TWO OPERATORS ON SANDPILE CONFIGURATIONS, THE SANDPILE MODEL ON THE COMPLETE BIPARTITE GRAPH, AND A CYCLIC LEMMA

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Abstract. We introduce two operators on stable configurations of the sandpile model that provide an algorithmic bijection between recurrent and parking configurations. This bijection preserves their equivalence classes with respect to the sandpile group. The study of these operators in the special case of the complete bipartite graph $K_{m,n}$ naturally leads to a generalization of the well known Cyclic Lemma of Dvoretsky and Motzkin, via pairs of periodic bi-infinite paths in the plane having slightly different slopes. We achieve our results by interpreting the action of these operators as an action on a point in the grid $\mathbb{Z}^2$ which is pointed to by one of these pairs of paths. Our Cyclic lemma allows us to enumerate several classes of polyominoes, and therefore builds on the work of Irving and Rattan (2009), Chapman et al. (2009), and Bonin et al. (2003).

1. Introduction

The abelian sandpile model is a cellular automaton on a graph. It was the first example of a dynamical system exhibiting a fascinating property called *self-organized criticality*; see [3]. This model has since proved to be a fertile ground from which many new and unlikely results have emerged. One popular example is the correspondence between recurrent configurations of the sandpile model on a graph and spanning trees of the same graph; see e.g. [9].

In the abelian sandpile model on an undirected connected loop-free graph, states are vectors which indicate the number of grains present at every vertex of the graph. A vertex may be toppled when the number of grains at that vertex is not less than the degree of that vertex. When a vertex is toppled, one grain of sand is sent along each incident edge to neighboring vertices. A *sink* is a distinguished vertex in the graph. A *configuration* is an assignment of grains to graph vertices, and a configuration is called *stable* if the number of grains at each vertex other than the sink is less than the degree of that vertex.

Two configurations are called *toppling equivalent* if there is a sequence of topplings of one of the configurations that results in the other. Given a configuration, the configurations that can be obtained from it by any finite sequence of topplings form the toppling equivalence class of this configuration. We study in particular the partition of stable configurations into toppling equivalence classes.

In Section 2 of this paper we consider two operators, ψ and φ, on stable sandpile configurations. These operators are, in a sense, dual to one another. We prove that the fixed points of the operator ψ are the recurrent sandpile configurations and the fixed points of φ are the G-parking sandpile configurations (an extension of the classical parking function to an arbitrary directed graph G). The motivation in introducing the operators ψ and φ was to produce an algorithm that allows one to go from recurrent configurations to G-parking configurations, and vice-versa, within the same toppling equivalence class. As a byproduct, we get two dual definitions of recurrent and G-parking configurations.

In Section 3, we consider pairs of periodic bi-infinite paths in the plane defined by a pair of binary words. These binary words describe their respective minimal periods. The two periods differ slightly since one period describes a lattice path from the origin to (m, n) while the other

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describes a path from the origin to $(m-1, n)$. For both of these finite paths, amend and prepend that same path to itself an infinite number of times to produce a pair of periodic paths with slightly different periods. We prove a result that we call the Cyclic Lemma (Lemma 3.1) which consists of two parts. The first part shows that the pairs of binary words can be partitioned into sets that each contain m elements – the sets consist of those pairs of binary words which define *up to translation* the same pair of bi-infinite paths. The second part shows that in every such set, there is precisely one pair of binary words that, when restricted to the rectangle of corners $(0, 0)$ and $m \times n$, form a parallelogram polyomino. An immediate corollary of our Cyclic Lemma is an enumeration of parallelogram polyominoes having an $m \times n$ bounding box.

Chottin [7] presented a result on words that is similar in spirit to our Cyclic Lemma. Our procedure is specifically designed to suit classes that are relevant in the context of the sandpile model. Moreover, our presentation is the perfect tool to deal with *labelled* parallelogram polyominoes, which were recently investigated in [1].

The paper [10] showed how configurations of the sandpile model on the complete bipartite graph $K_{m,n}$ that are both stable and sorted may be viewed as collections of cells in the plane. By sorted we mean that configuration heights are weakly increasing in each part with respect to vertex indices. This correspondence was shown to have the property that a configuration is recurrent if and only if the collection of corresponding cells in the plane forms a parallelogram polyomino whose bounding box is an $m \times n$ rectangle. In Section 4 we restrict our attention to the complete bipartite graph, and give an algorithm which computes φ for stable configurations on $K_{m,n}$.

In Section 5 we bring together the results of Sections 2, 3 and 4 while also building on the construction given in [10]. We will represent sandpile configurations on $K_{m,n}$ as bi-infinite pairs of paths in the plane. Configurations of the sandpile model may be read from these bi-infinite paths by placing a 'frame' at certain points of intersection and performing measurements to steps of the paths from this frame. We present and prove results which show how the algorithmic calculation of φ on sandpile configurations in Section 4 may be interpreted as the moving of the frame to a new point in the plane. We also give a similar interpretation for ψ, and we deduce several consequences of these results. Notably, we give a pictorial description of $K_{m,n}$-parking configurations, in analogy with the parallelogram polyominoes for recurrent configurations.

In Section 6 enumerative results about bi-infinite paths are given in the case when one of the bi-infinite paths has a particularly regular 'staircase shape'. Our work complements recent work of Irving and Rattan [15] and Chapman, Chow, Khetan, Moulton, Waters [6] concerning the enumeration of lattice paths with respect to a cyclically shifting boundary.

Finally, in Section 7 we show how results concerning the behavior of the operators φ and ψ on the complete graph K_n can be derived from those on $K_{m,n}$.

The results in this paper arose from studying statistics on parallelogram polyominoes and a symmetric functions interpretation of a bi-statistic generating function that relates these statistics to diagonal harmonics [2]. Throughout this paper, the phrase 'Cyclic Lemma' refers to our Lemma 3.1 unless stated otherwise.

2. Two operators on general sandpile configurations

In this section we define two operators on sandpile configurations for an undirected connected loop-free graph G. We will show that the fixed points of these operators correspond to recurrent configurations and G-parking configurations of the sandpile model on G. Following this we will make some observations concerning the injectivity of these operators, and present a relation between them in terms of an operator called β. These results are necessary in order to deal with the specialization of G to the graph $K_{m,n}$ from Section 4 onwards.

Consider an undirected connected loop-free graph G with vertex set $V = \{v_1, v_2, \ldots, v_{n+1}\}$. We call the vertex v_{n+1} the *sink*. Let d_i be the degree of the vertex v_i, and $e(i, j) \in \{0, 1\}$ the indicator function of an edge between v_i and v_j. Let $\alpha_i \in \mathbb{Z}^{n+1}$ be a vector with 1 in the

i-th position and 0 elsewhere. Define the *toppling* operators $\Delta_i := d_i\alpha_i - \sum_{j\neq i} e(i,j)\alpha_j$ for $i = 1, 2, \ldots, n+1$. Notice that $\sum_{j=1}^{n+1} \Delta_j = 0$.

A configuration on G is a vector $c = (c_1, \ldots, c_{n+1}) \in \mathbb{Z}^{n+1}$. We will consider configurations modulo the height of the sink v_{n+1}. Thus two configurations will be called equal or equivalent if their heights at all vertices, other than the sink, are the same. The number of grains at the sink is immaterial and we often record this number as '$*$'.

Let $c = (c_1, c_2, \ldots, c_{n+1})$ be a configuration on G. We call the sum $\sum_{i=1}^{n} c_i$ the *height* of the configuration. We say that c is *non-negative* if $c_i \geq 0$ for all $1 \leq i \leq n$ and that c is *semi-stable* if $c_i < d_i$ for all $1 \leq i \leq n$. If a configuration c is both non-negative and semi-stable, then we call it *stable*.

If c is a non-negative configuration such that $c - \Delta_i$ is still non-negative, then we say the vertex v_i is *unstable* and may be *toppled*. The act of toppling vertex v_i is equivalent to subtracting the toppling operator Δ_i from a configuration. By convention we can always topple the sink. Stable configurations are the ones for which there is no vertex that can be toppled except the sink. Let $\mathsf{Stable}(G)$ be the set of all stable configurations on G.

The following definition for recurrence is equivalent to the second sentence in the proof of Theorem 2.4.

Definition 2.1. *A configuration $c \in \mathsf{Stable}(G)$ is recurrent if, after toppling the sink, there is an order of the remaining vertices in which we can topple every vertex of G exactly once in that order, thereby arriving back to the original configuration c.*

Given $A \subseteq \{1, 2, \ldots, n, n+1\}$, we define $\Delta_A := \sum_{j\in A} \Delta_j$. By convention $\Delta_\emptyset$ is the zero vector.

Definition 2.2 (see [16])**.** *A non-negative configuration $c \in \mathsf{Stable}(G)$ is a G-parking configuration if the configuration $c - \Delta_A$ is not non-negative, for all non-empty $A \subseteq \{1, 2, \ldots, n\}$.*

In order to define the operator ψ on stable configurations of a graph G we need the following terminology concerning orders of vertices and sets thereof. Fix a total order $<_1$ on the vertices, for example $v_1 <_1 v_2 <_1 \cdots <_1 v_{n+1}$, which corresponds to the order $1 < 2 < \cdots < n+1$ on the indices. Next define the order $\prec$ on the subsets of $\{1, 2, \ldots, n\}$: if A and B are subsets $\{1, 2, \ldots, n\}$, then $A \prec B$ if (i) $|A| < |B|$, or (ii) if $|A| = |B|$ and A is smaller than B in the lexicographic order induced by the fixed order $<_1$ on the vertices.

Definition 2.3 (of ψ)**.** *Given $c \in \mathsf{Stable}(G)$, let*

$$\psi(c) = \begin{cases} c & \text{if } c + \Delta_A \notin \mathsf{Stable}(G) \text{ for all non-empty } A \subseteq \{1, 2, \ldots, n\} \\ c + \Delta_A & \text{otherwise, where } A \subseteq \{1, 2, \ldots, n\} \text{ is non-empty and minimal} \\ & \text{(w.r.t. } \prec\text{) such that } c + \Delta_A \in \mathsf{Stable}(G). \end{cases}$$

Theorem 2.4. *The fixed points of ψ are exactly the recurrent configurations of G.*

Proof. Suppose that $c \in \mathsf{Stable}(G)$ is recurrent. This means there exists some permutation $(a_1, a_2, \ldots, a_n)$ of $(1, 2, \ldots, n)$ such that $c - \Delta_{n+1} - \sum_{j=1}^{i} \Delta_{a_j}$ is non-negative for all $i = 0, 1, 2, \ldots, n$. Let $X := \{1, 2, \ldots, n, n+1\}$, and suppose by contradiction that there exists $A \subseteq \{1, 2, \ldots, n\}$ such that $c + \Delta_A = c - \Delta_{X\setminus A}$ is stable.

Let i be minimal such that $a_i \in A$. For $Y \subseteq X$ and v a vertex, let $\deg_Y v$ denote the number of edges from v to a vertex in Y.

Then, since $a_j \notin A$ for $j < i$, and by the fact that we could topple v_{a_i} in the configuration $c - \Delta_{n+1} - \sum_{j=1}^{i-1} \Delta_{a_j}$, we have

$$\begin{aligned} c_{a_i} + \deg_{X\setminus A} v_{a_i} &\geq c_{a_i} + e(a_i, n+1) + \sum_{j=1}^{i-1} e(a_i, a_j) \\ &\geq \deg_X v_{a_i}. \end{aligned}$$

But by the stability of $c+\Delta_A$ we must also have

$$c_{a_i} + \deg_X v_{a_i} - \deg_A v_{a_i} = c_{a_i} + \deg_{X\setminus A} v_{a_i} \leq \deg_X v_{a_i} - 1.$$

Putting these together, we get

$$\deg_A v_{a_i} \leq c_{a_i} \leq \deg_A v_{a_i} - 1,$$

a contradiction. Thus no such non-empty A exists, and $c+\Delta_A \notin \mathsf{Stable}(G)$ for all non-empty $A \subseteq \{1,\dots,n\}$. Therefore $\psi(c)=c$ from Definition 2.3.

Suppose now that c is a fixed point of ψ. This means that $c+\Delta_A = c-\Delta_{X\setminus A}$ is not stable for all non-empty $A \subseteq \{1,2,\dots,n\}$. Since $c-\Delta_{n+1} = c+\sum_{i\neq n+1}\Delta_i$ is not stable, it has a vertex, say $v_{a_1} \neq v_{n+1}$, that can be toppled. But then $c-\Delta_{n+1}-\Delta_{a_1}$ is also not stable and it has a vertex, v_{a_2} say, different from v_{a_1} and v_{n+1}, that can be toppled.

Iterating this argument we get a sequence $(a_1,a_2,\dots,a_n)$ of distinct indices, hence a permutation of $\{1,2,\dots,n\}$, such that, after toppling the sink, we can topple the other vertices in that order. In other words, c is recurrent. □

Notice that the condition in the theorem is related to the so-called *allowed configurations* (cf. [16]).

We now define another operator, φ, which acts as a dual operator to ψ. We use the same total order $<_1$ on the vertices that we used for ψ.

Definition 2.5 (of φ)**.** *Given $c \in \mathsf{Stable}(G)$, let*

$$\varphi(c) \quad = \quad \begin{cases} c & \text{if } c-\Delta_A \notin \mathsf{Stable}(G) \text{ for all non-empty } A \subseteq \{1,2,\dots,n\} \\ c-\Delta_A & \text{otherwise, and } A \subseteq \{1,2,\dots,n\} \text{ is non-empty and minimal} \\ & \text{(w.r.t.} \prec\text{) such that } c-\Delta_A \in \mathsf{Stable}(G). \end{cases}$$

Theorem 2.6. *The fixed points of φ are exactly the G-parking configurations.*

Proof. It is straightforward to show that a G-parking configuration is a fixed point of φ since stable configurations are non-negative.

Suppose the converse is not true. Let c be a fixed point of φ, and suppose that there exists a non-empty $A \subseteq \{1,2,\dots,n\}$ such that $c-\Delta_A$ is still non-negative. Since c is a fixed point of φ, $c-\Delta_A$ is also unstable, but non-negative. Therefore there must be an i such that we can topple v_i. There are two cases to consider.

Case $i \in A$: In this case, using the notation of Theorem 2.4, instability implies

$$c_i - \deg_X v_i + \deg_A v_i = c_i - \deg_{X\setminus A} v_i \geq \deg_X v_i.$$

However $c_i - \deg_{X\setminus A} v_i \leq c_i \leq \deg_X v_i - 1$, since c is stable. This gives a contradiction. Therefore i cannot be in A.

Case $i \notin A$: In this case consider $c-\Delta_{A\cup\{i\}}$. This configuration is non-negative, since $c-\Delta_A$ is non-negative, and we can topple v_i, but it is also unstable, since c is a fixed point of φ.

Iterating this argument, we can enlarge our set A until we get to the point where the second case does not occur. But this gives a contradiction. This completes the proof. □

Remark 2.7. *The last two theorems provide a perfect duality between the definitions of recurrent and G-parking configurations, a desirable fact that partially motivated our investigations.*

Set $\mathcal{G} := \mathbb{Z}^{n+1}/\langle \alpha_{n+1}\rangle$. We will call $\mathcal{G}/\langle \{\Delta_j\}_{j=1}^{n+1}\rangle$ the *sandpile group*. We call the cosets of the sandpile group *classes* so that we can talk about the class of a configuration. It is well known that in each class there is exactly one recurrent configuration and exactly one G-parking configuration (see for example [8, Theorem 1] and [4, Proposition 3.1]). There is an easy bijection between recurrent and G-parking configurations (cf. [4, Lemma 5.6]), but under this bijection configurations which correspond to one another do not necessarily lie in the same class: see Remark 2.11.

