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I. SUPPLEMENTARY NOTES

A. Supplementary Note 1 - Proof of Theorem 1

We will use a variant of Lemma 20 of [1].

Lemma 1. Consider a Hermitian matrix $L_{AB} \in \mathbb{B}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$, with $d_A \leq d_B$. Then

$$\|L_{AB}\|_1 \leq d_A^2 \max_{\mathcal{M}_B} \|\text{id}_A \otimes \mathcal{M}_B (L_{AB})\|_1,$$

where the maximum is taken over local measurement maps $\mathcal{M}_B(Y) = \sum_i \text{tr}(N_i Y)|i\rangle\langle i|$, with a POVM $\{N_i\}$ and orthonormal states $\{|i\rangle\}$. 

Proof. Write $L_{AB} = \sum_{i,j=1}^{d_A} |i\rangle\langle j| \otimes L_{ij}$ with $\{|i\rangle\}$ an orthonormal basis for $\mathbb{C}^{d_A}$. On one hand, thanks to the triangle inequality, we have

$$\|L_{AB}\|_1 = \left\| \sum_{i,j=1}^{d_A} |i\rangle\langle j| \otimes L_{ij} \right\|_1 \leq \sum_{i,j=1}^{d_A} \|L_{ij}\|_1,$$  

(1)

On the other hand,

$$\max_{\mathcal{M}_B} \|\text{id}_A \otimes \mathcal{M}_B (L_{AB})\|_1 = \max_{\mathcal{M}_B} \left\| \sum_{i,j=1}^{d_A} |i\rangle\langle j| \otimes \mathcal{M}_B(L_{ij}) \right\|_1$$

$$= \max_{\mathcal{M}_B} \left\| \sum_{i,j=1}^{d_A} |i\rangle\langle j| \otimes \mathcal{M}_B(L_{ij}) \right\|_1 \leq \max_{\mathcal{M}_B} \left\| \sum_{i,j=1}^{d_A} \text{tr}(K_{AB} \left( \sum_{i,j=1}^{d_A} |i\rangle\langle j| \otimes \mathcal{M}_B(L_{ij}) \right) ) \right\|_1$$

$$\geq \max_{\mathcal{M}_B} \left\| \sum_{i,j=1}^{d_A} |i\rangle\langle j| \otimes \mathcal{M}_B(L_{ij}) \right\|_1 \text{tr}(K_A \otimes K_B \left( \sum_{i,j=1}^{d_A} |i\rangle\langle j| \otimes \mathcal{M}_B(L_{ij}) \right) ) \right\|_1$$

$$\geq \max_{\mathcal{M}_B} \left\{ \max_{i,j} \|\mathcal{M}_B(L_{ii})\|_1, \max_{i,j} \|\mathcal{M}_B(L_{ij} + L_{ji})\|_1, \max_{i,j} \|\mathcal{M}_B(i(L_{ij} - L_{ji}))\|_1 \right\},$$

(2)

where we have repeatedly used the expression of the trace norm $\|X\|_1 = \max_{\|K\|_2 \leq 1} |\text{tr}(K X)|$, and the alternative choices $K_A = |i\rangle\langle i|$, $K_A = |i\rangle\langle j| + |j\rangle\langle i|$, or $K_A = i(|i\rangle\langle j| - |j\rangle\langle i|)$ to arrive to the last inequality.

It is clear that

$$\max_{\mathcal{M}_B} \|\mathcal{M}_B(L_{ii})\|_1 = \|L_{ii}\|_1$$

(3)

and similarly

$$\max_{\mathcal{M}_B} \|\mathcal{M}_B(L_{ij} + L_{ji})\|_1 = \|L_{ij} + L_{ji}\|_1,$$

$$\max_{\mathcal{M}_B} \|\mathcal{M}_B(i(L_{ij} - L_{ji}))\|_1 = \|L_{ij} - L_{ji}\|_1.$$  

(4)

To complete the proof it is enough to observe

$$\|L_{ij}\|_1 \leq \frac{1}{2} (\|L_{ij} + L_{ji}\|_1 + \|L_{ij} - L_{ji}\|_1) \leq \max\{\|L_{ij} + L_{ji}\|_1, \|L_{ij} - L_{ji}\|_1\}.$$  

(5)
A second lemma bounds the optimal distinguishability of two quantum channels (i.e. their diamond-norm distance) in terms of the distinguishability of their corresponding Choi-Jamiołkowski states.

