QUADRANT MARKED MESH PATTERNS IN 132-AVOIDING PERMUTATIONS III

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Abstract

Given a permutation $\sigma = \sigma_1 \dots \sigma_n$ in the symmetric group S_n, we say that σ_i matches the marked mesh pattern $\mathrm{MMP}(a,b,c,d)$ in σ if there are at least a points to the right of σ_i in σ which are greater than σ_i, at least b points to the left of σ_i in σ which are greater than σ_i, at least c points to the left of σ_i in σ which are smaller than σ_i, and at least d points to the right of σ_i in σ which are smaller than σ_i.

This paper is continuation of the systematic study of the distributions of quadrant marked mesh patterns in 132-avoiding permutations started in [9] and [10] where we studied the distribution of the number of matches of $\mathrm{MMP}(a,b,c,d)$ in 132-avoiding permutations where at most two elements of a,b,c,d are greater than zero and the remaining elements are zero. In this paper, we study the distribution of the number of matches of $\mathrm{MMP}(a,b,c,d)$ in 132-avoiding permutations where at least three of a,b,c,d are greater than zero. We provide explicit recurrence relations to enumerate our objects which can be used to give closed forms for the generating functions associated with such distributions. In many cases, we provide combinatorial explanations of the coefficients that appear in our generating functions.

1. Introduction

The notion of mesh patterns was introduced by Brändén and Claesson [2] to provide explicit expansions for certain permutation statistics as, possibly infinite, linear combinations of (classical) permutation patterns. This notion was further studied in [1, 3, 5, 6, 9, 12].

Kitaev and Remmel [6] initiated the systematic study of the distributions of quadrant marked mesh patterns on permutations. The study was extended to 132-avoiding permutations by Kitaev, Remmel and Tiefenbruck in [9, 10], and the present paper continues this line of research. Kitaev and Remmel also studied the distributions of quadrant marked mesh patterns in up-down and down-up permutations [7, 8].

Let $\sigma = \sigma_1 \dots \sigma_n$ be a permutation written in one-line notation. Then we will consider the *graph of* σ, $G(\sigma)$, to be the set of points (i, σ_i) for $i = 1, \dots, n$. For example, the graph of the permutation $\sigma = 471569283$ is pictured in Figure 1. Then if we draw a coordinate system centered at a point (i, σ_i), we will be interested in the points that lie in the four quadrants I, II, III, and IV of that coordinate system as pictured in Figure 1. Let $\mathbb{N} = \{1, 2, \dots\}$ denote the positive integers. For any $a, b, c, d \in \{0\} \cup \mathbb{N}$ and any $\sigma = \sigma_1 \dots \sigma_n \in S_n$, the set of all permutations of length n, we say that σ_i *matches the quadrant marked mesh pattern* $\mathrm{MMP}(a, b, c, d)$ in σ if, in $G(\sigma)$ relative to the coordinate system which has the point (i, σ_i) as its origin, there are at least a points in quadrant I, at least b points in quadrant II, at least c points in quadrant III, and at least d points in quadrant IV. For example, if $\sigma = 471569283$, the point $\sigma_4 = 5$ matches the marked mesh pattern $\mathrm{MMP}(2, 1, 2, 1)$ since, in $G(\sigma)$ relative to the coordinate system with the origin at $(4, 5)$, there are 3 points in quadrant I, 1 point in quadrant II, 2 points in quadrant III, and 2 points in quadrant IV. Note that if a coordinate in $\mathrm{MMP}(a, b, c, d)$ is 0, then there is no condition imposed on the points in the corresponding quadrant.

In addition, we considered patterns $\mathrm{MMP}(a, b, c, d)$ where $a, b, c, d \in \{\emptyset\} \cup \{0\} \cup \mathbb{N}$. Here when a coordinate of $\mathrm{MMP}(a, b, c, d)$ is the empty set, then for σ_i to match $\mathrm{MMP}(a, b, c, d)$ in $\sigma = \sigma_1 \dots \sigma_n \in S_n$, it must be the case that there are no points in $G(\sigma)$ relative to the coordinate system with the origin at (i, σ_i) in the corresponding quadrant. For example, if $\sigma = 471569283$, the point $\sigma_3 = 1$ matches the marked mesh pattern $\mathrm{MMP}(4, 2, \emptyset, \emptyset)$ since in $G(\sigma)$ relative to the coordinate system with the origin at $(3, 1)$, there are 6 points in quadrant I, 2 points in quadrant II, no points in quadrants III and IV. We let $\mathrm{mmp}^{(a,b,c,d)}(\sigma)$ denote the number of i such that σ_i matches $\mathrm{MMP}(a, b, c, d)$ in σ.

Note how the (two-dimensional) notation of Úlfarsson [12] for *marked mesh patterns* corresponds to our (one-line) notation for quadrant marked mesh patterns. For example,

Given a sequence $w = w_1 \dots w_n$ of distinct integers, let $\mathrm{red}(w)$ be the permutation found by replacing the i-th smallest integer that appears in σ by i. For example, if $\sigma = 2754$, then $\mathrm{red}(\sigma) = 1432$. Given a permutation $\tau = \tau_1 \dots \tau_j$ in the symmetric group S_j, we say that the pattern τ *occurs* in $\sigma = \sigma_1 \dots \sigma_n \in S_n$ provided there exist $1 \leq i_1 < \cdots < i_j \leq n$ such that $\mathrm{red}(\sigma_{i_1} \dots \sigma_{i_j}) = \tau$. We say that a permutation

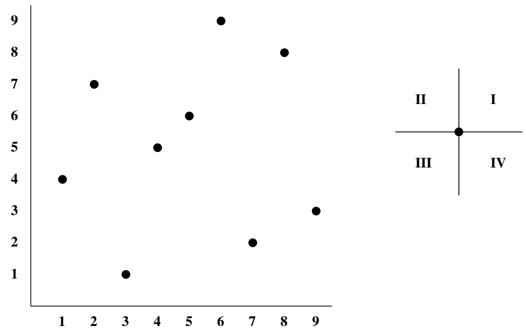

Figure 1: The graph of $\sigma = 471569283$.

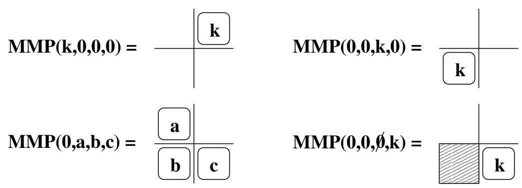

Figure 2: Úlfarsson notation for quadrant marked mesh patterns.

σ *avoids* the pattern τ if τ does not occur in σ. Let $S_n(\tau)$ denote the set of permutations in S_n which avoid τ. In the theory of permutation patterns, τ is called a *classical pattern*. See [4] for a comprehensive introduction to patterns in permutations.

It has been a rather popular direction of research in the literature on permutation patterns to study permutations avoiding a 3-letter pattern subject to extra restrictions (see [4, Subsection 6.1.5]). In [9], we started the study of the generating functions

$$Q_{132}^{(a,b,c,d)}(t,x) = 1 + \sum_{n\geq 1} Q_{n,132}^{(a,b,c,d)}(x)t^n$$

where for any $a,b,c,d \in \{\emptyset\} \cup \mathbb{N}$,

$$Q_{n,132}^{(a,b,c,d)}(x) = \sum_{\sigma \in S_n(132)} x^{\mathrm{mmp}^{(a,b,c,d)}(\sigma)}.$$

For any a,b,c,d, we will write $Q_{n,132}^{(a,b,c,d)}(x)|_{x^k}$ for the coefficient of x^k in $Q_{n,132}^{(a,b,c,d)}(x)$.

For any fixed (a,b,c,d), we know that $Q_{n,132}^{(a,b,c,d)}(1)$ is the number of 132-avoiding permutations in S_n which is the nth Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. Thus the coefficients in the polynomial $Q_{n,132}^{(a,b,c,d)}(x)$ represent a refinement of the nth Catalan number. It is then a natural question to ask whether (i) we can give explicit formulas for the coefficients that appear in $Q_{n,132}^{(a,b,c,d)}(x)$ or (ii) whether such coefficients count other interesting classes of combinatorial objects. Of course, there is an obvious answer to question (ii). That is, if one has a bijection from $S_n(132)$ to other classes of combinatorial objects which are counted by the Catalan numbers such as Dyck paths or binary trees, then one can use that bijection to give an interpretation of the pattern $\mathrm{MMP}(a,b,c,d)$ in the other setting. We shall see that in many cases, there are interesting connections with the coefficients that arise in our polynomials $Q_{n,132}^{(a,b,c,d)}(x)$ and other sets of combinatorial objects that do not just arise by such bijections.

In particular, it is natural to try to understand $Q_{n,132}^{(a,b,c,d)}(0)$ which equals the number of $\sigma \in S_n(132)$ that have no occurrences of the pattern $\mathrm{MMP}(a,b,c,d)$ as well as the coefficient of the highest power of x that occurs in $Q_{n,132}^{(a,b,c,d)}(x)$ since that coefficient equals the number of $\sigma \in S_n(132)$ that have the maximum possible number of occurrences of the pattern $\mathrm{MMP}(a,b,c,d)$. We shall see that in many cases, $Q_{n,132}^{(a,b,c,d)}(x)|_x$ and $Q_{n,132}^{(a,b,c,d)}(x)|_{x^2}$, the number of $\sigma \in S_n(132)$ with exactly one occurrence and two occurrences, respectively, of the pattern $\mathrm{MMP}(a,b,c,d)$ also have interesting combinatorics associated with them. There are many more interesting questions of this type that can be pursued, but due to space considerations, we shall mostly restrict ourselves to trying to understand the four coefficients in $Q_{n,132}^{(a,b,c,d)}(x)$ described above. We should note, however, that there is a uniform way to compute generating functions of the form

$$F_k^{(a,b,c,d)}(t) = \sum_{n\geq 0} Q_{n,132}^{(a,b,c,d)}(x)|_{x^k} t^n.$$

That is, $F_k^{(a,b,c,d)}(t)$ is just the result of setting $x=0$ in the generating function $\frac{1}{k!}\frac{\partial^k}{\partial x^k} Q_{132}^{(a,b,c,d)}(t,x)$. Due to space considerations, we will not pursue the study of the functions $F_k^{(a,b,c,d)}(t)$ for $k \geq 2$ in this paper.

There is one obvious symmetry for such generating functions which is induced by the fact that if $\sigma \in S_n(132)$, then $\sigma^{-1} \in S_n(132)$. That is, the following lemma was proved in [9].

Lemma 1. ([9]) *For any* $a,b,c,d \in \mathbb{N} \cup \{0\} \cup \{\emptyset\}$,

$$Q_{n,132}^{(a,b,c,d)}(x) = Q_{n,132}^{(a,d,c,b)}(x).$$

In [9], we studied the generating functions $Q_{132}^{(k,0,0,0)}(t,x)$, $Q_{132}^{(0,k,0,0)}(t,x)$, and $Q_{132}^{(0,0,k,0)}(t,x)$, where k can be either the empty set or a positive integer, as well as the generating functions $Q_{132}^{(k,0,\emptyset,0)}(t,x)$ and $Q_{132}^{(\emptyset,0,k,0)}(t,x)$. In [10], we studied $Q_{n,132}^{(k,0,\ell,0)}(t,x)$, $Q_{n,132}^{(k,0,0,\ell)}(t,x)$, $Q_{n,132}^{(0,k,\ell,0)}(t,x)$, and $Q_{n,132}^{(0,k,0,\ell)}(t,x)$, where $k,\ell \geq 1$. We also showed that sequences of the form $(Q_{n,132}^{(a,b,c,d)}(x)|_{x^r})_{n\geq s}$ count a variety of combinatorial objects that appear in the *On-line Encyclopedia of Integer Sequences*

(OEIS) [11]. Thus, our results gave new combinatorial interpretations of certain classical sequences such as the Fine numbers and the Fibonacci numbers as well as provided certain sequences that appear in the OEIS with a combinatorial interpretation where none had existed before. Another particular result of our studies in [9] is enumeration of permutations avoiding simultaneously the patterns 132 and 1234, while in [10], we made a link to the *Pell numbers*.

The main goal of this paper is to continue the study of $Q_{132}^{(a,b,c,d)}(t,x)$ and combinatorial interpretations of sequences of the form $(Q_{n,132}^{(a,b,c,d)}(x)|_{x^r})_{n\geq s}$ in the case where $a,b,c,d \in \mathbb{N}$ and at least three of these parameters are non-zero.

Next we list the key results from [9] and [10] which we need in this paper.

Theorem 1. *([9, Theorem 3.1])*

$$Q_{132}^{(0,0,0,0)}(t,x) = C(xt) = \frac{1-\sqrt{1-4xt}}{2xt}$$

and, for $k \geq 1$,

$$Q_{132}^{(k,0,0,0)}(t,x) = \frac{1}{1-tQ_{132}^{(k-1,0,0,0)}(t,x)}.$$

Hence

$$Q_{132}^{(1,0,0,0)}(t,0) = \frac{1}{1-t}$$

and, for $k \geq 2$,

$$Q_{132}^{(k,0,0,0)}(t,0) = \frac{1}{1-tQ_{132}^{(k-1,0,0,0)}(t,0)}.$$

Theorem 2. *([9, Theorem 4.1]) For* $k \geq 1$,

$$\begin{aligned} Q_{132}^{(0,0,k,0)}(t,x) &= \frac{1+(tx-t)(\sum_{j=0}^{k-1} C_j t^j) - \sqrt{(1+(tx-t)(\sum_{j=0}^{k-1} C_j t^j))^2 - 4tx}}{2tx} \\ &= \frac{2}{1+(tx-t)(\sum_{j=0}^{k-1} C_j t^j) + \sqrt{(1+(tx-t)(\sum_{j=0}^{k-1} C_j t^j))^2 - 4tx}} \end{aligned}$$

and

$$Q_{132}^{(0,0,k,0)}(t,0) = \frac{1}{1-t(C_0+C_1t+\cdots+C_{k-1}t^{k-1})}.$$

Theorem 3. *([10, Theorem 5]) For all* $k,\ell \geq 1$,

$$Q_{132}^{(k,0,\ell,0)}(t,x) = \frac{1}{1-tQ_{132}^{(k-1,0,\ell,0)}(t,x)}. \tag{1}$$

Theorem 4. *([10, Theorem 11]) For all $k, \ell \geq 1$,*

$$Q_{132}^{(k,0,0,\ell)}(t,x) = \frac{C_\ell t^\ell + \sum_{j=0}^{\ell-1} C_j t^j 1 - tQ_{132}^{(k-1,0,0,0)}(t,x) + t(Q_{132}^{(k-1,0,0,\ell-j)}(t,x) - \sum_{s=0}^{\ell-j-1} C_s t^s))}{1 - tQ_{132}^{(k-1,0,0,0)}(t,x)}. \quad (2)$$

Theorem 5. *([10, Theorem 14]) For all $k, \ell \geq 1$,*

$$Q_{132}^{(0,k,\ell,0)}(t,x) = \frac{C_{k-1} t^{k-1} + \sum_{j=0}^{k-2} C_j t^j (1 - tQ_{132}^{(0,0,\ell,0)}(t,x) + t(Q_{132}^{(0,k-i-1,\ell,0)}(t,x) - \sum_{s=0}^{k-i-2} C_s t^s))}{1 - tQ_{132}^{(0,0,\ell,0)}(t,x)}. \quad (3)$$

Theorem 6. *([10, Theorem 18]) For all $k, \ell \geq 1$,*

$$Q_{132}^{(0,k,0,\ell)}(t,x) = \frac{\Phi_{k,\ell}(t,x)}{1-t}, \quad (4)$$

where

$$\begin{aligned}
\Phi_{k,\ell}(t,x) = & \sum_{j=0}^{k+\ell-1} C_j t^j - \sum_{j=0}^{k+\ell-2} C_j t^{j+1} + \\
& t\left(\sum_{j=0}^{k-2} C_j t^j \left(Q_{132}^{(0,k-j-1,0,\ell)}(t,x) - \sum_{s=0}^{k+\ell-j-2} C_s t^s\right)\right) + \\
& t\left(Q_{132}^{(0,k,0,0)}(t,x) - \sum_{u=0}^{k-2} C_u t^u\right)\left(Q_{132}^{(0,0,0,\ell)}(t,x) - \sum_{v=0}^{\ell-1} C_v t^v\right) + \\
& t\left(\sum_{j=1}^{\ell-1} C_j t^j \left(Q_{132}^{(0,k,0,\ell-j)}(t,x) - \sum_{w=0}^{k+\ell-j-2} C_w t^w\right)\right).
\end{aligned}$$

As it was pointed out in [9], *avoidance* of a marked mesh pattern without quadrants containing the empty set can always be expressed in terms of multi-avoidance of (possibly many) classical patterns. Thus, among our results we will re-derive several known facts in permutation patterns theory. However, our main goals are more ambitious aimed at finding distributions in question.

