financial losses or rewards to the owner, and impacts on society or on the environment due to structural failure. An action a leads to a given outcome based on the state S of the structure, therefore $z = z(a,S)$, i.e., the outcome z can be seen as a function of the chosen action a and the state S. The set of all the outcomes is denoted by Ω_z and the subset of outcomes resulting from the i-th decision stage is denoted by $\Omega_z^{(i)}$.

B. Structural Health Monitoring

From a logical standpoint, SHM can be considered as an inference problem where we have to gain information on the state S of a structure, based on observations $\mathbf{y}$ provided by the sensors installed on the structure [1,19,20]. In order to accomplish this goal we typically state, or understand, the relationship between observations and state, usually described by a mechanical or heuristic model. When the relationship between the state S and the observations $\mathbf{y}$ is set deterministically, we use propositional logic to determine the state (or to conclude that the state is indeterminate).

Usually, the relationship between the observations and the state includes uncertainties, which stem from the inherent uncertain nature of measurement (metrological uncertainty), from the approximations of the data analysis, or from the incompleteness of the model (model uncertainty). In this case, the description of the state is provided by the statistical distribution of the state variable S, rather than a determinate value, and inference is made using the well-known Bayes' theorem [21,22], rather than propositional logic. When a vector of continuous parameters $\boldsymbol{\theta}$ defines the state, Bayes' rule reads:

$$\mathrm{p}(\boldsymbol{\theta} \mid \mathbf{y}) = \frac{\mathrm{p}(\mathbf{y} \mid \boldsymbol{\theta}) \cdot \mathrm{p}(\boldsymbol{\theta})}{\mathrm{p}(\mathbf{y})}, \tag{1}$$

where $\mathrm{p}(\boldsymbol{\theta} \mid \mathbf{y})$ is the *posterior distribution* of the state parameters (i.e., after the acquisition of the SHM observations $\mathbf{y}$), $\mathrm{p}(\boldsymbol{\theta})$ is the *prior distribution* (i.e., before the acquisition of $\mathbf{y}$); $\mathrm{p}(\mathbf{y} \mid \boldsymbol{\theta})$ is the *likelihood function*, which encodes the probabilistic relationship between observations and state; and $\mathrm{p}(\mathbf{y})$ is a normalization constant, called *evidence,* calculated as:

$$\mathrm{p}(\mathbf{y}) = \int_{\Omega_{\boldsymbol{\theta}}} \mathrm{p}(\mathbf{y} \mid \boldsymbol{\theta}) \cdot \mathrm{p}(\boldsymbol{\theta}) \cdot \mathrm{d}\boldsymbol{\theta}. \tag{2}$$

The prior distribution reflects the initial knowledge of the individual in charge of the data interpretation, and is usually set based on the literature, design codes, or subjective belief and experience [22]. The observations $\mathbf{y}$ and the model enter the Bayesian approach through the likelihood function $\mathrm{p}(\mathbf{y}|\boldsymbol{\theta})$, which gives the probability of observing $\mathbf{y}$, given a value of the state parameters $\boldsymbol{\theta}$. The posterior distribution $\mathrm{p}(\boldsymbol{\theta}|\mathbf{y})$ represents our best estimate of $\boldsymbol{\theta}$ and combines the prior knowledge with the information from the sensor data $\mathbf{y}$.

When the state variable can assume only discrete values, i.e. $\Omega_S = \{S_1,\dots,S_i,\dots,S_M\}$, Bayes' theorem allows calculation of the posterior probability $\mathrm{P}(S_i \mid \mathbf{y})$ of state S_i, from its prior probability $\mathrm{P}(S_i)$ through:

$$\mathrm{P}(S_i \mid \mathbf{y}) = \frac{\mathrm{p}(\mathbf{y} \mid S_i) \cdot \mathrm{P}(S_i)}{\mathrm{p}(\mathbf{y})}. \tag{3}$$

where $\mathrm{p}(\mathbf{y} \mid S_i)$ is the likelihood, and the evidence is now expressed as:

$$\mathrm{p}(\mathbf{y}) = \sum_{i=1}^{M} \mathrm{p}(\mathbf{y} \mid S_i) \cdot \mathrm{P}(S_i). \tag{4}$$

When each discrete state is further characterized by a set of parameters, the likelihood assumes the following form:

$$\mathrm{p}(\mathbf{y} \mid S_i) = \int_{\Omega_{\boldsymbol{\theta}}} \mathrm{p}(\mathbf{y} \mid S_i, \boldsymbol{\theta}) \cdot \mathrm{p}(\boldsymbol{\theta} \mid S_i) \cdot \mathrm{d}\boldsymbol{\theta}, \tag{5}$$

where $\mathrm{p}(\mathbf{y}|S_i,\boldsymbol{\theta})$ and $\mathrm{p}(\boldsymbol{\theta}|S_i)$ are respectively the likelihood function and the prior distribution of $\boldsymbol{\theta}$, for the state S_i. We note that the direct calculation of the posterior is usually computationally demanding, and should be solved numerically—see for instance [22–25].

C. Decision-making

When a decision maker faces a decision problem, he/she is supposed to choose an action $a^{(1)}$ defined in the space $\Omega_a^{(1)}$, which may be either continuous or discrete. Here, the superscript '(1)' is to indicate the objects involved in the first stage of the decision-making process, as opposed to later stages that we introduce below. The chosen action $a^{(1)}$ may affect the future state of the system S, and lead to an outcome $z^{(1)}$ defined in the space $\Omega_z^{(1)}$, which again may be either continuous or discrete. The probability of S after the decision $a^{(1)}$ is $\mathrm{P}(S|a^{(1)})$.

This decision process can be graphically represented with a decision tree. As an example, Fig. 1 depicts the decision tree of a single-stage decision process, in which the actions and the states are discrete variables. Squares represent decision nodes and circles represent chance nodes. In the process described by this figure, the decision maker chooses the action $a_2^{(1)}$ from a set of $J^{(1)}$ alternatives $\Omega_a^{(1)} = \left\{a_1^{(1)},\ldots,a_{J^{(1)}}^{(1)}\right\}$. Then, the state S may become a different state S_i, leading to the outcome $z_i^{(1)}$ and the consequent utility $u\left(z_i^{(1)}\right)$. The impact of an outcome $z^{(1)}$ is measured and ranked after transformation through a function $u(z^{(1)})$ known as the *utility function*.

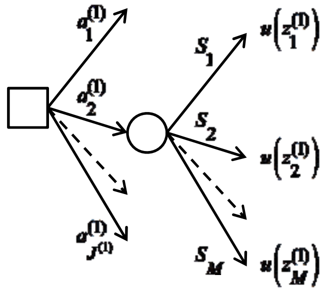

Fig. 1. Decision tree of a single-stage decision process, in which the actions $a_i^{(1)}$ and the states S_i are discrete variables. Squares represent decision nodes; circles represent chance nodes.