Lemma 2. Let $\Phi_{AA'} = \frac{1}{d_A} \sum_{k,k'} |k,k\rangle\langle k',k'|$ be a $d_A$-dimensional maximally entangled state. For any cptp map $\Lambda: D(A) \to D(B)$ we define the Choi-Jamiołkowski state of $\Lambda$ as $J(\Lambda) := id_A \otimes \Lambda(\Phi_{AA'})$. For two cptp maps $\Lambda_0$ and $\Lambda_1$ it then holds

$$\frac{1}{d_A} \|\Lambda_0 - \Lambda_1\|_\diamond \leq \|J(\Lambda_0) - J(\Lambda_1)\|_1 \leq \|\Lambda_0 - \Lambda_1\|_\diamond. \quad (6)$$

Proof. The second inequality in (6) is trivial, as the diamond norm between two cptp maps is defined through a maximization over input states, while $\|J(\Lambda_0) - J(\Lambda_1)\|_1$ corresponds to the bias in distinguishing the two operations $\Lambda_0$ and $\Lambda_1$ by using the maximally entangled state $\Phi_{AA'}$ as input. The first inequality can be derived as follows.

Any pure state $|\psi\rangle_{AA'}$ can be obtained by means of a local filtering of the maximally entangled state, i.e.,

$$|\psi\rangle_{AA'} = (\sqrt{d_A} C \otimes id_B)|\Phi\rangle_{AA'}$$

for a suitable $C \in \mathbb{B}(C^{d_A})$, which, for a normalized $|\psi\rangle_{AA'}$ satisfies $tr(C^\dagger C) = 1$. From the latter condition, we have that $\|C\|_\infty \leq 1$. Let $|\psi\rangle_{AA'}$ be a normalized pure state optimal for the sake of the diamond norm between $\Lambda_0$ and $\Lambda_1$. We find

$$\|\Lambda_0 - \Lambda_1\|_\diamond = \|id_A \otimes (\Lambda_0 - \Lambda_1)|\psi\rangle\langle \psi|\|_1$$

$$= \|id_A \otimes (\Lambda_0 - \Lambda_1) (\sqrt{d_A} C \otimes id_B) \Phi_{AA'} (\sqrt{d_A} C \otimes id_B)^\dagger\|_1$$

$$= \|\sqrt{d_A} C \otimes id_B (id_A \otimes (\Lambda_0 - \Lambda_1)) \Phi_{AA'} (\sqrt{d_A} C \otimes id_B)^\dagger\|_1$$

$$\leq d_A \|C\|_\infty^2 \|id_A \otimes (\Lambda_0 - \Lambda_1) \Phi_{AA'}\|_1$$

$$\leq d_A \|J(\Lambda_0) - J(\Lambda_1)\|_1,$$

where we used (twice) Hölder’s inequality $\|MN\|_1 \leq \min\{\|M\|_\infty \|N\|_1, \|M\|_1 \|N\|_\infty\}$ in the first inequality, and $\|C\|_\infty \leq 1$ in the second inequality.

We are in position to prove the main theorem, which we restate for the convenience of the reader.

Theorem 1 (restatement). Let $\Lambda: D(A) \to D(B_1 \otimes \ldots \otimes B_n)$ be a cptp map. Define $\Lambda_j := tr_{B_j} \circ \Lambda$ as the effective dynamics from $D(A)$ to $D(B_j)$ and fix a number $1 > \delta > 0$. Then there exists a measurement $\{M_k\}_k$ ($M_k \geq 0$, $\sum_k M_k = I$) and a set $S \subseteq \{1, \ldots, n\}$ with $|S| \geq (1 - \delta)n$ such that for all $j \in S$,

$$\|\Lambda_j - E_j\|_\diamond \leq \left(\frac{27 \ln(2)(d_A)^9 \log(d_A)}{n \delta^3}\right)^{1/3},$$