2. $Q_{n,132}^{(k,0,m,\ell)}(x) = Q_{n,132}^{(k,\ell,m,0)}(x)$ **where** $k, \ell, m \geq 1$

By Lemma 1, we know that $Q_{n,132}^{(k,0,m,\ell)}(x) = Q_{n,132}^{(k,\ell,m,0)}(x)$. Thus, we will only consider $Q_{n,132}^{(k,\ell,m,0)}(x)$ in this section.

Throughout this paper, we shall classify the 132-avoiding permutations $\sigma = \sigma_1 \ldots \sigma_n$ by the position of n in σ. That is, let $S_n^{(i)}(132)$ denote the set of $\sigma \in S_n(132)$ such that $\sigma_i = n$. Clearly each $\sigma \in S_n^{(i)}(132)$ has the structure pictured in Figure 3. That is, in the graph of σ, the elements to the left of n, $A_i(\sigma)$, have the structure of a 132-avoiding permutation, the elements to the right of n, $B_i(\sigma)$, have the structure of a 132-avoiding permutation, and all the elements in $A_i(\sigma)$ lie above all the elements in $B_i(\sigma)$. It is well-known that the number of 132-avoiding permutations in S_n is the *Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$ and the generating function for the C_n's is given by

$$C(t) = \sum_{n\geq 0} C_n t^n = \frac{1-\sqrt{1-4t}}{2t} = \frac{2}{1+\sqrt{1-4t}}.$$

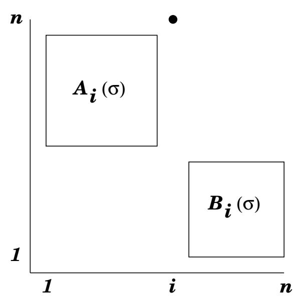

Figure 3: The structure of 132-avoiding permutations.

Suppose that $n \geq \ell$. It is clear that n cannot match the pattern $\mathrm{MMP}(k,\ell,m,0)$ for $k \geq 1$ in any $\sigma \in S_n(132)$. For $1 \leq i \leq n$, it is easy to see that as we sum over all the permutations σ in $S_n^{(i)}(132)$, our choices for the structure for $A_i(\sigma)$ will contribute a factor of $Q_{i-1,132}^{(k-1,\ell,m,0)}(x)$ to $Q_{n,132}^{(k,\ell,m,0)}(x)$. Similarly, our choices for the structure for $B_i(\sigma)$ will contribute a factor of $Q_{n-i,132}^{(k,\ell-i,m,0)}(x)$ to $Q_{n,132}^{(k,\ell,m,0)}(x)$ if $i < \ell$ since $\sigma_1 \ldots \sigma_i$ will automatically be in the second quadrant relative to the coordinate system with the origin at (s, σ_s) for any $s > i$. However if $i \geq \ell$, then our choices for the structure for $B_i(\sigma)$ will contribute a factor of $Q_{n-i,132}^{(k,0,m,0)}(x)$ to $Q_{n,132}^{(k,\ell,m,0)}(x)$. It follows that for $n \geq \ell$,

$$Q_{n,132}^{(k,\ell,m,0)}(x) = \sum_{i=1}^{\ell-1} Q_{i-1,132}^{(k-1,\ell,m,0)}(x) Q_{n-i,132}^{(k,\ell-i,m,0)}(x) + \sum_{i=\ell}^{n} Q_{i-1,132}^{(k-1,\ell,m,0)}(x) Q_{n-i,132}^{(k,0,m,0)}(x).$$

Note that for $i < \ell$, $Q^{(k-1,\ell,m,0)}_{i-1,132}(x) = C_{i-1}$. Thus, for $n \geq \ell$,

$$Q^{(k,\ell,m,0)}_{n,132}(x) = \sum_{i=1}^{\ell-1} C_{i-1} Q^{(k,\ell-i,m,0)}_{n-i,132}(x) + \sum_{i=\ell}^{n} Q^{(k-1,\ell,m,0)}_{i-1,132}(x) Q^{(k,0,m,0)}_{n-i,132}(x). \tag{5}$$

Multiplying both sides of (5) by t^n and summing for $n \geq \ell$, we see that for $k, \ell \geq 1$,

$$\begin{aligned}
Q^{(k,\ell,m,0)}_{132}(t,x) = &\sum_{j=0}^{\ell-1} C_j t^j + \sum_{i=1}^{\ell-1} C_{i-1} t^i \sum_{u \geq \ell-i} Q^{(k,\ell-i,m,0)}_{u,132}(x) t^u + \\
&t \sum_{n \geq \ell} \sum_{i=1}^{n} Q^{(k-1,\ell,m,0)}_{i-1,132}(x) t^{i-1} Q^{(k,0,m,0)}_{n-i,132}(x) t^{n-i} \\
= &\sum_{j=0}^{\ell-1} C_j t^j + \sum_{i=1}^{\ell-1} C_{i-1} t^i (Q^{(k,\ell-i,m,0)}_{132}(t,x) - \sum_{j=0}^{\ell-i-1} C_j t^j) + \\
&t Q^{(k,0,m,0)}_{132}(t,x) (Q^{(k-1,\ell,m,0)}_{132}(t,x) - \sum_{s=0}^{\ell-2} C_s t^s) \\
= &C_{\ell-1} t^{\ell-1} + t Q^{(k,0,m,0)}_{132}(t,x) Q^{(k-1,\ell,m,0)}_{132}(t,x) + \\
&\sum_{s=0}^{\ell-2} C_s t^s (1 + t Q^{(k,\ell-1-s,m,0)}_{132}(t,x) - t Q^{(k,0,m,0)}_{132}(t,x) - t \sum_{j=0}^{\ell-2-s} C_j t^j).
\end{aligned}$$

Thus, we have the following theorem.

Theorem 7.

$$\begin{aligned}
Q^{(k,\ell,m,0)}_{132}(t,x) = &C_{\ell-1} t^{\ell-1} + t Q^{(k,0,m,0)}_{132}(t,x) Q^{(k-1,\ell,m,0)}_{132}(t,x) + \\
&\sum_{s=0}^{\ell-2} C_s t^s (1 + t Q^{(k,\ell-1-s,m,0)}_{132}(t,x) - t Q^{(k,0,m,0)}_{132}(t,x) - t \sum_{j=0}^{\ell-2-s} C_j t^j).
\end{aligned} \tag{6}$$

Note that since we can compute $Q^{(k,0,m,0)}_{132}(t,x)$ by Theorem 3 and $Q^{(0,\ell,m,0)}_{132}(t,x)$ by Theorem 5, we can use (6) to compute $Q^{(k,\ell,m,0)}_{132}(t,x)$ for any $k, \ell, m \geq 1$.

2.1. Explicit formulas for $Q^{(k,\ell,m,0)}_{n,132}(x)|_{x^r}$

It follows from Theorem 7 that

$$\begin{aligned}
&Q^{(k,1,m,0)}_{132}(t,x) = 1 + t Q^{(k,0,m,0)}_{132}(t,x) Q^{(k-1,1,m,0)}_{132}(t,x) \text{ and} \\
&Q^{(k,2,m,0)}_{132}(t,x) = 1 + t Q^{(k,0,m,0)}_{132}(t,x) (Q^{(k-1,2,m,0)}_{132}(t,x) - 1) + t Q^{(k,1,m,0)}_{132}(t,x).
\end{aligned} \tag{7}$$

Note that it follows from Theorems 3 and 5 that

$$\begin{aligned} Q_{132}^{(1,1,1,0)}(t,0) =& 1 + tQ_{132}^{(1,0,1,0)}(t,0)Q_{132}^{(0,1,1,0)}(t,0) \\ =& 1 + t\frac{1-t}{1-2t} \cdot \frac{1-t}{1-2t} = \frac{1-3t+2t^2+t^3}{(1-2t)^2}. \end{aligned}$$

Thus, the generating function of the sequence $\{Q_{n,132}^{(1,1,1,0)}(0)\}_{n\geq 1}$ is $\left(\frac{1-t}{1-2t}\right)^2$ which is the generating function of the sequence A045623 in the OEIS. The n-th term a_n of this sequence has many combinatorial interpretations including the number of 1s in all partitions of $n+1$ and the number of 132-avoiding permutations of S_{n+2} which contain exactly one occurrence of the pattern 213. We note that for a permutation σ to avoid the pattern $\mathrm{MMP}(1,1,1,0)$, it must simultaneously avoid the patterns 3124, 4123, 1324, and 1423. Thus, the number of permutations $\sigma \in S_n(132)$ which avoid $\mathrm{MMP}(1,1,1,0)$ is the number of permutations in S_n that simultaneously avoid the patterns 132, 3124, and 4123.

Problem 1. *Find simple bijections between the set of permutations* $\sigma \in S_n(132)$ *which avoid* $\mathrm{MMP}(1,1,1,0)$ *and the other combinatorial interpretations of the sequence A045623 in the OEIS.*

Note that it follows from Theorem 3 and our previous results that

$$\begin{aligned} Q_{132}^{(2,1,1,0)}(t,0) =& 1 + tQ_{132}^{(2,0,1,0)}(t,0)Q_{132}^{(1,1,1,0)}(t,0) \\ =& 1 + t\frac{1-2t}{1-3t+t^2} \cdot \frac{1-3t+2t^2+t^3}{(1-2t)^2} \\ =& \frac{1-4t+4t^2+t^4}{1-5t+7t^2-2t^3}. \end{aligned}$$

The sequence $(Q_{n,132}^{(2,1,1,0)}(0))_{n\geq 1}$ is the sequence A142586 in the OIES which has the generating function $\frac{1-3t+2t^2+t^3}{(1-3t+t^2)(1-2t)}$. That is, $\frac{1-4t+4t^2+t^4}{1-5t+7t^2-2t^3} - 1 = \frac{t(1-3t+2t^2+t^3)}{(1-3t+t^2)(1-2t)}$. This sequence has no listed combinatorial interpretation so we have found a combinatorial interpretation of this sequence.

Similarly,

$$\begin{aligned} Q_{132}^{(3,1,1,0)}(t,0) =& 1 + tQ_{132}^{(3,0,1,0)}(t,0)Q_{132}^{(2,1,1,0)}(t,0) \\ =& 1 + t\frac{1-3t+t^2}{1-4t+3t^2} \cdot \frac{1-4t+4t^2+t^4}{1-5t+7t^2-2t^3} \\ =& \frac{1-5t+7t^2-2t^3+t^5}{1-6t+11t^2-6t^3}. \end{aligned}$$

$$\begin{aligned}
Q_{132}^{(1,1,2,0)}(t,0) =& 1+tQ_{132}^{(1,0,2,0)}(t,0)Q_{132}^{(0,1,2,0)}(t,0)\\
=& 1+t\frac{1-t-t^2}{1-2t-t^2}\cdot\frac{1-t-t^2}{1-2t-t^2}\\
=& \frac{1-3t+3t^3+3t^4+t^5}{(1-2t-t^2)^2}.
\end{aligned}$$

$$\begin{aligned}
Q_{132}^{(2,1,2,0)}(t,0) =& 1+tQ_{132}^{(2,0,2,0)}(t,0)Q_{132}^{(1,1,2,0)}(t,0)\\
=& 1+t\frac{1-2t-t^2}{1-3t+t^3}\cdot\frac{1-3t+3t^3+3t^4+t^5}{(1-2t-t^2)^2}\\
=& \frac{1-4t+2t^2+4t^3+t^4+2t^5+t^6}{(1-2t-t^2)(1-3t+t^3)}.
\end{aligned}$$

Using (7) and Theorem 3, we have computed the following.

$$\begin{aligned}
&Q_{132}^{(1,1,1,0)}(t,x) = 1+t+2t^2+5t^3+(12+2x)t^4+\left(28+12x+2x^2\right)t^5+\\
&\left(64+48x+18x^2+2x^3\right)t^6+\left(144+160x+97x^2+26x^3+2x^4\right)t^7+\\
&\left(320+480x+408x^2+184x^3+36x^4+2x^5\right)t^8+\\
&\left(704+1344x+1479x^2+958x^3+327x^4+48x^5+2x^6\right)t^9+\cdots.
\end{aligned}$$

$$\begin{aligned}
&Q_{132}^{(1,1,2,0)}(t,x) = 1+t+2t^2+5t^3+14t^4+(38+4x)t^5+\left(102+26x+4x^2\right)t^6+\\
&\left(271+120x+34x^2+4x^3\right)t^7+\left(714+470x+200x^2+42x^3+4x^4\right)t^8+\\
&\left(1868+1672x+964x^2+304x^3+50x^4+4x^5\right)t^9+\cdots.
\end{aligned}$$

$$\begin{aligned}
&Q_{132}^{(1,1,3,0)}(t,x) = 1+t+2t^2+5t^3+14t^4+42t^5+(122+10x)t^6+\\
&\left(351+68x+10x^2\right)t^7+\left(1006+326x+88x^2+10x^3\right)t^8+\\
&\left(2868+1364x+512x^2+108x^3+10x^4\right)t^9+\cdots.
\end{aligned}$$

We can explain the highest and second highest coefficients of x in these series. That is, we have the following theorem.

Theorem 8.

(i) *For all $m \geq 1$ and $n \geq 3+m$, the highest power of x that occurs in $Q_{n,132}^{(1,1,m,0)}(x)$ is x^{n-2-m} which appears with a coefficient of $2C_m$.*

(ii) *For $n \geq 5$, $Q_{n,132}^{(1,1,1,0)}(x)|_{x^{n-4}} = 6+2\binom{n-2}{2}$.*

(iii) *For $m \geq 2$ and $n \geq 4+m$, $Q_{n,132}^{(1,1,m,0)}(x)|_{x^{n-3-m}} = 2C_{m+1}+8C_m+4C_m(n-m-4)$.*

Proof. It is easy to see that for the maximum number of MMP$(1,1,m,0)$-matches in a $\sigma \in S_n(132)$, the permutation must be of the form $(n-1)\tau(m+1)\dots(n-2)n$ or $n\tau(m+1)\dots(n-2)(n-1)$ where $\tau \in S_m(132)$. Thus, the highest power of x occurring in $Q^{(1,1,m,0)}_{n,132}(x)$ is x^{n-2-m} which occurs with a coefficient of $2C_m$.

For parts (ii) and (iii), by (5) we have the recursion that

$$Q^{(1,1,m,0)}_{n,132}(x) = \sum_{i=1}^{n} Q^{(0,1,m,0)}_{i-1,132}(x) Q^{(1,0,m,0)}_{n-i,132}(x). \tag{8}$$

We proved in [10, Theorem 15] and [10, Theorem 6] that for $n \geq m+2$ the highest power of x which occurs in either $Q^{(0,1,m,0)}_{n,132}(x)$ or $Q^{(1,0,m,0)}_{n,132}(x)$ is x^{n-1-m} and

$$Q^{(0,1,m,0)}_{n,132}(x)|_{x^{n-1-m}} = Q^{(1,0,m,0)}_{n,132}(x)|_{x^{n-1-m}} = C_m.$$

It is then easy to check that the highest power of x in $Q^{(0,1,m,0)}_{i-1,132}(x) Q^{(1,0,m,0)}_{n-i,132}(x)$ is less than x^{n-3-m} for $i = 3, \dots, n-2$.

We also proved in [10, Theorems 9,10, and 15] that

$$\begin{aligned} Q^{(1,0,1,0)}_{n,132}(x)|_{x^{n-3}} =& Q^{(0,1,1,0)}_{n,132}(x)|_{x^{n-3}} = 2 + \binom{n-1}{2} \text{ for } n \geq 4 \text{ and} \\ Q^{(1,0,m,0)}_{n,132}(x)|_{x^{n-m-2}} =& Q^{(0,1,m,0)}_{n,132}(x)|_{x^{n-m-2}} \\ =& C_{m+1} + C_m + 2C_m(n-2-m) \text{ for } n \geq 3+m \text{ and } m \geq 2. \end{aligned}$$

For $m = 1$, we are left with 4 cases to consider in the recursion (8).