However, as we stated in the introduction, our motivation in introducing the operators ψ and φ was to produce an algorithm that allows one to pass from a recurrent configuration to a G-parking configuration in the same class, and vice versa.

For a configuration $c = (c_1, \ldots, c_{n+1})$ on an undirected connected loop-free graph G, let $D(c) = (d_0, d_1, \ldots)$ be the distribution of the distances of grains to the sink v_{n+1}. The distance of v_i to v_{n+1} is the minimal number of edges on a path from v_i to v_{n+1} in $E(G)$. In other words $d_k = \sum_{v_i} c_i$ where v_i runs over vertices whose distance from the sink v_{n+1} is k. Note that $d_0 = c_{n+1}$.

Definition 2.8 (of $<_2$). *Let G be a graph with $V(G) = \{v_1, \ldots, v_{n+1}\}$. Let $c = (c_1, \ldots, c_{n+1})$ and $c' = (c'_1, \ldots, c'_{n+1})$ be two configurations in $\mathsf{Stable}(G)$ with $D(c) = (d_0, d_1, \ldots)$ and $D(c') = (d'_0, d'_1, \ldots)$. If $D(c)$ is lexicographically smaller then $D(c')$ then we write $c <_2 c'$.*

Observe that when applied to any configuration which is not recurrent, the operator ψ is strictly decreasing with respect to the order $<_2$. Therefore, if we start with a G-parking configuration, iterating the operator ψ will get a recurrent configuration in finitely many steps. In the same way, when applied to a any configuration which is not G-parking, the operator φ is strictly increasing with respect to the order $<_2$. Thus, starting from a recurrent configuration, and iterating the operator φ we will get a G-parking configuration in finitely many steps. As a consequence, we have a bijection between the recurrent configurations of G and the G-parking configurations that clearly preserves the classes.

Remark 2.9. *The operator ψ (resp. φ) is, in general, not injective even if restricted to the stable configurations that are not recurrent (resp. G-parking). However, we will see that both in the case of K_{n+1} and in the case of $K_{m,n}$, if c is a stable configuration which is not recurrent, then $\varphi(\psi(c)) = c$, and if c is a stable configuration which is not parking, then $\psi(\varphi(c)) = c$.*

Moreover, in these cases the operators ψ and φ are inverses of each others in the sense of semigroups, i.e. $\psi(\varphi(\psi(c))) = \psi(c)$ and $\varphi(\psi(\varphi(c))) = \varphi(c)$ for all stable configurations c. See Remark 5.16.

Example 2.10. In this example we illustrate two applications of Definition 2.3 to sandpile configurations on a graph. These examples will then be used to show that ψ need not be injective **even** if restricted to non-recurrent configurations. Consider the graph $G = (V, E)$ with $V = \{v_1, \ldots, v_7\}$ and

$$E = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_4, v_6\}, \{v_5, v_6\}, \{v_5, v_7\}, \{v_6, v_7\}\}.$$

Let vertex v_7 be the sink. This graph is illustrated by Figure 1.

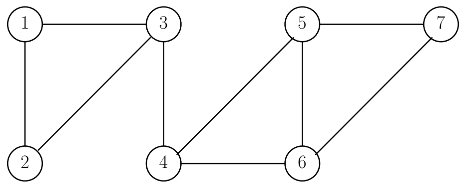


FIGURE 1. The graph G of Example 2.10.

Consider the sandpile configuration $c' = (0, 0, 2, 0, 2, 2, *)$ on G. Applying Definition 2.3 one finds that $A = \{1, 2\}$ is the minimal non-empty subset (w.r.t. $\prec$) of $\{1, 2, \ldots, 7\}$ such that $c' + \Delta_A \in \mathsf{Stable}(G)$. Thus we have

$$\begin{aligned} \psi(c') &= c' + \Delta_1 + \Delta_2 = (0, 0, 2, 0, 2, 2, *) + (2, -1, -1, 0, 0, 0, *) + (-1, 2, -1, 0, 0, 0, *) \\ &= (1, 1, 0, 0, 2, 2, *). \end{aligned}$$

For this example $n = 6$ and $c' = (c_1, ..., c_{n+1}) = (0, 0, 2, 0, 2, 2, *)$. We have $D(c) = (d_0, d_1, ...)$ where $d_0 = c_{n+1} = *$ and $d_1 = c_5 + c_6 = 4$, $d_2 = c_4 = 0$, $d_3 = c_3 = 2$, $d_4 = c_1 + c_2 = 0$. Therefore $D(c) = (*, 4, 0, 2, 0)$.

Next, consider the configuration $c'' = (1,1,0,2,0,0,*)$. Applying Definition 2.3 we find that the minimal subset A for which $c'' + \Delta_A$ is stable is $A = \{5,6\}$. Therefore

$$\begin{aligned}\psi(c'') &= c'' + \Delta_5 + \Delta_6 = (1,1,0,2,0,0,*) + (0,0,0,-1,3,-1,*) + (0,0,0,-1,-1,3,*) \\ &= (1,1,0,0,2,2,*).\end{aligned}$$

Finally, since $\psi(c') = \psi(c'')$ but $c' \neq c''$, the operator ψ is not injective **even** if restricted to non-recurrent configurations.

We conclude this section by showing how operators φ and ψ are *conjugate*. Let us introduce a well-known involution β (see e.g. [12]) defined on any configuration c whose i-th component is

$$\beta(c)_i = (d_i - 1) - c_i.$$

Remark 2.11. *β maps non-negative configurations to semi-stable configurations. Of course β is also an involution on stable configurations. Moreover, this mapping induces a bijection from parking to recurrent configurations* that does not preserve classes.

Proposition 2.12. *One has the following relations between operators φ, ψ and β:*

$$\varphi \cdot \beta = \beta \cdot \psi.$$

Proof. This proposition readily comes from the observation that for any subset A of $\{1,2,\dots,n\}$

$$\beta(c + \Delta_A) = \beta(c) - \Delta_A$$

which implies that $c + \Delta_A$ is stable if and only if $\beta(c) - \Delta_A$ is also stable. □

3. A cyclic lemma counting parallelogram polyominoes in a $m \times n$ rectangle

The aim of this section is to present and prove our *Cyclic Lemma* (Lemma 3.1) for pairs of paths in the plane. This lemma tells us how pairs of infinite paths in the plane, which are formed from two binary words having almost identical parameters in terms of the fixed global parameters m and n, can be partitioned into different classes with respect to some points of intersection. The lemma shows us that each of these classes have exactly the same size. Further to this, we prove that every class corresponds to a unique parallelogram polyomino whose bounding box is an $m \times n$ rectangle.

We first introduce some terminology pertinent to the remainder of the paper. This terminology will be illustrated in the example of Subsection 3.1. Some readers may prefer to skip directly to this example.

An (m,n)-*binomial word* is any word w over the alphabet $\{N, E\}$ consisting of m letter E's and n letter N's. We let $B_{m,n}$ be the set of all (m,n)-binomial words, of which there are $\binom{m+n}{n}$ many. A vertex $x = (x_1, x_2) \in \mathbb{Z}^2$ of the square lattice together with a binomial word w defines a path $[w]_x$: this paths starts at x and is made up of unit steps given by the letters of w wherein N corresponds to a north step $(0,1)$ and E to an east step $(1,0)$. By abuse of notation we will use the terms step $\leftrightarrow$ letter and path $\leftrightarrow$ word interchangeably. For a step s in a path, we denote by $(X_1(s), X_2(s))$ the coordinates of the starting vertex of this step. We will sometimes index, non-ambiguously, the steps of a binomial path: a step N is called N_i where i is $X_1(N)$, the ordinate of the starting vertex of this step, and a step E is called E_j where $j = X_2(E)$, the abscissa of the starting vertex of this step. In a path $[w]_x$, we let $[w]_{y|k}$ be the factor of this path that starts from vertex y in $[w]_x$ and consists of the k steps that follow it in $[w]_x$.

Employing this terminology to define parallelogram polyominoes, we have the following: A polyomino P is a $m \times n$ *parallelogram polyomino* iff it is the set of unit cells of a square lattice enclosed by a pair (u,v) of (m,n)-binomial paths intersecting only at their endpoints where $u = [Nu''E]_{(0,0)}$ and $v = [Ev''N]_{(0,0)}$. If one adds a final red east step to the red path in Figure 8(d) then the enclosed region is a 4×6 parallelogram polyomino. Let $\mathsf{Polyo}_{m,n}$ be the set of all parallelogram polyominoes in an $m \times n$ rectangle, i.e. having an $m \times n$ bounding box.

The number of primes on a sub-path of a path helps us to remember the number of deleted letters/steps. For example u'' means that two steps (or letters) have been removed from the path u, and ℓ' means that only one letter has been removed from ℓ.

In this definition, the steps N_0 and E_{m-1} in u and E_0 and N_{n-1} are forced so that counting the polyominoes of $\mathsf{Polyo}_{m,n}$ is equivalent to counting the number of non-intersecting pairs of $(m-1, n-1)$-binomial paths $([u'']_{(0,1)}, [v'']_{(1,0)})$ which end at positions $((m-1, n), (m, n-1))$. Let us note that $u'' = [u]_{(0,1)|m+n-2}$. This remark allows us to recognize the framework of the classical LGV-lemma (see [13]) and then to count the polyominoes via the following determinant:

$$|\mathsf{Polyo}_{m,n}| = \begin{vmatrix} \binom{m+n-2}{m-1} & \binom{m+n-2}{m} \\ \binom{m+n-2}{m-2} & \binom{m+n-2}{m-1} \end{vmatrix} = \frac{1}{m}\binom{m+n-2}{m-1}\binom{m+n-1}{m-1}.$$

The rightmost expression in the preceding equation bears a resemblance to the Catalan numbers $\frac{1}{2n+1}\binom{2n+1}{n}$. Catalan numbers count the number of Dyck words of semi-length n. The Dvoretzky-Motzkin ([11]) proof that the number of Dyck words is given by the Catalan numbers involved using a cyclic lemma that acted on partitions of all binomials words in $B_{n,n+1}$, and showed that there was precisely one Dyck word with an additional final east step in each of the partitions. Every part of the partition had $2n+1$ elements, and this explains the factor $1/(2n+1)$ in the expression for the Catalan numbers.

Their approach, combined with the similarity of the expressions, suggests it may be possible to employ similar machinery in our setting, and therefore reprove the expression for $|\mathsf{Polyo}_{m,n}|$ above. Indeed, this is exactly what we will do next to a partition $\Pi_{m,n}$ of non-constrained pairs of binomial paths in $B_{m-1,n-1} \times B_{m-1,n}$. Our partition of $B_{m-1,n-1} \times B_{m-1,n}$ will have each part containing exactly m pairs of paths and one polyomino.

3.1. **An example illustrating the terminology.** The example given in this subsection illustrates the terminology of this section. In the example we compute a part $\pi(u'', \ell')$ in our Cyclic Lemma (Lemma 3.1). There is some yet to be explained information contained in these figures, since this example will later allow us to read the iterates of the operator φ of a configuration on $K_{4,6}$ which will run from the recurrent to the parking configuration of all stable configurations of a toppling class.

In this example $m = 4$ and $n = 6$ and we choose the binomial paths $u'' = ENNENENENN \in B_{3,5}$ and $\ell' = NNNEEENENN \in B_{3,6}$. We draw a factor of two yet to be defined red and green bi-infinite periodic paths. The red path $(Nu'')^{\mathbb{Z}}$ contains the factor $[Nu''.Nu''.Nu'']_{(0,0)|23}$ that starts from the origin $(0,0)$ and is made up of 23 steps. Similarly the green path $(E\ell')^{\mathbb{Z}}$ contains the factor $[E\ell'.E\ell'.E\ell']_{(0,0)|24}$. The origin $z^{(3)} = (0,0)$ is marked with an orange disk which indicates that it is the bottom left corner of an $m \times n$ rectangle, also drawn in orange, and called the $z^{(3)}$-*rectangle*.

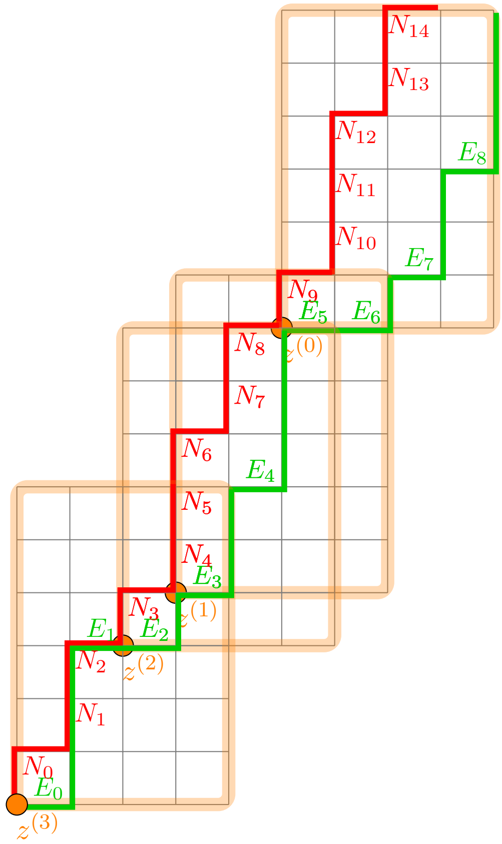


FIGURE 2. The relevant part of $((Nu'')^{\mathbb{Z}}, (E\ell')^{\mathbb{Z}})$ to compute $\pi(u'', \ell')$ and the iterations of $\varphi^k(u)$ for binomials paths $u'' = ENNENENN$, $\ell' = NNNEENENN$ and configuration $c = \binom{1,2,2,3,3,3,}{0,3,5,*}$ on $K_{4,6}$. The other pairs of binomial paths that stem from the stable intersections in the diagram are $(u'', \ell') = (ENNNENNE, NENNENNNE)$, $(NNENNENE, NNENNNEEN)$, and $(ENNNENNE, ENENNENNN)$.

Notice that, by definition, removing the first north red step N_0 of the factor made up of red steps included in the $z^{(3)}$-rectangle gives the path u''. Similarly, removing the east step E_0 of the green factor included in the $z^{(3)}$-rectangle gives the path ℓ'. In this example, there are $m = 4$ orange vertices $(z^{(i)})_{i=0,\dots,3}$ which correspond to stable intersections, i.e. those intersections of the red and green paths which are continued with a red north step and green east step.

A key property of our Cyclic Lemma is that there are exactly m stable intersections for any choice of u'' and ℓ'. As for the $z^{(3)}$-rectangle, we can extract from each $z^{(i)}$-rectangle a pair of binomials paths in $B_{3,5} \times B_{3,6}$. This is done by deleting the first red step and first green step in the $z^{(i)}$-rectangle. These $z^{(i)}$-rectangles are also illustrated in Figure 8. Note that every $z^{(i)}$-rectangle is $m \times n$, and every stable point is distinct.

The proof of our Cyclic Lemma relies on the key parameter $\mathrm{pos}(E_i)$. This parameter is defined for every east step of the green path of Figure 2. We describe it here geometrically so that the reader may bypass the formal symbolic definition: $\mathrm{pos}(E_8) = 8 - 6 = 2$ because the starting point of step E_8 has abscissa 8 and the starting point of step N_{12} (chosen because this is the unique north step which has the same ordinate as E_8) has abscissa 6.

An equivalent way to define the orange stable intersections $z^{(i)}$ is: a point is an orange stable intersection if it is the starting point of east green step E_j such that $\mathrm{pos}(E_j) = 0$. In a $z^{(i)}$-rectangle, the region between the lines is a parallelogram polyomino if and only if $\mathrm{pos}(E_j) > 0$

for every east green E_j in the $z^{(i)}$-rectangle, except the first E_k for which $\text{pos}(E_k) = 0$. In the example the parallelogram polyomino is in the $z^{(0)}$-rectangle.

