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# 1. Appendix: Integral Forms

This appendix contains the full expansion of the integrals that appear in the paper. Their complete expressions are as follows:

$$I_{11}(L_0, L, P_{10}, P_{20}) = -\frac{2}{B} \left[ \Lambda_0(\mathcal{L}) \right]_{L_0}^L \quad [1]$$

$$I_{12}(L_0, L, P_{10}, P_{20}) = \left[ -\frac{2}{B^3} \Lambda_0(\mathcal{L}) - \frac{1}{P_{20} B^2} \frac{P_{10} + (P_{10}^2 + P_{20}^2) \sin \mathcal{L}}{\Phi_0(\mathcal{L})} \right]_{L_0}^L \quad [2]$$

$$I_{13}(L_0, L, P_{10}, P_{20}) = \frac{1}{2} \left[ -\frac{2(2 + P_{10}^2 + P_{20}^2)}{B_0^5} \Lambda_0(\mathcal{L}) + \frac{1}{P_{20} B_0^2} \left( \frac{P_{10} + (P_{10}^2 + P_{20}^2) \sin \mathcal{L}}{\Phi_0^2(\mathcal{L})} + \frac{P_{10}(2 + P_{10}^2 + P_{20}^2) + 3(P_{10}^2 + P_{20}^2) \sin \mathcal{L}}{B_0 \Phi_0(\mathcal{L})} \right) \right]_{L_0}^L \quad [3]$$

$$I_{c2}(L_0, L, P_{10}, P_{20}) = \left[ \frac{2P_{20}}{B_0^3} \Lambda_0(\mathcal{L}) + \frac{1}{B_0^2} \left( \frac{P_{10} + \sin \mathcal{L}}{\Phi_0(\mathcal{L})} \right) \right]_{L_0}^L \quad [4]$$

$$I_{s2}(L_0, L, P_{10}, P_{20}) = \left[ \frac{2P_{10}}{B_0^3} \Lambda_0(\mathcal{L}) - \frac{1}{P_{20} B_0^2} \left( \frac{P_{20}^2 - 1 - P_{10} \sin \mathcal{L}}{\Phi_0(\mathcal{L})} \right) \right]_{L_0}^L \quad [5]$$

$$I_{c3}(L_0, L, P_{10}, P_{20}) = \frac{1}{2} \left[ \frac{6P_{20}}{B_0^5} \Lambda_0(\mathcal{L}) + \frac{1}{B_0^2} \left( \frac{P_{10} + \sin \mathcal{L}}{\Phi_0^2(\mathcal{L})} + \frac{3P_{10} + (1 + 2(P_{10}^2 + P_{20}^2)) \sin \mathcal{L}}{B_0^2 \Phi_0(\mathcal{L})} \right) \right]_{L_0}^L \quad [6]$$

$$I_{s3}(L_0, L, P_{10}, P_{20}) = \frac{1}{2} \left[ \frac{6P_{10}}{B_0^5} \Lambda_0(\mathcal{L}) + \frac{1}{P_{20} B_0^2} \left( \frac{1 - P_{20}^2 + P_{10} \sin \mathcal{L}}{\Phi_0^2(\mathcal{L})} + \frac{P_{10} (3P_{10} + (1 + 2(P_{10}^2 + P_{20}^2)) \sin \mathcal{L})}{B_0^2 \Phi_0(\mathcal{L})} \right) \right]_{L_0}^L \quad [7]$$

$$I_{t1} = \left( I_{13} - \frac{I_{12}}{\Phi_0(L_0)} \right) \quad [8]$$

$$I_{t2}(L_0, L, P_{10}, P_{20}) = \frac{1}{2(1-P_{20})B_0^3} \left[ \begin{array}{l} 2(P_{20}-1)\Lambda_0^2(\mathcal{L}) + \\ \cos(2\Lambda_0(\mathcal{L}))\left(2P_{10}B_0\Lambda_0(\mathcal{L}) - (P_{10}^2 + P_{20}^2) + P_{20}\right) + \\ -\sin(2\Lambda_0(\mathcal{L}))\left(2P_{10}B_0 + ((P_{10}^2 + P_{20}^2) - P_{20})\Lambda_0(\mathcal{L})\right) \end{array} \right]_{L_0}^L \quad [9]$$