Case 1. $i = 1$. In this case, $Q^{(0,1,1,0)}_{i-1,132}(x) Q^{(1,0,1,0)}_{n-i,132}(x)|_{x^{n-4}} = Q^{(1,0,1,0)}_{n-1,132}(x)|_{x^{n-4}}$ and

$$Q^{(1,0,1,0)}_{n-1,132}(x)|_{x^{n-4}} = 2 + \binom{n-2}{2} \text{ for } n \geq 5.$$

Case 2. $i = 2$. In this case, $Q^{(0,1,1,0)}_{i-1,132}(x) Q^{(1,0,1,0)}_{n-i,132}(x)|_{x^{n-4}} = Q^{(1,0,1,0)}_{n-2,132}(x)|_{x^{n-4}}$ and

$$Q^{(1,0,1,0)}_{n-2,132}(x)|_{x^{n-4}} = 1 \text{ for } n \geq 5.$$

Case 3. $i = n-1$. In this case, $Q^{(0,1,1,0)}_{i-1,132}(x) Q^{(1,0,1,0)}_{n-i,132}(x)|_{x^{n-4}} = Q^{(0,1,1,0)}_{n-2,132}(x)|_{x^{n-4}}$ and

$$Q^{(0,1,1,0)}_{n-2,132}(x)|_{x^{n-4}} = 1 \text{ for } n \geq 5.$$

Case 4. $i = n$. In this case, $Q^{(0,1,1,0)}_{i-1,132}(x) Q^{(1,0,1,0)}_{n-i,132}(x)|_{x^{n-4}} = Q^{(0,1,1,0)}_{n-1,132}(x)|_{x^{n-4}}$ and

$$Q^{(0,1,1,0)}_{n-1,132}(x)|_{x^{n-4}} = 2 + \binom{n-2}{2} \text{ for } n \geq 5.$$

Thus, $Q^{(1,1,1,0)}_{n,132}(x)|_{x^{n-4}} = 6 + 2\binom{n-2}{2}$ for $n \geq 5$.

Next we consider the case when $m \geq 2$. Again we have 4 cases.

Case 1. $i = 1$. We have $Q^{(0,1,m,0)}_{i-1,132}(x)Q^{(1,0,m,0)}_{n-i,132}(x)|_{x^{n-3-m}} = Q^{(1,0,m,0)}_{n-1,132}(x)|_{x^{n-3-m}}$, and

$$Q^{(1,0,m,0)}_{n-1,132}(x)|_{x^{n-3-\ell}} = C_{m+1} + 3C_m + 2C_m(n-4-m) \text{ for } n \geq 4+m.$$

Case 2. $i = 2$. We have $Q^{(0,1,m,0)}_{i-1,132}(x)Q^{(1,0,m,0)}_{n-i,132}(x)|_{x^{n-3-m}} = Q^{(1,0,m,0)}_{n-2,132}(x)|_{x^{n-3-m}}$, and

$$Q^{(1,0,m,0)}_{n-2,132}(x)|_{x^{n-3-m}} = C_m \text{ for } n \geq 4+m.$$

Case 3. $i = n-1$. Here, $Q^{(0,1,m,0)}_{i-1,132}(x)Q^{(1,0,m,0)}_{n-i,132}(x)|_{x^{n-3-m}} = Q^{(0,1,m,0)}_{n-2,132}(x)|_{x^{n-3-m}}$, and

$$Q^{(0,1,m,0)}_{n-2,132}(x)|_{x^{n-3-m}} = C_m \text{ for } n \geq 4+m.$$

Case 4. $i = n$. We have $Q^{(0,1,m,0)}_{i-1,132}(x)Q^{(1,0,m,0)}_{n-i,132}(x)|_{x^{n-3-m}} = Q^{(0,1,m,0)}_{n-1,132}(x)|_{x^{n-3-m}}$, and

$$Q^{(0,1,m,0)}_{n-1,132}(x)|_{x^{n-3-m}} = C_{m+1} + 3C_m + 2C_m(n-4-m) \text{ for } n \geq 4+m.$$

Thus, for $n \geq 4+m$,

$$Q^{(1,1,m,0)}_{n,132}(x)|_{x^{n-3-m}} = 2C_{m+1} + 8C_m + 4C_m(n-4-m).$$

Thus, when $m = 2$, we obtain that

$$Q^{(1,1,2,0)}_{n,132}(x)|_{x^{n-5}} = 26 + 8(n-6) \text{ for } n \geq 6$$

and, for $m = 3$, we obtain that

$$Q^{(1,1,3,0)}_{n,132}(x)|_{x^{n-6}} = 68 + 20(n-7) \text{for } n \geq 7$$

which agrees with our computed series. □

One can also find the coefficient of the highest power of x in $Q^{(k,1,1,0)}_{n}(x)$ for $k \geq 2$ and $Q^{(1,\ell,1,0)}_{n}(x)$ for $\ell \geq 2$.

Theorem 9.

(i) For all $k \geq 1$ and $n \geq 3+k$, the highest power of x that occurs in $Q^{(k,1,1,0)}_{n,132}(x)$ is x^{n-2-k} which appears with a coefficient of $k+1$.

(ii) For all $\ell \geq 1$ and $n \geq 3+\ell$, the highest power of x that occurs in $Q^{(1,\ell,1,0)}_{n,132}(x)$ is $x^{n-2-\ell}$ which appears with a coefficient of $C_{\ell+1}$.

Proof. For (i), it is easy to see that the permutations in $S_n(132)$ which have the most occurrences of $\text{MMP}(k,1,1,0)$ start with $n-j$ for some $0 \le j \le k$ followed by an increasing sequence which will have $n-2-k$ elements matching $\text{MMP}(k,1,1,0)$.

For (ii), it is easy to see that the way to construct a permutation σ of $S_n(132)$ which has the maximum number of occurrences of $\text{MMP}(1,\ell,1,0)$ is to start with a rearrangement $\tau = \tau_1 \dots \tau_{\ell+1}$ of $\{n-\ell, n-\ell+1, \dots, n\}$ such that $\text{red}(\tau) \in S_{\ell+1}(132)$ and then let $\sigma = \tau_1 \dots \tau_\ell \; 1\; 2 \dots (n-\ell-1)\tau_{\ell+1}$, which will have $n-2-\ell$ elements that match $\text{MMP}(1,\ell,1,0)$. □

One can also find a formula for the second highest coefficient of x in $Q_n^{(k,1,1,0)}(x)$ for any $k \ge 1$.

Theorem 10. *For any $k \ge 1$ and $n \ge 4+k$,*

$$Q_{n,132}^{(k,1,1,0)}(x)|_{x^{n-k-3}} = 3\binom{k+1}{2} + k + 2 + (k+1)\binom{n-k-1}{2}. \tag{9}$$

Proof. We proceed by induction on k. Note that our formula reduces to part (ii) of Theorem 8 when $k=1$.

Thus assume that $k>1$ and our formula holds for $k-1$. By recursion (5), we see that

$$Q_{n,132}^{(k,1,1,0)}(x) = \sum_{i=1}^{n} Q_{i-1,132}^{(k-1,1,1,0)}(x) Q_{n-i,132}^{(k,0,1,0)}(x). \tag{10}$$

From [10, Theorem 6], the highest power of x that can occur in $Q_{n,132}^{(k,0,1,0)}(x)$ is x^{n-k-1}, which occurs with a coefficient of 1 for $n \ge k+3$. Similarly, from Theorem 9, we know that the highest power of x that can occur in $Q_{n,132}^{(k,1,1,0)}(x)$ is x^{n-k-2} which occurs with a coefficient of $k+1$. One can then easily check that for $n \ge k+4$, there are only four terms on the right-hand side of (10) that can contribute to the coefficient of $Q_{n,132}^{(k,1,1,0)}(x)|_{x^{n-k-3}}$.

Case 1. $i=1$. In this case, $Q_{i-1,132}^{(k-1,1,1,0)}(x) Q_{n-i,132}^{(k,0,1,0)}(x)|_{x^{n-k-3}} = Q_{n-1,132}^{(k,0,1,0)}(x)|_{x^{n-k-3}}$ and by [10, Theorem 9], we know that

$$Q_{n-1,132}^{(k,0,1,0)}(x)|_{x^{n-k-3}} = 2k + \binom{n-1-k}{2} \text{ for } n \ge k+4.$$

Case 2. $i=2$. In this case, $Q_{i-1,132}^{(k-1,1,1,0)}(x) Q_{n-i,132}^{(k,0,1,0)}(x)|_{x^{n-k-3}} = Q_{n-2,132}^{(k,0,1,0)}(x)|_{x^{n-k-3}}$ and by [10, Theorem 6], we know that

$$Q_{n-2,132}^{(k,0,1,0)}(x)|_{x^{n-k-3}} = 1 \text{ for } n \ge k+4.$$

Case 3. $i=n-1$. Here, $Q_{i-1,132}^{(k-1,1,1,0)}(x) Q_{n-i,132}^{(k,0,1,0)}(x)|_{x^{n-4}} = Q_{n-2,132}^{(k-1,1,1,0)}(x)|_{x^{n-k-3}}$, and by Theorem 9, we know that

$$Q_{n-2,132}^{(k-1,1,1,0)}(x)|_{x^{n-k-3}} = k \text{ for } n \ge k+4.$$

Case 4. $i = n$. We have $Q_{i-1,132}^{(k-1,1,1,0)}(x)Q_{n-i,132}^{(k,0,1,0)}(x)|_{x^{n-4}} = Q_{n-1,132}^{(k-1,1,1,0)}(x)|_{x^{n-k-3}}$, and by induction,

$$Q_{n-1,132}^{(0,1,1,0)}(x)|_{x^{n-4}} = 3\binom{k}{2} + (k-1) + 2 + k\binom{n-1-k}{2} \text{ for } n \geq k+4.$$

Summing up these four cases, we find that for $n \geq k+4$,

$$Q_{n,132}^{(k,1,1,0)}(x)|_{x^{n-k-3}} = 3\binom{k+1}{2} + k + 2 + (k+1)\binom{n-k-1}{2}.$$

For example, for $k = 2$ and $k = 3$, we obtain that

$$Q_{n,132}^{(2,1,1,0)}(x)|_{x^{n-5}} = 13 + 3\binom{n-3}{2} \text{ for } n \geq 6$$

and

$$Q_{n,132}^{(3,1,1,0)}(x)|_{x^{n-6}} = 23 + 4\binom{n-4}{2} \text{ for } n \geq 7,$$

which agrees with the expansions of the series for $Q_{132}^{(2,1,1,0)}(t,x)$ and $Q_{132}^{(3,1,1,0)}(t,x)$ given below. □

One can ask whether there is a similar formula for the second highest coefficient of x in $Q_{n,132}^{(1,\ell,1,0)}(x)$ as a function of ℓ. We conjecture that it is possible to find such a formula but it will be more complicated because it is no longer the case that a fixed number of terms in the recursion (5) contribute to the coefficient of the second highest power of x that occurs in $Q_{n,132}^{(1,\ell,1,0)}(x)$. That is, as ℓ grows, the number of terms in the recursion (5) that contribute to the coefficient of the second highest power of x that occurs in $Q_{n,132}^{(1,\ell,1,0)}(x)$ grows. We can, however, give an explicit formula in the case where $\ell = 2$.

Theorem 11. *For $n \geq 6$,*

$$Q_{n,132}^{(1,2,1,0)}(x)|_{x^{n-5}} = 18 + 5\binom{n-3}{2}. \tag{11}$$

Proof. In this case, the recursion (5) becomes

$$Q_{n,132}^{(1,2,1,0)}(x) = Q_{n-1,132}^{(1,1,1,0)}(x) + \sum_{i=2}^{n} Q_{i-1,132}^{(0,2,1,0)}(x)Q_{n-i,132}^{(1,0,1,0)}(x). \tag{12}$$

One can then easily check that for $n \geq 6$, there are only five terms on the right-hand side of (12) that can contribute to the coefficient of $Q_{n,132}^{(1,2,1,0)}(x)|_{x^{n-5}}$.

Case 1. In part (ii) of Theorem 8, we proved that

$$Q_{n-1,132}^{(1,1,1,0)}(x)|_{x^{n-5}} = 6 + 2\binom{n-3}{2} \text{ for } n \geq 6.$$

Case 2. $i = 2$. In this case, $Q^{(0,2,1,0)}_{i-1,132}(x)Q^{(1,0,1,0)}_{n-i,132}(x)|_{x^{n-5}} = Q^{(1,0,1,0)}_{n-2,132}(x)|_{x^{n-5}}$ and by [10, Theorem 9], we know that

$$Q^{(1,0,1,0)}_{n-2,132}(x)|_{x^{n-5}} = 2 + \binom{n-3}{2} \text{ for } n \geq 6.$$

Case 3. $i = 3$. In this case, $Q^{(0,2,1,0)}_{i-1,132}(x)Q^{(1,0,1,0)}_{n-i,132}(x)|_{x^{n-5}} = 2Q^{(1,0,1,0)}_{n-3,132}(x)|_{x^{n-5}}$ and by [10, Theorem 6], we know that

$$2Q^{(1,0,1,0)}_{n-2,132}(x)|_{x^{n-5}} = 2 \text{ for } n \geq 6.$$

Case 4. $i = n-1$. In this case, $Q^{(0,2,1,0)}_{i-1,132}(x)Q^{(1,0,1,0)}_{n-i,132}(x)|_{x^{n-5}} = Q^{(0,2,1,0)}_{n-2,132}(x)|_{x^{n-5}}$ and by[10, Theorem 16 (i)], we know that

$$Q^{(0,2,1,0)}_{n-2,132}(x)|_{x^{n-5}} = 2 \text{ for } n \geq 6.$$

Case 5. $i = n$. In this case, $Q^{(0,2,1,0)}_{i-1,132}(x)Q^{(1,0,1,0)}_{n-i,132}(x)|_{x^{n-5}} = Q^{(0,2,1,0)}_{n-1,132}(x)|_{x^{n-5}}$ and by [10, Theorem 16 (ii)],

$$Q^{(0,2,1,0)}_{n-1,132}(x)|_{x^{n-5}} = 6 + 2\binom{n-3}{2} \text{ for } n \geq 6.$$

Thus, for $n \geq 6$,

$$Q^{(1,2,1,0)}_{n,132}(x)|_{x^{n-5}} = 18 + 5\binom{n-3}{2}.$$

□

The formulas for the series $Q^{(k,\ell,m,0)}_{132}(t,x)$ become increasingly complicated. For example, we have computed that

$$Q^{(1,1,1,0)}_{132}(t,x) = 1 + \frac{4tx^2}{\left(-1+t+2x-tx+\sqrt{(1+t(-1+x))^2-4tx}\right)^2},$$

$$Q^{(2,1,1,0)}_{132}(t,x) = 1 + \frac{t\left(1+\frac{4tx^2}{\left(-1+t+2x-tx+\sqrt{(1+t(-1+x))^2-4tx}\right)^2}\right)}{1-\frac{2tx}{-1+t+2x-tx+\sqrt{(1+t(-1+x))^2-4tx}}},$$

$$Q^{(3,1,1,0)}_{132}(t,x) = 1 + \frac{t\left(1+\frac{t\left(1+\frac{4tx^2}{\left(-1+t+2x-tx+\sqrt{(1+t(-1+x))^2-4tx}\right)^2}\right)}{1-\frac{2tx}{-1+t+2x-tx+\sqrt{(1+t(-1+x))^2-4tx}}}\right)}{1-\frac{t}{1-\frac{2tx}{-1+t+2x-tx+\sqrt{(1+t(-1+x))^2-4tx}}}},$$

$$Q^{(1,1,2,0)}_{132}(t,x) = 1 + \frac{t}{\left(1-\frac{1+t(1+t)(-1+x)-\sqrt{(1+t(1+t)(-1+x))^2-4tx}}{2x}\right)^2},$$

$$Q_{132}^{(1,1,3,0)}(t,x) = 1 + \frac{t}{\left(1 - \frac{1+t(1+t+2t^2)(-1+x)-\sqrt{(1+t(1+t+2t^2)(-1+x))^2-4tx}}{2x}\right)^2}, \text{ and}$$

$$Q_{132}^{(1,2,1,0)}(t,x) = 1 + t - \frac{4t^2x^2(1-t+tx-4x-\sqrt{1+t^2(-1+x)^2-2t(1+x)})}{(-1+t+2x-tx+\sqrt{1+t^2(-1+x)^2-2t(1+x)})^3}.$$