If the decision maker has to face two consecutive decision problems, the process becomes *multi-stage*. In this case, the second action is defined by $a^{(2)}$ in the domain $\Omega_a^{(2)}$, and it may lead to another different state S and the corresponding outcome $z^{(2)}$, with a probability $\mathrm{P}(S|a^{(2)},a^{(1)})$. As a second example, Fig. 2 depicts the decision tree of a two-stage decision process, in which the actions and the states are discrete variables. In the process described by Fig. 2, the decision maker chooses first in stage 1 the action $a_2^{(1)}$ from a set of $J^{(1)}$ alternatives $\Omega_a^{(1)} = \left\{a_1^{(1)},\ldots,a_{J^{(1)}}^{(1)}\right\}$. Then, in stage 2 the state S becomes S_2, and the decision maker is required to take a second choice $a^{(2)}$. The decision $a_2^{(2)}$ is made and, finally, the state S may turn to a different state S_i, leading to the outcome $z_i^{(2)}$ and the consequent utility $u\left(z_i^{(2)}\right)$.

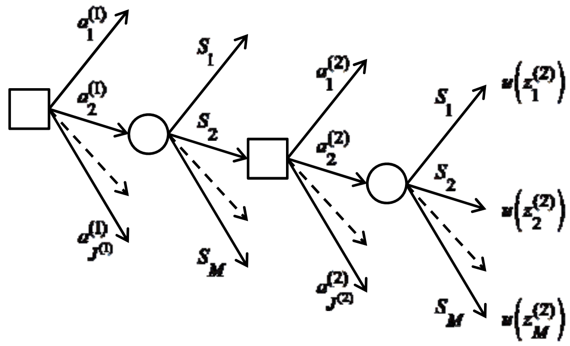

Fig. 2. Decision tree of a two-stage decision process in which the actions $a_i^{(1)}$, $a_i^{(2)}$ and states S_i are discrete variables. Squares represent decision nodes; circles represent chance nodes.

Decision trees allow, for any action, the identification of the possible outcomes and their probability of occurring. In order to make decisions, we need a principle that enables us to identify the optimal actions, based on the probabilities calculated by the process of Bayesian inference. The framework of EUT by von Neumann and Morgenstern [2] provides the basis of rational decision-making [8].

Generally speaking, the utility function $u(z)$ represents how the decision maker 'quantitatively perceives' the outcome z. The reader can find further considerations on the definition of the utility function in the literature [4,26,27]. The simplest way to state the utility is through a linear model: for example, if z is a financial loss and the decision maker is ideally rational and risk-neutral, $u(z)$ can be defined as

$$u(z) = -z \,. \tag{6}$$

In practice, the utility function in (6) represents the real utility perceived when the amount of money z involved is 'small' and there are no follow-up consequences. However, (6) might not be appropriate if z is very 'large' and the decision maker is highly risk-seeking or risk-averse. Moreover, as pointed out by Bernoulli [28], the utility given by the outcome z should be proportional to the wealth of the stakeholders that may be affected by the outcome.

According to the von Neumann-Morgenstern utility theory, we can calculate the expected utility of an action a, denoted by u^*, based on the utility $u(z)$ of each possible outcome z, and the probability $\mathrm{p}(S|a)$ of each possible state S, as

$$u^*(a,\mathbf{y}) = \mathrm{E}_{S|a}\left[u(z \mid S,a)\right], \tag{7}$$

where $\mathrm{E}_{S|a}$ is the expected value operator, and the argument of $u(\ldots)$ emphasizes that the outcome z actually may depend on the action a and on the state S following a. The operator $\mathrm{E}_{S|a}$ calculates the expected value (the mean) of its argument using the probability of S occurring after the action a. So, in mathematical terms, the expected utility is

$$u^*(a,\mathbf{y}) = \int_{\Omega_S} u(z\,|\,S,a)\cdot \mathrm{p}(S\,|\,a,\mathbf{y})\cdot \mathrm{d}S\,, \tag{8}$$

when S is defined in a continuous space Ω_S, or

$$u^*(a,\mathbf{y}) = \sum_{i=1}^{M} u(z\,|\,S_i,a)\cdot \mathrm{P}(S_i\,|\,a,\mathbf{y})\,, \tag{9}$$

when S is defined in a discrete space of dimension M.

Although von Neumann-Morgenstern's EUT is today almost universally accepted and adopted in decision making, it has been noted that in general there is no guarantee that the optimal decision resulting from EUT is the decision that would be taken by decision makers without the aid of that framework. Heuristic decisions are indeed affected by *cognitive biases*. Baron and Ritov [29], and Samuelson and Zeckhauser [30], found that there are heuristics that may lead the decision maker either away from the 'rational' choice in favor of doing nothing or to maintain a previous choice, and also heuristics that lead the decision maker to act when the status quo should be preferred instead. Kahneman and Tversky [31] present the issues that arise when EUT is employed to describe the behavior of a real person under risk. In their alternative framework, called *prospect theory*, they propose a process of decision evaluation where the structure of the rational tree is modified and the probabilities involved are weighted in order to model the effects of the cognitive biases.

Hereinafter, we assume true the EUT axioms, and we further assume that the utility function $u(z)$ is linear, as given by (6). In order to demonstrate the implementation and ease the understanding of EUT as it was proposed by von Neumann and Morgenstern, we present the solution of single- and multi-stage decision problems through two applications in the next two sections. Each solution provides one optimal decision for each stage, with the assumption that the utility function is linear and the decision maker is 'rational', i.e. having an objective perception of the probabilities and outcomes involved. Taking into account the decision maker's risk aversion by adjusting the utility function $u(z)$ would have complicated the analysis of the decision problems and would have provided a subjective solution, which is beyond the scope of this contribution.

III. Single-stage decisions

We have already stated that the SHM-based decision problem is to provide the optimal action based on a set of observations from the monitoring system. In practice, we wish to define a map $a_{\mathrm{opt}}(\mathbf{y})$ between the domain of the observations $\Omega_{\mathbf{y}}$ and the set of possible actions Ω_a. To start, let us consider a single-stage decision tree, i.e. one that includes only one decision node.

A. Formulation

In a single-stage decision problem, the decision maker has to choose one terminal action a that may affect the future state of the structure S and finally lead to an outcome z, which depends on both S and a. The choice is based on the observations $\mathbf{y}$, and may also be affected by prior information taken into account using the Bayesian framework. Equations (7) to (9) are used to calculate the expected utility of each available action a. Then, according to the von Neumann-Morgenstern framework, we calculate the maximum expected utility of the decision node as

$$u^*_{\max}(\mathbf{y}) = \max_{a\in\Omega_a} u^*(a,\mathbf{y})\,, \tag{10}$$

and the optimal action is that corresponding to the maximum expected utility:

$$a_{\mathrm{opt}}(\mathbf{y}) = \arg\max_{a\in\Omega_a} u^*(a,\mathbf{y})\,. \tag{11}$$

The action a is defined in the domain Ω_a, which may be either continuous or discrete. We seek an optimal value of a in a continuous space when we are interested in the optimal threshold of a continuous parameter involved in an SHM problem.

Equation (11) is a function of the observation vector $\mathbf{y}$ and maps each possible set of observations $\mathbf{y}$ to an optimal action. It implicitly defines, for each action a, a domain of the possible observations in which that action is optimal. This map is a *classifier* that, given an observation set $\mathbf{y}$ from the monitoring system, automatically provides the optimal action $a_{\mathrm{opt}}(\mathbf{y})$ to the decision maker. Fig. 3 shows as an example a classifier that provides the optimal action $a_{\mathrm{opt}}(\mathbf{y})$ out of three options a_1, a_2 and a_3, given the joint observation of two measurements $\mathbf{y} = \{y_1,y_2\}$.

