Convergence Rates of the Truncated Euler–Maruyama Method for Stochastic Differential Equations

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Abstract

Influenced by Higham, Mao and Stuart [9], several numerical methods have been developed to study the strong convergence of the numerical solutions to stochastic differential equations (SDEs) under the local Lipschitz condition. These numerical methods include the tamed Euler–Maruyama (EM) method, the tamed Milstein method, the stopped EM, the backward EM, the backward forward EM, etc. Recently, we developed a new explicit method in [23], called the truncated EM method, for the nonlinear SDE $dx(t) = f(x(t))dt + g(x(t))dB(t)$ and established the strong convergence theory under the local Lipschitz condition plus the Khasminskii-type condition $x^T f(x) + \frac{p-1}{2}|g(x)|^2 \le K(1 + |x|^2)$. However, due to the page limit there, we did not study the convergence rates for the method, which is the aim of this paper. We will, under some additional conditions, discuss the rates of L^q-convergence of the truncated EM method for $2 \le q < p$ and show that the order of L^q-convergence can be arbitrarily close to $q/2$.

Key words: Stochastic differential equation, local Lipschitz condition, Khasminskii-type condition, truncated Euler-Maruyama method, convergence rate.

1 Introduction

This is the continuation of our recent paper [23], where we developed a new explicit method, called the truncated EM method, for the multi-dimensional nonlinear SDE

$$dx(t) = f(x(t))dt + g(x(t))dB(t)$$

and established the strong convergence theory under the local Lipschitz condition plus the Khasminskii-type condition

$$x^T f(x) + \frac{p-1}{2}|g(x)|^2 \le K(1 + |x|^2).$$

However, we did not study the convergence rates for the method there. The key aim of this paper is to discuss the rates of L^q-convergence for $2 \le q < p$.

In [23], we have reviewed the developments of numerical methods for SDEs for the past twenty years. In summary, up to 2002, most of the existing strong convergence theory for numerical methods requires the coefficients of the SDEs to be globally Lipschitz continuous (see, e.g., [18, 21, 27]). In 2002, Higham, Mao and Stuart published a very influential paper [9] (Google citation 318 on 6 September 2015) which opened a new chapter in the study of numerical

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solutions of SDEs—to study the strong convergence question for numerical approximations under the local Lipschitz condition. For example, implicit methods have been used to study the numerical solutions to SDEs without the linear growth condition (see, e.g., [24, 34, 35] and for the background on the implicit methods, we refer the reader to the papers [2, 4, 9, 12, 11, 17, 26, 31] and the book [18]). Methods with variable stepsize also attract a lot of attention [5, 29, 36, 39]. Other weak forms of convergence, say weak convergence, convergence in probability and pathwise convergence, are discussed in [1, 7, 16, 18, 22, 25, 28, 37], just to mention a few. More significantly, some modified EM methods have recently been developed for the nonlinear SDEs without the linear growth condition. For example, the tamed EM method was developed in [14] to approximate SDEs with one-sided Lipschitz drift coefficient and the linear growth diffusion coefficient. This method was further developed in [33] while the tamed Milstein method was developed in [38]. Moreover, the stopped EM method was developed in [20] for nonlinear SDEs as well. Very recently, another new explicit method—the truncated EM method was developed in [23]. These new explicit EM methods have shown their abilities to approximate the solutions of nonlinear SDEs.

In this paper, we will investigate the convergence rates of the truncated EM method. For the convenience of the reader, we will, in section 2, make a quick review on the main results in [23], where the truncated EM method was initiated. We will then study the rates of convergence at a single time in section 3 and over a finite time interval in section 4. A number of examples will be discussed throughout sections 3 and 4 to illustrate our theory and to motivate further developments. We will conclude our paper in section 5.

2 The Truncated EM Method

Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (that is, it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets), and let $\mathbb{E}$ denote the expectation corresponding to $\mathbb{P}$. Let $B(t)$ be an m-dimensional Brownian motion defined on the space. If A is a vector or matrix, its transpose is denoted by A^T. If $x \in \mathbb{R}^d$, then $|x|$ is the Euclidean norm. If A is a matrix, we let $|A| = \sqrt{\text{trace}(A^T A)}$ be its trace norm. If A is a symmetric matrix, denote by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ its largest and smallest eigenvalue, respectively. Moreover, for two real numbers a and b, we use $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$. If G is a set, its indicator function is denoted by I_G, namely $I_G(x) = 1$ if $x \in G$ and 0 otherwise.

Consider a d-dimensional SDE

$$dx(t) = f(x(t))dt + g(x(t))dB(t) \tag{2.1}$$

on $t \geq 0$ with the initial value $x(0) = x_0 \in \mathbb{R}^d$, where

$$f : \mathbb{R}^d \to \mathbb{R}^d \quad \text{and} \quad g : \mathbb{R}^d \to \mathbb{R}^{d\times m}.$$

We impose two standing hypotheses in this paper.

Assumption 2.1 *Assume that the coefficients f and g satisfy the local Lipschitz condition: For any $R > 0$, there is a $K_R > 0$ such that*

$$|f(x) - f(y)| \vee |g(x) - g(y)| \leq K_R|x - y| \tag{2.2}$$

for all $x, y \in \mathbb{R}^d$ with $|x| \vee |y| \leq R$.

Assumption 2.2 *We also assume that the coefficients satisfy the Khasminskii-type condition: There is a pair of constants $p > 2$ and $K > 0$ such that*

$$x^T f(x) + \frac{p-1}{2}|g(x)|^2 \le K(1+|x|^2) \tag{2.3}$$

for all $x \in \mathbb{R}^d$.

We state a known result (see, e.g., [21, 22, 32]) as a lemma for the use of this paper.

Lemma 2.3 *Under Assumptions 2.1 and 2.2, the SDE (2.1) has a unique global solution $x(t)$ and, moreover,*

$$\sup_{0\le t\le T} \mathbb{E}|x(t)|^p < \infty, \quad \forall T > 0. \tag{2.4}$$

Let us now review the truncated EM method initiated in [23]. To define the truncated EM numerical solutions, we first choose a strictly increasing continuous function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\mu(u) \to \infty$ as $u \to \infty$ and

$$\sup_{|x|\le u} \big(|f(x)| \vee |g(x)|\big) \le \mu(u), \quad \forall u \ge 1. \tag{2.5}$$

Denote by μ^{-1} the inverse function of μ and we see that μ^{-1} is a strictly increasing continuous function from $[\mu(0), \infty)$ to $\mathbb{R}_+$. We also choose a number $\Delta^* \in (0, 1]$ and a strictly decreasing function $h : (0, \Delta^*] \to (0, \infty)$ such that

$$h(\Delta^*) \ge \mu(2), \quad \lim_{\Delta\to 0} h(\Delta) = \infty \quad \text{and} \quad \Delta^{1/4} h(\Delta) \le 1, \quad \forall \Delta \in (0, \Delta^*]. \tag{2.6}$$

For a given stepsize $\Delta \in (0, \Delta^*]$, let us define the truncated functions

$$f_\Delta(x) = f\Big((|x| \wedge \mu^{-1}(h(\Delta)))\frac{x}{|x|}\Big) \quad \text{and} \quad g_\Delta(x) = g\Big((|x| \wedge \mu^{-1}(h(\Delta)))\frac{x}{|x|}\Big) \tag{2.7}$$

for $x \in \mathbb{R}^d$, where we set $x/|x| = 0$ when $x = 0$. It is easy to see that

$$|f_\Delta(x)| \vee |g_\Delta(x)| \le \mu(\mu^{-1}(h(\Delta))) = h(\Delta), \quad \forall x \in \mathbb{R}^d. \tag{2.8}$$

That is, both truncated functions f_Δ and g_Δ are bounded although both f and g may not. It was shown in [23] that these truncated functions preserve nicely the Khasminskii-type condition for all $\Delta \in (0, \Delta^*]$ as described in the following lemma.