We also have computed the following.

$$Q_{132}^{(2,1,1,0)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + (39+3x)t^5 + \left(107+22x+3x^2\right)t^6 + \left(290+105x+31x^2+3x^3\right)t^7 + \left(779+415x+190x^2+43x^3+3x^4\right)t^8 + \left(2079+1477x+909x^2+336x^3+58x^4+3x^5\right)t^9 + \left(5522+4922x+3765x^2+1938x^3+570x^4+76x^5+3x^6\right)t^{10} + \cdots.$$

$$Q_{132}^{(3,1,1,0)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + (128+4x)t^6 + \left(390+35x+4x^2\right)t^7 + \left(1184+195x+47x^2+4x^3\right)t^8 + \left(3582+888x+325x^2+63x^3+4x^4\right)t^9 + \left(19808+3616x+1743x^2+542x^3+83x^4+4x^5\right)t^{10} + \cdots.$$

$$Q_{132}^{(2,1,2,0)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + (126+6x)t^6 + \left(376+47x+6x^2\right)t^7 + \left(1115+250x+59x^2+6x^3\right)t^8 + \left(3289+1110x+386x^2+71x^3+6x^4\right)t^9 + \left(9660+4444x+2045x^2+558x^3+83x^4+6x^5\right)t^{10} + \cdots.$$

$$Q_{132}^{(2,1,3,0)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + 132t^6 + (414+15x)t^7 + \left(1293+122x+15x^2\right)t^8 + \left(4025+670x+152x^2+15x^3\right)t^9 + \left(12486+3124x+989x^2+182x^3+15x^4\right)t^{10} + \cdots.$$

Again one can easily explain the highest coefficient in $Q_{n,132}^{(2,1,m,0)}(x)$. That is, to have the maximum number of MMP$(2,1,m,0)$-matches in a $\sigma \in S_n(132)$, the permutation must be of the form

$$\begin{aligned} &(n-2)\tau(m+1)\ldots(n-3)(n-1)n,\\ &(n-1)\tau(m+1)\ldots(n-3)(n-2)n, \text{or}\\ &n\tau(m+1)\ldots(n-3)(n-2)(n-1) \end{aligned}$$

where $\tau \in S_m(132)$. Thus, the highest power of x occurring in $Q_{n,132}^{(2,1,m,0)}(x)$ is x^{n-3-m} which occurs with a coefficient of $3C_m$.

We have computed the following.

$Q_{132}^{(1,2,1,0)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + (37+5x)t^5 + \left(94 + 33x + 5x^2\right)t^6 +$
$\left(232 + 144x + 48x^2 + 5x^3\right)t^7 + \left(560 + 520x + 277x^2 + 68x^3 + 5x^4\right)t^8 +$
$\left(1328 + 1680x + 1248x^2 + 508x^3 + 93x^4 + 5x^5\right)t^9 + \cdots.$

$Q_{132}^{(1,2,2,0)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + (122+10x)t^6 +$
$\left(348 + 71x + 10x^2\right)t^7 + \left(978 + 351x + 91x^2 + 10x^3\right)t^8 +$
$\left(2715 + 1463x + 563x^2 + 111x^3 + 10x^4\right)t^9 + \cdots.$

$Q_{132}^{(1,2,3,0)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + 132t^6 + (404+25x)t^7 +$
$\left(1220 + 185x + 25x^2\right)t^8 + \left(3655 + 947x + 235x^2 + 25x^3\right)t^9 + \cdots.$

Again, one can easily explain the highest coefficient in $Q_{n,132}^{(1,2,m,0)}(x)$. That is, to have the maximum number of MMP$(1,2,m,0)$-matches in a $\sigma \in S_n(132)$, one must be of the form

$$\begin{aligned}
&(n-2)(n-1)\tau(m+1)\ldots(n-3)n,\\
&(n-1)(n-2)\tau(m+1)\ldots(n-3)n,\\
&n(n-2)\tau(m+1)\ldots(n-3)(n-1),\\
&n(n-1)\tau(m+1)\ldots(n-3)(n-2), \text{ or}\\
&(n-1)n\tau(m+1)\ldots(n-3)(n-2)
\end{aligned}$$

where $\tau \in S_m(132)$. Thus, the highest power of x occurring in $Q_{n,132}^{(1,2,m,0)}(x)$ is x^{n-3-m} which occurs with a coefficient of $5C_m$.

Finally, we have computed the following.

$Q_{132}^{(2,2,1,0)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + (123+9x)t^6 +$
$(351 + 69x + 9x^2)t^7 + (982 + 343x + 96x^2 + 9x^3)t^8 +$
$(2707 + 1405x + 609x^2 + 132x^3 + 9x^4)t^9 + \cdots.$

$Q_{132}^{(2,2,2,0)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + 132t^6 + (411+18x)t^7 +$
$(1265 + 147x + 18x^2)t^8 + (3852 + 809x + 183x^2 + 18x^3)t^9 +$
$(11626 + 3704x + 1229x^2 + 219x^3 + 18x^4)t^{10} + \cdots.$

$Q_{132}^{(2,2,3,0)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + 132t^6 + 429t^7 + (1385+45x)t^8 +$
$(4436 + 381x + 45x^2)t^9 + (14118 + 2162x + 471x^2 + 45x^3)t^{10} +$
$(44670 + 10361x + 3149x^2 + 561x^3 + 45x^4)t^{11} + \cdots.$

Again, one can easily explain the highest coefficient in $Q_{n,132}^{(2,2,m,0)}(x)$. That is, to have the maximum number of MMP$(2,2,m,0)$-matches in a $\sigma \in S_n(132)$, one must be of the form

$$\begin{aligned}
&n(n-1)\tau(m+1)\dots(n-4)(n-3)(n-2),\\
&(n-1)n\tau(m+1)\dots(n-4)(n-3)(n-2),\\
&n(n-2)\tau(m+1)\dots(n-4)(n-3)(n-1),\\
&n(n-3)\tau(m+1)\dots(n-4)(n-2)(n-1),\\
&(n-1)(n-2)\tau(m+1)\dots(n-4)(n-3)n,\\
&(n-2)(n-1)\tau(m+1)\dots(n-4)(n-3)n,\\
&(n-1)(n-3)\tau(m+1)\dots(n-4)(n-2)n,\\
&(n-2)(n-3)\tau(m+1)\dots(n-4)(n-1)n, \text{ or}\\
&(n-3)(n-2)\tau(m+1)\dots(n-4)(n-1)n
\end{aligned}$$

where $\tau \in S_m(132)$. Thus, the highest power of x occurring in $Q_{n,132}^{(2,2,m,0)}(x)$ is x^{n-4-m} which occurs with a coefficient of $9C_m$.

3. $Q_{n,132}^{(0,k,\ell,m)}(x)$ where $k,\ell,m \geq 1$

Suppose that $k,\ell,m \geq 1$ and $n \geq k+m$. It is clear that n cannot match the pattern MMP$(0,k,\ell,m)$ for $k,\ell,m \geq 1$ in any $\sigma \in S_n(132)$. If $\sigma = \sigma_1 \dots \sigma_n \in S_n(132)$ and $\sigma_i = n$, then we have three cases, depending on the value of i.

Case 1. $i < k$. It is easy to see that as we sum over all the permutations σ in $S_n^{(i)}(132)$, our choices for the structure for $A_i(\sigma)$ will contribute a factor of C_{i-1} to $Q_{n,132}^{(0,k,\ell,m)}(x)$ since none of the elements σ_j for $j \leq k$ can match MMP$(0,k,\ell,m)$ in σ. Similarly, our choices for the structure for $B_i(\sigma)$ will contribute a factor of $Q_{n-i,132}^{(0,k-i,\ell,m)}(x)$ to $Q_{n,132}^{(0,k,\ell,m)}(x)$ since $\sigma_1 \dots \sigma_i$ will automatically be in the second quadrant relative to the coordinate system with the origin at (s,σ_s) for any $s > i$. Thus, the permutations in Case 1 will contribute

$$\sum_{i=1}^{k-1} C_{i-1} Q_{n-i,132}^{(0,k-i,\ell,m)}(x)$$

to $Q_{n,132}^{(0,k,\ell,m)}(x)$.

Case 2. $k \leq i \leq n-m$. It is easy to see that as we sum over all the permutations σ in $S_n^{(i)}(132)$, our choices for the structure for $A_i(\sigma)$ will contribute a factor of $Q_{i-1,132}^{(0,k,\ell,0)}(x)$ to $Q_{n,132}^{(0,k,\ell,m)}(x)$ since the elements in $B_i(\sigma)$ will all be in the fourth quadrant relative to a coordinate system centered at (r, σ_r) for $r \leq i$ in this case. Similarly, our choices for the structure for $B_i(\sigma)$ will contribute a factor of $Q_{n-i,132}^{(0,0,\ell,m)}(x)$ to $Q_{n,132}^{(0,k,\ell,m)}(x)$ since $\sigma_1 \ldots \sigma_i$ will automatically be in the second quadrant relative to the coordinate system with the origin at (s, σ_s) for any $s > i$. Thus, the permutations in Case 2 will contribute

$$\sum_{i=k}^{n-m} Q_{i-1,132}^{(0,k,\ell,0)}(x) Q_{n-i,132}^{(0,0,\ell,m)}(x)$$

to $Q_{n,132}^{(0,k,\ell,m)}(x)$.

Case 3. $i \geq n-m+1$. It is easy to see that as we sum over all the permutations σ in $S_n^{(i)}(132)$, our choices for the structure for $A_i(\sigma)$ will contribute a factor of $Q_{i-1,132}^{(0,k,\ell,m-(n-i))}(x)$ to $Q_{n,132}^{(0,k,\ell,m)}(x)$ since the elements in $B_i(\sigma)$ will all be in the fourth quadrant relative to a coordinate system centered at (r, σ_r) for $r \leq i$ in this case. Similarly, our choices for the structure for $B_i(\sigma)$ will contribute a factor of C_{n-i} to $Q_{n,132}^{(0,k,\ell,m)}(x)$ since the elements in $B_i(\sigma)$ do not have enough elements to the right to match $\mathrm{MMP}(0,k,\ell,m)$ in σ. Thus, the permutations in Case 3 will contribute

$$\sum_{i=n-m+1}^{n} Q_{i-1,132}^{(0,k,\ell,m-(n-i))}(x) C_{n-i}$$

to $Q_{n,132}^{(0,k,\ell,m)}(x)$. Hence, for $n \geq k+m$,

$$Q_{n,132}^{(0,k,\ell,m)}(x) = \sum_{i=1}^{k-1} C_{i-1} Q_{n-i,132}^{(0,k-i,\ell,m)}(x) + \sum_{i=k}^{n-m} Q_{i-1,132}^{(0,k,\ell,0)}(x) Q_{n-i,132}^{(0,0,\ell,m)}(x) + \sum_{i=n-m+1}^{n} Q_{i-1,132}^{(0,k,\ell,m-(n-i))}(x) C_{n-i}. \tag{13}$$

Multiplying (13) by t^n and summing, it is easy to compute that

$$\begin{aligned} Q_{132}^{(0,k,\ell,m)}(t,x) = & \sum_{p=0}^{k+m-1} C_p t^p + \\ & \sum_{i=0}^{k-2} C_i t^i \left(t Q_{132}^{(0,k-1-i,\ell,m)}(t,x) - t \sum_{r=0}^{k-i+m-2} C_r t^r \right) + \\ & t \left(Q_{132}^{(0,k,\ell,0)}(t,x) - \sum_{a=0}^{k-2} C_a t^a \right) \left(Q_{132}^{(0,0,\ell,m)}(t,x) - \sum_{b=0}^{m-1} C_b t^b \right) + \\ & \sum_{j=0}^{m-1} C_j t^j \left(t Q_{132}^{(0,k,\ell,m-j)}(t,x) - t \sum_{s=0}^{k+m-j-2} C_s t^s \right). \end{aligned} \tag{14}$$

Note that the $j = 0$ term in the last sum is $tQ_{132}^{(0,k,\ell,m)}(t,x) - t\sum_{s=0}^{k+m-2} C_s t^s$. Thus, taking the term $tQ_{132}^{(0,k,\ell,m)}(t,x)$ over to the other side and combining the sum $-t\sum_{s=0}^{k+m-2} C_s t^s$ with the sum $\sum_{p=0}^{k+m-1} C_p t^p$ to obtain $C_{k+m-1}t^{k+m-1} + (1-t)\sum_{p=0}^{k+m-2} C_p t^p$ and then dividing both sides by $1-t$ will yield the following theorem.

Theorem 12.

$$\begin{aligned}Q_{132}^{(0,k,\ell,m)}(t,x) = &\sum_{p=0}^{k+m-2} C_p t^p + \frac{C_{k+m-1}t^{k+m-1}}{1-t} + \\ &\frac{t}{1-t}\sum_{i=0}^{k-2} C_i t^i \left(Q_{132}^{(0,k-1-i,\ell,m)}(t,x) - \sum_{r=0}^{k-i+m-2} C_r t^r\right) + \\ &\frac{t}{1-t}\left(Q_{132}^{(0,k,\ell,0)}(t,x) - \sum_{a=0}^{k-2} C_a t^a\right)\left(Q_{132}^{(0,0,\ell,m)}(t,x) - \sum_{b=0}^{m-1} C_b t^b\right) + \\ &\frac{t}{1-t}\sum_{j=1}^{m-1} C_j t^j \left(Q_{132}^{(0,k,\ell,m-j)}(t,x) - \sum_{s=0}^{k+m-j-2} C_s t^s\right). \end{aligned} \tag{15}$$

Note that since we can compute $Q_{132}^{(0,k,\ell,0)}(t,x) = Q_{132}^{(0,0,\ell,k)}(t,x)$ by Theorem 5, we can compute $Q_{132}^{(0,k,\ell,m)}(t,x)$ for all $k, \ell, m \geq 1$.