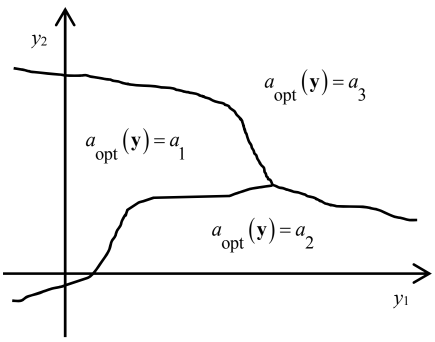

Fig. 3. A classifier that provides the optimal action a_{opt} out of three options a_1, a_2 and a_3, given the joint observation of two measurements $\mathbf{y} = \{y_1, y_2\}$.

B. Case study

The reference [1] describes the 'case of Tom,' a fictitious maintenance manager who wants to understand the damage state of Streicker Bridge, based on a single strain observation ε, recorded at the midspan of the bridge by a long-gauge fiber-optic strain sensor. Streicker Bridge, depicted in Fig. 4a and 4b, is a steel-concrete structure located at the Princeton University campus. It is a pedestrian bridge over Washington

Road, one of the busiest roads in Princeton. Further details about the bridge and its monitoring system are published in [1,32–34].

In 'Tom's problem,' the structural state is defined by S, which can assume one of the two mutually exclusive and exhaustive values: 'undamaged' U and 'damaged' D, defined as follows.

Undamaged, U. The structure is 'undamaged', U, if it is either truly undamaged or has some minor damage, negligible in terms of its effect on structural capacity.

Damaged, D. The structure is 'damaged', D, if it is still standing, but has suffered major damage to the steel arch sub-structure. Although the bridge is standing, Tom estimates that there is a significant chance of collapse of the bridge under a live load.

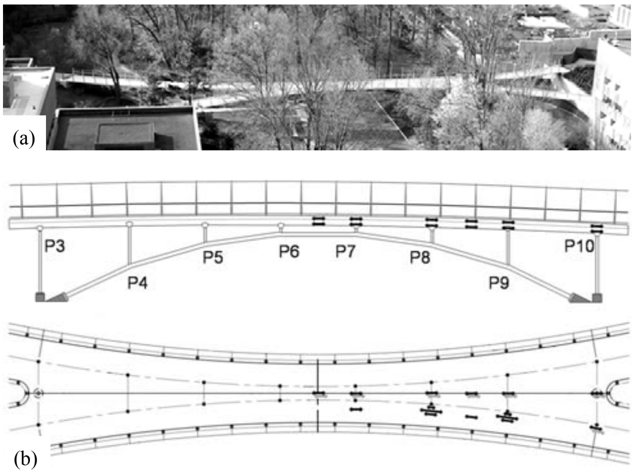

Fig. 4. Aerial photo of Streicker Bridge (a) and its structural scheme (b), from [1]. Streicker Bridge is made of a concrete prestressed slab, supported by steel hollow columns and a steel arch.

Using the formalism presented above, S can be represented as a discrete variable defined in the domain $\Omega_S = \{U,D\}$.

Tom expects that, if the bridge is 'undamaged', the change in strain is close to zero. He is also aware of the natural fluctuation of the midspan curvature, mainly due to thermal effects. The provider of Tom's monitoring system told him that the thermal fluctuation of the strain measurement ε might be in the order of ±300 με. Formally, we can encode this knowledge in a likelihood function that provides the probability density $p(\varepsilon|U)$ of ε for the undamaged state U. The probability density function $p(\varepsilon|U)$ is a Gaussian distribution with mean equal to zero and standard deviation $\sigma_{\varepsilon|U}$ = 300 με, and it represents Tom's expectation of the system response in the undamaged state U. Conversely, on the assumption that the bridge is 'damaged', Tom expects a significant change in strain. We can model the likelihood of ε for the damaged state $p(\varepsilon|D)$ as a Gaussian distribution with mean value of 1000 με, and standard deviation of $\sigma_{\varepsilon|D}$ = 600 με, which reflects Tom's uncertainty of expectation. The two likelihood functions $p(\varepsilon|U)$ and $p(\varepsilon|D)$ are shown in Fig. 5a.

Moreover, Tom guesses that after a damaging event, such as a road accident, scenario U is more likely than scenario D and he estimates prior probabilities P(D) = 0.30 and P(U) = 0.70. Using his prior judgment Tom can also predict the distribution of the observation ε, before these data are available, by marginalizing the system states through:

$$\mathrm{p}(\varepsilon)=\mathrm{p}(\varepsilon \mid D)\cdot \mathrm{P}(D)+\mathrm{p}(\varepsilon \mid U)\cdot \mathrm{P}(U). \tag{12}$$

When the monitoring observation ε is available to the manager, Tom can update his estimate of the probability of U or D state using Bayes' theorem:

$$\mathrm{P}(S|\varepsilon)=\frac{\mathrm{p}(\varepsilon|S)\cdot P(S)}{\mathrm{p}(\varepsilon)}, \tag{13}$$

where S is either U or D. The two posterior probabilities are depicted in Fig. 5b. Note that there is a threshold of elongation $\overline{\varepsilon}_P$ = 569 με whereby the two posterior probabilities satisfy $\mathrm{P}(U|\varepsilon)=\mathrm{P}(D|\varepsilon)$, meaning that, when $\varepsilon > \overline{\varepsilon}_P$, the bridge is more likely to be damaged than undamaged, and vice versa when $\varepsilon < \overline{\varepsilon}_P$.

In this simplified example, Tom is concerned by a single specific scenario in which, for example, a truck driving beneath the bridge collides with the steel arch supporting the concrete deck. We also assume that after the event Tom is immediately informed and his only concern is to understand the extent of damage and then make a decision. Before Tom's decision, no traffic is allowed to cross or go below the bridge.

In this first example, let us assume that Tom has to choose one of the two following actions.

Do nothing, DN. No special restriction is applied to the pedestrian traffic over the bridge or to road traffic under the bridge; repair and maintenance works with minimal cost can be carried out, but this would not interfere with the normal bridge use.

Close the bridge, CB. Both the bridge and the road underneath are closed to pedestrians and road traffic, respectively; access to the nearby area is restricted for the time required by structural rehabilitation, which Tom estimates to be one month.

Formally, this defines a domain of the possible actions $\Omega_a = \{DN,CB\}$. Tom logically understands that choosing to close the bridge will automatically prevent any effects due to a possible collapse of the bridge. The loss incurred from this action is the Daily Road User Cost (DRUC), stemming from the one-month downtime of road and bridge. This cost was estimated by Glišić and Adriaenssens [35] to be \$4,660/day, or z_{CB} = \$4,660/day × 30 days = \$139,800 for the one-month period [36,37]. In order to avoid this outcome, Tom can decide to do nothing (DN) and consequently to allow the traffic to flow as usual. If he decides to do nothing and the bridge state is undamaged U, he pays z_U = \$0. On the other hand, if the bridge state is damaged D, it may collapse and Tom has to pay the consequences. These consequences are the DRUC for a two-month period and the costs due to fatalities and injuries. The probability of a fatality F and of non-fatal injury I are estimated to be respectively equal to $\mathrm{P}(F|D) = 0.15$ and $\mathrm{P}(I|D) = 0.50$. Glišić and Adriaenssens [35] also calculated, based on the National Safety Council records [38], the average comprehensive cost of a pedestrian fatality and injury to be \$3,840,000 and \$52,000 respectively. These amounts cover medical expenses, victim's loss of work, public services and the loss of quality of life. Considering those values, Tom estimates for bridge collapse an overall cost of z_D = \$881,600, as specified in detail in Table I. Using a similar formulation as in [39], reference [1] also shows how to evaluate the costs over the bridge lifespan.