Lemma 2.4 *Let Assumption 2.2 hold. Then, for all $\Delta \in (0, \Delta^*]$, we have*

$$x^T f_\Delta(x) + \frac{p-1}{2}|g_\Delta(x)|^2 \le 2K(1+|x|^2), \quad \forall x \in \mathbb{R}^d. \tag{2.9}$$

The discrete-time truncated EM numerical solutions $X_\Delta(t_k) \approx x(t_k)$ for $t_k = k\Delta$ are formed by setting $X_\Delta(0) = x_0$ and computing

$$X_\Delta(t_{k+1}) = X_\Delta(t_k) + f_\Delta(X_\Delta(t_k))\Delta + g_\Delta(X_\Delta(t_k))\Delta B_k, \tag{2.10}$$

for $k = 0, 1, \cdots$, where $\Delta B_k = B(t_{k+1}) - B(t_k)$. There are two versions of the continuous-time truncated EM solutions. The first one is defined by

$$\bar{x}_\Delta(t) = \sum_{k=0}^{\infty} X_\Delta(t_k) I_{[t_k, t_{k+1})}(t), \quad t \ge 0. \tag{2.11}$$

This is a simple step process so its sample paths are not continuous. We will refer this as the continuous-time step-process truncated EM solution. The other one is defined by

$$x_\Delta(t) = x_0 + \int_0^t f_\Delta(\bar{x}_\Delta(s))ds + \int_0^t g_\Delta(\bar{x}_\Delta(s))dB(s) \tag{2.12}$$

for $t \geq 0$. We will refer this as the continuous-time continuous-sample truncated EM solution. We observe that $x_\Delta(t_k) = \bar{x}_\Delta(t_k) = X_\Delta(t_k)$ for all $k \geq 0$. Moreover, $x_\Delta(t)$ is an Itô process with its Itô differential

$$dx_\Delta(t) = f_\Delta(\bar{x}_\Delta(t))dt + g_\Delta(\bar{x}_\Delta(t))dB(t). \tag{2.13}$$

The truncated EM solutions have a number of nice properties established in [23]. We will cite a number of them here for the use of this paper.

Lemma 2.5 *For any $\Delta \in (0, \Delta^*]$ and any $\hat{p} > 0$, we have*

$$\mathbb{E}|x_\Delta(t) - \bar{x}_\Delta(t)|^{\hat{p}} \leq c_{\hat{p}} \Delta^{\hat{p}/2} (h(\Delta))^{\hat{p}}, \quad \forall t \geq 0, \tag{2.14}$$

where $c_{\hat{p}}$ is a positive constant dependent only on $\hat{p}$. Consequently

$$\lim_{\Delta \to 0} \mathbb{E}|x_\Delta(t) - \bar{x}_\Delta(t)|^{\hat{p}} = 0, \quad \forall t \geq 0. \tag{2.15}$$

It should be pointed out that this lemma was proved only for $\hat{p} \geq 2$ in [23]. However, it is easy to see that this lemma holds for any $\hat{p} \in (0, 2)$ as well. In fact, by the Hölder inequality, for any $\hat{p} \in (0, 2)$, we have

$$\mathbb{E}|x_\Delta(t) - \bar{x}_\Delta(t)|^{\hat{p}} \leq \Big(\mathbb{E}|x_\Delta(t) - \bar{x}_\Delta(t)|^2\Big)^{\hat{p}/2} \leq \Big(c_2 \Delta (h(\Delta))^2\Big)^{\hat{p}/2} = c_{\hat{p}} \Delta^{\hat{p}/2} (h(\Delta))^{\hat{p}}$$

as desired. This is useful as in our proofs later we will use this lemma for any $\hat{p} > 0$. For example, in the proof of Lemma 3.3, this lemma will be applied on the expression $\mathbb{E}|x_\Delta(t) - \bar{x}_\Delta(t)|^{pq/(2p-pr)}$ in (3.10) and our conditions there only ensure that $pq/(2p - pr) > 0$.

Lemma 2.6 *Let Assumptions 2.1 and 2.2 hold. Then*

$$\sup_{0 < \Delta \leq \Delta^*} \sup_{0 \leq t \leq T} \mathbb{E}|x_\Delta(t)|^p \leq C, \quad \forall T > 0, \tag{2.16}$$

where, and from now on, C stands for generic positive real constants dependent on T, p, K, x_0 etc. but independent of Δ, R (appeared in the next lemmas) and its values may change between occurrences.

Lemma 2.7 *Let Assumptions 2.1 and 2.2 hold. For any real number $R > |x_0|$, define the stopping time*

$$\tau_R = \inf\{t \geq 0 : |x(t)| \geq R\},$$

where throughout this paper we set $\inf \emptyset = \infty$ (and as usual $\emptyset$ denotes the empty set). Then

$$\mathbb{P}(\tau_R \leq T) \leq \frac{C}{R^p}. \tag{2.17}$$

Lemma 2.8 *Let Assumptions 2.1 and 2.2 hold. For any real number $R > |x_0|$ and $\Delta \in (0, \Delta^*)$, define the stopping time*

$$\rho_{\Delta,R} = \inf\{t \geq 0 : |x_\Delta(t)| \geq R\}.$$

Then

$$\mathbb{P}(\rho_{\Delta,R} \leq T) \leq \frac{C}{R^p}. \tag{2.18}$$

3 Convergence Rates at Time T

In [23], we established the theory of the strong L^q-convergence for $2 \le q < p$, where p is a parameter in Assumption 2.2. However, the convergence was in the asymptotic form without the convergence rate. Starting from this section we will discuss the convergence rate. Our study on the convergence rate will also reveal a strong relation between functions $\mu(\cdot)$ and $h(\cdot)$ that are used to define the truncated EM method. We first discuss the convergence rate at time T in this section and then discuss the path convergence rate in the next section. We need some additional conditions.

Assumption 3.1 *Assume that there is a pair of constants $q \ge 2$ and $H_1 > 0$ such that*

$$(x-y)^T(f(x)-f(y)) + \frac{q-1}{2}|g(x)-g(y)|^2 \le H_1|x-y|^2 \tag{3.1}$$

for all $x, y \in \mathbb{R}^d$.

Assumption 3.2 *Assume that there is a pair of positive constants r and H_2 such that*

$$|f(x)| \le H_2(1+|x|^r), \quad \forall x \in \mathbb{R}^d. \tag{3.2}$$

The following lemma will play a key role in the proof of the convergence rate.

Lemma 3.3 *Let Assumptions 2.1, 2.2, 3.1 and 3.2 hold and assume that $2p > qr$ and $p > q$. Let $R > |x_0|$ be a real number and let $\Delta \in (0, \Delta^*)$ be sufficiently small such that $\mu^{-1}(h(\Delta)) \ge R$. Let τ_R and $\rho_{\Delta,R}$ be the same as defined in Lemmas 2.7 and 2.8, respectively. Set*

$$\theta_{\Delta,R} = \tau_R \wedge \rho_{\Delta,R} \quad \text{and} \quad e_\Delta(t) = x_\Delta(t) - x(t) \quad \text{for } t \ge 0.$$

Then

$$\mathbb{E}|e_\Delta(T\wedge\theta_{\Delta,R})|^q \le C\Delta^{q/4}(h(\Delta))^{q/2}, \quad \forall T > 0. \tag{3.3}$$

Proof. We write $\theta_{\Delta,R} = \theta$ for simplicity. By the Itô formula [21, 30], we can show that for $0 \le t \le T$,

$$\begin{aligned} &\mathbb{E}|e_\Delta(t\wedge\theta)|^q \\ \le\ & \mathbb{E}\int_0^{t\wedge\theta} q|e_\Delta(s)|^{q-2}\Big(e_\Delta^T(s)[f(x(s)) - f_\Delta(\bar{x}_\Delta(s))] + \frac{q-1}{2}|g(x(s)) - g_\Delta(\bar{x}_\Delta(s))|^2\Big)ds. \end{aligned} \tag{3.4}$$

We observe that for $0 \le s \le t\wedge\theta$, $|\bar{x}_\Delta(s)| \le R$. But we have condition $\mu^{-1}(h(\Delta)) \ge R$, so $|\bar{x}_\Delta(s)| \le \mu^{-1}(h(\Delta))$. Recalling the definition of the truncated functions f_Δ and g_Δ, we hence have that

$$f_\Delta(\bar{x}_\Delta(s)) = f(\bar{x}_\Delta(s)) \quad \text{and} \quad g_\Delta(\bar{x}_\Delta(s)) = g(\bar{x}_\Delta(s)) \quad \text{for } 0 \le s \le t\wedge\theta.$$