Notice that any green east step E_i in the $z^{(3)}$-rectangle satisfies $\text{pos}(E_i) \leq 0$. Theorem 5.11 will show that moving from the $z^{(i)}$-rectangle to the $z^{(i+1)}$-rectangle is equivalent to one application of the operator φ to a stable configuration on $K_{m,n}$.

A final remark: the orange $z^{(i)}$ vertices are defined using the green east steps and the red north steps. We call such steps *frame steps*. Green north steps and red east steps are used to define the sorted stable configurations on $K_{m,n}$ so we call those the *configurations steps*.

3.2. **Partitioning paths and a Cyclic Lemma.** We now define the partition $\Pi_{m,n}$. This definition relies on some pairs of periodic bi-infinite paths. The elements of a generic part π_k will be exactly those for which the pairs of bi-infinite paths differ only by a geometric translation. Given an (m,n)-binomial word w, we define the bi-infinite path $w^{\mathbb{Z}}$ as the concatenation of the infinite sequence of paths $([w]_{(mi,ni)})_{i\in\mathbb{Z}}$. This concatenation is well-defined since the last vertex of $[w]_{(mi,ni)}$ is $(mi+m, ni+n)$ which is the starting vertex of $[w]_{(m(i+1),n(i+1))}$.

We define the bi-infinite pair of a pair $(u'', \ell') \in B_{m-1,n-1} \times B_{m-1,n}$ to be $((Nu'')^{\mathbb{Z}}, (E\ell')^{\mathbb{Z}})$. A vertex x at an intersection of $(Nu'')^{\mathbb{Z}}$ and $(E\ell')^{\mathbb{Z}}$ is called a *stable intersection* if $[(Nu'')^{\mathbb{Z}}]_{x|1} = [N]_x$ and $[(E\ell')^{\mathbb{Z}}]_{x|1} = [E]_x$, i.e. the intersection is followed by a north step in $(Nu'')^{\mathbb{Z}}$ and an east step in $(E\ell')^{\mathbb{Z}}$. Instead of using the equivalence by translation, we use the stable intersection to define the part $\pi_{(u'',\ell')}$ of a generic pair (u'', ℓ') in the partition $\Pi_{m,n}$:

$$\pi_{(u'',\ell')} = \left\{ \left([(Nu'')^{\mathbb{Z}}]_{(y_1,y_2+1)|m+n-2},\ \ [(E\ell')^{\mathbb{Z}}]_{(y_1+1,y_2)|m+n-1} \right) \right\}$$

where $y = (y_1, y_2)$ runs over all stable intersections of $((Nu'')^{\mathbb{Z}}, (E\ell')^{\mathbb{Z}})$. In other words, we are considering the factors of a path just after the stable intersection which follows the forced initial steps.

Lemma 3.1 (Cyclic Lemma). *The well-defined partition $\Pi_{m,n} = \bigcup_k \pi_k$ of all pairs (u'', ℓ') that are made up from an $(m-1, n-1)$-binomial path u'' and an $(m-1, n)$-binomial path ℓ' satisfies:*

- *The cardinality $|\pi_k|$ of every part π_k is m.*
- *In each part π_k there is exactly one pair (u'', ℓ') such that $\big([Nu''E]_{(0,0)}, [E\ell']_{(0,0)}\big)$ describes a polyomino in $\mathsf{Polyo}_{m,n}$.*

The enumeration of polyominoes in $\mathsf{Polyo}_{m,n}$ is an immediate corollary: the pairs (u'', ℓ') are an interpretation of $\binom{m+n-2}{m-1}\binom{m+n-1}{m-1}$ and the properties of the partition allow one to select one pair for every part π_k, i.e. divide the total number by m. This partition is different to the one given in Huq [14, 3.1.2] but the same as the one given in Chottin [7]. In addition, it has a deep relation with the algorithm we study in Section 4 for computing the operator φ on $K_{m,n}$. Aval et al. [1] use this cyclic lemma to calculate the Frobenius characteristic of the action of the symmetric group on labelled parallelogram polyominoes.

Proof of Lemma 3.1. We show that the binary relation R on pairs of words in $B_{m-1,n-1} \times B_{m-1,n}$ defined by

$$(v'', k')R(u'', \ell') \iff (v'', k') \in \pi_{(u'',\ell')}$$

is an equivalence relation whose classes are the parts of $\Pi_{m,n}$ and then that each class contains exactly m elements. First we will verify the three defining properties of an equivalence relation.

Reflexivity: By definition of $((Nu'')^{\mathbb{Z}}, (E\ell')^{\mathbb{Z}})$, the point $(0,0)$ is a stable intersection and so

$$\left([(Nu'')^{\mathbb{Z}}]_{(0,1)|m+n-2},\ \ [(E\ell')^{\mathbb{Z}}]_{(1,0)|m+n-1} \right) = (u'', \ell')$$

belongs to $\pi_{(u'',\ell')}$. Therefore R is reflexive.

Symmetry: Let $(v'', k') \in \pi_{(u'',\ell')}$ which, by definition, occurs as factors from a stable intersection (y_1, y_2). Since Nv'' and Nu'' describe, up to some cyclic conjugate, exactly the complete periodic pattern of $(Nu'')^{\mathbb{Z}}$, we remark that $(Nu'')^{\mathbb{Z}}$ is precisely the image of $(Nv'')^{\mathbb{Z}}$ under the vector translation $(-y_1, -y_2)$. This translation sends the stable intersection (y_1, y_2) to $(0, 0)$. We have exactly the same relation for $(E\ell')^{\mathbb{Z}}$ and $(Ek')^{\mathbb{Z}}$ so $((Nv'')^{\mathbb{Z}}, (Ek')^{\mathbb{Z}})$ is the image of $((Nu'')^{\mathbb{Z}}, (E\ell')^{\mathbb{Z}})$ by the vector translation $(-y_1, -y_2)$. This implies, in particular, that as the image of the stable intersection $(0, 0)$ in $((Nu'')^{\mathbb{Z}}, (E\ell')^{\mathbb{Z}})$ defining (u'', ℓ'), the vertex $(-y_1, -y_2)$ is a stable intersection in $((Nv'')^{\mathbb{Z}}, (Ek')^{\mathbb{Z}})$ from which we deduce that $(u'', \ell') \in \pi_{(v'',k')}$. Therefore R is symmetric.

Transitivity: These translations between the bi-infinite pairs of paths also imply transitivity of R. We consider three pairs of words in $B_{m-1,n-1} \times B_{m-1,n}$: (u'', ℓ'), (v'', k') and (w'', j') such that $(u'', \ell') \in \pi_{(v'',k')}$ and $(v'', k') \in \pi_{(w'',j')}$. More precisely, (u'', ℓ') appears in $((Nv'')^{\mathbb{Z}}, (Ek')^{\mathbb{Z}})$ from the stable intersection (x_1, x_2) and (v'', k') appears in $((Nw'')^{\mathbb{Z}}, (Ej')^{\mathbb{Z}})$ from the stable intersection (y_1, y_2). The translations between the bi-infinite paths imply that (u'', ℓ') appears in $((Nw'')^{\mathbb{Z}}, (Ej')^{\mathbb{Z}})$ from the stable intersection $(x_1 + y_1, x_2 + y_2)$. Therefore R is transitive.

The equivalence classes of R are described by the parts π_k of the well-defined partition $\Pi_{m,n}$. We have three things left to show:

(i) *Every part π_k of $\Pi_{m,n}$ contains exactly m elements:* Our proof that each part π_k contains exactly m elements relies on the following key parameter. For any step E_i in $(E\ell')^{\mathbb{Z}}$ we define its relative position $\mathrm{pos}(E_i)$ to be the only step N_j in $(Nu'')^{\mathbb{Z}}$, where $j = X_2(E_i)$, that may start from the common stable intersection:

$$\mathrm{pos}(E_i) = X_1(E_i) - X_1\left(N_{X_2(E_i)}\right).$$

Since $E\ell'$ contains exactly one more east step than Nu'' we have the relation

$$\mathrm{pos}(E_{i+m}) = \mathrm{pos}(E_i) + 1.$$

This unit increase in the pos statistic allows us to determine when it may take certain values. This shows that the equation $\mathrm{pos}(E_{mi+k}) = 0$ defining a stable intersection admits exactly one solution for each $k = 0, 1, \ldots, m-1$, thereby giving the m stable intersections from which we extract the factors giving the element of $\pi_{(u'',\ell')}$.

(ii) *Each of the extracted elements are distinct:* To show this we introduce a strictly increasing parameter on the frame east steps: the cumulated relative position $\mathrm{cumuledpos}(E_j)$ of a step E_j in $(E\ell')^{\mathbb{Z}}$ is

$$\mathrm{cumuledpos}(E_j) = \sum_{k=0}^{m-1} \mathrm{pos}(E_{j+k}).$$

From the previous relation between $\mathrm{pos}(E_{i+m})$ and $\mathrm{pos}(E_i)$, we have

$$\mathrm{cumuledpos}(E_{j+1}) = \mathrm{cumuledpos}(E_j) + 1.$$

For each stable intersection followed by the east frame step E_i, $\mathrm{cumuledpos}(E_i)$ is also a function of the extracted pair of paths (u'', ℓ') since this parameter can be computed using $Nu''E$ and $E\ell'$. Two different stable intersections which are followed by E_i and E_j, where $i < j$, lead to two distinct pairs since $\mathrm{cumuledpos}(E_i) \neq \mathrm{cumuledpos}(E_j)$.

(iii) The solitary parallelogram polyomino in a class π_k is also extracted via the relative position of the east frame steps. Let E_j be the maximal $j \in \mathbb{Z}$ such that $\mathrm{pos}(E_j) = 0$. This defines the "maximum y-coordinate" stable intersection y. By choice of E_j, $\mathrm{pos}(E_{j+k}) > 0$ for $k = 1, \ldots, m$ and this shows that the factor $[(E\ell')^{\mathbb{Z}}]_{y|m+n}$ is below $[(Nu'')^{\mathbb{Z}}]_{y|m+n-1}$ and intersects it only at y, whereas this is not the case for any other stable intersection. Thus only this pair of factors defines a polyomino in this part $\pi_{(u'',\ell')}$ by adding an final east step to the shorter path. For any other stable intersection $z = (z_1, z_2)$, the fact that $\mathrm{pos}(E_j) \leq 0$ for some east step of $[(E\ell')^{\mathbb{Z}}]_{(z_1+1,z_2)|m+n-1}$ implies that $[(E\ell')^{\mathbb{Z}}]_{z|m+n}$ intersects $[(Nu'')^{\mathbb{Z}}]_{z|m+n-1}$ outside their endpoints and does not define a parallelogram polyomino. □

Remark 3.2. *In the example of Figure 2, we have chosen* (u'', l') *such that* $(0,0)$ *is the first stable intersection and all the other stable intersections have non-negative coordinates. However, the choice of any of the other three stable intersections would have led to some stable intersections having negative coordinates. For example, the second stable intersection of coordinates* $(2,3)$ *defines* $(u'', l') = (ENNNENNE, NENNENNNE)$*. If one chooses this pair, then the first stable intersection has coordinates* $(-2,-3)$*.*

4. An algorithm to compute φ for stable configurations on $K_{m,n}$.

In this section we will give an algorithm which computes φ for stable configurations on $K_{m,n}$. On $K_{m,n}$, the edge set consists of single edges between vertices v_i and v_j such that $i \leq n < j$. Thus the vertex set $V = \{v_1, \ldots, v_{n+m}\}$ may be split into two sets that we, albeit abusively, call the *non-sink component*, $C^{\leq n}_{m,n} = \{v_1, \ldots, v_n\}$, and the *sink component*, $C^{>n}_{m,n} = \{v_{n+1}, \ldots, v_{m+n}\}$, since the sink is v_{n+m}. Let c be a generic configuration on $K_{m,n}$. The (partial) non-sink configuration $c^{\leq n}$ is $(c_i)_{1 \leq i \leq n} = (c_1, \ldots, c_n)$, the restriction of c to the non-sink component. The (partial) sink configuration $c^{>n}$ is $(c_i)_{n<i<n+m}$, the restriction of c to the sink component but *excluding the sink* v_{n+m}.

Due to the symmetries of $K_{m,n}$ with its distinguished vertex v_{n+m}, it is natural to consider the two symmetric group actions S_n and S_{m-1} on the non-sink configurations and sink configurations, respectively:

$$\sigma.c^{\leq n} = \left(c_{\sigma(i)}\right)_{1 \leq i \leq n} \qquad \text{for every } \sigma \in S_n$$
$$\tau.c^{>n} = \left(c_{\tau(i)}\right)_{n<i<n+m} \qquad \text{for every } \tau \in S_{m-1}.$$

We will call a configuration c *sorted* if both its sink and non-sink configurations are weakly increasing: $c_1 \leq c_2 \leq \ldots \leq c_n$ and $c_{n+1} \leq c_{n+2} \leq \ldots \leq c_{n+m-1}$. Under the action of $S_n \times S_{m-1}$ given above, the computations are equivalent up to permutations of this group. Without loss of generality, we will henceforth work at the level of orbits using the sorted configurations as representatives.

The interaction between permutations and toppling at this level of orbits suggests the introduction of toppling and permuting equivalence. Two configurations u and v are *toppling and permuting equivalent* if there exists a finite sequence of topplings followed by the action of a permutation which turns u into v. The sorted recurrent configuration are canonical elements of the classes of this equivalence.

We consider topplings which start from stable configurations and preserve the following helpful assumption. A configuration c satisfies the *compact range assumption* if

$$\max(c^{\leq n}) - \min(c^{\leq n}) \leq m \quad \text{ and } \quad \max(c^{>n}) - \min(c^{>n}) \leq n,$$

where $\max(c^{\leq n}) = \max(c_1, \ldots, c_n)$ etc. Given a sorted configuration c, let $T^{\leq n}(c)$ be the result of first toppling in the non-sink component by toppling v_n, and then sorting all entries in the non-sink component so that the resulting configuration is once again sorted. Similarly, given a sorted configuration c, let $T^{>n}(c)$ be the result of first toppling in the sink component by toppling v_{m+n-1} and then sorting all entries in the sink component so that the resulting configuration is once again sorted.

Lemma 4.1. *If a sorted configuration c satisfies the compact range assumption, then*

$$T^{\leq n}(c) = (c_n - m, c_1, \ldots, c_{n-1}, 1 + c_{n+1}, \ldots, 1 + c_{n+m-1})$$

and

$$T^{>n}(c) = (1 + c_1, \ldots, 1 + c_n, c_{n+m-1} - n, c_{n+1}, \ldots, c_{n+m-2}),$$

both of which satisfy the compact range assumption.

Proof. The toppling of the vertex v_n in configuration $c = (c_1, \ldots c_n, c_{n+1}, \ldots, c_{n+m-1})$ leads to the configuration $c' = (c_1, \ldots, c_{n-1}, c_n - m, c_{n+1} + 1, \ldots, c_{n+m-1} + 1)$. Since c satisfies the compact range assumption, $c_n - c_1 \leq m$ is equivalent to $c_n - m \leq c_1$ and so $(c_n - m, c_1, \ldots, c_{n-1}, 1+$

$c_{n+1}, \ldots 1+c_{n+m-1})$ is the sorted configuration representing the orbit of c' which, by definition, is $T^{\leq n}(c)$.

Proof of the expression for $T^{>n}(c)$ is analogous and differs only in using the compact range assumption $c_{m+n-1} - c_{n+1} \leq n$. □

In the case of $K_{m,n}$, the operator φ has some additional regularities on sorted configurations that we describe in the following proposition.