3.1. Explicit formulas for $Q_{n,132}^{(0,k,\ell,m)}(x)|_{x^r}$

It follows from Theorem 12 that

$$\begin{aligned}Q_{132}^{(0,1,\ell,1)}(t,x) =&1 + \frac{t}{1-t} + \frac{t}{1-t}Q_{132}^{(0,1,\ell,0)}(t,x)(Q_{132}^{(0,0,\ell,1)}(t,x) - 1)\\ =&\frac{1}{1-t} + \frac{t}{1-t}Q_{132}^{(0,1,\ell,0)}(t,x)(Q_{132}^{(0,0,\ell,1)}(t,x) - 1),\end{aligned}$$

$$\begin{aligned}Q_{132}^{(0,1,\ell,2)}(t,x) =&1 + t + \frac{2t^2}{1-t} + \frac{t}{1-t}Q_{132}^{(0,1,\ell,0)}(t,x)(Q_{132}^{(0,0,\ell,2)}(t,x) - (1+t)) + \\ &\frac{t^2}{1-t}(Q_{132}^{(0,1,\ell,1)}(t,x) - 1), \text{ and}\end{aligned}$$

$$\begin{aligned}Q_{132}^{(0,2,\ell,2)}(t,x) =&1 + t + 2t^2 + \frac{5t^3}{1-t} + \frac{t}{1-t}(Q_{132}^{(0,1,\ell,2)}(t,x) - (1+t+2t^2)) + \\ &\frac{t}{1-t}(Q_{132}^{(0,2,\ell,0)}(t,x) - 1)(Q_{132}^{(0,0,\ell,2)}(t,x) - (1+t)) + \\ &\frac{t^2}{1-t}(Q_{132}^{(0,2,\ell,1)}(t,x) - (1+t)).\end{aligned}$$

Again the formula for $Q_{132}^{(0,k,\ell,m)}(t,x)$ quickly become quite complicated. For example,

$$Q_{132}^{(0,1,1,1)}(t,x) = \frac{1 + \frac{2tx\left(1-t+tx-\sqrt{(1+t(-1+x))^2-4tx}\right)}{\left(-1+t+2x-tx+\sqrt{(1+t(-1+x))^2-4tx}\right)^2}}{1-t},$$

$$Q_{132}^{(0,1,2,1)}(t,x) = \frac{1 + \frac{2tx\left(1-t-t^2+tx+t^2x-\sqrt{(1+t(1+t)(-1+x))^2-4tx}\right)}{\left(-1+t+t^2+2x-tx-t^2x+\sqrt{(1+t(1+t)(-1+x))^2-4tx}\right)^2}}{1-t}, \text{ and}$$

$$Q_{132}^{(0,1,3,1)}(t,x) = \frac{1 + \frac{t\left(-1+\frac{1}{1-\frac{1+t\left(1+t+2t^2\right)(-1+x)-\sqrt{\left(1+t\left(1+t+2t^2\right)(-1+x)\right)^2-4tx}}{2x}}\right)}{1-\frac{1+t\left(1+t+2t^2\right)(-1+x)-\sqrt{\left(1+t\left(1+t+2t^2\right)(-1+x)\right)^2-4tx}}{2x}}}{1-t}.$$

We used these formulas to compute the following.

$$\begin{aligned}
&Q_{132}^{(0,1,1,1)}(t,x) = 1+t+2t^2+5t^3+(13+x)t^4+(33+8x+x^2)t^5+\\
&(81+39x+11x^2+x^3)t^6+(193+150x+70x^2+15x^3+x^4)t^7+\\
&(449+501x+337x^2+122x^3+20x^4+x^5)t^8+\\
&(1025+1524x+1363x^2+719x^3+204x^4+26x^5+x^6)t^9+\\
&(2305+4339x+4891x^2+3450x^3+1450x^4+327x^5+33x^6+x^7)t^{10}+\cdots.
\end{aligned}$$

$$\begin{aligned}
&Q_{132}^{(0,1,2,1)}(t,x) = 1+t+2t^2+5t^3+14t^4+2(20+x)t^5+(113+17x+2x^2)t^6+\\
&(314+92x+21x^2+2x^3)t^7+(859+404x+140x^2+25x^3+2x^4)t^8+\\
&(2319+1567x+745x^2+200x^3+29x^4+2x^5)t^9+\\
&(6192+5597x+3438x^2+1262x^3+272x^4+33x^5+2x^6)t^{10}+\cdots.
\end{aligned}$$

$$\begin{aligned}
&Q_{132}^{(0,1,3,1)}(t,x) = 1+t+2t^2+5t^3+14t^4+42t^5+(127+5x)t^6+\\
&(380+44x+5x^2)t^7+(1125+246x+54x^2+5x^3)t^8+\\
&(3299+1135x+359x^2+64x^3+5x^4)t^9+\\
&(9592+4691x+1942x^2+492x^3+74x^4+5x^5)t^{10}+\cdots.
\end{aligned}$$

Our next theorem will explain the coefficient of the highest and second highest powers of x that appear in $Q_{n,132}^{(0,1,\ell,1)}(x)$ in these series.

Theorem 13.

(i) For $n \geq 1+k+\ell+m$, the highest power of x that occurs in $Q_{n,132}^{(0,k,\ell,m)}(x)$ is $x^{n-k-\ell-m}$ which appears with a coefficient of $C_kC_\ell C_m$.

(ii) For $n \geq 5$, $Q_{n,132}^{(0,1,1,1)}(x)|_{x^{n-4}} = 5 + \binom{n-2}{2}$.

(iii) For all $\ell \geq 2$ *and* $n \geq 4+\ell$, $Q_{n,132}^{(0,1,\ell,1)}(x)|_{x^{n-3-\ell}} = C_{\ell+1} + 6C_\ell + 2C_\ell(n-4-\ell)$.

Proof. It is easy to see that the maximum number of matches of $\mathrm{MMP}(0,k,\ell,m)$ that are possible in a 132-avoiding permutation is a permutation of the form $\alpha\beta\gamma\delta$ where α is a rearrangement of $\{n-k+1,\ldots,n\}$ such that $\mathrm{red}(\alpha) \in S_k(132)$, β is a rearrangement of $\{m+1,\ldots,m+\ell\}$ such that $\mathrm{red}(\beta) \in S_\ell(132)$, γ is the increasing sequence $m+\ell+1, m+\ell+2, \ldots, n-k$, and $\delta \in S_m(132)$. Thus, the highest power in $Q_n^{(0,k,\ell,m)}$ is $x^{n-k-\ell-m}$ which has a coefficient of $C_k C_\ell C_m$.

For parts (ii) and (iii), we note that it follows from (13) that

$$Q_{n,132}^{(0,1,\ell,1)}(x) = Q_{n-1,132}^{(0,1,\ell,1)}(x) + \sum_{i=1}^{n-1} Q_{i-1,132}^{(0,1,\ell,0)}(x) Q_{n-i,132}^{(0,0,\ell,1)}(x).$$

We proved in [10, Theorems 6 and 15] that the highest power of x that appears in $Q_{n,132}^{(0,1,\ell,0)}(x) = Q_{n,132}^{(0,0,\ell,1)}(x)$ is $x^{n-\ell-1}$ which appears with a coefficient of C_ℓ for $n \geq \ell+2$. This implies that the highest power of x appearing in $Q_{i-1,132}^{(0,1,\ell,0)}(x) Q_{n-i,132}^{(0,0,\ell,1)}(x)$ is less than $x^{n-\ell-3}$ for $i = 3, \ldots, n-2$.

Hence we have four cases to consider when we are computing $Q_{n,132}^{(0,1,1,1)}(x)|_{x^{n-4}}$.

Case 1. $i=1$. In this case, $Q_{i-1,132}^{(0,1,1,0)}(x) Q_{n-i,132}^{(0,0,1,1)}(x)|_{x^{n-4}} = Q_{n-1,132}^{(0,0,1,1)}(x)|_{x^{n-4}}$ and we proved in [10, Theorems 9 and 15] that

$$Q_{n-1,132}^{(0,0,1,1)}(x)|_{x^{n-4}} = Q_{n-1,132}^{(0,1,1,0)}(x)|_{x^{n-4}} = 2 + \binom{n-2}{2} \text{ for } n \geq 5.$$

Case 2. $i=2$. In this case, $Q_{i-1,132}^{(0,1,1,0)}(x) Q_{n-i,132}^{(0,0,1,1)}(x)|_{x^{n-4}} = Q_{n-2,132}^{(0,0,1,1)}(x)|_{x^{n-4}}$ and we proved in [10, Theorems 6 and 15] that

$$Q_{n-2,132}^{(0,0,1,1)}(x)|_{x^{n-4}} = Q_{n-2,132}^{(0,1,1,0)}(x)|_{x^{n-4}} = 1 \text{ for } n \geq 5.$$

Case 3. $i=n-1$. In this case, $Q_{i-1,132}^{(0,1,1,0)}(x) Q_{n-i,132}^{(0,0,1,1)}(x)|_{x^{n-4}} = Q_{n-2,132}^{(0,1,1,0)}(x)|_{x^{n-4}}$ and we proved in [10, Theorems 6 and 15] that

$$Q_{n-2,132}^{(0,1,1,0)}(x)|_{x^{n-4}} = 1 \text{ for } n \geq 5.$$

Case 4. $Q_{n-1,132}^{(0,1,1,1)}(x)|_{x^{n-4}}$. By part (i), we know that $Q_{n-1,132}^{(0,1,1,1)}(x)|_{x^{n-4}} = 1$ for $n \geq 5$.

Thus, $Q_{n,132}^{(0,1,1,1)}(x)|_{x^{n-4}} = 5 + \binom{n-2}{2}$ for $n \geq 5$.

Again there are four cases to consider when computing $Q_{n,132}^{(0,1,\ell,1)}(x)|_{x^{n-3-\ell}}$ for $\ell \geq 2$.

Case 1. $i = 1$. In this case, $Q^{(0,1,\ell,0)}_{i-1,132}(x)Q^{(0,0,\ell,1)}_{n-i,132}(x)|_{x^{n-3-\ell}} = Q^{(0,0,\ell,1)}_{n-1,132}(x)|_{x^{n-3-\ell}}$ and we proved in [10, Theorems 10 and 15] that

$$\begin{aligned}Q^{(0,0,\ell,1)}_{n-1,132}(x)|_{x^{n-3-\ell}} =& Q^{(0,1,\ell,0)}_{n-1,132}(x)|_{x^{n-3-\ell}}\\ =& C_{\ell+1} + 3C_\ell + 2C_\ell(n-4-\ell) \text{ for } n \geq 4+\ell.\end{aligned}$$

Case 2. $i = 2$. In this case, $Q^{(0,1,\ell,0)}_{i-1,132}(x)Q^{(0,0,\ell,1)}_{n-i,132}(x)|_{x^{n-3-\ell}} = Q^{(0,0,\ell,1)}_{n-2,132}(x)|_{x^{n-3-\ell}}$ and we proved in [10, Theorems 6 and 15] that

$$Q^{(0,0,\ell,1)}_{n-2,132}(x)|_{x^{n-3-\ell}} = Q^{(0,1,\ell,0)}_{n-2,132}(x)|_{x^{n-3-\ell}} = C_\ell \text{ for } n \geq 4+\ell.$$

Case 3. $i = n-1$. In this case, $Q^{(0,1,\ell,0)}_{i-1,132}(x)Q^{(0,0,\ell,1)}_{n-i,132}(x)|_{x^{n-3-\ell}} = Q^{(0,1,\ell,0)}_{n-2,132}(x)|_{x^{n-3-\ell}}$ and we proved in [10, Theorems 6 and 15] that

$$Q^{(0,1,\ell,0)}_{n-2,132}(x)|_{x^{n-3-\ell}} = C_\ell \text{ for } n \geq 4+\ell.$$

Case 4. $Q^{(0,1,\ell,1)}_{n-1,132}(x)|_{x^{n-3-\ell}}$. By part (i), we know that

$$Q^{(0,1,\ell,1)}_{n-1,132}(x)|_{x^{n-3-\ell}} = C_\ell \text{ for } n \geq 4+\ell.$$

Thus,

$$Q^{(0,1,\ell,1)}_{n,132}(x)|_{x^{n-3-\ell}} = C_{\ell+1} + 6C_\ell + 2C_\ell(n-4-\ell) \text{ for } n \geq 4+\ell.$$

For example, when $\ell = 2$, we have that

$$Q^{(0,1,2,1)}_{n,132}(x)|_{x^{n-5}} = 17 + 4(n-6) \text{ for } n \geq 6$$

and, for $\ell = 3$, we have that

$$Q^{(0,1,3,1)}_{n,132}(x)|_{x^{n-6}} = 44 + 10(n-7) \text{ for } n \geq 7,$$

which agrees with the series we computed. □

We can also find the formulas for $Q^{(0,1,1,1)}_{n,132}(0)$ and $Q^{(0,1,1,1)}_{n,132}(x)|_x$.

Theorem 14.

(i) $Q^{(0,1,1,1)}_{n,132}(0) = (n-1)2^{n-2} + 1$ *for* $n \geq 1$.

(ii) $Q^{(0,1,1,1)}_{n,132}(x)|_x = (n^2 - 9n + 24)2^{n-3} - 3 - n$ *for* $n \geq 4$.

Proof. In this case, the recursion (13) becomes

$$Q_{n,132}^{(0,1,1,1)}(x) = Q_{n-1,132}^{(0,1,1,1)}(x) + \sum_{i=1}^{n-1} Q_{i-1,132}^{(0,1,1,0)}(x) Q_{n-i,132}^{(0,0,1,1)}(x).$$

It was proved in [10, Theorem 15] that $Q_{132}^{(0,1,1,0)}(t,x) = Q_{132}^{(1,0,1,0)}(t,x)$ so that $Q_{n,132}^{(0,0,1,1)}(x) = Q_{n,132}^{(0,1,1,0)}(x) = Q_{n,132}^{(1,0,1,0)}(x)$ for all n. Thus we have that

$$Q_{n,132}^{(0,1,1,1)}(x) = Q_{n-1,132}^{(0,1,1,1)}(x) + \sum_{i=1}^{n-1} Q_{i-1,132}^{(1,0,1,0)}(x) Q_{n-i,132}^{(1,0,1,0)}(x). \tag{16}$$

By [10, Theorems 7 and 8], we know that $Q_{n,132}^{(1,0,1,0)}(0) = 2^{n-1}$ for $n \geq 1$ and that $Q_{n,132}^{(1,0,1,0)}(x)|_x = (n-3)2^{n-2}+1$ for $n \geq 3$. Our calculations show that our formulas hold for $n \leq 5$. Then for $n \geq 6$, we have by induction that

$$\begin{aligned}
Q_{n,132}^{(0,1,1,1)}(0) =& Q_{n-1,132}^{(0,1,1,1)}(0) + Q_{n-1,132}^{(1,0,1,0)}(0) + \sum_{i=2}^{n-1} Q_{i-1,132}^{(1,0,1,0)}(0) Q_{n-i,132}^{(1,0,1,0)}(0) \\
=& (n-2)2^{n-3} + 1 + 2^{n-2} + \sum_{i=2}^{n-1} 2^{i-2} 2^{n-i-1} \\
=& (n-2)2^{n-3} + 1 + 2^{n-2} + (n-2)2^{n-3} \\
=& (n-1)2^{n-2} + 1.
\end{aligned}$$

Similarly, if we separate out the $i = 1, 2, 3, n-2, n-1$ terms from the sum in recursion (16), we find by induction that

$$\begin{aligned}
Q_{n,132}^{(0,1,1,1)}(x)|_x =& Q_{n-1,132}^{(0,1,1,1)}(x)|_x + Q_{n-1,132}^{(1,0,1,0)}(x)|_x + Q_{n-2,132}^{(1,0,1,0)}(x)|_x + 2Q_{n-3,132}^{(1,0,1,0)}(x)|_x + \\
& Q_{n-2,132}^{(1,0,1,0)}(x)|_x + 2Q_{n-3,132}^{(1,0,1,0)}(x)|_x + \\
& \sum_{i=4}^{n-3} Q_{i-1,132}^{(1,0,1,0)}(x)|_x Q_{n-i,132}^{(1,0,1,0)}(0) + \sum_{i=4}^{n-3} Q_{i-1,132}^{(1,0,1,0)}(0) Q_{n-i,132}^{(1,0,1,0)}(x)|_x \\
=& ((n-1)^2 - 9(n-1) + 24)2^{n-4} - 3 - (n-1) + ((n-4)2^{n-3} + 1) + \\
& 2((n-5)2^{n-4} + 1) + 4((n-6)2^{n-5} + 1) + \\
& \sum_{i=4}^{n-3} ((i-4)2^{i-3} + 1)2^{n-i-1} + \sum_{i=4}^{n-3} 2^{i-2}((n-i-3)2^{n-i-2} + 1) \\
=& ((n-1)^2 - 9(n-1) + 24)2^{n-4} - 3 - (n-1) + (6n-30)2^{n-4} + \\
& 7 + 2^{n-4} \left(\sum_{i=4}^{n-3} (i-4) + \sum_{i=4}^{n-3} (n-i-3) \right) + \sum_{i=4}^{n-3} 2^{n-i-1} + \sum_{i=4}^{n-3} 2^{i-2}
\end{aligned}$$

$$
\begin{aligned}
=&2^{n-4}\left((n-1)^2-9(n-1)+24+6n-30+2\binom{n-6}{2}\right)+\\
&-3-(n-1)+7+2(2^{n-4}-4)\\
=&(n^2-9n+24)2^{n-3}-3-n.
\end{aligned}
$$

□

The sequence $(Q_{n,132}^{(0,1,1,1)}(0))_{n\geq 1}$ starts out $1, 2, 5, 13, 33, 81, 193, 449, \ldots$. This is the sequence A005183 in OEIS and it counts the number of permutations of length n which avoids the patterns 132 and 4312. We obtain the interpretation that this sequence also counts the number of permutations of length n which avoid the patterns 132 and 4231 (which are precisely the permutations counted by $Q_{n,132}^{(0,1,1,1)}(0)$, since an occurrence of the pattern $\mathrm{MMP}(0, 1, 1, 1)$ in a 132-avoiding permutation implies an occurrence of the pattern 4231, and vice versa).

Problem 2. *Find a bijection between permutations of length n avoiding the patterns 132 and 4312, and permutations of length n avoiding the patterns 132 and 4231.*

The sequence $(Q_{n,132}^{(0,1,1,1)}(x)|_x)_{n\geq 4}$ starts out $1, 8, 39, 150, 501, 1524\ldots$ is sequence A055581 which counts the number of *directed column-convex polyominoes* of area $n + 5$ having along the lower contour exactly 2 *reentrant corners*.