(7)

with

$$E_j(X) := \sum_k tr(M_k X) \sigma_j,k,$$

(8)

for states $\sigma_j,k \in D(B_j)$. Here $d_A$ is the dimension of the space $A$. 
Proof. Let \( \Phi_{AA'} = \sum_{k,k'} |k,k\rangle \langle k',k'| \) be a \( d_A \)-dimensional maximally entangled state and 
\[ \rho_{AB_1,\ldots,B_n} := 1 \otimes \Lambda(\Phi_{AA'}) \] be the Choi-Jamiołkowski state of \( \Lambda \) [2]. Define \( \pi := 1 \otimes M_1 \otimes \cdots \otimes M_n(\rho) \) for quantum-classical channels \( M_1,\ldots,M_n \) defined as 
\[ M_i(X) := \sum_l \text{tr}(N_{i,l}X)|l\rangle \langle l|, \]
for a POVM \( \{N_{i,l}\}_l \).

We will proceed in two steps. In the first we show that conditioned on measuring a few of the \( B_i \)'s of \( \rho_{AB_1,\ldots,B_n} \), the conditional mutual information of \( A \) and \( B_i \) (on average over \( i \)) is small. In the second we show that this implies that the reduced state \( \rho_{AB_i} \) is close to a separable state 
\[ \sum_z p(z) \rho_{z,A} \otimes \rho_{B_i,z} \] with the ensemble \( \{p(z),\rho_{z,A}\} \) independent of \( i \). We will conclude showing that by the properties of the Choi-Jamiołkowski isomorphism, this implies that the effective channel from \( A \) to \( B_i \) is close to a measure-and-prepare channel with a POVM independent of \( i \).

Let \( \mu \) be the uniform distribution over \( [n] \) and define \( \mu^\land_k \) as the distribution on \( [n]^k \) obtained by sampling \( m \) times without replacement according to \( \mu \); i.e.
\[ \mu^\land_k(i_1,\ldots,i_k) = \begin{cases} 0 & \text{if } i_1,\ldots,i_k \text{ are not all distinct} \\ \frac{\mu(i_1)\cdots\mu(i_k)}{\sum_{j_1,\ldots,j_k \text{ distinct}} \mu(j_1)\cdots\mu(j_k)} & \text{otherwise} \end{cases} \]

Then
\[ \log d_A \geq \mathbb{E} \max_{\mu^\land_k M_{j_1},\ldots,M_{j_k}} I(A : B_{j_1},\ldots,B_{j_k})_\pi \] (10)
\[ = \mathbb{E} \max_{\mu^\land_k M_{j_1},\ldots,M_{j_k}} \left( I(A : B_{j_1})_\pi + \cdots + I(A : B_{j_k} | B_{j_1},\ldots,B_{j_{k-1}})_\pi \right) \]
\[ =: f(k). \]

The inequality comes from the fact that \( \pi \) is separable between \( A \) and \( B_1B_2\ldots B_n \) because of the action of the quantum-classical channels \( M_1,\ldots,M_n \). The second line follows from the chain rule of mutual information given by Eq. (54) in Supplementary Methods.

Define \( J_k := \{j_1,\ldots,j_{k-1}\} \). We have
\[ f(k) \overset{(i)}{=} \mathbb{E} \max_{\mu^\land_k M_{j_1},\ldots,M_{j_{k-1}}} \left( I(A : B_{j_1})_\pi + \cdots + \max_{M_{j_k}} I(A : B_{j_k} | B_{j_1},\ldots,B_{j_{k-1}})_\pi \right) \] (11)
\[ \overset{(ii)}{\geq} \mathbb{E} \max_{\mu^\land_{k-1} M_{j_1},\ldots,M_{j_{k-1}}} \mathbb{E}_{j_k \notin J_k} \left( I(A : B_{j_1})_\pi + \cdots + \max_{M_{j_k}} I(A : B_{j_k} | B_{j_1},\ldots,B_{j_{k-1}})_\pi \right) \]
\[ \overset{(iii)}{=} \mathbb{E} \max_{\mu^\land_{k-1} M_{j_1},\ldots,M_{j_{k-1}}} \left( I(A : B_{j_1})_\pi + \cdots + \mathbb{E}_{j_k \notin J_k} \max_{M_{j_k}} I(A : B_{j_k} | B_{j_1},\ldots,B_{j_{k-1}})_\pi \right) \]
\[ \overset{(iv)}{\geq} \mathbb{E} \max_{\mu^\land_{k-1} M_{j_1},\ldots,M_{j_{k-1}}} \left( I(A : B_{j_1})_\pi + \cdots + I(A : B_{j_{k-1}} | B_{j_1},\ldots,B_{j_{k-2}})_\pi \right) \]
\[ + \mathbb{E} \min_{\mu^\land_{k-1} M_{j_1},\ldots,M_{j_{k-1}}} \mathbb{E}_{j_k \notin J_k} \max_{M_{j_k}} I(A : B_{j_k} | B_{j_1},\ldots,B_{j_{k-1}})_\pi, \]
\[ \overset{(v)}{=} f(k - 1) + \mathbb{E} \min_{\mu^\land_{k-1} M_{j_1},\ldots,M_{j_{k-1}}} \mathbb{E}_{j_k \notin J_k} \max_{M_{j_k}} I(A : B_{j_k} | B_{j_1},\ldots,B_{j_{k-1}})_\pi, \]