It therefore follows from (3.4) that

$$\begin{aligned} &\mathbb{E}|e_\Delta(t\wedge\theta)|^q \\ \le\ & \mathbb{E}\int_0^{t\wedge\theta} q|e_\Delta(s)|^{q-2}\Big(e_\Delta^T(s)[f(x(s)) - f(\bar{x}_\Delta(s))] + \frac{q-1}{2}|g(x(s)) - g(\bar{x}_\Delta(s))|^2\Big)ds. \end{aligned} \tag{3.5}$$

Re-arranging this gives

$$\mathbb{E}|e_\Delta(t\wedge\theta)|^q \le J_1 + J_2, \tag{3.6}$$

where

$$J_1 = \mathbb{E}\int_0^{t\wedge\theta} q|e_\Delta(s)|^{q-2}\Big((x(s)-\bar{x}_\Delta(s))^T[f(x(s))-f(\bar{x}_\Delta(s))] + \frac{q-1}{2}|g(x(s))-g(\bar{x}_\Delta(s))|^2\Big)ds \tag{3.7}$$

and

$$J_2 = \mathbb{E}\int_0^{t\wedge\theta} q|e_\Delta(s)|^{q-2}(\bar{x}_\Delta(s)-x_\Delta(s))^T[f(x(s))-f(\bar{x}_\Delta(s))]ds. \tag{3.8}$$

By Assumption 3.1, the Young inequality and Lemma 2.5, we derive that

$$\begin{aligned}
J_1 &\le qH_1\mathbb{E}\int_0^{t\wedge\theta} |e_\Delta(s)|^{q-2}|x(s)-\bar{x}_\Delta(s)|^2ds \\
&\le 2qH_1\mathbb{E}\int_0^{t\wedge\theta}\Big(|e_\Delta(s)|^q + |e_\Delta(s)|^{q-2}|x_\Delta(s)-\bar{x}_\Delta(s)|^2\Big)ds \\
&\le 4(q-1)H_1\mathbb{E}\int_0^{t\wedge\theta}|e_\Delta(s)|^q ds + 4H_1\mathbb{E}\int_0^{t\wedge\theta}|x_\Delta(s)-\bar{x}_\Delta(s)|^q ds \\
&\le 4(q-1)H_1\int_0^{t}\mathbb{E}|e_\Delta(s\wedge\theta)|^q ds + 4H_1\int_0^{T}\mathbb{E}|x_\Delta(s)-\bar{x}_\Delta(s)|^q ds \\
&\le 4(q-1)H_1\int_0^{t}\mathbb{E}|e_\Delta(s\wedge\theta)|^q ds + C\Delta^{q/2}(h(\Delta))^q.
\end{aligned} \tag{3.9}$$

Moreover, by Assumption 3.2 and the Hölder inequality as well as Lemmas 2.3, 2.5 and 2.6, we derive that

$$\begin{aligned}
J_2 &\le \mathbb{E}\int_0^{t\wedge\theta}\Big((q-2)|e_\Delta(s)|^q + 2|\bar{x}_\Delta(s)-x_\Delta(s)|^{q/2}|f(x(s))-f(\bar{x}_\Delta(s))|^{q/2}\Big)ds \\
&\le (q-2)\mathbb{E}\int_0^{t}|e_\Delta(s\wedge\theta)|^q ds \\
&+ C\int_0^{T}\mathbb{E}\Big(|\bar{x}_\Delta(s)-x_\Delta(s)|^{q/2}(1+|x(s)|^{qr/2}+|\bar{x}_\Delta(s)|^{qr/2})\Big)ds \\
&\le (q-2)\int_0^{t}\mathbb{E}|e_\Delta(s\wedge\theta)|^q ds \\
&+ C\int_0^{T}\Big(\mathbb{E}|\bar{x}_\Delta(s)-x_\Delta(s)|^{pq/(2p-qr)}\Big)^{(2p-qr)/2p}\Big((1+\mathbb{E}|x(s)|^p+\mathbb{E}|\bar{x}_\Delta(s)|^p)\Big)^{qr/2p}ds \\
&\le (q-2)\int_0^{t}\mathbb{E}|e_\Delta(s\wedge\theta)|^q ds + C\Delta^{q/4}(h(\Delta))^{q/2}.
\end{aligned} \tag{3.10}$$

Substituting (3.9) and (3.10) into (3.6) yields

$$\mathbb{E}|e_\Delta(t\wedge\theta)|^q \le C\int_0^t \mathbb{E}|e_\Delta(s\wedge\theta)|^q ds + C\Delta^{q/4}(h(\Delta))^{q/2}.$$

By the Gronwall inequality, we obtain the required assertion (3.3). □

Let us now state our first result on the convergence rate, where we reveal a strong relation between functions $\mu(\cdot)$ and $h(\cdot)$, which are used to define the truncated EM method.

Theorem 3.4 *Let Assumptions 2.1, 2.2, 3.1 and 3.2 hold with $2p > qr$ and $p > q$. If*

$$h(\Delta) \ge \mu\big((\Delta^{q/4}(h(\Delta))^{q/2})^{-1/(p-q)}\big) \tag{3.11}$$

for all sufficiently small $\Delta \in (0, \Delta^*)$, then, for every such small Δ,

$$\mathbb{E}|x(T) - x_\Delta(T)|^q \le C\Delta^{q/4}(h(\Delta))^{q/2} \quad \text{and} \quad \mathbb{E}|x(T) - \bar{x}_\Delta(T)|^q \le C\Delta^{q/4}(h(\Delta))^{q/2} \tag{3.12}$$

for all $T > 0$.

Proof. Let τ_R, $\rho_{\Delta,R}$, $\theta_{\Delta,R}$ and $e_\Delta(t)$ be the same as before. Using the Young inequality, we derive that for any $\delta > 0$,

$$\begin{aligned} \mathbb{E}|e_\Delta(T)|^q &= \mathbb{E}\Big(|e_\Delta(T)|^q I_{\{\theta_{\Delta,R} > T\}}\Big) + \mathbb{E}\Big(|e_\Delta(T)|^q I_{\{\theta_{\Delta,R} \le T\}}\Big) \\ &\le \mathbb{E}\Big(|e_\Delta(T)|^q I_{\{\theta_{\Delta,R} > T\}}\Big) + \frac{q\delta}{p}\mathbb{E}|e_\Delta(T)|^p + \frac{p-q}{p\delta^{q/(p-q)}}\mathbb{P}(\theta_{\Delta,R} \le T). \end{aligned} \tag{3.13}$$

By Lemmas 2.3 and 2.6, we have

$$\mathbb{E}|e_\Delta(T)|^p \le C$$

while by Lemmas 2.7 and 2.8,

$$\mathbb{P}(\theta_{\Delta,R} \le T) \le \mathbb{P}(\tau_R \le T) + \mathbb{P}(\rho_{\Delta,R} \le T) \le \frac{C}{R^p}.$$

We hence have

$$\mathbb{E}|e_\Delta(T)|^q \le \mathbb{E}\Big(|e_\Delta(T)|^q I_{\{\theta_{\Delta,R} > T\}}\Big) + \frac{Cq\delta}{p} + \frac{C(p-q)}{pR^p\delta^{q/(p-q)}}. \tag{3.14}$$

Consequently

$$\mathbb{E}|e_\Delta(T)|^q \le \mathbb{E}\Big(|e_\Delta(T \wedge \theta_{\Delta,R})|^q\Big) + \frac{Cq\delta}{p} + \frac{C(p-q)}{pR^p\delta^{q/(p-q)}} \tag{3.15}$$

holds for any $\Delta \in (0, \Delta^*)$, $R > |x_0|$ and $\delta > 0$. We can therefore choose $\delta = \Delta^{q/4}(h(\Delta))^{q/2}$ and $R = (\Delta^{q/4}(h(\Delta))^{q/2})^{-1/(p-q)}$ to get

$$\mathbb{E}|e_\Delta(T)|^q \le \mathbb{E}|e_\Delta(T \wedge \theta_{\Delta,R})|^q + C\Delta^{q/4}(h(\Delta))^{q/2}. \tag{3.16}$$