In summary, Tom's decision problem can be represented with the single-stage decision tree of Fig. 6.

TABLE I
COSTS DUE TO STREICKER BRIDGE COLLAPSE [35]

Cause	Cost
Two-month downtime	\$279,600
Cost of fatality	\$3,840,000 × 0.15 = \$576,000
Cost of injury	\$52,000 × 0.50 = \$26,000
Total failure cost z_D	\$881,600

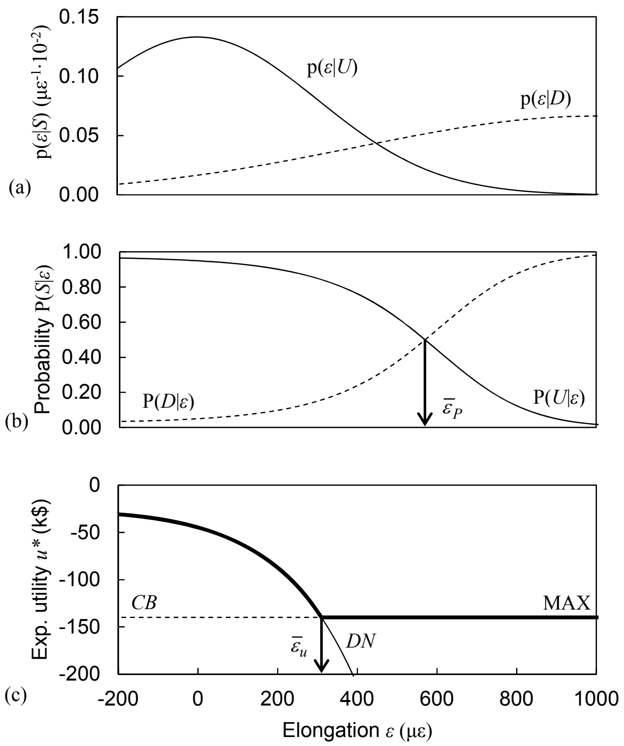

Fig. 5. Probabilities and expected utilities involved in the solution of Tom's single-stage decision problem. (a) shows the likelihood functions p(ε|U) and p(ε|D); (b) shows the posterior distributions P(U|ε) and P(D|ε) of the two possible states; (c) shows the expected utilities of the two possible actions 'do nothing' DN and 'close the bridge' CB.

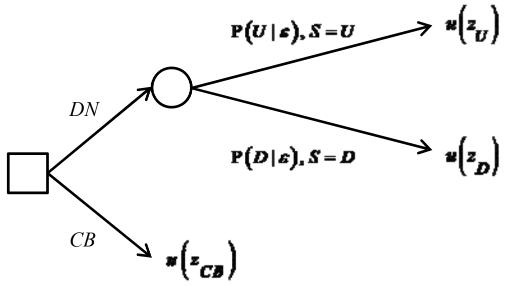

Fig. 6. Decision tree of Tom's single-stage decision problem. The two available actions DN and CB lead to different expected utilities.

The above information is both necessary and sufficient to calculate how Tom's decision depends on the measurement ε provided by the monitoring system. Formally, we want to calculate two domains in the space of ε where the two actions—do nothing (DN) and close the bridge (CB)—are respectively the optimum. Since we assumed that $u(z) = -z$, the estimates of the expected utilities associated with actions DN or CB are:

$$u^*(DN,\varepsilon) = -z_U \cdot \mathrm{P}(U|\varepsilon) - z_D \cdot \mathrm{P}(D|\varepsilon),$$
$$u^*(CB,\varepsilon) = -z_{CB}. \tag{14a,b}$$

If Tom's decision is driven by the expected utility principles, he will choose the least expensive option, so the loss associated with the decision, in the posterior condition, is:

$$u^*_{\max}(\varepsilon) = \max\left\{u^*(DN,\varepsilon), u^*(CB,\varepsilon)\right\}. \tag{15}$$

The expected utility functions calculated for each action as in (13) are represented in Fig. 5c, along with the maximum expected utility of (15). In this particular case, $u^*(CB,\varepsilon)$ is constant, while $u^*(DN,\varepsilon)$ decreases with the elongation. The two expected utilities meet for the threshold $\bar{\varepsilon}_u$ = 310 με. Closing the bridge is more convenient than doing nothing whenever $\varepsilon > \bar{\varepsilon}_u$. This automatically defines two separate regions in the observation domain, each mapped to a different optimal action:

$$a_{\text{opt}}(\varepsilon) = \begin{cases} DN & \varepsilon \le \bar{\varepsilon}_u, \\ CB & \varepsilon > \bar{\varepsilon}_u. \end{cases} \tag{16}$$

Note that the decision threshold $\bar{\varepsilon}_u$ does not coincide with the state probability threshold $\bar{\varepsilon}_P$. The manager will decide to close the bridge not when the elongation ε shows that the state 'damaged' D is more likely than 'undamaged' U, but when the expected cost for doing nothing z_{DN} is higher than the cost of closing the bridge. It should also be noted that there is range of values $\bar{\varepsilon}_u < \varepsilon < \bar{\varepsilon}_P$ in which, even if the state 'undamaged' U is more likely than state 'damaged' D, it is still more suitable to close the bridge. This example should shed some light on the different natures of SHM and decision-making: whereas

SHM aims to correlate the observation ε to the most likely state S_i of the structure; SHM-based decision making aims to correlate the observation ε to the optimal action.

IV. Multi- stage decisions

The decision problem described and solved in the preceding section is relatively simple. In real-life settings, decision makers can choose among multiple options, some of which may lead to other decision nodes. In this section, we address the solution of multi-stage decision problems. For these purposes, Tom's problem is expanded by adding an additional decision stage.

A. Formulation

In multi-stage decision problems, the decision maker has to take multiple actions. The first action $a^{(1)}$ may not be the terminal one. In practice, the decision maker may not know the actions that will be available in the next stages, nor the possible outcomes. However, analysing multi-stage decision processes with the framework of expected utility requires us to make an assumption about the configuration of the decision tree and to predict the probabilities of the future states S. In the formulation below, we denote the predicted values of probability with a superscript '(n)', where n is the stage in which the prediction occurs. For example, $\mathrm{p}^{(1)}(S|a^{(2)},a^{(1)},\mathbf{y})$ is the probability of the state S predicted in stage 1. In stage 2, the value of $\mathrm{p}^{(2)}(S|a^{(2)},a^{(1)},\mathbf{y})$ may be influenced by new data, depending on the outcome of the decision made in stage 1, and therefore it may be different from the value of $\mathrm{p}^{(1)}(S|a^{(2)},a^{(1)},\mathbf{y})$. We use the same superscript '(n)' on the expected value operator E[…] to denote the stage n in which the expected value of the argument is calculated. In order to solve a multi-stage decision problem, we also need to set a *decision rule* [3]. The decision rule prescribes how the decision maker acts after the first stage. We assume that the decision maker always chooses the action corresponding to the highest expected utility, calculated with the information available in his/her current stage.