$$\Lambda_0(L) = \text{atan} \left( \frac{1}{B} \left( -P_{10} + (P_{20}-1) \tan \left( \frac{L}{2} \right) \right) \right) \quad [10]$$

$$I_{2c3}(L_0, L, P_{10}, P_{20}) = \frac{1}{B_0^4} \left[ \begin{array}{l} \frac{P_{10}^2 - 2P_{20}^2 - 1}{B_0} \Lambda_0(\mathcal{L}) - \\ \left( \Phi_0(\mathcal{L})(P_{10}(2 - 3P_{10}^2 + P_{10}^4 + (5P_{10}^2 - 3)P_{20}^2 + 4P_{20}^4) + \right. \\ \left. (P_{10}^2 - P_{10}^4 - P_{20}^2 + 3P_{10}^2P_{20}^2 + 4P_{20}^4) \sin \mathcal{L}) + \right. \\ \left. B_0^2(P_{10}(-1 + P_{10}^2 - 2P_{20}^2) + (P_{20}^2 - P_{10}^2B_0^2) \sin \mathcal{L}) \right) \\ \left. \frac{4P_{20}(P_{10}^2 + P_{20}^2)\Phi_0^2(\mathcal{L})}{4P_{20}(P_{10}^2 + P_{20}^2)\Phi_0^2(\mathcal{L})} \right]_{L_0}^L \quad [11]$$

$$I_{1c1s3}(L_0, L, P_{10}, P_{20}) = \frac{1}{B_0^2} \left[ \begin{array}{l} -\frac{3P_{10}P_{20}}{B_0^3} \Lambda_0(\mathcal{L}) + \\ \left( (-1 + P_{20}^2 + P_{10}(-2 + P_{10}^2 + P_{20}^2) \sin \mathcal{L}) - \right. \\ \left. \Phi_0(\mathcal{L})(-P_{10}^4 + P_{10}^2(-4 + P_{20}^2) + \right. \\ \left. \frac{2(P_{20}^2 - 1)^2 - P_{10}(-2 + 5(P_{10}^2 + P_{20}^2)) \sin \mathcal{L}}{2\Phi_0^2(\mathcal{L})} \right) \end{array} \right]_{L_0}^L \quad [12]$$

$$I_{2s3}(L_0, L, P_{10}, P_{20}) = \left[ \begin{array}{l} \frac{P_{20}^2 - 2P_{10}^2 - 1}{B_0^5} \Lambda_0(\mathcal{L}) - \\ \left( \Phi_0(\mathcal{L})(P_{10}(2 - 5P_{10}^2 + (3P_{10}^2 - 5)P_{20}^2 + 3P_{20}^4) + \right. \\ \left. (P_{10}^2 - 4P_{10}^4 - P_{20}^2 - 3P_{10}^2P_{20}^2 + P_{20}^4) \sin \mathcal{L}) - \right. \\ \left. \frac{B_0^2(P_{10}(1 - P_{20}^2) + (P_{10}^2 - P_{20}^2B_0^2) \sin \mathcal{L})}{2P_{20}B_0^2(P_{10}^2 + P_{20}^2)\Phi_0^2(\mathcal{L})} \right) \end{array} \right]_{L_0}^L \quad [13]$$

$$F(\varphi, k) = \int_0^\varphi \frac{d\theta}{\sqrt{1 - k \sin^2 \theta}} \quad [14]$$