But, by condition (3.11), we have

$$\mu^{-1}(h(\Delta)) \ge (\Delta^{q/4}(h(\Delta))^{q/2})^{-1/(p-q)} = R.$$

We can hence apply Lemma 3.3 to obtain

$$\mathbb{E}|e_\Delta(T \wedge \theta_{\Delta,R})|^q \le C\Delta^{q/4}(h(\Delta))^{q/2}. \tag{3.17}$$

Substituting this into (3.16) yields the first inequality in (3.12). The second inequality there follows from the first one and Lemma 2.5. $\Box$

Example 3.5 Let us illustrate this theorem by an example before we discuss a better convergence rate under stronger conditions. Consider the scalar SDE

$$dx(t) = (x(t) - x^3(t))dt + |x(t)|^{3/2}dB(t), \tag{3.18}$$

where $B(t)$ is a scalar Brownian motion. This is a specified Lewis stochastic volatility model [19]. The reason we only consider this specified model is to keep it simple while our theory is illustrated fully. Of course, our theory works for the general Lewis stochastic volatility model. Clearly, its coefficients $f(x) = x - x^3$ and $g(x) = |x|^{3/2}$ are locally Lipschitz continuous (i.e., satisfy Assumption 2.1). Also, for any $p > 3$, we have

$$xf(x) + \frac{p-1}{2}|g(x)|^2 = |x|^2 - |x|^4 + \frac{p-1}{2}|x|^3,$$

which is bounded above, say by K, for $x \in \mathbb{R}$. That is, Assumption 2.2 is satisfied for any $p > 3$. Moreover, by the mean-value theorem, it is easy to show that

$$|g(x) - g(y)| \le \frac{3}{2}(|x|^{1/2} + |y|^{1/2})|x - y|, \quad \forall x, y \in \mathbb{R}. \tag{3.19}$$

We can then further show that

$$(x - y)(f(x) - f(y)) + \frac{1}{2}|g(x) - g(y)|^2 \le 4|x - y|^2.$$

In other words, Assumption 3.1 is satisfied with $q = 2$. Furthermore, it is obvious that

$$|f(x)| \le |x| + |x|^3 \le 2(1 + |x|^3), \quad \forall x \in \mathbb{R}.$$

Namely, Assumption 3.2 holds with $r = 3$. So far, we have verified that Assumptions 2.1, 2.2, 3.1, 3.2 hold for $q = 2$, $r = 3$ and any $p > 3$. To apply Theorem 3.4, we still need to design functions μ and h in order for (3.11) to hold for all sufficiently small Δ. Noting that

$$\sup_{|x| \le u} (|f(x)| \vee |g(x)|) \le 2u^3, \quad \forall u \ge 1,$$

we can have $\mu(u) = 2u^3$ and its inverse function $\mu^{-1}(u) = (u/2)^{1/3}$ for $u \ge 0$. For $\varepsilon \in (0, 1/4]$, we define $h(\Delta) = \Delta^{-\varepsilon}$ for $\Delta > 0$. Letting $\Delta^* \in (0, 1]$ be sufficiently small, we can make (2.6) hold. Now, inequality (3.11) becomes

$$\Delta^{-\varepsilon} \ge 2(\Delta^{1/2-\varepsilon})^{-3/(p-2)}, \text{ namely, } 1 \ge 2\Delta^{\varepsilon - 3(1/2-\varepsilon)/(p-2)}. \tag{3.20}$$

But, for any $\varepsilon \in (0, 1/4]$, we can choose sufficiently large p such that $\varepsilon - 3(1/2 - \varepsilon)/(p - 2) > 0$ and hence (3.20) holds for all sufficiently small Δ. We can therefore conclude by Theorem 3.4 that the truncated EM solutions of the SDE (3.18) satisfy

$$\mathbb{E}|x(T) - x_\Delta(T)|^2 = O(\Delta^{1/2-\varepsilon}) \quad \text{and} \quad \mathbb{E}|x(T) - \bar{x}_\Delta(T)|^2 = O(\Delta^{1/2-\varepsilon}).$$

This example shows that when the truncated EM method is applied to the SDE (3.18), the order of L^2-convergence is close to 1/2 (or, the order of L^1-convergence is close to 1/4). Can we improve the order?

The answer is yes. In the remaining of this section, we will establish a new result which shows the order of L^q-convergence is close to $q/2$. This is almost optimal if we recall that the classical EM method has order $q/2$ of L^q-convergence under the global Lipschitz condition. For this almost optimal result, we need slightly stronger condition than Assumption 3.2.

Assumption 3.6 *Assume that there is a pair of positive constants ρ and H_3 such that*

$$|f(x) - f(y)|^2 \vee |g(x) - g(y)|^2 \le H_3(1 + |x|^\rho + |y|^\rho)|x - y|^2 \tag{3.21}$$

for all $x, y \in \mathbb{R}^d$.

The following is another key lemma.

Lemma 3.7 *Let Assumptions 2.1, 2.2, 3.1 and 3.6 hold and assume that $p > q > 2$ and $2p > q\rho$. Let $R > |x_0|$ be a real number and let $\Delta \in (0, \Delta^*)$ be sufficiently small such that $\mu^{-1}(h(\Delta)) \ge R$. Let $\theta_{\Delta,R}$ and $e_\Delta(t)$ be the same as before. Then, for any $\bar{q} \in [2, q)$,*

$$\mathbb{E}|e_\Delta(T \wedge \theta_{\Delta,R})|^{\bar{q}} \le C\Delta^{\bar{q}/2}(h(\Delta))^{\bar{q}}, \quad \forall T > 0. \tag{3.22}$$

Proof. We use the same notation as in the proof of Lemma 3.3. Clearly, (3.5) holds when q there is replaced with $\bar{q}$. Namely, we have

$$\begin{aligned} &\mathbb{E}|e_\Delta(t\wedge\theta)|^{\bar{q}} \\ \le\ &\mathbb{E}\int_0^{t\wedge\theta} \bar{q}|e_\Delta(s)|^{\bar{q}-2}\Big(e_\Delta^T(s)[f(x(s))-f(\bar{x}_\Delta(s))] + \frac{\bar{q}-1}{2}|g(x(s))-g(\bar{x}_\Delta(s))|^2\Big)ds. \end{aligned} \tag{3.23}$$

Noting

$$\begin{aligned} &\frac{\bar{q}-1}{2}|g(x(s))-g(\bar{x}_\Delta(s))|^2 \\ \le\ &\frac{\bar{q}-1}{2}\Big[\Big(1+\frac{q-\bar{q}}{\bar{q}-1}\Big)|g(x(s))-g(x_\Delta(s))|^2 + \Big(1+\frac{\bar{q}-1}{q-\bar{q}}\Big)|g(x_\Delta(s))-g(\bar{x}_\Delta(s))|^2\Big] \\ =\ &\frac{q-1}{2}|g(x(s))-g(x_\Delta(s))|^2 + \frac{(\bar{q}-1)(q-1)}{2(q-\bar{q})}|g(x_\Delta(s))-g(\bar{x}_\Delta(s))|^2, \end{aligned}$$

we get from (3.23) that

$$\mathbb{E}|e_\Delta(t\wedge\theta)|^{\bar{q}} \le J_3 + J_4, \tag{3.24}$$

where

$$J_3 = \mathbb{E}\int_0^{t\wedge\theta} \bar{q}|e_\Delta(s)|^{\bar{q}-2}\Big(e_\Delta^T(s)[f(x(s))-f(x_\Delta(s))] + \frac{q-1}{2}|g(x(s))-g(x_\Delta(s))|^2\Big)ds \tag{3.25}$$

and

$$\begin{aligned} J_4 = \mathbb{E}\int_0^{t\wedge\theta} \bar{q}|e_\Delta(s)|^{\bar{q}-2}\Big(&e_\Delta^T(s)[f(x_\Delta(s))-f(\bar{x}_\Delta(s))] \\ &+ \frac{(\bar{q}-1)(q-1)}{2(q-\bar{q})}|g(x_\Delta(s))-g(\bar{x}_\Delta(s))|^2\Big)ds. \end{aligned} \tag{3.26}$$