In mathematical terms, when the decision tree has multiple stages we can obtain the optimal decision of stage n as

$$a_{\mathrm{opt}}^{(n)}(\mathbf{y}) = \underset{a^{(n)}\in\Omega_a^{(n)}}{\arg\max}\, u^*\left(a^{(n)},\mathbf{y}\right). \tag{17}$$

Unlike the single step case, here the complexity of the problem is in calculating $u^*\left(a^{(n)},\mathbf{y}\right)$, because other stages may be nested in each branch of the decision tree. The solution to this problem is based on the principle of *backward induction* [4]. First, for each t-th terminal decision node we calculate the expected utility, as a function of each $a^{(t)}$ as presented in Section III-A, but using the probabilities $\mathrm{p}^{(n)}(S|a^{(t)},\ldots,\mathbf{y})$ expected in stage n instead of the actual probabilities. Then, we assign to each t-th terminal decision node a value of utility $u_{\mathrm{node}}^{(t)}(\mathbf{y})$ equal to the expected utility of its optimal action:

$$u_{\mathrm{node}}^{(t)} = \max_{a^{(t)}\in\Omega_a^{(t)}} \mathrm{E}_{S|a^{(t)},\ldots}^{(n)}\left[u\left(z^{(t)}\mid S,a^{(t)},\ldots\right)\right]. \tag{18}$$

This operation is possible thanks to the assumption that the decision maker always chooses the action corresponding to the highest expected utility. If we assume that each terminal decision node is an outcome leading to a value of utility $u_{\mathrm{node}}^{(t)}(\mathbf{y})$, we can now solve in the same way each stage connected to the terminal decision nodes and repeat this process until we obtain the expected value of the actions $a^{(n)}$, required to solve (17). This approach is illustrated in the next section.

B. Case study

The example presented in Section III-A includes all the principal features of a decision model, but it is not fully realistic. In that case, simplified to ease the presentation, Tom is forced to make a decision between two extreme actions (just 'do nothing' or 'close the bridge' for two months), without any intermediate option. More credibly, Tom would consider the chance of delaying his final decision until he has acquired more knowledge about the state S of the bridge. For example, he can decide to send an inspector to the bridge, with the task of rapidly giving an expert judgment based on his/her experience. In that case, in addition to the two actions 'do nothing' DN or 'close the bridge' CB, we have a third option, which is defined as follows.

Send inspector, SI. The bridge and the road underneath are closed respectively to pedestrians and road traffic, while the access to the nearby area is restricted for the time necessary to perform an inspection, expected to last two days.

Formally, this redefines the domain of the possible actions $a^{(1)}$ that Tom can undertake as $\Omega_a^{(1)} = \{DN,SI,CB\}$, where the superscript '(1)' denotes that the decision is made at stage 1. Once the inspection is completed, the inspector, call him Harry, gives a judgment $\tilde{S}$ on the state of the bridge, which in summary could be either the bridge is 'undamaged' U or the bridge is 'damaged' D. In mathematical terms, we can represent Harry's judgment with a variable $\tilde{S}\in\{U,D\}$ defined in the same domain as the possible state Ω_S. Based on Harry's report, Tom can eventually make a second decision between the remaining two actions $a^{(2)}\in\Omega_a^{(2)} = \{DN^{(2)},CB^{(2)}\}$, where now the superscript '(2)' indicates a decision made at stage 2. This decision scheme is schematically represented in the tree in Fig. 7.

It is important to note that Harry's judgment $\tilde{S}$ does not necessarily coincide with the actual state of the bridge S, but is just an estimation of it. The evaluation is carried out based on a visual inspection, and therefore the chances of identifying or misidentifying the correct state depend significantly on Harry's subjective experience and skills. The nature of Harry's judgment is the same as that of the observation ε, i.e. a piece of information that we must relate to the actual state S through a probabilistic model. In the same way as the relation between the monitoring observation ε and the state S has been stated through the two likelihood functions p($\varepsilon|S$), the relationship between Harry's judgment $\tilde{S}$ and S is stated through the definition of the values of probability $\mathrm{P}\left(\tilde{S} \mid S\right)$, i.e. the probabilities of identification or misidentification of the state S. The likelihood $\mathrm{P}\left(\tilde{S} \mid S\right)$ is represented by the two-by-two matrix of (19), usually called the *confusion matrix* [40], where the element in column i and row j represents the probability that Harry evaluates the bridge in the j-th state while it is actually in the i-th state:

$$\mathrm{P}\left(\tilde{S} \mid S\right)=\begin{bmatrix} \mathrm{P}\left(\tilde{U} \mid U\right) & \mathrm{P}\left(\tilde{U} \mid D\right) \\ \mathrm{P}\left(\tilde{D} \mid U\right) & \mathrm{P}\left(\tilde{D} \mid D\right) \end{bmatrix}, \tag{19}$$

where, for convenience, we use '~' on the states U and D when they are an estimation provided by the inspector, Harry. We use this notation below too.

The individual values of the confusion matrix essentially represent Tom's judgment about Harry's capacity to correctly identify the bridge state. For the purpose of this presentation, we assume that, when the structural condition is actually 'undamaged' U, Harry identifies the state correctly in 80% of cases and misidentifies the state in 20% of cases. We also assume that, when the structural condition is actually 'damaged' D, Harry identifies the state correctly in 99% of cases and misidentifies the state in 1% of cases. Consequently, (19) transforms into (20) as follows:

$$P\left(\tilde{S} \mid S\right)=\begin{bmatrix} 0.80 & 0.01 \\ 0.20 & 0.99 \end{bmatrix}. \tag{20}$$

In order to complete the solution we can use the principle of backward induction. Put simply, we start from the last stage of the tree, assuming that Tom has already decided to send Harry to the bridge and Harry has already reported. The new posterior probability $\mathrm{P}\left(\mathrm{S}\middle|\varepsilon, \tilde{S}\right)$ of 'damaged' D or 'undamaged' U state perceived by Tom will also depend on Harry's judgment $\tilde{S}$. Similarly to (13), we can update the posterior probability using Bayes' rule:

$$\mathrm{P}\left(S\middle|\tilde{S},\varepsilon\right)=\frac{\mathrm{p}\left(\varepsilon\middle|S\right)\cdot \mathrm{P}\left(\tilde{S} \mid S\right)\cdot \mathrm{P}\left(S\right)}{\mathrm{p}\left(\tilde{S},\varepsilon\right)}, \tag{21}$$

where now the evidence is expressed as

$$\begin{aligned} &\mathrm{p}\left(\tilde{S},\varepsilon\right)=\mathrm{p}\left(\varepsilon\middle|\mathrm{U}\right)\cdot \mathrm{P}\left(\tilde{S} \mid \mathrm{U}\right)\cdot \mathrm{P}\left(\mathrm{U}\right)+ \\ &+\mathrm{p}\left(\varepsilon\middle|\mathrm{D}\right)\cdot \mathrm{P}\left(\tilde{S} \mid \mathrm{D}\right)\cdot \mathrm{P}\left(\mathrm{D}\right). \end{aligned} \tag{22}$$

The posteriors on the left-hand side of (21) are depicted in Fig. 8a and Fig. 8b for the cases in which Harry's judgment is $\tilde{S}=U$ and $\tilde{S}=D$, respectively.