Problem 3. *Find a bijective correspondence between the number of permutations in $S_n(132)$ which have exactly one occurrence of the pattern* $\mathrm{MMP}(0, 1, 1, 1)$ *and the polyominoes described in A055581 in the OEIS.*

We have computed the following.

$$
\begin{aligned}
&Q_{132}^{(0,1,1,2)}(t,x) = 1+t+2t^2+5t^3+14t^4+2(20+x)t^5+\left(111+19x+2x^2\right)t^6+\\
&\left(296+106x+25x^2+2x^3\right)t^7+\left(761+456x+178x^2+33x^3+2x^4\right)t^8+\\
&\left(1898+1677x+947x^2+295x^3+43x^4+2x^5\right)t^9+\\
&\left(4619+5553x+4191x^2+1901x^3+475x^4+55x^5+2x^6\right)t^{10}+\cdots.
\end{aligned}
$$

$$
\begin{aligned}
&Q_{132}^{(0,1,2,2)}(t,x) = 1+t+2t^2+5t^3+14t^4+42t^5+4(32+x)t^6+\\
&\left(385+40x+4x^2\right)t^7+\left(1135+243x+48x^2+4x^3\right)t^8+\\
&\left(3281+117x+351x^2+56x^3+4x^4\right)t^9+\\
&\left(9324+4905x+2016x^2+483x^3+64x^4+4x^5\right)t^{10}+\cdots.
\end{aligned}
$$

$Q_{132}^{(0,1,3,2)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + 132t^6 + (419 + 10x)t^7 +$
$\left(1317 + 103x + 10x^2\right) t^8 + \left(4085 + 644x + 123x^2 + 10x^3\right) t^9 +$
$\left(12514 + 3229x + 900x^2 + 143x^3 + 10x^4\right) t^{10} +$
$\left(37913 + 14282x + 5222x^2 + 1196x^3 + 163x^4 + 10x^5\right) t^{11} + \cdots .$

$Q_{132}^{(0,2,1,2)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + 4t^6(32 + x) +$
$\left(380 + 45x + 4x^2\right) t^7 + \left(1083 + 286x + 57x^2 + 4x^3\right) t^8 +$
$\left(2964 + 1368x + 453x^2 + 73x^3 + 4x^4\right) t^9 +$
$\left(7831 + 5501x + 2650x^2 + 717x^3 + 93x^4 + 4x^5\right) t^{10} +$
$\left(20092 + 19675x + 12749x^2 + 5035x^3 + 1114x^4 + 117x^5 + 4x^6\right) t^{11} + \cdots .$

$Q_{132}^{(0,2,2,2)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + 132t^6 + (421 + 8x)t^7 +$
$\left(1328 + 94x + 4x^2\right) t^8 + \left(4103 + 641x + 110x^2 + 8x^3\right) t^9 +$
$\left(12401 + 3376x + 885x^2 + 126x^3 + 8x^4\right) t^{10} +$
$\left(36740 + 15235x + 5484x^2 + 1177x^3 + 142x^4 + 8x^5\right) t^{11} +$
$\left(106993 + 62012x + 28872x^2 + 8452x^3 + 1517x^4 + 158x^5 + 8x^6\right) t^{12} + \cdots .$

$Q_{132}^{(0,2,3,2)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + 132t^6 + 429t^7 +$
$(1410 + 20x)t^8 + \left(4601 + 241x + 20x^2\right) t^9 + \left(14809 + 1686x + 281x^2 + 20x^3\right) t^{10} +$
$\left(46990 + 9187x + 2268x^2 + 321x^3 + 20x^4\right) t^{11} +$
$\left(147163 + 43394x + 14144x^2 + 2930x^3 + 361x^4 + 20x^5\right) t^{12} + \cdots .$

We can explain the coefficient of the highest power of x in $Q_{n,132}^{(0,k,\ell,m)}(x)$ for large enough n.

Theorem 15. *For $n \geq \ell + k + m + 1$, the highest power of x that appears in $Q_{n,132}^{(0,k,\ell,m)}(x)$ is $x^{n-k-\ell-m}$, which occurs with a coefficient of $C_k C_\ell C_m$.*

Proof. It is easy to see the maximum number of matches of $MMP(0,k,\ell,m)$ for $n > k+\ell+m$ that are possible in a 132-avoiding permutation occur in a permutation of the form $\alpha\beta\gamma$ where α is a permutation of $n-k+1,\ldots,n$ such that $\text{red}(\alpha) \in S_k(132)$, $\gamma \in S_m(132)$, and β is some permutation of $k+1, k+2, \ldots, n-m$, which has the maximum number of matches of $MMP(0,0,\ell,0)$. Such a β must be of the form $\alpha(\ell+a)(\ell+a+1)(\ell+a+2)\ldots(n-m)$ where α is a pemutation of $k+1,\ldots,k+\ell$ such that $\text{red}(\alpha) \in S_\ell(132)$. Thus the highest power of x that occurs in $Q_n^{(0,k,\ell,m)}(x)$ for $n > n-k-\ell-m$ is $x^{n-k-\ell-m}$ which occurs with a coefficient of $C_k C_\ell C_m$. For example, for $n > n-\ell-3$, the highest power of x in $Q_{n,132}^{(0,1,\ell,2)}(x)$ is $x^{n-\ell-3}$ which has a coefficient of $2C_\ell$ and for $n > n-\ell-4$, the highest power of x in $Q_{n,132}^{(0,2,\ell,2)}(x)$ is $x^{n-\ell-4}$ which has a coefficient of $4C_\ell$. This agrees with our computations. $\square$

For sufficiently large n, we can also explain the coefficient of the second highest term in $Q_{n,132}^{(0,1,\ell,2)}(x)$.

Theorem 16.

(i) For $n \geq 6$, $Q_{n,132}^{(0,1,1,2)}(x)|_{x^{n-5}} = 13 + 2\binom{n-3}{2}$.

(ii) For all $\ell \geq 2$ *and* $n \geq 5+\ell$, $Q_{n,132}^{(0,1,\ell,2)}(x)|_{x^{n-4-\ell}} = 2C_{\ell+1}+15C_\ell+4C_\ell(n-5-\ell)$.

Proof. For (i) and (ii), we note that the recursion for $Q_{n,132}^{(0,1,\ell,2)}(x)$ is

$$Q_{n,132}^{(0,1,\ell,2)}(x) = Q_{n-1,132}^{(0,1,\ell,2)}(x) + Q_{n-2,132}^{(0,1,\ell,1)}(x) + \sum_{i=1}^{n-2} Q_{i-1,132}^{(0,1,\ell,0)}(x) Q_{n-i,132}^{(0,0,\ell,2)}(x).$$

By [10, Theorems 6 and 15], the highest power of x that occurs in $Q_{n,132}^{(0,1,\ell,0)}(x)$ is $x^{n-1-\ell}$, and by [10, Theorem 17], the highest power of x that occurs in $Q_{n,132}^{(0,0,\ell,2)}(x)$ is $n-2-\ell$. It follows that the highest power of x that occurs in $Q_{i-1,132}^{(0,1,\ell,0)}(x)Q_{n-i,132}^{(0,0,\ell,2)}(x)$ is less than $x^{n-4-\ell}$ for $i = 3, \ldots, n-3$.

Thus, we have to consider five cases when computing $Q_{n,132}^{(0,1,\ell,2)}(x)|_{x^{n-4-\ell}}$.

Case 1. $Q_{n-1,132}^{(0,1,\ell,2)}(x)|_{x^{n-4-\ell}}$. By part Theorem 15,

$$Q_{n-1,132}^{(0,1,\ell,2)}(x)|_{x^{n-4-\ell}} = 2C_\ell \text{ for } n \geq \ell+5.$$

Case 2. $Q_{n-2,132}^{(0,1,\ell,1)}(x)|_{x^{n-4-\ell}}$. We have shown in Theorem 13 that

$$Q_{n-2,132}^{(0,1,\ell,1)}(x)|_{x^{n-4-\ell}} = C_\ell \text{ for } n \geq \ell+5.$$

Case 3. $i = n-2$. In this case, $Q_{i-1,132}^{(0,1,\ell,0)}(x)Q_{n-i,132}^{(0,0,\ell,2)}(x)$ equals $2Q_{n-3,132}^{(0,1,\ell,0)}(x)$. We have shown in [10, Theorems 6 and 15] that $Q_{n-3,132}^{(0,1,\ell,0)}(x)|_{x^{n-4-\ell}} = C_\ell$ for $n \geq \ell+5$ so that we get a contribution of $2C_\ell$ in this case.

Case 4. $i = 2$. In this case, $Q_{i-1,132}^{(0,1,\ell,0)}(x)Q_{n-i,132}^{(0,0,\ell,2)}(x)$ equals $Q_{n-2,132}^{(0,0,\ell,2)}(x)$. We have shown in [10, Theorem 17] that

$$Q_{n-2,132}^{(0,0,\ell,2)}(x)|_{x^{n-4-\ell}} = 2C_\ell \text{ for } n \geq \ell+5.$$

Case 5. $i = 1$. In this case, $Q_{i-1,132}^{(0,1,\ell,0)}(x)Q_{n-i,132}^{(0,0,\ell,2)}(x)$ equals $Q_{n-1,132}^{(0,0,\ell,2)}(x)$. We have shown in [10, Theorem 17] that for $n \geq \ell+5$,

$$Q_{n-1,132}^{(0,0,\ell,2)}(x)|_{x^{n-4-\ell}} = \begin{cases} 6 + 2\binom{n-3}{2} & \text{if } \ell = 1, \\ 2C_{\ell+1} + 8C_\ell + 4C_\ell(n-5-\ell) & \text{if } \ell \geq 2. \end{cases}$$

□

Thus, for $\ell = 1$, we get

$$Q_{n,132}^{(0,1,1,2)}(x)|_{x^{n-5}} = 13 + 2\binom{n-3}{2} \text{ for } n \geq 6$$

and, for $\ell \geq 2$,

$$Q_{n,132}^{(0,1,\ell,2)}(x)|_{x^{n-4-\ell}} = 2C_{\ell+1} + 15C_\ell + 4C_\ell(n-5-\ell) \text{ for } n \geq 5+\ell.$$

For example, when $\ell = 2$, we get

$$Q_{n,132}^{(0,1,2,2)}(x)|_{x^{n-6}} = 40 + 8(n-7) \text{ for } n \geq 7$$

and, for $\ell = 3$, we get

$$Q_{n,132}^{(0,1,3,2)}(x)|_{x^{n-7}} = 103 + 20(n-8) \text{ for } n \geq 8,$$

which agrees with the series that we computed.

4. $Q_{n,132}^{(\ell,k,0,m)}(x)$ where $k, \ell, m \geq 1$

Suppose that $k, \ell, m \geq 1$ and $n \geq k + m$. It is clear that n cannot match MMP$(\ell, k, 0, m)$ for $k, \ell, m \geq 1$ in any $\sigma \in S_n(132)$. If $\sigma = \sigma_1 \ldots \sigma_n \in S_n(132)$ and $\sigma_i = n$, then we have three cases, depending on the value of i.

Case 1. $i < k$. It is easy to see that as we sum over all the permutations σ in $S_n^{(i)}(132)$, our choices for the structure for $A_i(\sigma)$ will contribute a factor of C_{i-1} to $Q_{n,132}^{(\ell,k,0,m)}(x)$ since the elements in $A_i(\sigma)$ do not have enough elements to the left to match MMP$(\ell, k, 0, m)$ in σ. Similarly, our choices for the structure for $B_i(\sigma)$ will contribute a factor of $Q_{n-i,132}^{(\ell,k-i,0,m)}(x)$ to $Q_{n,132}^{(\ell,k,0,m)}(x)$ since $\sigma_1 \ldots \sigma_i$ will automatically be in the second quadrant relative to the coordinate system with the origin at (s, σ_s) for any $s > i$. Thus, the permutations in Case 1 will contribute

$$\sum_{i=1}^{k-1} C_{i-1} Q_{n-i,132}^{(\ell,k-i,0,m)}(x)$$

to $Q_{n,132}^{(\ell,k,0,m)}(x)$.

Case 2. $k \leq i \leq n - m$. It is easy to see that as we sum over all the permutations σ in $S_n^{(i)}(132)$, our choices for the structure for $A_i(\sigma)$ will contribute a factor of $Q_{i-1,132}^{(\ell-1,k,0,0)}(x)$ to $Q_{n,132}^{(\ell,k,0,m)}(x)$ since the elements in $B_i(\sigma)$ will all be in the fourth quadrant and $\sigma_i = n$ is in the first quadrant relative to a coordinate system centered at (r, σ_r) for $r < i$ in this case. Similarly, our choices for the structure for $B_i(\sigma)$ will contribute a factor of $Q_{n-i,132}^{(\ell,0,0,m)}(x)$ to $Q_{n,132}^{(\ell,k,0,m)}(x)$ since $\sigma_1 \ldots \sigma_i$ will automatically be in the second quadrant relative to the coordinate system with the origin at (s, σ_s) for any $s > i$. Thus, the permutations in Case 2 will contribute

$$\sum_{i=k}^{n-m} Q_{i-1,132}^{(\ell-1,k,0,0)}(x) Q_{n-i,132}^{(\ell,0,0,m)}(x)$$

to $Q_{n,132}^{(\ell,k,0,m)}(x)$.

Case 3. $i \geq n-m+1$. It is easy to see that as we sum over all the permutations σ in $S_n^{(i)}(132)$, our choices for the structure for $A_i(\sigma)$ will contribute a factor of $Q_{i-1,132}^{(\ell-1,k,0,m-(n-i))}(x)$ to $Q_{n,132}^{(\ell,k,0,m)}(x)$ since $\sigma_i = n$ will be in the first quadrant and the elements in $B_i(\sigma)$ will all be in the fourth quadrant relative to a coordinate system centered at (r, σ_r) for $r < i$ in this case. Similarly, our choices for the structure for $B_i(\sigma)$ will contribute a factor of C_{n-i} to $Q_{n,132}^{(\ell,k,0,m)}(x)$ since σ_j where $j > i$ does not have enough elements to its right to match $\mathrm{MMP}(\ell, k, 0, m)$ in σ. Thus, the permutations in Case 3 will contribute

$$\sum_{i=n-m+1}^{n} Q_{i-1,132}^{(\ell-1,k,0,m-(n-i))}(x) C_{n-i}$$

to $Q_{n,132}^{(\ell,k,0,m)}(x)$. Thus, we have the following. For $n \geq k+m$,

$$\begin{aligned} Q_{n,132}^{(\ell,k,0,m)}(x) = & \sum_{i=1}^{k-1} C_{i-1} Q_{n-i,132}^{(\ell,k-i,0,m)}(x) + \sum_{i=k}^{n-m} Q_{i-1,132}^{(\ell-1,k,0,0)}(x) Q_{n-i,132}^{(\ell,0,0,m)}(x) + \\ & \sum_{i=n-m+1}^{n} Q_{i-1,132}^{(\ell-1,k,0,m-(n-i))}(x) C_{n-i}. \end{aligned} \tag{17}$$

Multiplying (17) by t^n and summing over n will yield the following theorem.

Theorem 17. *For all $\ell, k, m \geq 1$,*

$$\begin{aligned} Q_{132}^{(\ell,k,0,m)}(t,x) = & \sum_{p=0}^{k+m-1} C_p t^p + \\ & t \sum_{i=0}^{k-2} C_i t^i \left(Q_{132}^{(\ell,k-1-i,0,m)}(t,x) - \sum_{r=0}^{k-i+m-2} C_r t^r \right) + \\ & t \left(Q_{132}^{(\ell-1,k,0,0)}(t,x) - \sum_{a=0}^{k-2} C_a t^a \right) \left(Q_{132}^{(\ell,0,0,m)}(t,x) - \sum_{b=0}^{m-1} C_b t^b \right) + \end{aligned}$$

$$t\sum_{j=0}^{m-1} C_j t^j \left(Q_{132}^{(\ell-1,k,0,m-j)}(t,x) - \sum_{s=0}^{k+m-j-2} C_s t^s \right). \tag{18}$$

Note that we can compute $Q_{132}^{(\ell,k,0,0)}(t,x) = Q_{132}^{(\ell,0,0,k)}(t,x)$ by Theorem 4 so that (18) allows us to compute $Q_{132}^{(\ell,k,0,m)}(t,x)$ for any $k,\ell,m \geq 0$.