By Assumption 3.1, we can show easily that

$$J_3 \le \bar{q}H_1\int_0^t \mathbb{E}|e_\Delta(s\wedge\theta)|^{\bar{q}}ds. \tag{3.27}$$

By Assumption 3.6, we also have

$$J_4 \le (\bar{q}/2)\int_0^t \mathbb{E}|e_\Delta(s\wedge\theta)|^{\bar{q}}ds + J_5, \tag{3.28}$$

where

$$J_5 = C\,\mathbb{E}\int_0^{t\wedge\theta}\Big(|e_\Delta(s)|^{\bar{q}-2}(1+|x_\Delta(s)|^\rho+|\bar{x}_\Delta(s)|^\rho)|x_\Delta(s)-\bar{x}_\Delta(s)|^2\Big)ds.$$

By the Young inequality etc., we derive that

$$\begin{aligned} J_5 \ \le\ & C\int_0^t \mathbb{E}|e_\Delta(s\wedge\theta)|^{\bar{q}}ds \\ +\ & C\int_0^T \mathbb{E}\Big((1+|x_\Delta(s)|^{\rho\bar{q}/2}+|\bar{x}_\Delta(s)|^{\rho\bar{q}/2})|x_\Delta(s)-\bar{x}_\Delta(s)|^{\bar{q}}\Big)ds. \end{aligned} \tag{3.29}$$

In the same way as (3.10) was proved, we can then show

$$J_5 \ \le\ C\int_0^t \mathbb{E}|e_\Delta(s\wedge\theta)|^{\bar{q}}ds + C\Delta^{\bar{q}/2}(h(\Delta))^{\bar{q}}. \tag{3.30}$$

Putting (3.30) into (3.28) and then substituting it and (3.27) into (3.24), we get

$$\mathbb{E}|e_\Delta(t \wedge \theta)|^{\bar{q}} \leq C \int_0^t \mathbb{E}|e(s \wedge \theta)|^{\bar{q}} ds + C\Delta^{\bar{q}/2}(h(\Delta))^{\bar{q}}.$$

By the Gronwall inequality, we obtain the required assertion (3.22). □

The following theorem shows a higher order of $L^{\bar{q}}$-convergence of the truncated EM method.

Theorem 3.8 *Let Assumptions 2.1, 2.2, 3.1 and 3.6 hold with $p > q > 2$ and $2p > q\rho$. Let $\bar{q} \in [2, q)$. If*

$$h(\Delta) \geq \mu\big([\Delta^{\bar{q}/2}(h(\Delta))^{\bar{q}}]^{-1/(p-\bar{q})}\big) \tag{3.31}$$

for all sufficiently small $\Delta \in (0, \Delta^*)$, then, for every such small Δ,

$$\mathbb{E}|x(T) - x_\Delta(T)|^{\bar{q}} \leq C\Delta^{\bar{q}/2}(h(\Delta))^{\bar{q}} \quad \textit{and} \quad \mathbb{E}|x(T) - \bar{x}_\Delta(T)|^2 \leq C\Delta^{\bar{q}/2}(h(\Delta))^{\bar{q}}. \tag{3.32}$$

Proof. We use the same notation as in the proof of Theorem 3.4. Clearly, (3.15) holds if q there is replaced with $\bar{q}$. Choosing $\delta = \Delta^{\bar{q}/2}(h(\Delta))^{\bar{q}}$ and $R = [\Delta^{\bar{q}/2}(h(\Delta))^{\bar{q}}]^{-1/(p-\bar{q})}$, we then obtain

$$\mathbb{E}|e_\Delta(T)|^{\bar{q}} \leq \mathbb{E}|e_\Delta(T \wedge \theta_{\Delta,R})|^{\bar{q}} + C\Delta^{\bar{q}/2}(h(\Delta))^{\bar{q}}. \tag{3.33}$$

But, by condition (3.31), we have

$$\mu^{-1}(h(\Delta)) \geq [\Delta^{\bar{q}/2}(h(\Delta))^{\bar{q}}]^{-1/(p-\bar{q})} = R.$$

We can hence apply Lemma 3.7 to obtain

$$\mathbb{E}|e_\Delta(T \wedge \theta_{\Delta,R})|^{\bar{q}} \leq C\Delta^{\bar{q}/2}(h(\Delta))^{\bar{q}}. \tag{3.34}$$

Substituting this into (3.33) yields the first inequality in (3.32). The second inequality there follows from the first one and Lemma 2.5. □

Example 3.9 (Continuation of Example 3.5) Let us now return to the SDE (3.18). Recalling (3.19), we can easily show that, for any $x, y \in \mathbb{R}$,

$$(x-y)(f(x)-f(y)) + |g(x)-g(y)|^2 \leq 12|x-y|^2$$

and

$$|f(x)-f(y)|^2 \vee |g(x)-g(y)|^2 \leq 12(1+|x|^4+|y|^4)|x-y|^2.$$

In other words, Assumptions 3.1 and 3.6 are satisfied with $q = 3$ and $\rho = 4$. We hence have that Assumptions 2.1, 2.2, 3.1, 3.6 hold for $q = 3$, $\rho = 4$ and any $p > 6$. To apply Theorem 3.8, we let $\bar{q} = 2$ and still choose $\mu(u) = 2u^3$ but let $h(\Delta) = \Delta^{-\varepsilon/2}$ for $\varepsilon \in (0, 1/2]$. Then, inequality (3.31) becomes

$$\Delta^{-\varepsilon/2} \geq 2\Delta^{-3(1-\varepsilon)/(p-2)}, \text{ namely, } 1 \geq 2\Delta^{\varepsilon/2 - 3(1-\varepsilon)/(p-2)}. \tag{3.35}$$

But, for any $\varepsilon \in (0, 1/2]$, we can choose sufficiently large p such that $\varepsilon/2 > 3/(p-2)$ and hence (3.35) holds for all sufficiently small Δ. We can therefore conclude by Theorem 3.8 that the truncated EM solutions of the SDE (3.18) satisfy

$$\mathbb{E}|x(T) - x_\Delta(T)|^2 = O(\Delta^{1-\varepsilon}) \quad \text{and} \quad \mathbb{E}|x(T) - \bar{x}_\Delta(T)|^2 = O(\Delta^{1-\varepsilon}).$$

That is, the order of L^2-convergence can be arbitrarily close to 1.

4 Convergence Rates over a Finite Time Interval

In the previous section, we showed that both truncated EM solutions $x_\Delta(T)$ and $\bar{x}_\Delta(T)$ will converge to the true solution $x(T)$ in $L^{\bar{q}}$ for any $T > 0$. This is sufficient for some applications e.g. when we need to approximate the European put or call option value (see, e.g., [8]). However, we sometimes need to approximate quantities that are path-dependent, for example, the European barrier option value. In these situations, we will need the strong convergence for a numerical solution to the true solution over a finite time interval (see, e.g., [7]). Let us now begin to discuss the convergence rates over the time interval $[0, T]$. We need a stronger assumption.

Assumption 4.1 *Assume that there is a pair of positive constants γ and H such that*

$$(x-y)^T(f(x)-f(y)) \le H|x-y|^2, \tag{4.1}$$

$$|f(x)-f(y)|^2 \le H(1+|x|^\gamma+|y|^\gamma)|x-y|^2, \tag{4.2}$$

$$|g(x)-g(y)|^2 \le H|x-y|^2 \tag{4.3}$$

for all $x, y \in \mathbb{R}^d$.

These conditions are the same as those imposed in [9, Assumptions 3.1 and 4.1]. Under these conditions, it was showed in [9] that either the split step backward Euler solution or the backward Euler solution converge to the true solution over the finite time interval $[0, T]$ in L^2 with order 1 (i.e., half in L^1). In this section, we will show that under these same conditions, the the truncated EM solution will converge to the true solution over the finite time interval $[0, T]$ in L^q with order close to $q/2$.