Once Harry has reported, the decision problem that Tom faces is the same as discussed in the single-stage example in Section III-B, however, with two differences: the posterior probabilities are now expressed by (20), and the utilities involved must take into account the monetary loss z_{SI}, i.e., Harry's salary for the inspection. For the purposes of presentation, we assume that the fee for his work is quoted at \$5,000. Furthermore, Harry's inspection requires two days of bridge closure corresponding to a DRUC of \$4,660 × 2 = \$9,320. Therefore, the overall cost to Tom for a visual inspection is z_{SI} = \$14,320. The expected utilities of stage 2 are then calculated as

$$\begin{aligned} &u^*\left(DN^{(2)},\tilde{S},\varepsilon\right)=-\left(z_U+z_{SI}\right)\cdot \mathrm{P}\left(U\middle|\tilde{S},\varepsilon\right)+ \\ &-\left(z_D+z_{SI}\right)\cdot \mathrm{P}\left(D\middle|\tilde{S},\varepsilon\right), \\ &u^*\left(CB^{(2)},\tilde{S},\varepsilon\right)=-\left(z_{CB}+z_I\right), \end{aligned} \tag{23a,b}$$

and therefore the maximum expected utility is

$$u^{*(2)}_{\max}\left(\tilde{S},\varepsilon\right)=\max\left\{u^*\left(DN^{(2)} \mid \tilde{S},\varepsilon\right), u^*\left(CB^{(2)} \mid \tilde{S},\varepsilon\right)\right\}. \tag{24}$$

Results from (23) and (24) are reported in Fig. 8c and Fig. 8d for the cases where the Harry's judgment is $\tilde{S}=U$ and $\tilde{S}=D$, respectively. The two thresholds that help in deciding when it is more convenient to close the bridge are: $\overline{\varepsilon}^{(2)}_u\left(\tilde{U}\right)$ = 877 με if $\tilde{S}=U$, and $\overline{\varepsilon}^{(2)}_u\left(\tilde{D}\right)$ = –161 με if $\tilde{S}=D$. The optimal action classifier at stage 2 is therefore:

$$a^{(2)}_{\mathrm{opt}}\left(\tilde{S},\varepsilon\right)=\begin{cases} DN^{(2)} & \varepsilon \le \overline{\varepsilon}^{(2)}_u\left(\tilde{S}\right), \\ CB^{(2)} & \varepsilon > \overline{\varepsilon}^{(2)}_u\left(\tilde{S}\right). \end{cases} \tag{25}$$

Equation (25) is the solution of the second stage of the tree, seen as a function of the monitoring observation ε and the inspector's judgment $\tilde{S}$.

Now we can move to the first level of our decision tree. We already know how to calculate the expected utility of actions 'do nothing' DN and 'close the bridge' CB for this stage, so we just have to obtain the expected utility of the action 'send inspector' SI. In order to do so, we have to calculate the probability that Harry's estimate is $\tilde{S}=U$ or $\tilde{S}=D$. Seen from Tom's point of view, Harry's judgment depends on the structure's state, which in turn is a function of the monitoring observation ε. In mathematical terms:

$$\mathrm{P}\left(\tilde{S} \mid \varepsilon\right)=\mathrm{P}\left(\tilde{S} \mid U\right)\cdot \mathrm{P}\left(U \mid \varepsilon\right)+\mathrm{P}\left(\tilde{S} \mid D\right)\cdot \mathrm{P}\left(D \mid \varepsilon\right). \tag{26}$$

In order to clarify the meaning of (26), note that Harry will provide $\tilde{S}=U$ either if he identifies the right state when the state is actually 'undamaged' U or if he makes a mistake when the bridge state is 'damaged' D. Equation (26) is depicted in Fig. 9.

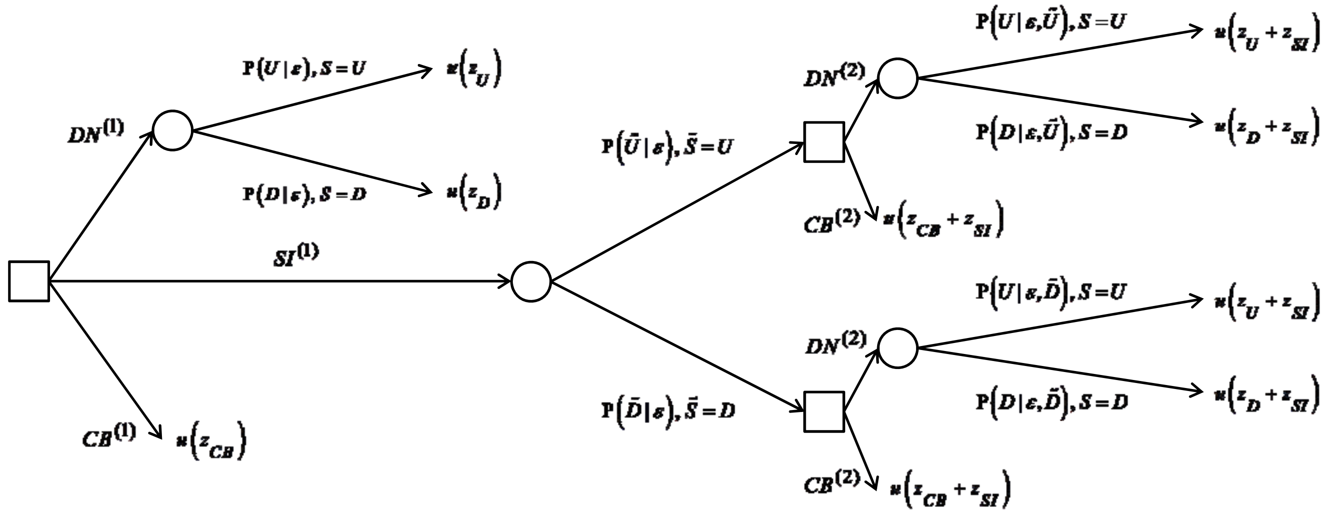

Fig. 7. Multi-stage decision tree of the problem in which Tom is supposed to choose whether to close Streicker Bridge after an accident.