$$E(\varphi, k) = \int_0^\varphi \sqrt{1 - k \sin^2 \theta} d\theta \quad [15]$$

$$I_a = \left[ \frac{1}{1-e_0} E \left( \frac{\vartheta}{2}, \frac{4e_0}{(1+e_0)^2} \right) + \frac{1}{1+e_0} F \left( \frac{\vartheta}{2}, \frac{4e_0}{(1+e_0)^2} \right) - \frac{\sqrt{1+e_0^2+2e_0 \cos \vartheta} e_0 \sin \vartheta}{(1-e_0)^2(1+e_0 \cos \vartheta)} \right]_{\theta_0}^\theta \quad [16]$$

$$I_{P_1} = \frac{1}{e_0(1-e_0)} \left[ E \left( \frac{\vartheta}{2}, \frac{4e_0}{(1+e_0)^2} \right) - \frac{1+e_0^2}{(1+e_0)^2} F \left( \frac{\vartheta}{2}, \frac{4e_0}{(1+e_0)^2} \right) + \frac{1}{2(1+e_0)} \log \left( \frac{\sqrt{1+e_0^2+2e_0 \cos \vartheta} + e_0 \sin \vartheta}{\sqrt{1+e_0^2+2e_0 \cos \vartheta} - e_0 \sin \vartheta} \right) - \frac{\sqrt{1+e_0^2+2e_0 \cos \vartheta} e_0 \sin \vartheta}{(1+e_0)(1+e_0 \cos \vartheta)} \right]_{\theta_0}^{\theta} \quad [17]$$

$$I_{P_2} = -\frac{1}{e_0(1-e_0^2)} \left[ \frac{2}{\sqrt{1-e_0^2}} \operatorname{atan} \left( \frac{\sqrt{1+e_0^2+2e_0 \cos \vartheta}}{\sqrt{1-e_0^2}} \right) + \frac{\sqrt{1+e_0^2+2e_0 \cos \vartheta}}{(1+e_0 \cos \vartheta)} \right]_{\theta_0}^{\theta} \quad [18]$$