Proposition 4.2. *Let c be a sorted stable configuration on $K_{m,n}$. The minimal set A, if it exists, which defines $\varphi(c) = c - \Delta_A$ is*

$$A = \{v_{n-k}, \ldots, v_n\} \cup \{v_{m+n-1-l}, \ldots, v_{n+m-1}\}$$

for some $0 \leq k \leq n-1$ and $0 \leq l \leq m-2$. In this event, $c_{n-k-1} < c_{n-k}$ and $c_{m+n-2-l} < c_{m+n-1-l}$.

This proposition is deduced from the following two lemmas:

Lemma 4.3. *Suppose two vertices v_i and v_j are in the same component of $K_{m,n}$. Further suppose that A is as in Definition 2.5. If $c_i \geq c_j$, then ($v_j \in A \implies v_i \in A$).*

Proof. Let d denote the degree of vertices in the component under consideration and let t be the number of vertices in the intersection of A and the other component. If $v_j \in A$, then $(\varphi(c))_j = c_j + t - d \geq 0$ since $\varphi(c)$ is non-negative. If $v_i \notin A$, then $d \leq c_j + t \leq c_i + t = (\varphi(c))_i$ which gives a contradiction since $\varphi(c)$ is stable (and $(\varphi(c))_i < d$) and so v_i must be in A. □

Lemma 4.4. *Let c be a sorted stable configuration on $K_{m,n}$. Suppose that the set A in Definition 2.5 exists. Then both v_n and v_{n+m-1} are members of A.*

Proof. Since A is non-empty, from Lemma 4.3 $v_n \in A$ or $v_{n+m-1} \in A$. Since the proof is symmetric in the two cases, assume without loss of generality that $v_n \in A$. As c is stable, $c_n \leq m-1$ and because $\varphi(c)$ is non-negative $(\varphi(c))_n = c_n - m + t \geq 0$ where t is the number of toppled vertices in the sink component. These two inequalities imply that $t \geq 1$ so at least one vertex of the sink component belongs to A which, by Lemma 4.3, implies $v_{m+n-1} \in A$. □

Proof of Proposition 4.2. From Lemma 4.4, the two following intersections

$$A_{\text{non-sink}} = A \cap \{v_1, \ldots, v_n\} \text{ and } A_{\text{sink}} = A \cap \{v_{n+1}, \ldots, v_{n+m-1}\}$$

are non-empty so let v_i, respectively v_j, be the vertex of minimal index of $A_{\text{non-sink}}$, respectively A_{sink}.

From Lemma 4.3 and the fact that c is sorted we have $c_{i-1} < c_i$, when c_{i-1} exists, and since $c_j \geq c_i$ for all $1 \leq j \leq n$ we also have

$$A_{\text{non-sink}} = \{v_i, v_{i+1}, \ldots, v_n\} = \{v_{n-k}, \ldots, v_n\}$$

where $k = n - i$. An analogous argument for A_{sink} gives

$$A_{\text{sink}} = \{v_{n+m-1-l}, \ldots, v_{n+m-1}\}.$$ □

These results culminate in Algorithm 1 which computes φ for any stable configuration. Some minor additional terminology is needed: The configuration 0 is the configuration c such that $c_i = 0$ at every vertex. The configuration δ is the stable configuration with the maximal number of grains at every vertex, i.e. $\delta_i^{\leq n} = m-1$ and $\delta_i^{>n} = n-1$ for all vertices v_i. Two (partial) configurations c and q satisfy $c \trianglelefteq q$ if $c_i \leq q_i$ for all i.

The variable *nloops* counts the number of while-loop iterations and is used exactly when v is a parking configuration. The procedure within the while-loop is executed at most $m+n$ times.

Algorithm 1 An algorithm that computes $\varphi(c)$ for stable configurations c on $K_{m,n}$

procedure $\varphi(c)$
    $c' \leftarrow T^{\leq n} \cdot T^{>n}(c)$
    $nloops \leftarrow 0$
    while not$(0 \trianglelefteq c' \trianglelefteq \delta)$ **do**
        if $nloops \geq m+n$ **then return** c **end if**
        if $c'^{\leq n} \ntrianglelefteq \delta^{\leq n}$ or $0^{>n} \ntrianglelefteq c'^{>n}$ **then** $c' \leftarrow T^{\leq n}(c')$ **end if**
        if $c'^{>n} \ntrianglelefteq \delta^{>n}$ or $0^{\leq n} \ntrianglelefteq c'^{\leq n}$ **then** $c' \leftarrow T^{>n}(c')$ **end if**
        $nloops \leftarrow nloops + 1$
    end while
    return c'
end procedure

5. Interpreting Algorithm 1 as a moving frame on pairs of paths

In this section we bring together the results of the previous sections. We will represent toppling and permuting equivalent classes of sandpile configurations on $K_{m,n}$ as bi-infinite pairs of paths in the plane, up to translation. The types of paths are precisely those that were used in Section 3. Sorted configurations of the sandpile model may be read from these bi-infinite paths by placing a 'frame' at certain points of intersection and performing measurements to steps of the paths from this frame. We present and prove results which show how the calculation of φ on sandpile configurations in Algorithm 1 may be interpreted as the moving of the frame to a new point in the plane.

The underlying theme of this section is graphic in nature and we encourage the reader to refer to the examples in the diagrams when attempting to interpret the results.

Let $(u'', \ell') \in B_{m-1,n-1} \times B_{m-1,n}$ be a pair of binomial words. We will assume that there are two bi-infinite paths, $(Nu'')^{\mathbb{Z}}$ which is coloured red and $(E\ell')^{\mathbb{Z}}$ which is coloured green, in the plane. We will label half of these steps as follows and refer to this labelling as a *NE labelling.*

- Label every N step in $(E\ell')^{\mathbb{Z}}$ with N_i where i is the ordinate of the lower point of the step.
- Label every E step in $(Nu'')^{\mathbb{Z}}$ with E_i where i is the abscissa of the leftmost point of the step.

See Figure 3 for an example of this labelling for the paths u'' and ℓ' used in Figure 2.

Definition 5.1. *Given $m, n \in \mathbb{N}$ and a point $y = (y_1, y_2) \in \mathbb{Z}^2$, a frame is a collection of coloured edges which are anchored about a point y, and have labels as shown in Figure 4. We denote this frame by* $\mathsf{Frame}^y_{m,n}$.

The frame is something we will use to measure distances to steps in the bi-infinite path from.

Definition 5.2. *Let $\mathcal{F}_y = \mathsf{Frame}^y_{m,n}$ and $(u'', \ell') \in B_{m-1,n-1} \times B_{m-1,n}$, where $m, n \in \mathbb{Z}$ and $y = (y_1, y_2) \in \mathbb{Z}^2$. Consider the pair of bi-infinite paths $\mathcal{P} = ((Nu'')^{\mathbb{Z}}, (E\ell')^{\mathbb{Z}})$ (coloured red and green, respectively) and suppose they are NE labelled. A measurement of $\mathcal{P}$ with respect to a frame $\mathcal{F}_y$ is a sequence of numbers describing the horizontal and vertical distances from steps of the frame to steps of the path which have the same label:*

$$\mathrm{Gauge}_{u'',\ell'}(y) = (a_0, \ldots, a_{n-1}, b_0, \ldots, b_{m-2})$$

where a_i is the horizontal distance (that can be negative) from step N_{y_2+i} of $\mathcal{F}_y$ to the corresponding step in $\mathcal{P}$ and b_j is the vertical distance from step E_{y_1+j} of $\mathcal{F}_y$ to the corresponding step in $\mathcal{P}$.

Example 5.3. *Let $m = 4$, $n = 6$ and $y = (2, 7)$. Suppose that the pair of bi-infinite paths $\mathcal{P}$ to be the same as in Figure 3. The frame* $\mathsf{Frame}^{(2,7)}_{4,6}$ *is illustrated in Figure 5. The horizontal distance*

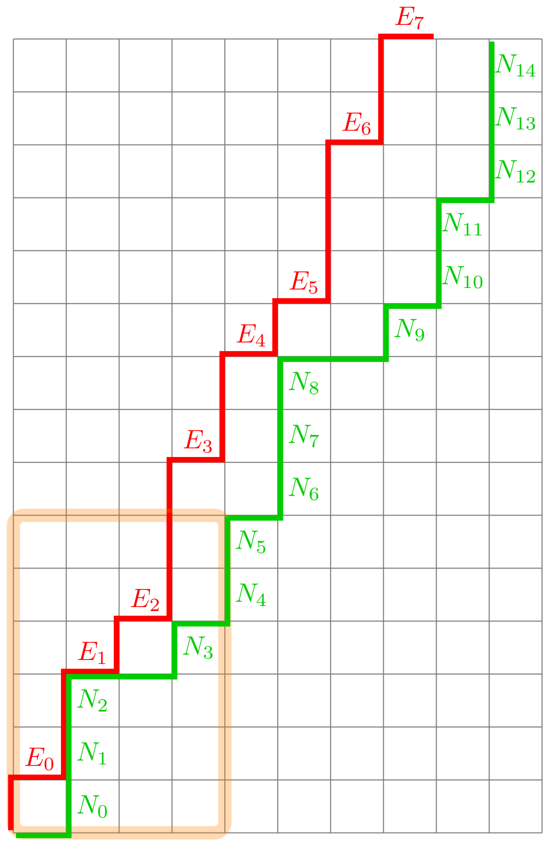


FIGURE 3. The NE labelling of the pair of bi-infinite paths $((Nu'')^{\mathbb{Z}}, (E\ell')^{\mathbb{Z}})$ where $u'' = ENNENENN$, $\ell' = NNNEENENN$. The bottom left corner is the origin. Notice that the steps that have been labelled are precisely those steps of Figure 2 that were not labelled. Note that $m = 4$ and $n = 6$.

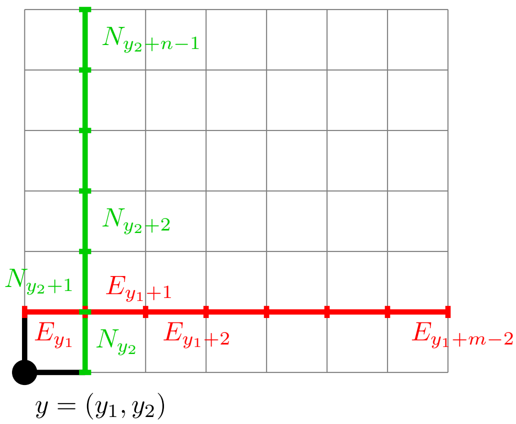


FIGURE 4. The frame $\mathsf{Frame}^{y}_{m,n}$.

from the frames lowest north step N_7 *to the* N_7 *on* $\mathcal{P}$ *is* 2*, so the first entry of* $\text{Gauge}_{u'',\ell'}(y)$ *is* 2*. The horizontal distance from the next lowest north step of the frame* N_8 *to the corresponding one on the path is also* 2*, so the second entry of* $\text{Gauge}_{u'',\ell'}(y)$ *is* 2*. For step* N_9*, the horizontal distance to step* N_9 *on the path* $\mathcal{P}$ *is* 4*, so the third entry of* $\text{Gauge}_{u'',\ell'}(y)$ *is* 4*. Doing the same for the steps* N_{10}*,* N_{11} *and* N_{12} *we get the values* 5*,* 5*, and* 6*, respectively. The values for* $\text{Gauge}_{u'',\ell'}(y)$ *are* $(2, 2, 4, 5, 5, 6)$ *so far.*

Next we consider the east steps of the frame from left to right. The vertical distance from step E_2 of the frame to step E_2 of $\mathcal{P}$ is -4. This means the next entry of $\text{Gauge}_{u'',\ell'}(y)$ is -4. For E_3, the vertical distance is -1 so the next entry of $\text{Gauge}_{u'',\ell'}(y)$ is -1. For steps E_4 that value is $+1$.

Therefore $\text{Gauge}_{u'',\ell'}(y) = (2, 2, 4, 5, 5, 6, -4, -1, 1)$.

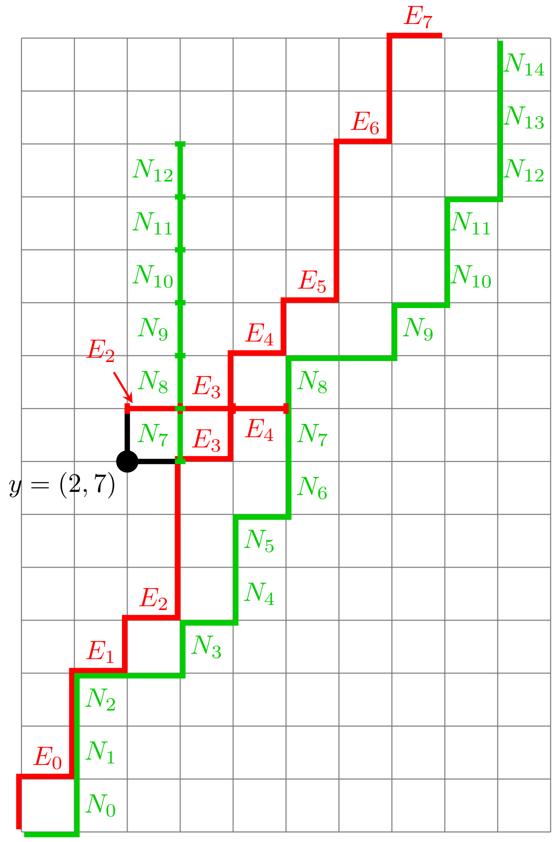


FIGURE 5. The frame and bi-infinite path of Example 5.3. Note here that $m = 4$ and $n = 6$.

5.1. **The operator φ on $K_{m,n}$.** The main result of this subsection is Theorem 5.11 which explains the behavior of φ on stable configurations in terms of stable intersections. We will require several technical lemmas in order to achieve this goal. In order to prove the required lemmas concerning the frame measurement of paths, we will need a more algebraic definition of a frame measurement.

Definition 5.4 (Equivalent to Definition 5.2). *Let $\mathcal{F}_y = \mathsf{Frame}^y_{m,n}$ and $(u'', \ell') \in B_{m-1,n-1} \times B_{m-1,n}$ where $m, n \in \mathbb{Z}$ and $y = (y_1, y_2) \in \mathbb{Z}^2$. Consider the pair of bi-infinite paths $\mathcal{P} = ((Nu'')^{\mathbb{Z}}, (E\ell')^{\mathbb{Z}})$ and suppose they are NE labelled. A measurement of $\mathcal{P}$ with respect to a frame $\mathcal{F}_y$ is a sequence of numbers describing the horizontal and vertical distances from steps of the frame to steps of the path which have the same label:*

$$\text{Gauge}_{u'',\ell'}(y) = (a_0, \ldots, a_{n-1}, b_0, \ldots, b_{m-2})$$

where

$$a_i = X_1(N_{y_2+i}) - y_1 - 1 \quad \text{and} \quad b_j = X_2(E_{y_1+j}) - y_2 - 1$$

for all $0 \le i < n$ and $0 \le j < m - 1$.

We will now use this idea of frame measurement to map to configurations of the sandpile model on $K_{m,n}$ and prove results concerning them.

Lemma 5.5. *For every $(u'', \ell') \in B_{m-1,n-1} \times B_{m-1,n}$ and $y \in \mathbb{Z}^2$, the configuration $c = (c_1, \ldots, c_{n+m-1}) = \text{Gauge}_{u'',\ell'}(y)$ is a sorted configuration satisfying the compact range assumption.*

Proof. The configuration c is sorted since the sequences $(X_1(N_i))_{i\in\mathbb{Z}}$ and $(X_2(E_i))_{i\in\mathbb{Z}}$ are weakly increasing in the bi-infinite binomial paths $(E\ell')^{\mathbb{Z}}$ and $(Nu'')^{\mathbb{Z}}$ respectively.