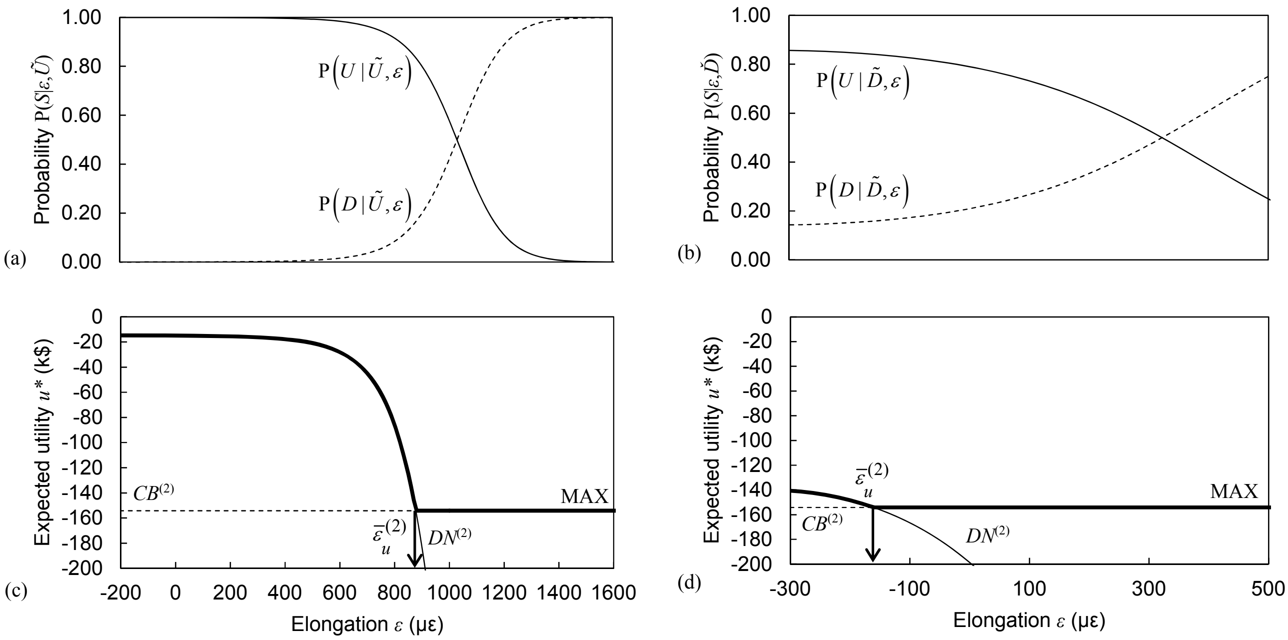

Fig. 8. Posterior probabilities for $\tilde{S}=U$ (a) and $\tilde{S}=D$ (b); expected utilities for $\tilde{S}=U$ (c) and $\tilde{S}=D$ (d) involved in stage 2 of Tom's multi-stage decision problem.

At this point, we can calculate the expected utility for action 'send inspector' *SI* and the other actions in stage 1 as

$$\begin{aligned} &u^*\left(DN^{(1)},\varepsilon\right) = -z_U \cdot \mathrm{P}\left(U\middle|\varepsilon\right) - z_D \cdot \mathrm{P}\left(D\middle|\varepsilon\right), \\ &u^*\left(CB^{(1)},\varepsilon\right) = -z_{CB}, \\ &u^*\left(SI^{(1)},\varepsilon\right) = u^{*(2)}_{\max}\left(\tilde{U},\varepsilon\right)\cdot \mathrm{P}\left(\tilde{U}\mid\varepsilon\right) + \\ &+ u^{*(2)}_{\max}\left(\tilde{D},\varepsilon\right)\cdot \mathrm{P}\left(\tilde{D}\mid\varepsilon\right). \end{aligned} \tag{27a,b,c}$$

Finally, the maximum expected utility is the maximum among $u^*\left(DN^{(1)},\varepsilon\right)$, $u^*\left(CB^{(1)},\varepsilon\right)$ and $u^*\left(SI^{(1)},\varepsilon\right)$. Fig. 10a depicts the three expected utilities as a function of the monitoring observation ε along with the maximum expected utility. We can easily recognize in the expected utility graph two thresholds, one, $\overline{\varepsilon}_u^{(1)}$ = 32 με, that separates $DN^{(1)}$ from $SI^{(1)}$, and the other, $\overline{\overline{\varepsilon}}_u^{(1)}$ = 750 με separating $SI^{(1)}$ from $CB^{(1)}$. In this specific case, the optimal action depends on the output of the monitoring system ε through:

$$a_{\mathrm{opt}}^{(1)}\left(\varepsilon\right) = \begin{cases} DN^{(1)} & \varepsilon \le \overline{\varepsilon}_u^{(1)}, \\ SI^{(1)} & \overline{\varepsilon}_u^{(1)} < \varepsilon \le \overline{\overline{\varepsilon}}_u^{(1)}, \\ CB^{(1)} & \varepsilon > \overline{\overline{\varepsilon}}_u^{(1)}. \end{cases} \tag{28}$$

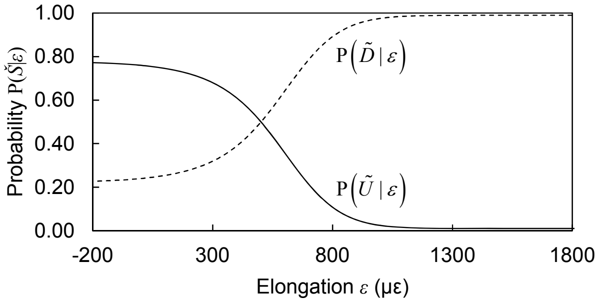

Fig. 9. Probabilities of having a judgment either $\tilde{S}=U$ or $\tilde{S}=D$, given a value of elongation ε, for Tom's multi-stage decision problem.

C. Discussion

The expected utilities corresponding to the three actions available, 'do nothing' $DN^{(1)}$, 'send inspector' $SI^{(1)}$ and 'close the bridge' $CB^{(1)}$ are shown in Fig. 10a. It can be noted that, in the first stage, when the elongation ε is lower than $\overline{\varepsilon}_u^{(1)}$, $DN^{(1)}$ is the optimal choice, whereas when the elongation ε is higher than $\overline{\overline{\varepsilon}}_u^{(1)}$, $CB^{(1)}$ is the optimal choice. The domain between $\overline{\varepsilon}_u^{(1)}$ and $\overline{\overline{\varepsilon}}_u^{(1)}$ is where $SI^{(1)}$ is the optimal choice. The expected utility of action $DN^{(1)}$ and $CB^{(1)}$ does not depend on the confusion matrix $P\left(\tilde{S}\mid S\right)$ of (19), and is therefore independent of the inspector. However, both $\overline{\varepsilon}_u^{(1)}$ and $\overline{\overline{\varepsilon}}_u^{(1)}$ depend on the expected utility of $SI^{(1)}$, which depends on $P\left(\tilde{S}\mid S\right)$. So, in order to understand the impact of $P\left(\tilde{S}\mid S\right)$ and the effectiveness of the monitoring system, we present below three additional cases. In the first two cases, the probabilities $P\left(\tilde{S}\mid S\right)$ are changed; in the last case, we changed the likelihood functions of the two possible structural states U and D.

The perfect inspector. We repeated the analysis of Tom's multi-stage decision problem in the same way as described above, but assuming a new confusion matrix:

$$P_{\text{pi}}\left(\tilde{S}\mid S\right)=\begin{bmatrix}1.00 & 0.00\\ 0.00 & 1.00\end{bmatrix}. \tag{29}$$

In other words, we assumed that the inspector cannot classify the bridge in the wrong state. Fig. 10b shows the expected utilities of the actions $DN^{(1)}$, $SI^{(1)}$ and $CB^{(1)}$ obtained in the case of a perfect inspector. In this case, the expected utility of $SI^{(1)}$ is always higher that the expected utility of $DN^{(1)}$, and therefore the action 'send inspector' $SI^{(1)}$ is optimal for any observation $\varepsilon < \overline{\varepsilon}_{u,\text{pi}}^{(1)} = 822$ με. However, this is not always the case: if the cost of the inspector z_{SI} changes, we may find a non-empty domain in which $DN^{(1)}$ is optimal. Moreover, although the inspector cannot fail, due to the inspector's fee there is still a domain $\varepsilon > \overline{\varepsilon}_{u,\text{pi}}^{(1)}$ in which the action 'close the bridge' $CB^{(1)}$ is optimal. This means that, if $\varepsilon > \overline{\varepsilon}_{u,\text{pi}}^{(1)}$, the probability of damage is so high that taking the chance to discover the actual state of the bridge is not worth the cost of the inspector z_{SI}.