$$I_{Ja}(L_0, L, P_{10}, P_{20}) = - \left[ \begin{aligned} & -3 \left( G_0^2 P_{20} (4 + P_{10}^2 + P_{20}^2) - 2P_{20} (18 + 3P_{10}^2 + 5P_{20}^2) Q_{10}^2 + \right. \\ & \left. 4P_{10} (6 + P_{10}^2 + 3P_{20}^2) Q_{10} Q_{20} - 2P_{20} (6 + 3P_{10}^2 + P_{20}^2) Q_{20}^2 \right) \cos \mathcal{L} + \\ & 6 \left( G_0^2 (P_{10} - P_{20}) (P_{10} + P_{20}) + 4(1 + 3P_{20}^2) Q_{10}^2 - 4(1 + 3P_{10}^2) Q_{20}^2 \right) \cos(2\mathcal{L}) + \\ & \left( G_0^2 (3P_{10}^2 P_{20} - P_{20}^3) + 3P_{20} (12 - 3P_{10}^2 + 5P_{20}^2) Q_{10}^2 + \right. \\ & \left. 18P_{10} (4 + P_{10}^2 + P_{20}^2) Q_{10} Q_{20} - 3P_{20} (12 + 9P_{10}^2 + P_{20}^2) Q_{20}^2 \right) \cos(3\mathcal{L}) + \\ & 18 (P_{20} (Q_{10} - Q_{20}) + P_{10} (Q_{10} + Q_{20})) (P_{10} (Q_{20} - Q_{10}) + P_{20} (Q_{10} + Q_{20})) \cos(4\mathcal{L}) + \\ & 3 \left( -2P_{10}^3 Q_{10} Q_{20} + 6P_{10} P_{20}^2 Q_{10} Q_{20} + \right. \\ & \left. P_{20}^3 (Q_{10} - Q_{20}) (Q_{10} + Q_{20}) + 3P_{10}^2 P_{20} (-Q_{10}^2 + Q_{20}^2) \right) \cos(5\mathcal{L}) + \\ & -3 \left( G_0^2 P_{10} (4 + P_{10}^2 + P_{20}^2) - 2P_{10} (6 + P_{10}^2 + 3P_{20}^2) Q_{10}^2 + \right. \\ & \left. 4P_{20} (6 + 3P_{10}^2 + P_{20}^2) Q_{10} Q_{20} - 2P_{10} (5P_{10}^2 + 3(6 + P_{20}^2)) Q_{20}^2 \right) \sin(\mathcal{L}) + \\ & -12 \left( G_0^2 P_{10} P_{20} + 6P_{10}^2 Q_{10} Q_{20} + \right. \\ & \left. 2(2 + 3P_{20}^2) Q_{10} Q_{20} - 6P_{10} P_{20} (Q_{10}^2 + Q_{20}^2) \right) \sin(2\mathcal{L}) + \\ & \left( G_0^2 (P_{10}^3 - 3P_{10} P_{20}^2) + 3P_{10} (12 + P_{10}^2 + 9P_{20}^2) Q_{10}^2 + \right. \\ & \left. -18P_{20} (4 + P_{10}^2 + P_{20}^2) Q_{10} Q_{20} - 3P_{10} (12 + 5P_{10}^2 - 3P_{20}^2) Q_{20}^2 \right) \sin(3\mathcal{L}) + \\ & 36 (P_{20} Q_{10} + P_{10} Q_{20}) (P_{20} Q_{20} - P_{10} Q_{10}) \sin(4\mathcal{L}) + \\ & 3 \left( 6P_{10}^2 P_{20} Q_{10} Q_{20} - 2P_{20}^3 Q_{10} Q_{20} + \right. \\ & \left. 3P_{10} P_{20}^2 (Q_{10} - Q_{20}) (Q_{10} + Q_{20}) + P_{10}^3 (-Q_{10}^2 + Q_{20}^2) \right) \sin(5\mathcal{L}) \end{aligned} \right]_{L_0}^L \quad [19]$$