The configuration c satisfies the compact range assumption on the non-sink component $c^{\leq n}$: N_{y_2} (respectively N_{y_2+n-1}) is the first (respectively last) north step of the periodic pattern, which is a conjugate to $E\ell'$ that is a binomial path of $B_{m,n}$. Between N_{y_2} and N_{y_2+n-1} there are at most m east steps so

$$m \geq X_1(N_{y_2+n-1}) - X_1(N_{y_2}) = (X_1(N_{y_2+n-1}) - y_1 - 1) - (X_1(N_{y_2}) - y_1 - 1) = c_n - c_1.$$

A similar argument about the east steps E_{y_1} and E_{y_1+m-2} of $(Nu'')^{\mathbb{Z}}$ alongside a consideration of the n north steps of a conjugate of Nu'' shows that $c_{m+n-1} - c_{n+1} \leq n$. Therefore the configuration c satisfies the compact range assumption. $\square$

The following lemma shows that every stable sorted configuration on $K_{m,n}$ can be described by at least one frame $\mathsf{Frame}^y_{m,n}$ and a pair of paths (u'', ℓ').

Lemma 5.6. *For any stable sorted configuration $c = (c_1, \ldots, c_{n+m})$ on $K_{m,n}$ there exists a triple $(u'', \ell', y) \in B_{m-1,n-1} \times B_{m-1,n} \times \mathbb{Z}^2$ such that $c = \text{Gauge}_{u'',\ell'}(y)$.*

Proof. A triple (u'', ℓ', y) for which $c = \text{Gauge}_{u'',\ell'}(y)$ is given by

$$\begin{aligned} u'' &= (N^{c_{n+1}}E)(N^{c_{n+2}-c_{n+1}}E)\ldots(N^{c_{n+m-1}-c_{n+m-2}}E)N^{n-1-c_{n+m-1}} \\ \ell' &= (E^{c_1}N)(E^{c_2-c_1}N)\ldots(E^{c_n-c_{n-1}}N)E^{m-1-c_n} \\ y &= (0,0). \end{aligned}$$

This triple is well-defined because the configuration is non-negative, sorted, and stable. We leave it to the reader to verify that $\text{Gauge}_{u'',\ell'}(y) = c$. $\square$

The effect of the toppling $T^{\leq n}$ (respectively $T^{>n}$) used in Algorithm 1 may be interpreted as a move of the frame one unit step to the south (respectively west) without changing the bi-infinite paths.

Lemma 5.7. *For every $(u'', \ell') \in B_{m-1,n-1} \times B_{m-1,n}$ and $y = (y_1, y_2) \in \mathbb{Z}^2$, we have*

$$\begin{aligned} T^{\leq n}\left(\text{Gauge}_{u'',\ell'}((y_1, y_2))\right) &= \text{Gauge}_{u'',\ell'}((y_1, y_2 - 1)) \\ T^{>n}\left(\text{Gauge}_{u'',\ell'}((y_1, y_2))\right) &= \text{Gauge}_{u'',\ell'}((y_1 - 1, y_2)). \end{aligned}$$

Proof. This proof is illustrated by an example in Figure 6. Let

$$\begin{aligned} \text{Gauge}_{u'',\ell'}((y_1, y_2)) &= c = (c_1, \ldots, c_{m+n-1}) \\ \text{Gauge}_{u'',\ell'}((y_1, y_2 - 1)) &= c' = (c'_1, \ldots, c'_{n+m-1}). \end{aligned}$$

We will describe the configuration c' in terms of the configuration c by analysing the move of the frames position from (y_1, y_2) to $(y_1, y_2 - 1)$.

The decrement of y_2 by unity implies that for $i > n$,

$$c'_i = X_2(E_{y_1+(i-n)-1}) - (y_2 - 1) - 1 = (X_2(E_{y_1+(i-n)-1}) + 1) - y_2 - 1 = c_i + 1.$$

For $i \leq n$, the north steps $(N_{y_2+i-1})_{1\leq i\leq n}$ of $(E\ell')^{\mathbb{Z}}$ defining the configuration c (see Figure 5) become the north steps $(N_{(y_2-1)+i-1})_{1\leq i\leq n}$ in c' (see Figure 6). All steps except the last step (N_{y_2+n-1}) are simply shifted to the next index in this sequence, and the new first index is N_{y_2-1}.

For all of these shifted steps $(N_{y_2+i-1})_{1\leq i<n}$, the number $X_1(N_{y_2+i-1}) - y_1 - 1$ is unchanged since neither y_1 nor $X_1(N_{y_2+i-1})$ changes from c to c'. This implies that $c'_i = c_{i-1}$ for $2 \leq i \leq n$.

For $i = 1$, we remark that the north step N_{y_2-1} that appears in $(E\ell')^{\mathbb{Z}}$ which defines c'_1 differs from the disappearing north step N_{y_2+n-1} defining c_n by n. This means that these two north

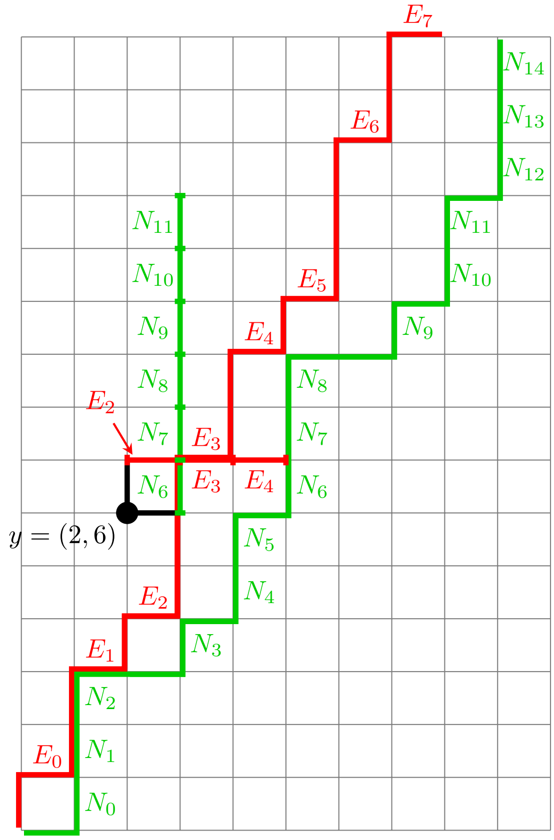


FIGURE 6. Example of moving the frame of Figure 5 one step south. Note that $m = 4$ and $n = 6$.

steps are the "same" step in the periodic pattern. This periodic pattern is an (m, n)-binomial path, so $X_1(N_{y_2-1}) = X_1(N_{y_2+n-1}) - m$ due to the m east steps of the periodic pattern. In terms of configurations, it means $c'_1 = c_n - m$.

To summarize this discussion we have

$$(c'_1, \dots, c'_{n+m-1}) = (c_n - m, c_1, \dots, c_{n-1}, c_{n+1} + 1, \dots, c_{n+m-1} + 1) = T^{\leq n}(c)$$

where the rightmost equality comes from Lemma 4.1 since, according to Lemma 5.5, the configuration c satisfies the compact range assumption. The proof for the operator $T^{>n}$ is similar: in particular the periodic pattern is a $(m-1, n)$-binomial path and we have to consider $X_2(E_{y_1-1}) = X_2(E_{y_1+(m-1)-1}) - n$. □

Notice that the operators $T^{\leq n}$ and $T^{>n}$ preserve toppling and permuting classes since both are the composition of a toppling and a (cyclic) permutation on one of the components. The following lemma shows that, with respect to the paths in the plane, there is only one equivalence class.

Lemma 5.8. *Let* $(u'', \ell') \in B_{m-1,n-1} \times B_{m-1,n}$, $x = (x_1, x_2) \in \mathbb{Z}^2$ *and* $y = (y_1, y_2) \in \mathbb{Z}^2$. *The configurations* $\mathrm{Gauge}_{u'',\ell'}(x)$ *and* $\mathrm{Gauge}_{u'',\ell'}(y)$ *are toppling and permuting equivalent.*

Proof. Let $z = (z_1, z_2) = (\min(x_1, y_1), \min(x_2, y_2))$. We show that both configurations of the lemma are toppling and permuting equivalent to the configuration $\mathrm{Gauge}_{u'',\ell'}(z)$ via some applications of the operators $T^{\leq n}$ and $T^{>n}$ which preserve the toppling and permuting classes.

Indeed, using Lemma 5.7, we have

$$\mathrm{Gauge}_{u'',\ell'}(z) = \left(T^{\leq n}\right)^{x_2 - z_2} \cdot \left(T^{>n}\right)^{x_1 - z_1} \left(\mathrm{Gauge}_{u'',\ell'}(x)\right)$$

and a similar expression exists for $\mathrm{Gauge}_{u'',\ell'}(y)$. □

To simulate Algorithm 1 using frames it remains to show that the test in the argument of the 'while' condition on line 4 of the algorithm can be realized in this setting.

Lemma 5.9. *Let* $(u'', \ell', y = (y_1, y_2)) \in B_{m-1,n-1} \times B_{m-1,n} \times \mathbb{Z}^2$ *and let* $c = \text{Gauge}_{u'',\ell'}(y)$. *Then*

(i) $c^{\leq n} \not\trianglelefteq \delta^{\leq n} \iff X_1(N_{y_2+n-1}) > y_1 + m$
(ii) $c^{\leq n} \not\trianglerighteq 0^{\leq n} \iff X_1(N_{y_2}) \leq y_1$
(iii) $c^{>n} \not\trianglelefteq \delta^{>n} \iff X_2(E_{y_1+m-2}) > y_2 + n$
(iv) $c^{>n} \not\trianglerighteq 0^{>n} \iff X_2(E_{y_1}) \leq y_2$.

Proof. Since the configuration $c = \text{Gauge}_{u'',\ell'}(y)$ is sorted, the four equivalences are respectively equivalent to $c_n > m - 1$, $c_1 < 0$, $c_{n+m-1} > n - 1$ and $c_{n+1} < 0$. The statements in (i)–(iv) give path-wise interpretations of these (simpler) inequalities. □

To complete the description of the algorithm in terms of a moving frame, it remains to show that the iterates of the operator φ visit all stable intersections. The following lemma shows that stable intersections of bi-infinite paths are exactly the bottom-left corner of frames defining stable sorted configurations.

Lemma 5.10. *Suppose that* $(u'', \ell', y) \in B_{m-1,n-1} \times B_{m-1,n} \times \mathbb{Z}^2$. *Then the configuration* $\text{Gauge}_{u'',\ell'}(y)$ *is stable if and only if* y *is a stable intersection of* $((Nu'')^{\mathbb{Z}}, (E\ell')^{\mathbb{Z}})$.

Proof. Let $c = (c_1, \ldots, c_{n+m-1}) = \text{Gauge}_{u'',\ell'}(y)$ and $y = (y_1, y_2)$.

- If the configuration c is stable, it means that $0 \trianglelefteq c \trianglelefteq \delta$. Let us first consider the steps of the path $(E\ell')^{\mathbb{Z}}$. Since $0 \leq c_1 \leq c_n \leq m - 1$ we deduce from Lemma 5.9 that

  $$X_1(N_{y_2}) > y_1 \text{ and } X_1(N_{y_2+n-1}) \leq y_1 + m.$$

  Since the periodic pattern $E\ell'$ contains m east steps and n north steps, we have

  $$X_1(N_{y_2-1}) = X_1(N_{y_2+n-1}) - m.$$

  From these two observations we have

  $$X_1(N_{y_2-1}) \leq y_1 < X_1(N_{y_2}).$$

  See Figure 7 for an illustation of why this must be do. These inequalities imply that y is a vertex of $(E\ell')^{\mathbb{Z}}$ and the strict inequality implies that y is followed by an east step.

  A similar discussion for the path $(Nu'')^{\mathbb{Z}}$ leads to the similar inequalities:

  $$X_2(E_{y_1-1}) \leq y_2 < X_2(E_{y_1}).$$

  This shows that y also belongs to $(Nu'')^{\mathbb{Z}}$ and it is followed by a north step. Therefore y is a stable intersection in $((Nu'')^{\mathbb{Z}}, (E\ell')^{\mathbb{Z}})$.
- Conversely, we assume that y is a stable intersection. Since y belongs to $(E\ell')^{\mathbb{Z}}$, we have the inequalities

  $$X_1(N_{y_2-1}) \leq y_1 < X_1(N_{y_2})$$

  where the strict equality comes from the east step following y. Since $X_1(N_{y_2-1+n}) = X_1(N_{y_2-1}) + m$ we have

  $$X_1(N_{y_2+n-1}) \leq y_1 + m.$$

  Using Lemma 5.9 we deduce that $0^{\leq n} \trianglelefteq c^{\leq n} \trianglelefteq \delta^{\leq n}$. Since y belongs to $(Nu'')^{\mathbb{Z}}$ we may deduce, in a similar manner, that $0^{>n} \trianglelefteq c^{>n} \trianglelefteq \delta^{>n}$. Thus c is a stable configuration. □

We can now finally state and prove that the computation of φ may be interpreted in term of a moving frame as a jump from a stable intersection to the preceding (closest in the south-west direction) stable intersection, if such an intersection exists.

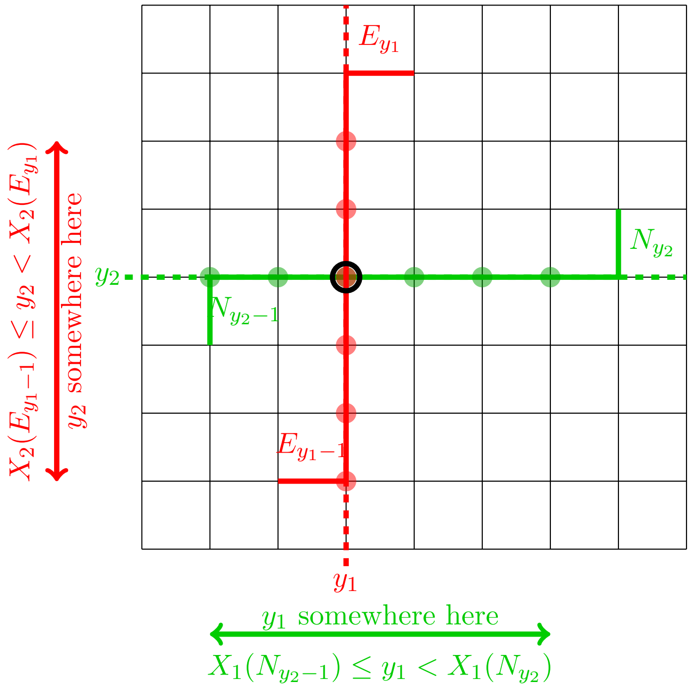


FIGURE 7.