Before we proceed to establish our theory, we need to make a remark which will make our proof of Theorem 4.6 below more clear.

Remark 4.2 We observe that Assumption 4.1 implies all assumptions we imposed so far in this paper. In fact, Assumption 2.1, 3.2 and 3.6 follows from Assumption 4.1 obviously. We now show that *Assumption 2.2 is satisfied for any $p > 2$*. In fact,

$$\begin{aligned}
& x^T f(x) + \frac{p-1}{2}|g(x)|^2 \\
\le\; & x^T f(0) + x^T(f(x)-f(0)) + (p-1)|g(0)|^2 + (p-1)|g(x)-g(0))|^2 \\
\le\; & 0.5|f(0)|^2 + p|g(0)|^2 + (0.5+pH)|x|^2 \\
\le\; & K(1+|x|^2),
\end{aligned}$$

where $K = (0.5|f(0)|^2 + p|g(0)|^2) \vee (0.5+pH)$. Similarly, we can show that *Assumption 3.1 holds for any $q \ge 2$*.

We therefore see that all the results in Sections 2 and 3 hold under Assumption 4.1. Of course, under the stronger Assumption 4.1, we will be able to show the convergence rate for the paths of the solution. The following lemma is a key.

Lemma 4.3 *Let Assumption 4.1 hold. Let $R > |x_0|$ be a real number and let $\Delta \in (0, \Delta^*)$ be sufficiently small such that $\mu^{-1}(h(\Delta)) \ge R$. Let $\theta_{\Delta,R}$ and $e_\Delta(t)$ be the same as before. Let $q \ge 2$ be arbitrary. Then*

$$\mathbb{E}\Big(\sup_{0\le u\le T\wedge\theta_{\Delta,R}} |e_\Delta(u)|^q\Big) \le C\Delta^{q/2}(h(\Delta))^q, \quad \forall T > 0. \tag{4.4}$$

Proof. Again write $\theta_{\Delta,R} = \theta$. By the Itô formula, we have that, for $0 \le t \le T$,

$$
\begin{aligned}
& \mathbb{E}\Big(\sup_{0\le u\le t\wedge\theta} |e_\Delta(u)|^q \Big) \\
\le\ & \mathbb{E}\Big(\sup_{0\le u\le t\wedge\theta} \int_0^u q|e_\Delta(s)|^{q-2}\Big[e_\Delta^T(s)[f(x(s)) - f_\Delta(\bar{x}_\Delta(s))] \\
& \qquad\qquad + \frac{q-1}{2}|g(x(s)) - g_\Delta(\bar{x}_\Delta(s))|^2\Big] ds\Big) \\
+\ & \mathbb{E}\Big(\sup_{0\le u\le t\wedge\theta} \int_0^u q|e_\Delta(s)|^{q-2} e_\Delta^T(s)[g(x(s)) - g_\Delta(\bar{x}_\Delta(s))]dB(s)\Big).
\end{aligned} \tag{4.5}
$$

As explained in the proof of Lemma 3.3, we see that

$$
f_\Delta(\bar{x}_\Delta(s)) = f(\bar{x}_\Delta(s)) \quad \text{and} \quad g_\Delta(\bar{x}_\Delta(s)) = g(\bar{x}_\Delta(s)) \quad \text{for } 0 \le s \le t\wedge\theta.
$$

It therefore follows from (4.5) that

$$
\mathbb{E}\Big(\sup_{0\le u\le t\wedge\theta} |e_\Delta(u)|^2 \Big) \le J_6 + J_7, \tag{4.6}
$$

where

$$
\begin{aligned}
J_6 \ =\ & \mathbb{E}\Big(\sup_{0\le u\le t\wedge\theta} \int_0^u q|e_\Delta(s)|^{q-2}\Big[e_\Delta^T(s)[f(x(s)) - f(\bar{x}_\Delta(s))] \\
& \qquad\qquad + \frac{q-1}{2}|g(x(s)) - g(\bar{x}_\Delta(s))|^2\Big] ds\Big)
\end{aligned}
$$

and

$$
J_7 = \mathbb{E}\Big(\sup_{0\le u\le t\wedge\theta} \int_0^u q|e_\Delta(s)|^{q-2} e_\Delta^T(s)[g(x(s)) - g(\bar{x}_\Delta(s))]dB(s)\Big).
$$

By Assumption 4.1, we derive that

$$
\begin{aligned}
J_6 \ \le\ & \mathbb{E}\Big(\sup_{0\le u\le t\wedge\theta} \int_0^u q|e_\Delta(s)|^{q-2}\Big[e_\Delta^T(s)[f(x(s)) - f(x_\Delta(s))] + e_\Delta^T(s)[f(x_\Delta(s)) - f(\bar{x}_\Delta(s))] \\
& \qquad\qquad + \frac{q-1}{2}|g(x(s)) - g(\bar{x}_\Delta(s))|^2\Big] ds\Big) \\
\le\ & \mathbb{E}\int_0^{t\wedge\theta} q|e_\Delta(s)|^{q-2}\Big[(H+1)|e_\Delta(s)|^2 + |f(x_\Delta(s)) - f(\bar{x}_\Delta(s))|^2 \\
& \qquad\qquad + H(q-1)(|e_\Delta(s)|^2 + |x_\Delta(s) - \bar{x}_\Delta(s)|^2)\Big] ds\Big) \\
\le\ & \mathbb{E}\int_0^{t\wedge\theta} q(Hq+1)|e_\Delta(s)|^{q-2}\Big[|e_\Delta(s)|^2 + (1 + |x_\Delta(s)|^\gamma + |\bar{x}_\Delta(s)|^\gamma)|x_\Delta(s) - \bar{x}_\Delta(s)|^2\Big) \\
\le\ & 2q(Hq+1)\int_0^t \mathbb{E}|e_\Delta(s\wedge\theta)|^q ds \\
+\ & q(Hq+1)\int_0^T \mathbb{E}\Big[(1 + |x_\Delta(s)|^\gamma + |\bar{x}_\Delta(s)|^\gamma)^{q/2}|x_\Delta(s) - \bar{x}_\Delta(s)|^q\Big] ds.
\end{aligned} \tag{4.7}
$$

In the same way as Lemmas 3.3 and 3.7 were proved, we can then further show that

$$
J_6 \ \le\ 2q(Hq+1)\int_0^t \mathbb{E}|e_\Delta(s\wedge\theta)|^q ds + C\Delta^{q/2}(h(D))^q. \tag{4.8}
$$

Moreover, by the Burkholder–Davis–Gundy inequality (see, e.g., [21]) and Assumption 4.1, we derive that

$$\begin{aligned}
J_7 &\leq 4\sqrt{2}q\mathbb{E}\Big(\Big[\int_0^{t\wedge\theta} |e_\Delta^T(s)|^{2(q-1)}|g(x(s)) - g(\bar{x}_\Delta(s))|^2 ds\Big]^{1/2}\Big) \\
&\leq 4\sqrt{2}qH\mathbb{E}\Big(\sup_{0\leq u\leq t\wedge\theta} |e_\Delta(u)|^q \Big[\int_0^{t\wedge\theta} |e_\Delta^T(s)|^{q-2}|x(s) - \bar{x}_\Delta(s)|^2 ds\Big]^{1/2}\Big) \\
&\leq \frac{1}{2}\mathbb{E}\Big(\sup_{0\leq u\leq t\wedge\theta} |e_\Delta(u)|^q\Big) + 16q^2H^2\mathbb{E}\int_0^{t\wedge\theta} |e_\Delta^T(s)|^{q-2}|x(s) - \bar{x}_\Delta(s)|^2 ds.
\end{aligned}$$

But, by the Young inequality,

$$|e_\Delta^T(s)|^{q-2}|x(s) - \bar{x}_\Delta(s)|^2 \leq \frac{q-2}{q}|e_\Delta^T(s)|^q + \frac{2}{q}|x(s) - \bar{x}_\Delta(s)|^q.$$