$$\begin{aligned}
I_{JP_1}(L_0, L, P_{10}, P_{20}) = & - \left[ \begin{aligned}
& -12P_{20} \left( G_0^2 + 2(-4 + Q_{10}^2 + Q_{20}^2)(Q_{10}^2 + Q_{20}^2) \right) L + \\
& 6 \left( \begin{aligned}
& G_0^2 P_{10} P_{20} - 10P_{10}^2 Q_{10} Q_{20} + \\
& 2P_{10} P_{20} (Q_{10}^4 + 2(-5 + 2Q_{10}^2)Q_{20}^2 + 3Q_{20}^4) \\
& - 2Q_{10} Q_{20} (2 + P_{20}^2(-5 + 2Q_{10}^2 + 2Q_{20}^2))
\end{aligned} \right) \cos L + \\
& 6 \left( \begin{aligned}
& G_0^2 P_{10} + 4(P_{10}(Q_{10}^2 - 4Q_{20}^2) + \\
& -P_{20} Q_{10} Q_{20} (-4 + Q_{10}^2 + Q_{20}^2))
\end{aligned} \right) \cos(2L) + \\
& 2 \left( \begin{aligned}
& G_0^2 P_{10} P_{20} + 17P_{10}^2 Q_{10} Q_{20} + \\
& P_{10} P_{20} (-5Q_{10}^2 + 2Q_{10}^4 - 7Q_{20}^2 - 2Q_{20}^4) \\
& + Q_{10} Q_{20} (28 + P_{20}^2(15 - 4Q_{10}^2 - 4Q_{20}^2))
\end{aligned} \right) \cos(3L) + \\
& 18(2P_{20} Q_{10} Q_{20} + P_{10}(-Q_{10}^2 + Q_{20}^2)) \cos(4L) + \\
& 6(P_{20} Q_{10} + P_{10} Q_{20})(-P_{10} Q_{10} + P_{20} Q_{20}) \cos(5L) + \\
& -3 \left( \begin{aligned}
& G_0^2(4 + P_{10}^2 + 3P_{20}^2) - 20Q_{10}^2 - 28Q_{20}^2 + \\
& 6P_{10}^2(Q_{10}^2 - 3Q_{20}^2) - 8P_{10} P_{20} Q_{10} Q_{20} (-6 + Q_{10}^2 + Q_{20}^2) \\
& + 2P_{20}^2(6Q_{10}^4 - 7Q_{20}^2 + 2Q_{20}^4 + Q_{10}^2(-19 + 8Q_{20}^2))
\end{aligned} \right) \sin(L) + \\
& -6(G_0^2 P_{20} + 20P_{10} Q_{10} Q_{20} + 2P_{20}(-7Q_{10}^2 + Q_{10}^4 + Q_{20}^2 - Q_{20}^4)) \sin(2L) + \\
& \left( \begin{aligned}
& G_0^2(P_{10} - P_{20})(P_{10} + P_{20}) + Q_{10}^2(28 + 11P_{10}^2 + P_{20}^2(21 - 4Q_{10}^2)) \\
& - 4P_{10} P_{20} (Q_{10} + 2Q_{10}^3) Q_{20} - (28 + 23P_{10}^2 + 9P_{20}^2) Q_{20}^2 + \\
& - 8P_{10} P_{20} Q_{10} Q_{20}^3 + 4P_{20}^2 Q_{20}^4
\end{aligned} \right) \sin(3L) + \\
& 18(2P_{10} Q_{10} Q_{20} + P_{20}(Q_{10}^2 - Q_{20}^2)) \sin(4L) + \\
& 3(P_{20}(Q_{10} - Q_{20}) + P_{10}(Q_{10} + Q_{20}))(P_{10}(-Q_{10} + Q_{20}) + P_{20}(Q_{10} + Q_{20})) \sin(5L)
\end{aligned} \right]
\end{aligned}$$

[20]