Theorem 5.11. *Let $(u'', \ell') \in B_{m-1,m-1} \times B_{m-1,n}$ and let y be a stable intersection of the pair $\big((Nu'')^{\mathbb{Z}}, (E\ell')^{\mathbb{Z}}\big)$. Let x be the closest stable intersection of $((Nu'')^{\mathbb{Z}}, (E\ell')^{\mathbb{Z}})$ north-east of y if such an intersection exists, and $x = y$ otherwise. Then*

$$\varphi\left(\mathrm{Gauge}_{u'',\ell'}(y)\right) = \mathrm{Gauge}_{u'',\ell'}(x).$$

Proof. Let $\left\{z^{(k)} = \left(z_1^{(k)}, z_2^{(k)}\right)\right\}_{0\leq k<m}$ be the collection of m stable intersections of the pair $((Nu'')^{\mathbb{Z}}, (E\ell')^{\mathbb{Z}})$ where $z_1^{(m-1)} < \ldots < z_1^{(0)}$ and $z_2^{(m-1)} < \ldots < z_2^{(0)}$. Consider the stable configuration of some stable intersection $z^{(j)}$:

$$c = \mathrm{Gauge}_{u'',\ell'}(z^{(j)}).$$

Recall that c is stable by Lemma 5.10. Since the operator φ may be interpreted as a sequence of applications of $T^{\leq n}$ and $T^{>n}$ corresponding to unit steps to the south or west, the resulting stable configuration $c' = \varphi(c)$ is defined by a stable intersection $z^{(i)}$ where $j \leq i$:

$$c' = \mathrm{Gauge}_{u'',\ell'}(z^{(i)}).$$

If $j = m-1$, then $i = m-1$ and we arrive at the $x = y$ case of the statement. Note that these c' are the G-parking functions.

Otherwise it remains to show that $i = j+1$. We obtain this fact by a contradiction that involves the minimality of the cardinality of A in the definition of φ. Assume that $i > j+1$ and consider the configuration related to the stable intersection $z^{(j+1)}$:

$$c'' = \mathrm{Gauge}_{u'',\ell'}(z^{(j+1)}).$$

From the definition of $\varphi(c)$ in our algorithm and Proposition 4.2, we have

$$A = \{v_{n-k}, \ldots, v_n\} \cup \{v_{m+n-1-l}, \ldots, v_{n+m-1}\}$$

where $k = z_2^{(j)} - z_2^{(i)}$ and $l = z_1^{(j)} - z_1^{(i)}$.

However, from Lemma 5.7, we have

$$c'' = (T^{\leq n})^{z_2^{(j)} - z_2^{(j+1)}} \cdot (T^{>n})^{z_1^{(j)} - z_2^{(j+1)}}(c).$$

Since c'' is a stable configuration we deduce that the set

$$A' = \{v_{n-k'}, \ldots, v_n\} \cup \{v_{m+n-1-l'}, \ldots, v_{n+m-1}\}$$

where $k' = z_2^{(j)} - z_2^{(j+1)} > 0$ and $l' = z_1^{(j)} - z_1^{(j+1)} > 0$, is non-empty and thus also a candidate for the definition of $\varphi(c)$. This means that $k' < k$ and $l' < l$ so that $A' \neq A$ and $A' \subset A$, and we arrive at a contradiction to the minimality of A as claimed in the definition of φ. Therefore $\varphi(c) = c''$ and we are done. □

We conclude this subsection by a proposition formalizing the description of all sorted configurations satisfying the compact range assumption in a given toppling and permuting equivalence class. We do not need this general proposition to bring us forward, but deem it worthy of a mention.

Proposition 5.12. *Let (u'', l') be fixed in $B_{m-1,m-1} \times B_{m-1,n}$. The map $P_{u'',l'} : y \mapsto \mathrm{Gauge}_{u'',l'}(y)$ is a bijection between $\mathbb{Z}^2$ and sorted configurations which satisfy the compact range assumption and are toppling and permuting equivalent to* $\mathrm{Gauge}_{u'',l'}((0,0))$.

Proof. To show the surjectivity of $P_{u'',l'}$, we consider u to be a sorted configuration satisfying the compact range assumption. We start by showing that there exists a sorted *stable* configuration v such that $v = \left(T^{\leq n}\right)^{\alpha} \cdot \left(T^{>n}\right)^{\beta}(u)$ for $(\alpha, \beta) \in \mathbb{Z}^2$. To do this, we first check that

$$w = \left[\left(T^{\leq n}\right)^{-n} \cdot \left(T^{>n}\right)^{1-m}\right]^{-u_1} \cdot \left\{\left[\left(T^{\leq n}\right)^{-n} \cdot \left(T^{>n}\right)^{1-m}\right]^{m} \cdot \left(T^{\leq n}\right)^{n}\right\}^{\lceil -\frac{u_{n+1}}{n} \rceil}(u)$$

is a non-negative configuration since $\left[\left(T^{\leq n}\right)^{-n} \cdot \left(T^{>n}\right)^{1-m}\right]^{-u_1}$ corresponds to $-u_1$ topplings of the sink, adding exactly $-u_1$ grains to each vertex of $u^{\leq n}$. In the same way, the remaining term corresponds to the addition of $n\lceil -\frac{u_{n+1}}{n} \rceil$ grains to each vertex of $u^{>n}$. Then, we may topple in this non-negative configuration w the unstable vertices of maximal value in each component, and obtain for $(\gamma, \delta) \in \mathbb{N}^2$, the stable configuration

$$v = \left(T^{\leq n}\right)^{\gamma} \cdot \left(T^{>n}\right)^{\delta} w = \left(T^{\leq n}\right)^{\alpha} \cdot \left(T^{>n}\right)^{\beta}(u)$$

with $(\alpha, \beta) \in \mathbb{Z}^2$.

Now, we apply φ^m (which is also a combination of operators $T^{\leq n}$ and $T^{>n}$) to this v. We get the (unique) sorted recurrent configuration r of the toppling class. Hence, any sorted configuration satisfying the compact range assumption is related to the sorted recurrent configuration r via the operators $T^{\leq n}$ and $T^{>n}$, and these operators are invertible when restricted to configurations satisfying the compact range assumption. Since r is unique, this implies that u and $\mathrm{Gauge}_{u'',l'}((0,0))$ are toppling and permuting equivalent and that there exists $(\alpha', \beta') \in \mathbb{Z}^2$ such that:

$$u = \left(T^{\leq n}\right)^{\alpha'} \cdot \left(T^{>n}\right)^{\beta'}(\mathrm{Gauge}_{u'',l'}((0,0))) = \mathrm{Gauge}_{u'',l'}(y)$$

with $y = (-\beta', -\alpha')$. This proves the surjectivity of $P_{u'',l'}$.

To prove the injectivity of $P_{u'',l'}$, we define the two following parameters on a configuration u:

$$I_1(u) = \sum_{i=1}^{n+m-1} u_i \text{ and } I_2(u) = \sum_{i=1}^{n} u_i.$$

The relations

$$I_1(T^{\leq n}(u)) = I_1(u) - 1,\ I_2(T^{\leq n}(u)) = I_2(u) - n,\ I_1(T^{>n}(u)) = I_1(u) \text{ and } I_2(T^{>n}(u)) = I_2(u) + m$$

show that all $\mathrm{Gauge}_{u'',l'}(y)$ are distinct when y runs over $\mathbb{Z}^2$. □

5.2. **The operator ψ on $K_{m,n}$.** We can give a similar pictorial description for the action of the operator ψ on sorted stable configurations.

In Proposition 2.12 the operators φ and ψ were shown to be conjugate. This conjugation used the involution β which sends a configuration c to the configuration $\delta - c$. To deal with sorted configurations, let us denote by ρ the element of $S_n \times S_k$ which reverses the order of entries of both the sink and the non-sink parts of a configuration.

Now we may write for every sorted configuration c on $K_{m,n}$:

$$\psi(c) = \rho \cdot \beta \cdot \varphi \cdot \rho \cdot \beta(c).$$

In this way, we may compute the action of ψ on sorted configurations through the action of φ on the same set.

Theorem 5.13. *Let $(u'', \ell') \in B_{m-1,n-1} \times B_{m-1,n}$ and let y be a stable intersection of the pair $\big((Nu'')^{\mathbb{Z}}, (E\ell')^{\mathbb{Z}}\big)$. Let x be the closest stable intersection of $((Nu'')^{\mathbb{Z}}, (E\ell')^{\mathbb{Z}})$ in the south-west direction if such an intersection exists, and $x = y$ otherwise. Then*

$$\psi\left(\mathrm{Gauge}_{u'',\ell'}(y)\right) = \mathrm{Gauge}_{u'',\ell'}(x).$$

Given a binomial word $u = u_1 u_2 \dots u_{k-1} u_k$ written as k letters, define the reverse of u to be $\rho(u) = u_k u_{k-1} \dots u_2 u_1$. The proof of the previous theorem relies on the following lemma.

Lemma 5.14. *For any triple $(u'', \ell', y) \in B_{m-1,n-1} \times B_{m-1,n} \times \mathbb{Z}^2$ we have*

$$\rho \cdot \beta \cdot \mathrm{Gauge}_{u'',\ell'}((y_1, y_2)) = \mathrm{Gauge}_{\rho(u''),\rho(\ell')}((-y_1, -y_2)).$$

Proof. If $\mathrm{Gauge}_{u'',\ell'}((0,0)) = (a_0, \dots, a_{n-1}, b_0, \dots, b_{m-2})$ then

$$\ell' = E^{a_0 - 0} N E^{a_1 - a_0} N \dots E^{a_{n-1} - a_{n-2}} N E^{m-1-a_{n-1}}$$
$$\implies \rho(\ell') = E^{m-1-a_{n-1}} N E^{a_{n-1} - a_{n-2}} N \dots E^{a_1 - a_0} N E^{a_0 - 0}$$

and

$$u'' = N^{b_0 - 0} E N^{b_1 - b_0} E \dots E N^{n-1-b_{m-2}}$$
$$\implies \rho(u'') = N^{n-1-b_{m-2}} E N^{b_{m-2} - b_{m-3}} E \dots E N^{b_0 - 0}.$$

From the paths for $N\rho(u'')$ and $E\rho(\ell')$ we see that

$$\mathrm{Gauge}_{\rho(u''),\rho(\ell')}((0,0)) = (m-1-a_{n-1}, m-1-a_{n-2}, \dots, m-1-a_0, n-1-b_{m-2}, \dots, n-1-b_0).$$

Applying β to $\mathrm{Gauge}_{u'',\ell'}((0,0))$ gives

$$\beta \cdot \mathrm{Gauge}_{u'',\ell'}((0,0)) = (m-1-a_0, \dots, m-1-a_{n-1}, n-1-b_0, \dots, n-1-b_{m-2}).$$

Finally, applying ρ to this configuration gives

$$\rho \cdot \beta \cdot \mathrm{Gauge}_{u'',\ell'}((0,0))$$
$$= (m-1-a_{n-1}, \dots, m-1-a_0, n-1-b_{m-2}, \dots, n-1-b_0) = \mathrm{Gauge}_{\rho(u''),\rho(\ell')}((0,0)).$$

Hence, the claimed formula is satisfied for $y = (0,0)$

We extend it next to any y as follows. Note that the operators $T^{\leq n}$ and $T^{>n}$ when restricted to configurations satisfying the compact range assumption have well-defined inverses and are mutually commutative. We obtain by inspection, similar to that given at the start of this proof, the following identities

$$(T^{\leq n})^{-1} = \rho \cdot \beta \cdot T^{\leq n} \cdot \rho \cdot \beta \quad \text{and} \quad (T^{>n})^{-1} = \rho \cdot \beta \cdot T^{>n} \cdot \rho \cdot \beta.$$

Then the following relation, deduced from Lemma 5.7, leads to the claim for any $y \in \mathbb{Z}^2$

$$\mathrm{Gauge}_{u'',\ell'}((y_1, y_2)) = (T^{\leq n})^{-y_2} \cdot (T^{>n})^{-y_1} (\mathrm{Gauge}_{u'',\ell'}((0,0))).$$

Indeed, using in addition the fact that $\rho \cdot \beta$ is an involution, we have

$$\begin{aligned}\rho\cdot\beta(\mathrm{Gauge}_{u'',\ell'}((y_1,y_2))) &= \rho\cdot\beta\cdot(T^{\leq n})^{-y_2}\cdot(T^{>n})^{-y_1}(\mathrm{Gauge}_{u'',\ell'}((0,0))\\ &= \rho\cdot\beta\cdot(T^{\leq n})^{-y_2}\cdot(T^{>n})^{-y_1}\cdot\rho\cdot\beta(\mathrm{Gauge}_{\rho(u''),\rho(\ell')}((0,0)))\\ &= (T^{\leq n})^{y_2}\cdot(T^{>n})^{y_1}(\mathrm{Gauge}_{\rho(u''),\rho(\ell')}((0,0)))\\ &= \mathrm{Gauge}_{\rho(u''),\rho(\ell')}((-y_1,-y_2)).\end{aligned}$$

□

Proof. (of Theorem 5.13) We have the following equivalences:

$$\begin{aligned}&y=(y_1,y_2) \text{ is a stable intersection of } ((Nu'')^{\mathbb{Z}},(E\ell')^{\mathbb{Z}})\\ \iff\quad &\mathrm{Gauge}_{u'',\ell'}((y_1,y_2)) \text{ is a stable configuration}\\ \iff\quad &\mathrm{Gauge}_{\rho(u''),\rho(\ell')}((-y_1,-y_2)) \text{ is a stable configuration}\\ \iff\quad &(-y_1,-y_2) \text{ is a stable intersection of } ((N\rho(u''))^{\mathbb{Z}},(E\rho(\ell'))^{\mathbb{Z}}),\end{aligned}$$

where in the second equivalence one uses the fact that the involution $\rho \cdot \beta$ is also an involution when restricted to stable configurations.

According to Theorem 5.11, we consider the $x = (x_1, x_2)$ stable intersection after $(-y_1, -y_2)$ in $((N\rho(u''))^{\mathbb{Z}}, (E\rho(\ell'))^{\mathbb{Z}})$, if any, and $(x_1, x_2) = (-y_1, -y_2)$ otherwise.

We have

$$\begin{aligned}\psi(\mathrm{Gauge}_{u'',\ell'}((y_1,y_2))) &= \rho\cdot\beta\cdot\varphi\cdot\rho\cdot\beta(\mathrm{Gauge}_{u'',\ell'}((y_1,y_2))\\ &= \rho\cdot\beta\cdot\varphi(\mathrm{Gauge}_{\rho(u''),\rho(\ell')}((-y_1,-y_2)))\\ &= \rho\cdot\beta(\mathrm{Gauge}_{\rho(u''),\rho(\ell')}((x_1,x_2)))\\ &= \mathrm{Gauge}_{u'',\ell'}((-x_1,-x_2))).\end{aligned}$$

To conclude we observe that $(-x_1, -x_2)$, if different from (y_1, y_2), is the stable intersection preceding (y_1, y_2) in $((Nu'')^{\mathbb{Z}}, (E\ell')^{\mathbb{Z}})$. □

5.3. **Consequences of the pictorial interpretations.** An interesting consequence of Theorem 5.13 is a pictorial characterization of the sorted $K_{m,n}$-parking configurations.

Corollary 5.15. *The sorted $K_{m,n}$-parking configurations are the stable ones described by a pair of periodic bi-infinite paths $((Nu'')^{\mathbb{Z}}, (El')^{\mathbb{Z}})$ and a stable intersection (y_1, y_2) such that for $i = 1, \ldots, m-1$, the east step E_{y_1+i} of $(El')^{\mathbb{Z}}$ satisfies* $\mathrm{pos}(E_{y_1+i}) \leq 0$.

Proof. Let u be a sorted stable configuration.

If there exists i such that $\mathrm{pos}(E_{y_1+i}) \geq 1$, then there exists a stable intersection (y'_1, y'_2) such that $y'_1 < y_1$ or $y'_2 < y_2$. Let $u' = \mathrm{Gauge}_{u'',l'}((y'_1, y'_2))$. Since $\varphi^k(u) = \left(T^{\leq n}\right)^{\alpha'(k)} \cdot (T^{>n})^{\beta'(k)}(u)$, by Lemma 5.7 we have $(\alpha'(k), \beta'(k)) = (y_2 - y'_2, y_1 - y'_1) \in \mathbb{N}^2 - \{(0, 0)\}$, and we deduce that $\varphi(u) \neq u$ is therefore not $K_{m,n}$-parking.