The non-informative inspector. In this case, we assume that the inspector cannot provide any information about the structure: regardless of the actual state, the outcome is always a state drawn randomly based on the confusion matrix

$$P_{\text{nii}}\left(\tilde{S}\mid S\right)=\begin{bmatrix}0.50 & 0.50\\ 0.50 & 0.50\end{bmatrix}. \tag{30}$$

In practice, we can imagine that in this case Harry estimates the structural state based on a coin flip, instead of performing the inspection. Fig. 10c shows the expected utilities of the actions $DN^{(1)}$, $SI^{(1)}$ and $CB^{(1)}$ obtained. Now, it can be noted that, due to the cost of the inspector, the expected utility of $SI^{(1)}$ is always lower than the expected utility of the other actions. Therefore, the domain in which $SI^{(1)}$ is optimal is empty and the decision maker should always avoid the action 'send inspector' $SI^{(1)}$. The threshold that defines the two remaining domains, i.e. those domains where $DN^{(1)}$ and $CB^{(1)}$ are respectively optimal, is in this case equal to the threshold calculated for the single-stage decision problem, $\overline{\varepsilon}_u = 310$ με.

Highly effective SHM. Finally, we decided to also investigate the effects of the monitoring system effectiveness. We assume that, thanks to better sensors or an improved data analysis process, the likelihood functions p($\varepsilon|U$) and p($\varepsilon|D$) of the two structural states 'undamaged' U and 'damaged' D are sharper. We assumed new values of the standard deviations $\sigma_{\varepsilon|U} = 50$ με and $\sigma_{\varepsilon|D} = 100$ με, while we employed the same other conditions as described in Section IV-B. The results are shown in Fig 10d. It should be noted that, by improving the reliability of the information provided by the monitoring system, we could reduce the domain where 'send inspector' $SI^{(1)}$ is the optimal choice. Indeed, the thresholds in this case resulted in $\overline{\varepsilon}_{u,\text{ii}}^{(1)} = 327$ με and $\overline{\overline{\varepsilon}}_{u,\text{ii}}^{(1)} = 348$ με. This result emphasizes that, if human inspection is an option that is available but that we wish to avoid, then we necessarily have to use a better monitoring system or perform a better data analysis process in order to increase the information provided by the measurement ε.

V. Conclusion

Although the purpose of SHM is to support infrastructure managers in making decisions, it is believed that the optimal action is a mere consequence of the most probable structural state, which is estimated based only on the monitoring data. In contrast, in other contexts like finance or medicine the evaluation of decisions is usually based on the expected utility framework. In this paper, we propose a bridge management strategy in which the evaluation of each decision is performed using EUT. However, in order to obtain the information from the real structure through SHM and select the optimal action, the decision model must interface with the results of the probabilistic model used to estimate the structural state. We showed how the combination of Bayesian and EUT frameworks provides a map between the possible sensor observations and the optimal actions.

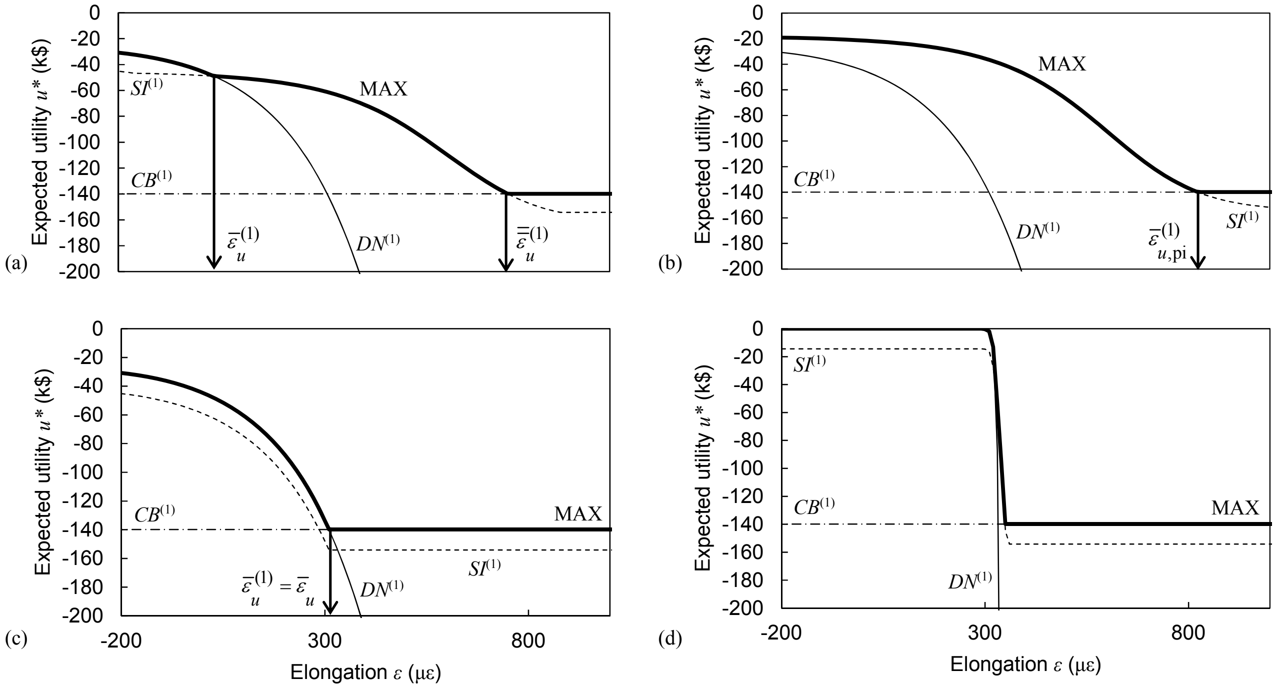

Fig. 10. Expected utilities of the actions available in stage 1 of Tom's multi-stage decision problem; (a) shows the results obtained using Harry's confusion matrix; (b) shows the results obtained using a perfect inspector; (c) shows the results obtained using a non-informative inspector; (d) shows the results obtained using Harry's confusion matrix and two sharper likelihood functions of the structural states.

With the aid of a real-life example, we investigated single-stage and multi-stage decision problems. By analyzing the relationship between observations and actions, we showed the components that affect the selection of optimal decisions and their effects. These components are: (1) the likelihood of the possible structural states, given the monitoring observations; (2) the probability of the states perceived a priori by the decision maker; (3) the expected financial consequences of the realization of each state; (4) the way financial costs are connected to the subjective utility of the decision maker. What is interesting to point out is that, of all these items, only (1) is linked to sensor measurements and the structural analysis. The remaining items encode the management policy and the socio-economic impact of actions, or simply reflect the subjective perception of the individual in charge of the decision.

The presented approach to decision making in SHM allows a rigorous identification of the financially most convenient action, based on the probabilities efficiently provided by Bayesian inference. The two examples presented showed the feasibility of the approach in real-life settings. If the probabilistic model is computationally inexpensive and the outcomes of each structural state are quantitatively assessed, the implementation of the decision model is straightforward and the solution computationally inexpensive. However, if the inference requires Monte Carlo methods [22], the calculation of expected utility may become slow and inaccurate. Moreover, there is still an inherent complexity in the calculation of outcomes. Solving decision problems involving major civil structures requires the quantification of the environmental and social impact of each state, but a standardized approach for this purpose is still missing.

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