$$\begin{aligned}
I_{JP_2}(L_0, L, P_{10}, P_{20}) = & - \left[ \begin{aligned}
& 12P_{10} \left( G_0^2 + 2(-4 + Q_{10}^2 + Q_{20}^2)(Q_{10}^2 + Q_{20}^2) \right) \mathcal{L} + \\
& -3 \left( \begin{aligned}
& G_0^2(4 + 3P_{10}^2 + P_{20}^2) - 28Q_{10}^2 + 2(-10Q_{20}^2 + 3P_{20}^2(-3Q_{10}^2 + Q_{20}^2) + \\
& -4P_{10}P_{20}Q_{10}Q_{20}(-6 + Q_{10}^2 + Q_{20}^2) + \\
& P_{10}^2(2Q_{10}^4 - 19Q_{20}^2 + 6Q_{20}^4 + Q_{10}^2(-7 + 8Q_{20}^2))
\end{aligned} \right) \cos \mathcal{L} + \\
& -6 \left( G_0^2P_{20} - 4(P_{20}(4Q_{10}^2 - Q_{20}^2) + P_{10}Q_{10}Q_{20}(-4 + Q_{10}^2 + Q_{20}^2)) \right) \cos(2\mathcal{L}) + \\
& \left( \begin{aligned}
& G_0^2(P_{10} - P_{20})(P_{10} + P_{20}) + Q_{10}^2(28 + 23P_{20}^2 + P_{10}^2(9 - 4Q_{10}^2)) + \\
& 4P_{10}P_{20}(Q_{10} + 2Q_{10}^3)Q_{20} - (28 + 21P_{10}^2 + 11P_{20}^2)Q_{20}^2 + \\
& 8P_{10}P_{20}Q_{10}Q_{20}^3 + 4P_{10}^2Q_{20}^4
\end{aligned} \right) \cos(3\mathcal{L}) + \\
& 18(2P_{10}Q_{10}Q_{20} + P_{20}(Q_{10}^2 - Q_{20}^2)) \cos(4\mathcal{L}) + \\
& 3(P_{20}(Q_{10} - Q_{20}) + P_{10}(Q_{10} + Q_{20}))(P_{10}(Q_{20} - Q_{10}) + P_{20}(Q_{10} + Q_{20})) \cos(5\mathcal{L}) + \\
& 6 \left( \begin{aligned}
& G_0^2P_{10}P_{20} - 2(2 + 5P_{20}^2)Q_{10}Q_{20} - 2P_{10}^2Q_{10}Q_{20}(-5 + 2Q_{10}^2 + 2Q_{20}^2) \\
& + 2P_{10}P_{20}(3Q_{10}^4 + Q_{20}^4 + 2Q_{10}^2(-5 + 2Q_{20}^2))
\end{aligned} \right) \sin(\mathcal{L}) + \\
& -6 \left( G_0^2P_{10} + 20P_{20}Q_{10}Q_{20} + 2P_{10}(-7Q_{20}^2 + Q_{20}^4 + Q_{10}^2 - Q_{10}^4) \right) \sin(2\mathcal{L}) + \\
& -2 \left( \begin{aligned}
& G_0^2P_{10}P_{20} + (28 + 17P_{20}^2)Q_{10}Q_{20} + P_{10}^2Q_{10}Q_{20}(15 - 4Q_{10}^2 - 4Q_{20}^2) \\
& - P_{10}P_{20}(7Q_{10}^2 + 2Q_{10}^4 + 5Q_{20}^2 - 2Q_{20}^4)
\end{aligned} \right) \sin(3\mathcal{L}) + \\
& 18(-2P_{20}Q_{10}Q_{20} + P_{10}(Q_{10}^2 - Q_{20}^2)) \sin(4\mathcal{L}) + \\
& 6(P_{20}Q_{10} + P_{10}Q_{20})(P_{10}Q_{10} - P_{20}Q_{20}) \sin(5\mathcal{L})
\end{aligned} \right]_{L_0}
\end{aligned}
\tag{21}$$

$$\begin{aligned}
I_{JQ_1}(L_0, L, P_{10}, P_{20}) = & - \left[ \begin{aligned}
& 6Q_{20}\mathcal{L} + \\
& 3(P_{20}Q_{10} - 3P_{10}Q_{20}) \cos \mathcal{L} + \\
& 3Q_{10} \cos(2\mathcal{L}) + \\
& (P_{20}Q_{10} + P_{10}Q_{20}) \cos(3\mathcal{L}) + \\
& 3(-P_{10}Q_{10} + P_{20}Q_{20}) \sin(\mathcal{L}) + \\
& -3Q_{20} \sin(2\mathcal{L}) + \\
& (P_{10}Q_{10} - P_{20}Q_{20}) \sin(3\mathcal{L})
\end{aligned} \right]_{L_0}
\end{aligned}
\tag{22}$$

$$I_{JQ2}(L_0, L, P_{10}, P_{20}) = - \left[ \begin{array}{l} -6Q_{10}\mathcal{L} + \\ 3(P_{10}Q_{10} - P_{20}Q_{20})\cos\mathcal{L} + \\ -3Q_{20}\cos(2\mathcal{L}) + \\ (P_{10}Q_{10} - P_{20}Q_{20})\cos(3\mathcal{L}) + \\ 3(-3P_{20}Q_{10} + P_{10}Q_{20})\sin(\mathcal{L}) + \\ -3Q_{10}\sin(2\mathcal{L}) + \\ -(P_{20}Q_{10} + P_{10}Q_{20})\sin(3\mathcal{L}) \end{array} \right]_{L_0}^L \quad [23]$$