If for all $i = 1, \ldots, m-1$, $\mathrm{pos}(E_{y_1+1}) \leq 0$ then there is no stable intersection strictly before (y_1, y_2) and the description of φ by positive powers of $\varphi^k(u) = \left(T^{\leq n}\right)^{\alpha(k)} \cdot (T^{>n})^{\beta(k)}(u)$ for some $(\alpha(k), \beta(k)) \in \mathbb{N}^2$ given by the algorithm implies that $\varphi(u) = u$, hence u is $K_{m,n}$-parking. □

Remark 5.16. *This remark is a sequel to Remark 2.9. Theorems 5.11 and 5.13 show, in particular, that the operators φ and ψ acting on the sorted stable configurations on $K_{m,n}$ are essentially inverses of each other.*

In fact, if c is a sorted stable configuration which is not recurrent, then $\varphi(\psi(c)) = c$, and if c is a sorted stable configuration which is not parking, then $\psi(\varphi(c)) = c$.

Moreover, in these cases the operators ψ and φ are inverses of each others in the sense of semigroups, i.e. $\psi(\varphi(\psi(c))) = \psi(c)$ and $\varphi(\psi(\varphi(c))) = \varphi(c)$ for all sorted stable configurations c.

Starting with any sorted stable configuration on $K_{m,n}$, Lemma 5.10 gives us a stable intersection of a pair of bi-infinite periodic paths. According to Theorem 5.11, we can act iteratively with ψ, moving between stable points until we get a sorted recurrent configuration. Then we can move back, acting with ϕ, according to Theorem 5.13, until we get a sorted $K_{m,n}$-parking configuration. In this way, we always pass through m distinct sorted stable configurations, since these configurations correspond to the stable intersections of the corresponding periodic bi-infinite paths counted in Lemma 3.1. Notice also that every sorted stable configuration occurs in one of these m-sets.

This discussion provides the following *graduated description* of all the m sorted stable configurations on $K_{m,n}$ in each toppling and permuting class.

Corollary 5.17. *Let c be the sorted recurrent configuration of a toppling and permuting class of the sandpile model on $K_{m,n}$. Let $(Nu''E, E\ell')$ be the parallelogram polyomino describing c. Let $(z^{(0)}, \ldots, z^{(m-1)})$ be the ordered stable intersections of $((Nu'')^{\mathbb{Z}}, (E\ell')^{\mathbb{Z}})$. Then the m sorted stable configurations toppling and permuting equivalent to c are described by*

$$\mathrm{Gauge}_{u'',\ell'}(z^{(k)}) = \varphi^k(c)$$

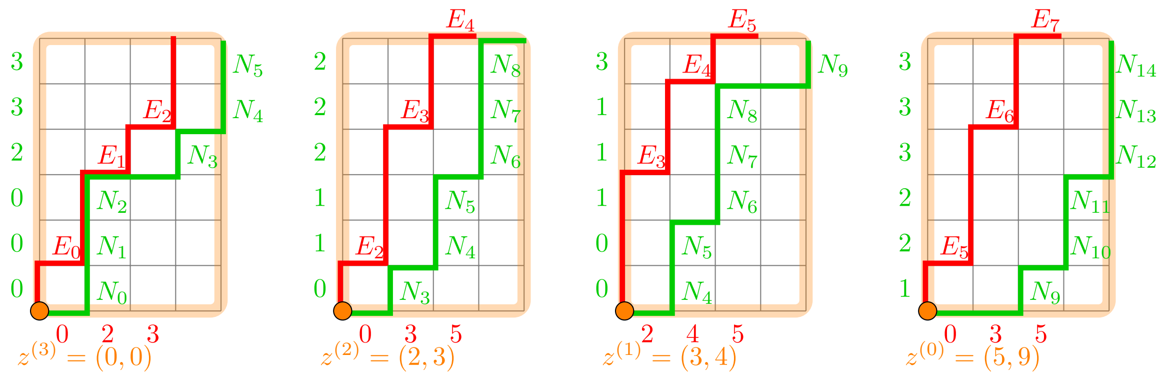
for all $0 \leq k \leq m-1$.

Similarly, let c be a sorted $K_{m,n}$-parking configuration of a toppling and permuting class of the sandpile model on $K_{m,n}$. Let $(Nu''E, E\ell')$ be the pair of binomial paths describing c. Let $(z^{(0)}, \ldots, z^{(m-1)})$ be the ordered stable intersections of $((Nu'')^{\mathbb{Z}}, (E\ell')^{\mathbb{Z}})$. Then the m sorted stable configurations toppling and permuting equivalent to c are described by

$$\mathrm{Gauge}_{u'',\ell'}(z^{(k)}) = \psi^{m-k-1}(c)$$

for all $0 \leq k \leq m-1$.

Figure 8 illustrates the 4 sorted stable configurations of the toppling and permuting class described by the example in Figure 2.



(a) $\varphi^3(u) = \binom{0,0,0,2,3,3,}{0,2,3,*}$ (b) $\varphi^2(u) = \binom{0,1,1,2,2,2,}{0,3,5,*}$ (c) $\varphi(u) = \binom{0,0,1,1,1,3,}{2,4,5,*}$ (d) $u = \binom{1,2,2,3,3,3,}{0,3,5,*}$

FIGURE 8. The toppling and permuting equivalent sorted stable configurations from parking to recurrent configurations. We remind the reader that φ^3 is a parking function.

As another corollary, we also recover the known bijection (see [10]) between sorted recurrent configurations on $K_{m,n}$ and parallelogram polyominoes, since those are exactly the fixed points of the (pictorial) operator ψ.

These descriptions of the extremal stable configurations in the graduation, be they parking or recurrent, are particular cases of the following general and *local* description of the grade of a stable configuration.

For any sorted stable configuration c, we have a pair of finite binomial paths (u'', ℓ') which occurs in a pair p of bi-infinite paths after a stable intersection $y = (y_1, y_2)$. We consider the m east steps of the (green) factor $E\ell'$, which are the $(E_{y_1+k})_{0 \leq k < m}$ in p. We define

$$P^{[y]}_{\geq 1}(c) = \{(y_1 + k) \mod m \;:\; 0 \leq k < m \text{ and } \mathrm{pos}(E_{y_1+k}) \geq 1\}$$

which describes the east steps in $E\ell'$ whose parameter pos is at least 1. The *grade* grade(c) of a stable configuration c is defined as the cardinality of $P^{[y]}_{\geq 1}(c)$.

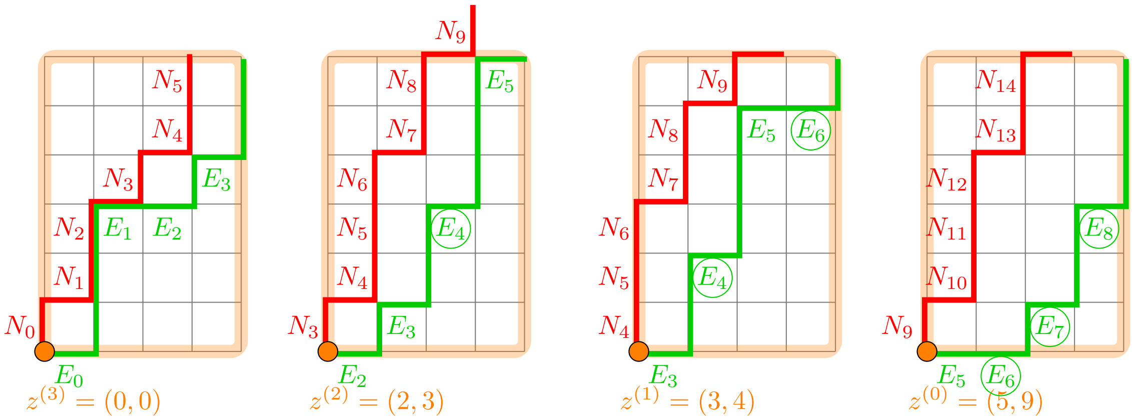
We remark that this definition of grade is not changed by a translation $t = (t_1, t_2) \in \mathbb{Z}^2$ of the pair p of bi-infinite path. Indeed, the stable intersection describing c becomes $y + t = (y_1 + t_1, y_2 + t_2)$ and

$$P^{[y+t]}_{\geq 1}(c) = \{(x + t_1) \mod m \ : \ x \in P^{[y]}_{\geq 1}(c)\}$$

hence $|P^{[y+t]}_{\geq 1}(c)| = |P^{[y]}_{\geq 1}(c)|$. So an equivalent and explicitly local definition is

$$\text{grade}(c) = |P^{[(0,0)]}_{\geq 1}(c)|.$$

In Figure 9 we reproduce the stable configurations in Figure 8, mentioning now the indices of east green steps and north red steps, used to compute the $\text{pos}(E_{y_1+k})$ in the definition of $P^{[y]}_{\geq 1}$. We draw a circle around the green east steps such that $\text{pos}(E_{y_i}) \geq 1$. By additional convention, the sorted parking configuration of a toppling and permuting equivalent class is described by a stable intersection at the origin $z^{(m-1)} = z^{(3)} = (0, 0)$. This additional convention induces a global choice of stable intersections for the stable configurations of this class. This convention will be used in the proof of the following Proposition 5.18.



(a) $P^{[z^{(3)}]}_{\geq 1}(\varphi^3(u)) = \{\}$ (b) $P^{[z^{(2)}]}_{\geq 1}(\varphi^2(u)) = \{0\}$ (c) $P^{[z^{(1)}]}_{\geq 1}(\varphi(u)) = \{0, 2\}$ (d) $P^{[z^{(0)}]}_{\geq 1}(u) = \{0, 2, 3\}$

FIGURE 9. Evaluation of $P^{[z^{(i)}]}_{\geq 1}$ from parking to recurrent configurations as in Figure 8.

Proposition 5.18. *Let c be a sorted stable configuration on $K_{m,n}$. Let* parking(c) *(resp.* recurrent(c)*) be the parking (resp. recurrent) configuration in the toppling and permuting class of c. We have*

$$\psi^{\text{grade}(c)}(\text{parking}(c)) = c = \varphi^{m-1-\text{grade}(c)}(\text{recurrent}(c)).$$

Proof. Let $(z^{(i)})_{i=0,\dots,m-1}$ the m stable intersections in the pair of paths related to c. By convention we assume without loss of generality that $z^{(m-1)} = (0, 0)$. We shall use the notation $z^{(i)} = (X_1(z^{(i)}), X_2(z^{(i)}))$ for stable intersections as we did for steps. For each green east step E_k, $k \in \mathbb{Z}$, the east green step $E_{k-m\cdot\text{pos}(E_k)}$ belongs to a stable intersection denoted $z^{(f(k))}$ since

$$\text{pos}(E_{k-m\cdot\text{pos}(E_k)}) = \text{pos}(E_k) - \text{pos}(E_k) = 0,$$

using the relation $\text{pos}(E_{k+m}) = \text{pos}(E_k) + 1$ induced by periodicities of paths. Hence, we have the equivalence

$$\text{pos}(E_k) \geq 1 \iff X_1(E_k) > X_1(z^{(f(k))}).$$

Using this equivalence the definition of $P^{[z^{(i)}]}_{\geq 1}(\varphi^i(\text{recurrent}(c)))$ becomes

$$P^{[z^{(i)}]}_{\geq 1}(\varphi^i(\text{recurrent}(c))) = \{X_1(z^{(j)}) \mod m \ : \ j = m-1, \dots, j+1\}$$

since $\{z^{(j)}\}_{j=0,\ldots,m-1} = \{z^{(f(E_{X_1(z^{(i)})+k}))}\}_{k=0,\ldots,m-1}$, $X_1(z^{(0)}) > X_1(z^{(1)}) > \ldots > X_1(z^{(m-1)})$ and for the step $E_{X_1(z^{(i)})+k}$

$$\begin{aligned}\mathrm{pos}(E_{X_1(z^{(i)})+k}) \geq 1 \quad &\iff \quad X_1(z^{(f(X_1(z^{(i)})+k))}) < X_1(z^{(i)}) + k \\ &\iff \quad X_1(z^{(f(X_1(z^{(i)})+k))}) \leq X_1(z^{(i)}) + k - m < X_1(z^{(i)})\end{aligned}$$

where the last equivalence uses $X_1(z^{(f(X_1(z^{(i)})+k))}) - (X_1(z^{(i)})+k) \mod m = 0$. This equivalent definition implies that

$$\mathrm{grade}(\varphi^i(\mathrm{recurrent}(c))) = m - 1 - i$$

and the proposition follows. □

6. Some enumerative results

In this section we will present some enumerative results that we can derive by considering pairs of bi-infinite paths in which one of the paths has a particularly regular step-like structure. Specializations of our Cyclic Lemma lead to lattice path enumerations that are new, e.g. Proposition 6.3, and already established, e.g. Proposition 6.1.

Let p be a binomial word on the alphabet $\{N, E\}$, and which we will call a pattern in this context. A binomial word w *cyclically matches* the pattern p if Ew may be decomposed as $Ew = fg$ where $gf = p$. Let $\mathrm{Cyc}[p]$ be the set of binomial words that cyclically match p. We denote by $\mathrm{Polyo}[p]$ the polyominoes whose lower path is p.

Let a, b and c be positive integers. Fix $p = (E^aN^b)^c$ and consider $\mathrm{Polyo}[(E^aN^b)^c]$, the set of parallelogram polyominoes having an $ac \times bc$ bounding box and such that the lower path is $(E^aN^b)^c$. It transpires that one can restrict the Cyclic Lemma to the pairs in the cartesian product $B_{cb-1,ca-1} \times \mathrm{Cyc}[(E^aN^b)^c]$ in order to count those parallelogram polyominoes in $\mathrm{Polyo}[(E^aN^b)^c]$. This gives us the following result which also appears in Irving and Rattan [15, Cor. 16] in 2009 and which can be further traced back to Bonin, de Mier and Noy [5, Thm. 8.3] in 2003. The proof of Bonin et al. [5, Thm. 8.3] is a specialization of our more general Cyclic Lemma.

Proposition 6.1. *For all $a, b, c \geq 1$, we have*

$$|\mathrm{Polyo}[(E^aN^b)^c]| = \frac{1}{c}\binom{c(b+a)-2}{ca-1}.$$

Proof. The set $\mathrm{Polyo}[(E^aN^b)^c]$, as a subset of parallelogram polyominoes having a $ca \times cb$ bounding box, corresponds to a subset of sorted recurrent configurations on $K_{ac,ab}$. To be able to apply the Cyclic Lemma, we have to identify in what follows the set $P^{a,b,c}$ of all possible pairs in $B_{ca-1,cb-1} \times B_{ca-1,cb}$ that come from stable intersections of pairs of paths $((Nu'')^{\mathbb{Z}}, (E\ell')^{\mathbb{Z}})$ related to the configurations in the subset of sorted recurrent configurations.

First we provide a necessary condition on $P^{a,b,c}$. Recall that $E\ell' = (E^aN^b)^c$. At a stable intersection y, we have the binomial word w such that $Ew = [((E^aN^b)^c)^{\mathbb{Z}}]_{y|c(a+b)}$ is any period of $((E^aN^b)^c)^{\mathbb{Z}}$ which starts with the letter E. So w necessarily cyclically matches $(E^aN^b)^c$, i.e. $w \in \mathrm{Cyc}[(E^aN^b)^c]$.