4.1. Explicit formulas for $Q_{n,132}^{(\ell,k,0,m)}(x)|_{x^r}$

In general, the formulas for $Q_{132}^{(\ell,k,0,m)}(t,x)$ are quite complicated. For example, in the simplest case,

$$Q_{132}^{(1,1,0,1)}(t,x) = \frac{R(t,x) + S(t,x)\sqrt{1-4xt}}{(1-t)(1-2t+\sqrt{1-4xt})^3}$$

where

$$\begin{aligned} R(t,x) =& 4(1-3t+4t^2-2t^3-t^4-3xt+6xt^2-3xt^3+xt^4) \text{ and} \\ S(t,x) =& 4(1-3t+4t^2-t^4-xt-xt^3+xt^4). \end{aligned}$$

It follows from Theorem 17 that

$$\begin{aligned} Q_{132}^{(\ell,1,0,1)}(t,x) =& 1+t+tQ_{132}^{(\ell-1,1,0,0)}(t,x)(Q_{132}^{(\ell,0,0,1)}(t,x)-1)+ \\ & t(Q_{132}^{(\ell-1,1,0,1)}(t,x)-1), \end{aligned} \tag{19}$$

$$\begin{aligned} Q_{132}^{(\ell,1,0,2)}(t,x) =& 1+t+2t^2+tQ_{132}^{(\ell-1,1,0,0)}(t,x)(Q_{132}^{(\ell,0,0,2)}(t,x)-(1+t))+ \\ & t(Q_{132}^{(\ell-1,1,0,2)}(t,x)-(1+t)+t(Q_{132}^{(\ell-1,1,0,1)}(t,x)-1)), \text{ and} \end{aligned}$$

$$\begin{aligned} Q_{132}^{(\ell,2,0,2)}(t,x) =& 1+t+2t^2+5t^3+t(Q_{132}^{(\ell,1,0,2)}(t,x)-(1+t+2t^2))+ \\ & t(Q_{132}^{(\ell-1,2,0,0)}(t,x)-1)(Q_{132}^{(\ell,0,0,2)}(t,x)-(1+t))+ \\ & t(Q_{132}^{(\ell-1,2,0,2)}(t,x)-(1+t+2t^2)+t(Q_{132}^{(\ell-1,2,0,1)}(t,x)-(1+t))). \end{aligned}$$

One can use these formulas to compute the following.

$$\begin{aligned} & Q_{132}^{(1,1,0,1)}(t,x) = 1+t+2t^2+5t^3+2(5+2x)t^4+(17+17x+8x^2)t^5+ \\ & (26+44x+42x^2+20x^3)t^6+(37+90x+129x^2+117x^3+56x^4)t^7+ \\ & (50+160x+305x^2+397x^3+350x^4+168x^5)t^8+ \\ & (65+259x+615x^2+1029x^3+1268x^4+1098x^5+528x^6)t^9+ \\ & (82+392x+1113x^2+2259x^3+3503x^4+4167x^5+3564x^6+1716x^7)t^{10}+\cdots. \end{aligned}$$

It is easy to explain the highest coefficient of x in $Q_{n,132}^{(1,k,0,1)}(x)$.

Theorem 18. *For $n \geq 3+k$, the highest power of x that occurs in $Q_{n,132}^{(1,k,0,1)}(x)$ is x^{n-2-k} which occurs with a coefficient of $2C_{k+1}C_{n-k-2}$.*

Proof. It is easy to see that the maximum number of MMP$(1,k,0,1)$ matches occurs in $\sigma \in S_n(132)$ when σ is of the form $\alpha_1 \dots \alpha_k \beta \alpha_{k+1} 1$ or $\alpha_1 \dots \alpha_k \beta 1 \alpha_{k+1}$ where $\alpha = \alpha_1 \dots \alpha_{k+1}$ is a rearrangement of $\{n-k, n-k+1, \dots, n\}$ such that $\text{red}(\alpha) \in S_{k+1}(132)$ and β is a 132-avoiding permutation on $2, \dots, n-k-1$. Thus, the highest power of x in $Q_{n,132}^{(1,k,0,1)}(x)$ for $n \geq k+3$ is x^{n-2-k} which occurs with a coefficient of $2C_{k+1}C_{n-k-2}$. □

We can also explain the second highest coefficient in $Q_{n,132}^{(1,1,0,1)}(x)$.

Theorem 19. *For $n \geq 5$,*

$$Q_{n,132}^{(1,1,0,1)}(x)|_{x^{n-4}} = 8C_{n-3} + C_{n-4}.$$

Proof. In this case, the recursion for $Q_{n,132}^{(1,1,0,1)}(x)$ is

$$Q_{n,132}^{(1,1,0,1)}(x) = Q_{n-1,132}^{(0,1,0,1)}(x) + \sum_{i=1}^{n-1} Q_{i-1,132}^{(0,1,0,0)}(x) Q_{n-i,132}^{(1,0,0,1)}(x).$$

It was proved in [9, Theorem 5.1] that for $n \geq 1$, the highest power of x that occurs in $Q_{n,132}^{(0,1,0,0)}(x)$ is x^{n-1} which occurs with a coefficient of C_{n-1}. It was proved in [10, Theorem 12] that for $n \geq 3$, the highest power of x that occurs in $Q_{n,132}^{(1,0,0,1)}(x)$ is x^{n-2} which occurs with a coefficient of $2C_{n-2}$. It follows that

$$\begin{aligned} Q_{n,132}^{(1,1,0,1)}(x)|_{x^{n-4}} = & Q_{n-1,132}^{(0,1,0,1)}(x)|_{x^{n-4}} + Q_{n-1,132}^{(1,0,0,1)}(x)|_{x^{n-4}} + Q_{n-2,132}^{(0,1,0,0)}(x)|_{x^{n-4}} + \\ & \sum_{i=2}^{n-2} Q_{i-1,132}^{(0,1,0,0)}(x)|_{x^{i-2}} Q_{n-i,132}^{(1,0,0,1)}(x)|_{x^{n-i-2}}. \end{aligned}$$

The following three equations follow from [10, Theorem 19], [10, Theorem 12] and [9, Theorem 5.1], respectively.

$$\begin{aligned} Q_{n-1,132}^{(0,1,0,1)}(x)|_{x^{n-4}} =& 2C_{n-3} + C_{n-4} \text{ for } n \geq 5, \\ Q_{n-1,132}^{(1,0,0,1)}(x)|_{x^{n-4}} =& 3C_{n-3} \text{ for } n \geq 5, \text{ and} \\ Q_{n-2,132}^{(0,1,0,0)}(x)|_{x^{n-4}} =& C_{n-3} \text{ for } n \geq 5. \end{aligned}$$

Thus, for $n \geq 5$,

$$\begin{aligned} Q_{n,132}^{(1,1,0,1)}(x)|_{x^{n-4}} =& 2C_{n-3} + C_{n-4} + 3C_{n-3} + C_{n-3} + \sum_{i=2}^{n-2} C_{i-2} \cdot 2C_{n-i-2} \\ =& 6C_{n-3} + C_{n-4} + 2\sum_{i=2}^{n-2} C_{i-2}C_{n-i-2} \\ =& 6C_{n-3} + C_{n-4} + 2C_{n-3} \\ =& 8C_{n-3} + C_{n-4}. \end{aligned}$$

□

The sequence $(Q_{n,132}^{(1,1,0,1)}(0))_{n\geq 1}$ starts out $1, 2, 5, 10, 17, 26, 37, 50, 65, 82, \ldots$ which is the sequence A002522 in the OEIS. The n-th element of the sequence has the formula $(n-1)^2 + 1$. This can be verified by computing the generating function $Q_{132}^{(1,1,0,1)}(t, 0)$. That is, we proved in [9] and [10] that

$$\begin{aligned} Q_{132}^{(0,1,0,0)}(t,0) =& \frac{1}{1-t}, \\ Q_{132}^{(1,0,0,1)}(t,0) =& \frac{1-2t+2t^2}{(1-t)^3}, \text{and} \\ Q_{132}^{(0,1,0,1)}(t,0) =& \frac{1}{1-t} + \frac{t^2}{(1-t)^2}. \end{aligned}$$

Plugging these formulas into (19), one can compute that

$$Q_{132}^{(1,1,0,1)}(t,0) = \frac{1-2t+2t^2+t^3}{(1-t)^3}.$$

It was pointed out to us by the anonymous referee how to prove directly that $Q_{n,132}^{(1,1,0,1)}(0) = (n-1)^2 + 1$ for $n \geq 1$. It can be shown by an easy case analysis according to for which i we have $\sigma_i = n$ in a 132-avoiding permutation $\sigma_1 \ldots \sigma_n$. For $i = n$ we get permutations avoiding both of the patterns 321 and 132, which are counted by $Q_{n-1,132}^{(0,1,0,1)}(0) = 1 + \binom{n-1}{2}$ by Theorem 21 of [10]. For $i < n$, we have only increasing permutation on the left and a concatenation of a decreasing (β) and an increasing (γ) permutation on the right, where all elements of β are larger than all elements of γ giving us $\sum_{i=1}^{n-1}(n-i) = \binom{n}{2}$ choices. Thus

$$Q_{n,132}^{(1,1,0,1)}(0) = 1 + \binom{n-1}{2} + \binom{n}{2} = (n-1)^2 + 1.$$

$$Q_{132}^{(2,1,0,1)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + (33+9x)t^5 + \left(71 + 43x + 18x^2\right)t^6 + \left(146 + 137x + 101x^2 + 45x^3\right)t^7 + \left(294 + 368x + 367x^2 + 275x^3 + 126x^4\right)t^8 + \left(587 + 906x + 1100x^2 + 1079x^3 + 812x^4 + 378x^5\right)t^9 + \left(1169 + 2125x + 2973x^2 + 3463x^3 + 3352x^4 + 2526x^5 + 1188x^6\right)t^{10} + \cdots$$

$$Q_{132}^{(3,1,0,1)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + (116+16x)t^6 + \left(308 + 89x + 32x^2\right)t^7 + \left(807 + 341x + 202x^2 + 80x^3\right)t^8 + \left(2108 + 1140x + 849x^2 + 541x^3 + 224x^4\right)t^9 + \left(5507 + 3583x + 3046x^2 + 2406x^3 + 1582x^4 + 672x^5\right)t^{10} + \left(14397 + 10897x + 10141x^2 + 9039x^3 + 7310x^4 + 4890x^5 + 2112x^6\right)t^{11} + \cdots .$$

We have computed that

$$Q_{132}^{(2,1,0,1)}(t,0) = \frac{1 - 4t + 6t^2 - 3t^3 + 2t^4 - 4t^5 + t^6}{(1-2t)(1-t)^3} \text{ and}$$
$$Q_{132}^{(3,1,0,1)}(t,0) = \frac{1 - 5t + 9t^2 - 7t^3 + 3t^4 + 2t^5 - 8t^6 + 4t^7}{(1-3t+t^2)(1-t)^3}.$$

We can find an explicit formula of the coefficient of the highest power of x in $Q_{n,132}^{(\ell,1,0,1)}(x)$.

Theorem 20. *For $n \geq \ell + 3$, the highest power or x that occurs in $Q_{n,132}^{(\ell,1,0,1)}(x)$ is $x^{n-\ell-2}$ which appears with a coefficient of $(\ell+1)^2 C_{n-k-2}$.*

Proof. It is easy to see that the maximum number of occurrences of MMP$(\ell,1,0,1)$ for a $\sigma \in S_n(132)$ occurs when σ is of the form $x\tau\beta$ where $x \in \{n-\ell,\ldots,n\}$, β is a shuffle of 1 with the increasing sequence which results from $(n-\ell)(n-\ell+1)\ldots n$ by removing x, and τ is a 132-avoiding permutation on $2,\ldots,n-\ell-1$. Thus we have $\ell+1$ choices for x and, once x is chosen, we have $\ell+1$ choices for β, and $C_{n-\ell-2}$ choices for τ. $\square$

$$Q_{132}^{(1,1,0,2)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + (32+10x)t^5 + \left(62 + 50x + 20x^2\right)t^6 + \left(107 + 149x + 123x^2 + 50x^3\right)t^7 + \left(170 + 345x + 433x^2 + 342x^3 + 140x^4\right)t^8 + \left(254 + 685x + 1154x^2 + 1327x^3 + 1022x^4 + 420x^5\right)t^9 + \left(362 + 1225x + 2589x^2 + 3868x^3 + 4228x^4 + 3204x^5 + 1320x^6\right)t^{10} + \cdots .$$

$$Q_{132}^{(2,1,0,2)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + (105+27x)t^6 + \left(235+140x+54x^2\right)t^7 + \left(494+470x+331x^2+135x^3\right)t^8 + \left(1004+1301x+1275x^2+904x^3+378x^4\right)t^9 + \left(2007+3248x+3960x^2+3773x^3+2674x^4+1134x^5\right)t^{10} + \cdots.$$

$$Q_{132}^{(3,1,0,2)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + 132t^6 + (373+56x)t^7 + \left(998+320x+112x^2\right)t^8 + \left(2615+1233x+734x^2+280x^3\right)t^9 + \left(6813+4092x+3131x^2+1976x^3+784x^4\right)t^{10} + \left(17749+12699x+11223x^2+8967x^3+5796x^4+2352x^5\right)t^{11} + \cdots.$$

$$Q_{132}^{(1,2,0,2)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + (107+25x)t^6 + \left(233+146x+50x^2\right)t^7 + \left(450+498x+357x^2+125x^3\right)t^8 + \left(794+1299x+1429x^2+990x^3+350x^4\right)t^9 + \left(1307+2869x+4263x^2+4353x^3+2954x^4+1050x^5\right)t^{10} + \cdots.$$

$$Q_{132}^{(2,2,0,2)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + 132t^6 + (348+81x)t^7 + \left(811+457x+162x^2\right)t^8 + \left(1747+1625x+1085x^2+405x^3\right)t^9 + \left(3587+4663x+4443x^2+2969x^3+1134x^4\right)t^{10} + \left(7167+11864x+14360x^2+13201x^3+8792x^4+3402x^5\right)t^{11} + \cdots.$$

$$Q_{132}^{(3,2,0,2)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + 132t^6 + 429t^7 + (1234+196x)t^8 + 2\left(1657+578x+196x^2\right)t^9 + \left(8643+4497x+2676x^2+980x^3\right)t^{10} + \left(22345+14839x+11622x^2+7236x^3+2744x^4\right)t^{11} + \cdots.$$

We can explain the highest power of x in $Q_{n,132}^{(k,\ell,0,2)}(x)$ for $\ell \in \{1,2\}$. But first we need to prove the following lemma.

Lemma 2. *The number of $\sigma = \sigma_1 \dots \sigma_{k+2} \in S_{k+2}(132)$ such that $\sigma_3 < \cdots < \sigma_{k+2}$ is $k+1+\binom{k+2}{2}$.*

Proof. Clearly for $k=1$, every $\sigma \in S_3(132)$ satisfies our condition so that there are $5 = 2 + \binom{3}{2}$ such permutations. Thus the lemma holds for $k=1$. Now assume that our lemma holds for k. Then we can classify the $\sigma = \sigma_1 \dots \sigma_{k+3} \in S_{k+3}(132)$ such that $\sigma_3 < \cdots < \sigma_{k+3}$ by the position of $k+3$. Clearly, we have only three cases to consider.

Case 1. $\sigma_1 = k+3$. In this case, $\sigma_2 \ldots \sigma_{k+3}$ is a permutation in $S_{k+2}(132)$ such that $\sigma_3 < \cdots < \sigma_{k+3}$. There are $k+2$ such permutations since in such a situation, σ_2 can be any element from $\{1, \ldots, k+2\}$.

Case 2. $\sigma_2 = k+3$. In this case $\sigma_1 = k+2$ and $\sigma_3 \ldots \sigma_{k+3} = 12 \ldots (k+1)$. Thus we have one choice in this case.

Case 3. $\sigma_{k+3} = k+3$. In this case $\sigma_1 \ldots \sigma_{k+2}$ is a permutation in $S_{k+2}(132)$ such that $\sigma_3 < \cdots < \sigma_{k+2}$. Thus by induction, we have $k+1+\binom{k+2}{2}$ choices for σ in this case.