Hence

$$\begin{aligned}
J_7 &\leq \frac{1}{2}\mathbb{E}\Big(\sup_{0\leq u\leq t\wedge\theta} |e_\Delta(u)|^q\Big) + 16q(q-2)H^2\,\mathbb{E}\int_0^{t\wedge\theta} |e_\Delta(s)|^q ds \\
&+ 32qH^2\int_0^T \mathbb{E}|x_\Delta(s) - \bar{x}_\Delta(s)|^q ds \\
&\leq \frac{1}{2}\mathbb{E}\Big(\sup_{0\leq u\leq t\wedge\theta} |e_\Delta(u)|^q\Big) + 16q(q-2)H^2\int_0^t \mathbb{E}\Big(\sup_{0\leq u\leq s\wedge\theta} |e_\Delta(u)|^q\Big)ds \\
&+ C\Delta^{q/2}(h(\Delta))^q, \qquad (4.9)
\end{aligned}$$

where we have used Lemma 2.5 in the last step. Substituting (4.8) and (4.9) into (4.6), we get

$$\mathbb{E}\Big(\sup_{0\leq u\leq t\wedge\theta} |e_\Delta(u)|^q\Big) \leq C\int_0^t \mathbb{E}\Big(\sup_{0\leq u\leq s\wedge\theta} |e_\Delta(u)|^q\Big)ds + C\Delta^{q/2}(h(\Delta))^q. \qquad (4.10)$$

Finally, the Gronwall inequality yields the required assertion (4.4). □

We also cite a couple of results from [23] as lemmas which we will need to prove our main theorem in this section.

Lemma 4.4 *Let Assumption 4.1 hold. Then, for any $p \geq 2$,*

$$\mathbb{E}\Big(\sup_{0\leq t\leq T} |x(t)|^p\Big) \leq C \qquad (4.11)$$

and

$$\sup_{0<\Delta\leq\Delta^*} \mathbb{E}\Big(\sup_{0\leq t\leq T} |x_\Delta(t)|^p\Big) \leq C. \qquad (4.12)$$

Lemma 4.5 *Let $q \geq 2$ and $\Delta \in (0, \Delta^*]$. Let n be a sufficiently large integer for which*

$$\Big(\frac{2n}{2n-1}\Big)^q (T+1)^{q/2n} \leq 2 \quad \text{and} \quad \frac{n-1}{2n} > \frac{1}{3}. \qquad (4.13)$$

We then have

$$\mathbb{E}\Big(\sup_{0\leq t\leq T} |x_\Delta(t) - \bar{x}_\Delta(t)|^q\Big) \leq 2^{q+1}n^{q/2}(h(\Delta))^q\Delta^{q(n-1)/2n}. \qquad (4.14)$$

Before we state our theorem in this section, let us remark that it is straightforward to see from Assumption 4.1 that

$$\sup_{|x|\le u}(|f(x)|\vee|g(x)|)\le \bar{H}|x|^{1+\gamma},\quad \forall u\ge 1, \tag{4.15}$$

where $\bar{H}=|f(0)|+|g(0)|+2\sqrt{H}$. It should be pointed out that this inequality holds if γ is replaced by $\gamma/2$ but this will not affect the proof of the following theorem. In fact, every thing in the statement and proof of the theorem will work if γ is replaced by $\gamma/2$.

Theorem 4.6 *Let Assumption 4.1 hold and $\varepsilon\in(0,1/2)$ be arbitrary. Define*

$$\mu(u)=\bar{H}u^{1+\gamma},\quad u\ge 0$$

and

$$h(\Delta)=\Delta^{-\varepsilon/2},\quad \Delta\in(0,1]$$

Letting $\Delta^*\in(0,1]$ be sufficiently small, we can make (2.6) hold. Then, for any $q\ge 2$, the truncated EM solutions satisfy

$$\mathbb{E}\Big(\sup_{0\le t\le T}|x_\Delta(t)-x(t)|^q\Big)=O(\Delta^{q(1-\varepsilon)/2}) \tag{4.16}$$

and

$$\mathbb{E}\Big(\sup_{0\le t\le T}|\bar{x}_\Delta(t)-x(t)|^q\Big)=O(\Delta^{q(1-\varepsilon)/2}). \tag{4.17}$$

Proof. Let $e_\Delta(t)$ and $\theta_{\Delta,R}$ be the same as defined in the proof of Lemma 3.3. Recalling Remark 4.2, we know that all of the assumptions in sections 2 and 3 are satisfied under Assumption 4.1. In particular, we can choose $p>2$ as large as we need for Assumption 2.2 to hold. For our proof, we choose $p>q\vee(1+\gamma)$ sufficiently large such that

$$\frac{\varepsilon}{2}>\frac{q(1+\gamma)}{2(p-q)}. \tag{4.18}$$

Using the Young inequality, we can show that, for any $\Delta\in(0,\Delta^*)$, $\delta>0$ and $R>|x_0|$,

$$\begin{aligned}&\mathbb{E}\Big(\sup_{0\le t\le T}|e_\Delta(t)|^q\Big)\\ \le\ &\mathbb{E}\Big(I_{\{\theta_{\Delta,R}>T\}}\sup_{0\le t\le T}|e_\Delta(t)|^q\Big)+\frac{q\delta}{p}\mathbb{E}\Big(\sup_{0\le t\le T}|e_\Delta(t)|^q\Big)+\frac{p-q}{p\delta^{q/(p-q)}}\mathbb{P}(\theta_{\Delta,R}\le T).\end{aligned} \tag{4.19}$$

By Lemmas 4.4, 2.7 and 2.8, we can then have

$$\mathbb{E}\Big(\sup_{0\le t\le T}|e_\Delta(t)|^q\Big)\le\mathbb{E}\Big(I_{\{\theta_{\Delta,R}>T\}}\sup_{0\le t\le T}|e_\Delta(t)|^q\Big)+\frac{Cq\delta}{p}+\frac{C(p-q)}{pR^p\delta^{q/(p-q)}}. \tag{4.20}$$

We therefore see that the inequality

$$\mathbb{E}\Big(\sup_{0\le t\le T}|e_\Delta(t)|^q\Big)\le\mathbb{E}\Big(\sup_{0\le t\le T}|e_\Delta(t\wedge\theta_{\Delta,R})|^q\Big)+\frac{Cq\delta}{p}+\frac{C(p-q)}{pR^p\delta^{q/(p-q)}} \tag{4.21}$$

holds for any $\Delta\in(0,\Delta^*)$, $\delta>0$ and $R>|x_0|$. Choosing $\delta=\Delta^{q(1-\varepsilon)/2}$ and $R=\Delta^{-q(1-\varepsilon)/2(p-q)}$, we then get

$$\mathbb{E}\Big(\sup_{0\le t\le T}|e_\Delta(t)|^q\Big)\le\mathbb{E}\Big(\sup_{0\le t\le T}|e_\Delta(t\wedge\theta_{\Delta,R})|^q\Big)+C\Delta^{q(1-\varepsilon)/2} \tag{4.22}$$

for any $\Delta \in (0, \Delta^*)$. On the other hand, by (4.18), we see that

$$\Delta^{-\varepsilon/2} \geq \bar{H}\Delta^{-q(1-\varepsilon)(1+\gamma)/2(p-2)}$$

for all sufficiently small Δ. For every such small Δ, we then have

$$\mu^{-1}(h(\Delta)) \geq \Delta^{-q(1-\varepsilon)/2(p-q)} = R.$$

By Lemma 4.3, we hence get from (4.22) that

$$\mathbb{E}\Big(\sup_{0\leq t\leq T}|e_\Delta(t)|^2\Big) \leq C\Delta^{q(1-\varepsilon)/2} \tag{4.23}$$

for every sufficiently small Δ. In other words, the required assertion (4.16) has been proved. To show (4.17), we choose an integer $n > 1/\varepsilon$ sufficiently large for (4.13) to hold and then, by Lemma 4.5, we have

$$\mathbb{E}\Big(\sup_{0\leq t\leq T}|x_\Delta(t) - \bar{x}_\Delta(t)|^2\Big) \leq C\Delta^{q(1-2\varepsilon)/2}, \quad \forall \Delta \in (0, \Delta^*).$$