Next we show that for $(v, w) \in B_{ca-1,cb-1} \times B_{ca-1,cb}$, the condition $w \in \mathrm{Cyc}[(E^aN^b)^c]$ is also a sufficient condition for $(v, w) \in P^{a,b,c}$. In this case we remark that the single parallelogram polyomino defined by a stable intersection of $((Nv)^{\mathbb{Z}}, (Ew)^{\mathbb{Z}})$ belongs to $\mathrm{Polyo}[(E^aN^b)^c]$. This is because its lower path is Ef where $f \in \mathrm{Cyc}[(E^aN^b)^c]$ and Ef ends in the letter N, as it does for any parallelogram polyomino, hence $Ef = (E^aN^b)^c$. This gives us the following description of pairs of paths involved in these instances of the cyclic lemma:

$$P^{a,b,c} = B_{ca-1,cb-1} \times \mathrm{Cyc}[(E^aN^b)^c].$$

We remark that $|\mathrm{Cyc}[(E^aN^b)^c]| = a$ since it is easily shown that this set is in bijection with marking one letter E in the factor E^aN^b. Therefore,

$$|\mathrm{Polyo}[(E^aN^b)^c]| = \frac{1}{ca}|B_{ca-1,cb-1}||\mathrm{Cyc}[(E^aN^b)^c]|. \qquad \square$$

We leave it to the reader to verify the following classical results obtained here as a corollary.

Corollary 6.2.

(i) $\mathrm{Polyo}[(EN)^{n+1}]$ *is in bijection with Dyck words of semi-length* n. *(In this case the restriction of the cyclic lemma is essentially the Dvoretsky-Motzkin cyclic lemma.)*

(ii) $\mathrm{Polyo}[(EN^m)^n]$ *is in bijection with paths in* $B_{n,mn}$ *consisting of* $n(m+1)$ *steps which are above the line of slope* m.

A symmetry on the parallelogram polyominoes allows us to extend these results and count parallelogram polyominoes whose lower path is $(E^aN^aE^bN^b)^c$. This kind of periodic conditions seems to be new, in particular it is not covered by Theorem 5 in the work of Chapman, Chow, Khetan, Petrie-Moulton and Waters [6].

Proposition 6.3. *For all* $a, b, c \geq 1$ *such that* $a \neq b$ *we have*

$$|\mathrm{Polyo}[(E^aN^aE^bN^b)^c]| = \frac{1}{2c}\binom{2c(a+b)-2}{c(a+b)-1}.$$

Proof. The proof is a variation on the previous proof of Proposition 6.1. We reuse the notation $P^{a,b,c}$ to denote similar but now different sets of objects.

In this case $\mathrm{Cyc}[(E^aN^aE^bN^b)^c]$ has cardinality $a+b$ (seen by marking a letter E in the factor $E^aN^aE^bN^b$). The pairs of paths involved in the restriction of the cyclic lemma are still described by the Cartesian product

$$P^{a,b,c} = B_{c(a+b)-1,c(a+b)-1} \times \mathrm{Cyc}[(E^aN^aE^bN^b)^c].$$

The main difference is that now the parallelogram polyominoes involved in the cyclic lemma have two possible fixed lower paths. More precisely, these polyominoes are defined as

$$\mathrm{Polyo}^{a,b,c} = \mathrm{Polyo}[(E^aN^aE^bN^b)^c] \cup \mathrm{Polyo}[(E^bN^bE^aN^a)^c]$$

which is a disjoint union since $a \neq b$. Using this in conjunction with the cyclic lemma we get

$$|\mathrm{Polyo}^{a,b,c}| = \frac{1}{c(a+b)}\binom{2c(a+b)-2}{c(a+b)-1}(a+b).$$

Let κ by the involutive word morphism defined on letters by $\kappa(E) = N$ and $\kappa(N) = E$. When $\rho \cdot \kappa$ is applied to upper and lower paths of $\mathrm{Polyo}[(E^aN^aE^bN^b)^c]$ we obtain an involution which maps to $\mathrm{Polyo}[(E^bN^bE^aN^a)^c]$. Hence

$$|\mathrm{Polyo}[(E^aN^aE^bN^b)^c]| = |\mathrm{Polyo}[(E^bN^bE^aN^a)^c]|,$$

and so

$$|\mathrm{Polyo}[(E^aN^aE^bN^b)^c]| = \frac{1}{2}|\mathrm{Polyo}^{a,b,c}|. \qquad \square$$

7. Operators φ and ψ on K_n

In this section, we show how we can derive a description of the operators φ and ψ in the case of the complete graph K_n from our results for $K_{m,n}$ by setting $m = n$. As we shall show in Proposition 7.2, the operators φ and ψ on K_n may be simulated by (variable) powers of the operators φ and ψ acting on special configurations on $K_{n,n}$.

As in the previous sections, all computations are equivalent up to permutations of the entries of the configurations. Therefore, and without loss of generality, we will be able to work at the level of orbits, ie. to use the *sorted* configurations as representatives.

We start with some definitions and notations. For any vector $v = (v_i)_{i \in I}$, we denote $v \oplus 1 = (v_i + 1)_{i \in I}$ and $v \ominus 1 = (v_i - 1)_{i \in I}$. A configuration u on $K_{n,n}$ is called *staircase* if $u^{\leq n}$ is a permutation of $0, 1, \ldots, n-1$. A configuration u on $K_{n,n}$ is said to be 0-*free* if $u_i \geq 1$ for all $i = n+1, \ldots, 2n-1$, *i.e.* all entries of $u^{>n}$ are at least equal to 1.

Now we give a lemma which links topplings in K_n to topplings in $K_{n,n}$. Roughly speaking, this lemma says that a given toppling in K_n corresponds to *two* topplings in $K_{n,n}$.

Lemma 7.1. *Let v be a configuration on K_n, and u be any staircase configuration on $K_{n,n}$ such that $u^{>n} = v$. Let j be the (unique) vertex of the non-sink component (i.e. $1 \leq j \leq n$) of u such that $u_j = n-1$. Then for any $i \in \{n+1, \ldots, 2n-1, 2n\}$ (including the sink), we have:*

$$u - \Delta_i - \Delta_j = (\eta(u^{\leq n}), v - \Delta_{i-n})$$

where for any vector $w = (w_i)_{i=1,\ldots,n}$, we denote $\eta(w) = (w_i + 1 \mod n)_{i=1,\ldots,n}$. In particular, $u - \Delta_i - \Delta_j$ is also staircase.

Proof. The toppling Δ_i sends n grains to the non-sink component of u. Because u is staircase, this induces exactly one toppling Δ_j that sends back n grains to the sink component (including the vertex i). Thus in the sink component, the configuration is the one obtained by performing Δ_{i-n} to v in K_n: v_i is decreased by $n-1$, the other $v_{i'}$'s are increased by 1. For what concerns the non-sink component, the vertex j loses its $n-1$ grains, the other vertices get 1 grain each. By observing that $0 = (n-1) + 1 \mod n$, we conclude that $u^{\leq n}$ is mapped to $\eta(u^{\leq n})$. □

Proposition 7.2. *Let v be a stable configuration on K_n, and let $u = (u^{\leq n}, v \oplus 1)$ be a staircase (0-free) configuration on $K_{n,n}$. We have*

$$\varphi(v) = \left(\varphi^k(u)\right)^{>n} \ominus 1 \text{ and } \psi(v) = \left(\psi^l(u)\right)^{>n} \ominus 1$$

where k is the minimal positive integer such that $\varphi^k(u)$ is 0-free and distinct from u, if any, otherwise $k = 0$, and l is the minimal positive integer such that $\psi^l(u)$ is 0-free and distinct from u, if any, otherwise $l = 0$.

Proof. Let us first examine the assertion on φ.

We start by observing that the configuration v on K_n is stable if and only if a staircase configuration of the form $u = (u^{\leq n}, v \oplus 1)$ is stable and 0-free. Let v be a stable configuration on K_n, and let $u = (u^{\leq n}, v \oplus 1)$ be a staircase configuration on $K_{n,n}$.

Suppose first that $\varphi(v) \neq v$, and let $\varphi(v) = v - \Delta_A$, so that A is the minimal non-empty subset of $\{1, 2, \ldots, n\}$ such that $\varphi(v) = v - \Delta_A$ is stable. Let $B = B^{\leq n} \cup B^{>n}$ where $B^{\leq n} = \{j \mid j \leq n \text{ and } u_j + |A| \geq n\}$ and $B^{>n} = \{n + i \mid i \in A\}$. First notice that, since $u^{\leq n}$ is a permutation of $\{0, 1, 2, \ldots, n-1\}$, we have $|B^{>n}| = |A| = |B^{\leq n}|$. For each $i \in B^{>n}$, we can consider $j \leq n$ such that $u_j = n-1$ and apply Lemma 7.1, getting

$$u - \Delta_i - \Delta_j = \left(\eta(u^{\leq n}), (v \oplus 1) - \Delta_{i-n}\right) = \left(\eta(u^{\leq n}), (v - \Delta_{i-n}) \oplus 1\right).$$

Now we can iterate this application of the lemma with $B \setminus \{i, j\} = (B^{\leq n} \setminus \{j\}) \cup (B^{>n} \setminus \{i\})$, taking some $k \in B^{>n} \setminus \{i\}$ and $h \in B^{\leq n} \setminus \{j\}$ such that the h-th component of $\eta(u^{\leq n})$ is equal to $n-1$. At the end of the iteration, we get

$$u - \Delta_B = \left(\eta^{|A|}(u^{\leq n}), (v - \Delta_A) \oplus 1\right),$$

so $(u - \Delta_B)^{>n} = (v - \Delta_A) \oplus 1$, or equivalently $(u - \Delta_B)^{>n} \ominus 1 = v - \Delta_A$. In particular $u - \Delta_B$ is stable and 0-free.

Using the properties of φ on $K_{m,n}$ (with $m = n$) that are given in Proposition 5.18, the configuration $u - \Delta_B$ is toppling (and permuting) equivalent to u and both are sorted and stable configurations. Hence, up to permutation, $u - \Delta_B = \varphi^{k'}(\text{parking}(u))$ and $u = \varphi^{k''}(\text{parking}(u))$ so $u - \Delta_B = \varphi^k(u)$ where $k = k'' - k'$. So it remains to show that such a k is minimal with the property that $\varphi^k(u)$ is 0-free.

If this is not the case, let i be such that $0 < i < k$, and $u' = \varphi^i(u) = u - \Delta_C$ is stable and 0-free, where $C = C^{\leq n} \cup C^{>n}$ is the partition of C given by the intersections with the non-sink and the sink components of $K_{n,n}$. By definition, in the non-sink component of the staircase configuration u any height *modulo* n appears exactly once. Notice that this property is preserved by any toppling. Since the resulting configuration u' is stable, u' must be also staircase. So, after the topplings, $u' = u - \Delta_C$ has preserved the number of grains in the non-sink component, therefore we must have $|C^{\leq n}| = |C^{>n}|$.

As we already observed, by Lemma 7.1

$$u' = \left(\eta^{|D|}(u^{\leq n}), (v - \Delta_D) \oplus 1\right)$$

where $D = \{c - n \ : \ c \in C^{>n}\}$. This definition of D is such that $C^{>n} = \{d + n : d \in D\}$, i.e. the toppling of $d \in D$ corresponds to the topplings of $d + n$ in $C^{>n}$ and the vertex in $C^{\leq n}$ of value n after this first toppling. As u' is 0-free, $v - \Delta_D$ is stable, but this contradicts the minimality of A, since $D \subsetneq A$. This shows that k is minimal.

If instead $\varphi(v) = v$, then there is no non-empty $A \subseteq \{1, 2, \ldots, n\}$ such that $\varphi(v) = v - \Delta_A$ is stable. If there is a positive integer k such that $\varphi^k(u) = u - \Delta_C$ is 0-free, then, by what we observed earlier, $|C^{\leq n}| = |C^{>n}|$ and

$$\varphi^k(u) = \left(\eta^{|D|}(u^{\leq n}), (v - \Delta_D) \oplus 1\right)$$

where $D = \{c - n \ : \ c \in C^{>n}\}$. As $\varphi^k(u)$ is 0-free, $v - \Delta_D$ is stable, which implies $D = \emptyset$. Therefore $C = \emptyset$ and $\varphi^k(u) = u$, which implies $\varphi(u) = u$, i.e. u is $K_{n,n}$-parking. But this is a contradiction, since a $K_{n,n}$-parking staircase configuration cannot be 0-free.

Therefore there is no such positive k, hence by definition $\varphi(v) = v = (\varphi^0(u))^{>n} \ominus 1$ as claimed.

The assertion for the operator ψ is proved analogously. First of all there is an analogue of Lemma 7.1, that is proved in the same way: if v is a configuration on K_n, and u is any staircase configuration on $K_{n,n}$ such that $u^{>n} = v$, let j be the (unique) vertex of the non-sink component (i.e. $1 \leq j \leq n$) of u such that $u_j = 0$; then for any $i \in \{n+1, \ldots, 2n-1, 2n\}$ (including the sink), we have:

$$u + \Delta_i + \Delta_j = \left(\widetilde{\eta}(u^{\leq n}), v + \Delta_{i-n}\right)$$

where for any vector $w = (w_i)_{i=1,\ldots,n}$, we denote $\widetilde{\eta}(w) = (w_i - 1 \mod n)_{i=1,\ldots,n}$. In particular, $u + \Delta_i + \Delta_j$ is also staircase.

Using this, if $\psi(v) \neq v$, then let $\psi(v) = v + \Delta_A$, so that A is the minimal non-empty subset of $\{1, 2, \ldots, n\}$ such that $\psi(v) = v + \Delta_A$ is stable. Let $B = B^{\leq n} \cup B^{>n}$ where $B^{\leq n} = \{j \mid j \leq n \text{ and } u_j - |A| \lneq 0\}$ and $B^{>n} = \{n + i \mid i \in A\}$. First notice that, since $u^{\leq n}$ is a permutation of $\{0, 1, 2, \ldots, n-1\}$, we have $|B^{>n}| = |A| = |B^{\leq n}|$. For each $i \in B^{>n}$, we can consider $j \leq n$ such that $u_j = 0$ and apply the previous lemma, getting

$$u + \Delta_i + \Delta_j = \left(\widetilde{\eta}(u^{\leq n}), (v \oplus 1) + \Delta_{i-n}\right) = \left(\widetilde{\eta}(u^{\leq n}), (v + \Delta_{i-n}) \oplus 1\right).$$

Now, as we did for φ, we can iterate this application of the lemma, getting in the end

$$u + \Delta_B = \left(\widetilde{\eta}^{|A|}(u^{\leq n}), (v + \Delta_A) \oplus 1\right),$$

so $(u + \Delta_B)^{>n} = (v + \Delta_A) \oplus 1$, or equivalently $(u + \Delta_B)^{>n} \ominus 1 = v + \Delta_A$. In particular $u + \Delta_B$ is stable and 0-free.

By the properties of ψ on $K_{m,n}$ (with $m = n$) that we proved in this paper, there exists some $k > 0$ such that $u + \Delta_B = \psi^k(u)$. The proof that such k is minimal with the property that $\psi^k(u)$ is 0-free is analogous to what we have done with φ, and it is omitted.

If instead $\psi(v) = v$, i.e. v is recurrent, then it can be shown as we did for φ that u is also recurrent, i.e. $\psi(u) = u$. So in this case $k = 1$, which is clearly minimal, and this completes the proof. □

Example 7.3. Let $n = 5$, and consider the stable configuration $v = (0, 2, 2, 3, *)$ on K_n. Then in this case we can take $u = (0, 1, 2, 3, 4; 1, 3, 3, 4, *)$. Now $\varphi(u) = (1, 2, 3, 4, 0; 2, 4, 4, 0, *)$, which is not 0-free, but $\varphi^2(u) = (3, 4, 0, 1, 2; 4, 1, 1, 2, *)$. And indeed $\varphi(v) = (3, 0, 0, 1, *) = (\varphi^2(u))^{>5} \ominus 1$ as predicted.

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