It follows that we have $k+2+1+k+1+\binom{k+2}{2} = (k+2)+\binom{k+3}{2}$ choices for such σ. $\square$

Theorem 21.

(i) The highest power of x that occurs in $Q_{n,132}^{(k,1,0,2)}(x)$ is x^{n-k-3} and

$$Q_{n,132}^{(k,1,0,2)}(x)|_{x^{n-k-3}} = (k+1)\left(k+1+\binom{k+2}{2}\right)C_{n-k-3}.$$

(ii) The highest power of x that occurs in $Q_{n,132}^{(k,2,0,2)}(x)$ is x^{n-k-4} and

$$Q_{n,132}^{(k,2,0,2)}(x)|_{x^{n-k-4}} = \left(k+1+\binom{k+2}{2}\right)^2 C_{n-k-4}.$$

Proof. For (i), we see that the maximum number of matches of $\text{MMP}(k,1,0,2)$ occurs in permutations $\sigma \in S_n(132)$ of the form $\alpha_1\beta\gamma$ where $\alpha = \alpha_1\alpha_2 \ldots \alpha_{k+1}$ is a rearrangement of $\{n-k, \ldots, n\}$ such that $\text{red}(\alpha) \in S_{k+1}(132)$ and $\alpha_2 < \cdots < \alpha_{k+1}$, β is a rearrangement of $\{3, \ldots, n-k-1\}$ such that $\text{red}(\beta) \in S_{n-k-3}$ and γ is a shuffle of $\alpha_2 \ldots \alpha_{k+1}$ with either 12 or 21 that avoids 132. First, we claim that the number of choices for γ is $k+1+\binom{k+2}{2}$. That is, if a shuffle of $\alpha_2 \ldots \alpha_{k+1}$ and 12 is to avoid 132, then we cannot have one of the α_is lie between 1 and 2. Thus 1 and 2 must form a consecutive sequence in any such shuffle so that we have $k+1$ choices in this case since we can either place 12 in front of $\alpha_2 \ldots \alpha_{k+1}$ or immediately after any α_i for $i = 2, \ldots, k+1$. On the other hand, any shuffle of $\alpha_2 \ldots \alpha_{k+1}$ and 21 will avoid 132 so that we have $\binom{k+2}{2}$ choices in this case. But clearly we have $k+1$ choices for α since we have $k+1$ choices for α_1. Thus the highest power of x that can occur in $Q_{n,132}^{(k,1,0,2)}(x)$ is x^{n-k-3} and $Q_{n,132}^{(k,1,0,2)}(x)|_{x^{n-k-3}} = (k+1)\left(k+1+\binom{k+2}{2}\right)C_{n-k-3}$.

Similarly, for (ii), it is not difficult to see that the maximum number of matches of $\text{MMP}(k,2,0,2)$ occurs in permutations $\sigma \in S_n(132)$ of the form $\alpha_1\alpha_2\beta\gamma$ where $\alpha = \alpha_1\alpha_2 \ldots \alpha_{k+2}$ is a rearrangement of $\{n-k-1, \ldots, n\}$ such that $\text{red}(\alpha) \in$

$S_{k+2}(132)$ and $\alpha_3 < \cdots < \alpha_{k+2}$, β is a rearrangement of $\{3,\ldots,n-k-2\}$ such that $\text{red}(\beta) \in S_{n-k-4}$ and γ is a shuffle of $\alpha_3 \ldots \alpha_{k+2}$ with either 12 or 21 that avoids 132. We have shown that the number of choices for γ is $k+1+\binom{k+2}{2}$. The number of choices for β is C_{n-k-4}. Thus to complete the proof, we only need to use Lemma 2. □

Problem 4. *In general, one can use similar arguments to the ones in Theorem 21 to show that for $n \geq \ell+k+m+1$, the highest power of x in $Q_{n,132}^{(\ell,k,0,m)}(x)$ is $x^{n-k-\ell-m}$ which appears with a coefficient of $a_{\ell,k,m}C_{n-\ell-k-m}$ for some constant $a_{\ell,k,m}$. Find a formula for $a_{\ell,k,m}$.*

5. $Q_{n,132}^{(a,b,c,d)}(x)$ where $a,b,c,d \geq 1$

Suppose that $a,b,c,d \geq 1$ and $n \geq b+d$. It is clear that n cannot match the pattern $\text{MMP}(a,b,c,d)$ for $a,b,c,d \geq 1$ in any $\sigma \in S_n(132)$. If $\sigma = \sigma_1 \ldots \sigma_n \in S_n(132)$ and $\sigma_i = n$, then we have three cases, depending on the value of i.

Case 1. $i < b$. It is easy to see that as we sum over all the permutations σ in $S_n^{(i)}(132)$, our choices for the structure for $A_i(\sigma)$ will contribute a factor of C_{i-1} to $Q_{n,132}^{(a,b,c,d)}(x)$ since the elements in $A_i(\sigma)$ do not have enough elements to the left to match $\text{MMP}(a,b,c,d)$ in σ. Similarly, our choices for the structure for $B_i(\sigma)$ will contribute a factor of $Q_{n-i,132}^{(a,b-i,c,d)}(x)$ to $Q_{n,132}^{(a,b,c,d)}(x)$ since $\sigma_1 \ldots \sigma_i$ will automatically be in the second quadrant relative to the coordinate system with the origin at (s,σ_s) for any $s > i$. Thus, the permutations in Case 1 will contribute

$$\sum_{i=1}^{b-1} C_{i-1} Q_{n-i,132}^{(a,b-i,c,d)}(x)$$

to $Q_{n,132}^{(a,b,c,d)}(x)$.

Case 2. $b \leq i \leq n-d$. It is easy to see that as we sum over all the permutations σ in $S_n^{(i)}(132)$, our choices for the structure for $A_i(\sigma)$ will contribute a factor of $Q_{i-1,132}^{(a-1,b,c,0)}(x)$ to $Q_{n,132}^{(a,b,c,d)}(x)$ since the elements in $B_i(\sigma)$ will all be in the fourth quadrant and $\sigma_i = n$ is in the first quadrant relative to a coordinate system centered at (r,σ_r) for $r < i$ in this case. Similarly, our choices for the structure for $B_i(\sigma)$ will contribute a factor of $Q_{n-i,132}^{(a,0,c,d)}(x)$ to $Q_{n,132}^{(a,b,c,d)}(x)$ since $\sigma_1 \ldots \sigma_i$ will automatically be in the second quadrant relative to the coordinate system with the origin at (s,σ_s) for any $s > i$. Thus, the permutations in Case 2 will contribute

$$\sum_{i=b}^{n-d} Q_{i-1,132}^{(a-1,b,c,0)}(x) Q_{n-i,132}^{(a,0,c,d)}(x)$$

to $Q_{n,132}^{(a,b,c,d)}(x)$.

Case 3. $i \geq n-d+1$. It is easy to see that as we sum over all the permutations σ in $S_n^{(i)}(132)$, our choices for the structure for $A_i(\sigma)$ will contribute a factor of $Q_{i-1,132}^{(a-1,b,c,d-(n-i))}(x)$ to $Q_{n,132}^{(a,b,c,d)}(x)$, since $\sigma_i = n$ will be in the first quadrant and the elements in $B_i(\sigma)$ will all be in the fourth quadrant relative to a coordinate system centered at (r, σ_r) for $r < i$ in this case. Similarly, our choices for the structure for $B_i(\sigma)$ will contribute a factor of C_{n-i} to $Q_{n,132}^{(a,b,c,d)}(x)$ since σ_j, where $j > i$, does not have enough elements to its right to match MMP(a,b,c,d) in σ. Thus, the permutations in Case 3 will contribute

$$\sum_{i=n-d+1}^{n} Q_{i-1,132}^{(a-1,b,c,d-(n-i))}(x) C_{n-i}$$

to $Q_{n,132}^{(a,b,c,d)}(x)$. Thus, we have the following. For $n \geq b+d$,

$$Q_{n,132}^{(a,b,c,d)}(x) = \sum_{i=1}^{b-1} C_{i-1} Q_{n-i,132}^{(a,b-i,c,d)}(x) + \sum_{i=b}^{n-d} Q_{i-1,132}^{(a-1,b,c,0)}(x) Q_{n-i,132}^{(a,0,c.d)}(x) + \sum_{i=n-d+1}^{n} Q_{i-1,132}^{(a-1,b,c,d-(n-i))}(x) C_{n-i}. \tag{20}$$

Multiplying (20) by t^n and summing, we obtain the following theorem.

Theorem 22. *For all $a, b, c, d \geq 1$,*

$$\begin{aligned} Q_{132}^{(a,b,c,d)}(t,x) = & \sum_{p=0}^{b+d-1} C_p t^p + \\ & t \sum_{i=0}^{b-2} C_i t^i \left(Q_{132}^{(a,b-1-i,c,d)}(t,x) - \sum_{r=0}^{b-i+d-2} C_r t^r \right) + \\ & t \left(Q_{132}^{(a-1,b,c,0)}(t,x) - \sum_{i=0}^{b-2} C_i t^i \right) \left(Q_{132}^{(a,0,c,d)}(t,x) - \sum_{j=0}^{d-1} C_j t^j \right) + \\ & t \sum_{j=0}^{d-1} C_j t^j \left(Q_{132}^{(a-1,b,c,d-j)}(t,x) - \sum_{s=0}^{b+d-j-2} C_s t^s \right). \end{aligned} \tag{21}$$

Thus, for example,

$$\begin{aligned} Q_{132}^{(1,1,1,1)}(t,x) = & 1 + t + t Q_{132}^{(0,1,1,0)}(t,x) \left(Q_{132}^{(1,0,1,1)}(t,x) - 1 \right) + \\ & t(Q_{132}^{(0,1,1,1)}(t,x) - 1), \end{aligned}$$

and, for $k \geq 2$,

$$Q_{132}^{(k,1,1,1)}(t,x) = 1 + t + tQ_{132}^{(k-1,1,1,0)}(t,x)\left(Q_{132}^{(k,0,1,1)}(t,x) - 1\right) + t(Q_{132}^{(k-1,1,1,1)}(t,x) - 1).$$

Again the formulas for $Q_{132}^{(k,1,1,1)}(t,x)$ are quite complex. For example,

$$Q_{132}^{(1,1,1,1)}(t,x) = 1 + t + \frac{8t^2x^3}{\left(-1+t+2x-tx+\sqrt{(1+t(-1+x))^2-4tx}\right)^3} + t\left(-1 + \frac{1}{1-t} - \frac{2tx\left(1-t+tx-\sqrt{(1+t(-1+x))^2-4tx}\right)}{(-1+t)\left(-1+t+2x-tx+\sqrt{(1+t(-1+x))^2-4tx}\right)^2}\right).$$

Nevertheless, we can use the recursion above to compute the following.

$$Q_{132}^{(1,1,1,1)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + (38+4x)t^5 + \left(99+29x+4x^2\right)t^6 + \left(249+135x+41x^2+4x^3\right)t^7 + \left(609+510x+250x^2+57x^3+4x^4\right)t^8 + \left(1457+1701x+1177x^2+446x^3+77x^4+4x^5\right)t^9 + \left(3425+5220x+4723x^2+2564x^3+759x^4+101x^5+4x^6\right)t^{10} + \cdots.$$

$$Q_{132}^{(2,1,1,1)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + (123+9x)t^6 + \left(350+70x+9x^2\right)t^7 + \left(974+350x+97x^2+9x^3\right)t^8 + \left(2667+1433x+620x^2+133x^3+9x^4\right)t^9 + \left(7218+5235x+3079x^2+1077x^3+178x^4+9x^5\right)t^{10} + \cdots.$$

$$Q_{132}^{(3,1,1,1)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + 132t^6 + (413+16x)t^7 + \left(1277+137x+16x^2\right)t^8 + \left(3909+752x+185x^2+16x^3\right)t^9 + \left(11881+3383x+1267x^2+249x^3+16x^4\right)t^{10} + \cdots.$$

It is easy to explain the coefficient of the highest power term that appears in $Q_{n,132}^{(k,1,1,1)}(x)$ for $k \geq 1$. That is, the maximum number of matches of MMP$(k,1,1,1)$ for $\sigma \in S_n(132)$ is when σ is of the form $x\ \alpha\ \beta$ where $x \in \{n-k,\ldots,n\}$, β is a shuffle of 1 with the sequence $(n-k)(n-k+1)\ldots n$ with x removed, and $\alpha = 23\ldots(n-k-1)$. Note that we have $k+1$ choices for x and, once we have chosen x, we have $k+1$ choices for β. Thus, the highest power of x that occurs in $Q_{n,132}^{(k,1,1,1)}(x)$ is x^{n-k-3} which occurs with a coefficient of $(k+1)^2$ for $n \geq k+4$.

We also have

$$\begin{aligned} Q_{132}^{(0,1,1,0)}(t,0) =& \frac{1-t}{1-2t}, \\ Q_{132}^{(1,0,1,1)}(t,0) =& 1+t\left(\frac{1-t}{1-2t}\right)^2, \text{ and} \\ Q_{132}^{(0,1,1,1)}(t,0) =& \frac{1-4t+5t^2-t^3}{(1-2t)^2(1-t)}, \end{aligned}$$

to compute that

$$Q_{132}^{(1,1,1,1)}(t,0) = \frac{1-6t+13t^2-11t^3+3t^4-2t^5+t^6}{(1-t)(1-2t)^3}.$$

Note that $Q_{132}^{(1,1,1,1)}(t,0)$ is the generating function of the permutations that avoid the patterns from the set $\{132, 52314, 52341, 42315, 42351\}$.

Finally, we can also determine the second highest coefficient of x in $Q_{n,132}^{(1,1,1,1)}(x)$.

Theorem 23. *For all $n \geq 6$,*

$$Q_{n,132}^{(1,1,1,1)}(x)|_{x^{n-5}} = 17 + 4\binom{n-3}{2}.$$

Proof. The recursion of $Q_{n,132}^{(1,1,1,1)}(x)$ is

$$Q_{n,132}^{(1,1,1,1)}(x) = Q_{n-1,132}^{(0,1,1,1)}(x) + \sum_{i=1}^{n-1} Q_{i-1,132}^{(0,1,1,0)}(x) Q_{n-i,132}^{(1,0,1,1)}(x).$$

For $n \geq 3$, the highest power of x which occurs in $Q_{n,132}^{(0,1,1,0)}(x)$ is x^{n-2} and for $n \geq 4$, the highest power of x that occurs in $Q_{n,132}^{(1,0,1,1)}(x)$ is x^{n-3}. It follows that for $i = 3, \ldots, n-3$, the highest power of x that occurs in $Q_{i-1,132}^{(0,1,1,0)}(x) Q_{n-i,132}^{(1,0,1,1)}(x)$ is x^{n-6}. It follows that

$$\begin{aligned} Q_{n,132}^{(1,1,1,1)}(x)|_{x^{n-5}} =& Q_{n-1,132}^{(1,0,1,1)}(x)|_{x^{n-5}} + Q_{n-2,132}^{(1,0,1,1)}(x)|_{x^{n-5}} + 2Q_{n-3,132}^{(0,1,1,0)}(x)|_{x^{n-5}} + \\ & Q_{n-2,132}^{(0,1,1,0)}(x)|_{x^{n-5}} + Q_{n-1,132}^{(0,1,1,1)}(x)|_{x^{n-5}}. \end{aligned}$$

But for $n \geq 6$, we have proved that

$$\begin{aligned}
Q_{n-1,132}^{(1,0,1,1)}(x)|_{x^{n-5}} =& 6 + 2\binom{n-3}{2} \text{ (by Theorem 8)},\\
Q_{n-2,132}^{(1,0,1,1)}(x)|_{x^{n-5}} =& 2 \text{ (by Theorem 8)},\\
2Q_{n-3,132}^{(0,1,1,0)}(x)|_{x^{n-5}} =& 2C_1 = 2 \text{ (by [10, Theorem 15])},\\
Q_{n-2,132}^{(0,1,1,0)}(x)|_{x^{n-5}} =& 2 + \binom{n-3}{2} \text{ (by [10, Theorem 15]), and}\\
Q_{n-1,132}^{(0,1,1,1)}(x)|_{x^{n-5}} =& 5 + \binom{n-3}{2} \text{ (by Theorem 13).}
\end{aligned}$$

Thus, $Q_{n,132}^{(1,1,1,1)}(x)|_{x^{n-5}} = 17 + 4\binom{n-3}{2}$. □

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