This, together with (4.23), implies

$$\mathbb{E}\Big(\sup_{0\leq t\leq T}|\bar{x}_\Delta(t) - x(t)|^2\Big) \leq C\Delta^{q(1-2\varepsilon)/2} \tag{4.24}$$

for all sufficiently small Δ. In other words,

$$\mathbb{E}\Big(\sup_{0\leq t\leq T}|\bar{x}_\Delta(t) - x(t)|^2\Big) = O(\Delta^{q(1-2\varepsilon)/2}).$$

As $\varepsilon \in (0, 1/2)$ is arbitrary, we must therefore have (4.17) as desired. $\Box$

Example 4.7 To illustrate this theorem, let us consider the scalar SDE

$$dx(t) = ax(t)(b - x^2(t))dt + cx(t)dB(t), \quad t \geq 0, \quad x(0) = x_0 \in \mathbb{R}, \tag{4.25}$$

where a, b, c are all positive numbers. This is known as the stochastic Ginzburg–Landau equation (see, e.g., [6, 18]) or the power logistic model (see, e.g., [3]). It is known (see, e.g., [10, 13]) that the second moment of the EM solution to the SDE (4.25) will blow up so will not converge to the true solution in L^2. However, by our theory, we will see that the truncated EM solutions will converge to the true solution in L^2 with order close to 1. In fact, it is easy to verify that the coefficients $f(x) = ax(b - x^2)$ and $g(x) = cx$ satisfy Assumption 4.1 and

$$\sup_{|x|\leq u}(|f(x)| \vee |g(x)|) \leq \bar{H}|x|^3, \quad \forall u \geq 1,$$

where $\bar{H} = a(b+1) \vee c$. Let $\varepsilon \in (0, 1/2]$ be arbitray. Define $\mu(u) = \bar{H}u^3$ and $h(\Delta) = \Delta^{-\varepsilon/2}$. By Theorem 4.6, we can then conclude that the truncated EM solutions of the SDE (4.25) satisfy

$$\mathbb{E}\Big(\sup_{0\leq t\leq T}|x_\Delta(t) - x(t)|^2\Big) = O(\Delta^{1-\varepsilon}) \quad \text{and} \quad \mathbb{E}\Big(\sup_{0\leq t\leq T}|\bar{x}_\Delta(t) - x(t)|^2\Big) = O(\Delta^{1-\varepsilon}).$$

To support our theoretical results, we perform numerical simulations for the SDE

$$dx(t) = 0.1x(t)(1 - x^2(t))dt + 0.2x(t)dB(t), \quad 0 \leq t \leq 1, \quad x(0) = 2. \tag{4.26}$$

It was proved in [9] that the backward EM method applied to the SDE (4.26) has the order 0.5 in L^1-convergence (or 1 in L^2). It is therefore sufficient to compare our new truncated EM

(TEM) with the backward EM (BEM). We will hence use the TEM and BEM to carry out the numerical simulations (and we choose $\varepsilon = 0.5$ for the TEM). Figure 5.1 shows the computer simulations of the sample paths of $x(t)$ by the TEM and BEM, respectively, with stepsize 10^{-5}. We see that both sample paths are almost identical. We also perform 1000 sample paths of the TEM and BEM solutions for each of stepsizes 10^{-3},10^{-4}, 10^{-5} and 10^{-6}. The log-log plot of the strong errors against the stepsizes is shown in Figure 5.2. Comparing it with the dashed reference line of slop 1, we observe that the order of the strong error between the TEM and BEM is 1. But the BEM method has order 0.5 in L^1-convergence so we see that our TEM also has the order 0.5 in L^1-convergence. This supports our theoretical results.

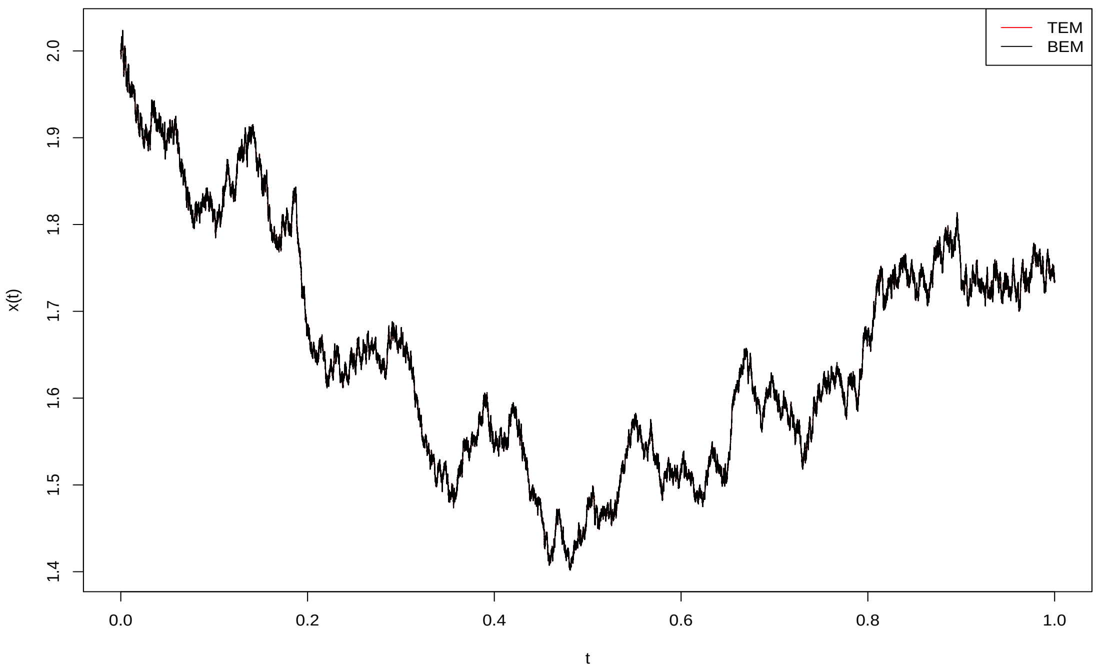

Figure 5.1: Computer simulations of a sample path of $x(t)$ by the TEM and BEM with stepsize 10^{-5}: red for TEM and black for BEM.

5 Conclusions

This is the continuation of our recent paper [23], where the truncated EM method was initiated for the multi-dimensional nonlinear SDEs. Under some additional conditions to those imposed in [23], we have discussed the L^q-convergence rates of the truncated EM method and showed that the order of L^q-convergence could be arbitrarily close to $q/2$. Several examples have been discussed to illustrate our theory. The computer simulations also support our theoretical results.

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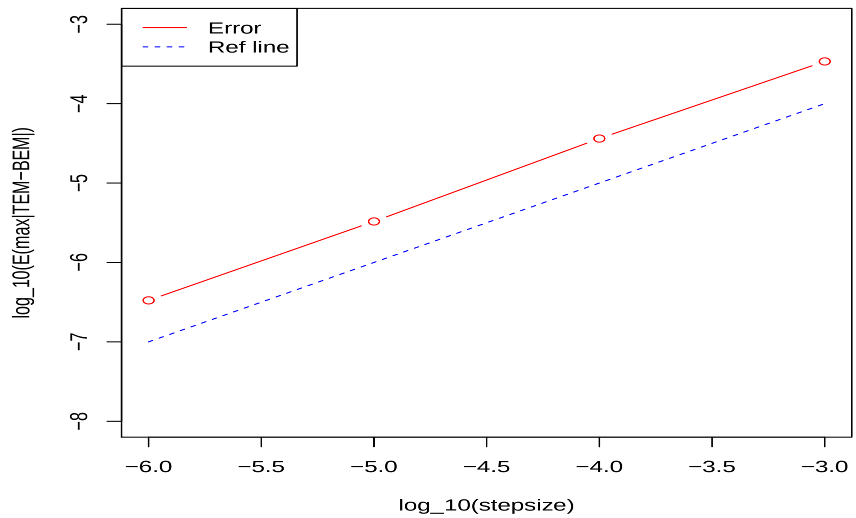

Figure 5.2: The strong errors between TEM and BEM. The dashed reference line has slop 1.

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