where (i) follows since only \( I(A : B_{j_k} | B_{j_1},\ldots,B_{j_{k-1}})_\pi \) depends on \( M_{j_k} \); (ii) by convexity of the maximum function; (iii) again because all the other terms in the sum are independent of \( j_k \); (iv) directly by inspection and linearity of expectation; and (v) by the definition of \( f(k) \) in Eq. (10).

From Eqs. (10) and (11), we obtain
\[ \log d_A \geq \sum_{q=1}^k \mathbb{E} \min_{\mu^\land_{q-1} M_{j_1},\ldots,M_{j_{q-1}}} \mathbb{E} \max_{M_{j_q}} I(A : B_{j_q} | B_{j_1},\ldots,B_{j_{q-1}})_\pi, \] (12)
and so there exists a \( q \leq k \) such that

\[
\mathbb{E} \min_{(j_1, \ldots, j_{q-1}) \sim \mathcal{M}^{q-1}} \mathbb{E} \max_{M_{j_1}, \ldots, M_{j_{q-1}}} I(A : B_j | B_{j_1}, \ldots, B_{j_{q-1}}) \pi \leq \frac{\log d_A}{k},
\]

where we relabelled \( j_q \to j \). Thus there exists a \((q-1)\)-tuple \( J := (j_1, \ldots, j_{q-1}) \) and measurements \( M_{j_1}, \ldots, M_{j_{q-1}} \) such that

\[
\mathbb{E} \max_{j \notin J} I(A : B_j | B_{j_1}, \ldots, B_{j_{q-1}}) \pi \leq \frac{\log d_A}{k}.
\]

Let \( \rho_{AB_j}^z \) be the post-measurement state on \( AB_j \) conditioned on obtaining \( z \) — a short-hand notation for the ordered collection of the local results — when measuring \( M_{j_1}, \ldots, M_{j_{q-1}} \) in the subsystems \( B_{j_1}, \ldots, B_{j_{q-1}} \) of \( \rho \). Note that \( \rho_A^z \) is independent of \( B_j \) (for \( j \notin J \)). By Pinsker’s inequality (55) in Supplementary Methods, convexity of \( x \mapsto x^2 \), and Eq. (56) in Supplementary Methods,

\[
\left\| \text{id}_A \otimes M_j \left( \rho_{AB_j} - \mathbb{E}_{z} \rho_A^z \otimes \rho_{B_j}^z \right) \right\|_1^2 = \left( \left\| \text{id}_A \otimes M_j \left( \mathbb{E}_{z} \rho_{AB_j}^z - \mathbb{E}_{z} \rho_A^z \otimes \rho_{B_j}^z \right) \right\|_1 \right)^2 \\
\leq \mathbb{E}_z \left\| \text{id}_A \otimes M_j \left( \rho_{AB_j}^z - \rho_A^z \otimes \rho_{B_j}^z \right) \right\|_1^2 \\
\leq 2 \ln(2) I(A : B_j | B_{j_1}, \ldots, B_{j_{q-1}}) \pi.
\]

By Eq. (14) and convexity of \( x \mapsto x^2 \),

\[
\mathbb{E} \max_{j \notin J} \left\| \text{id}_A \otimes M_j \left( \rho_{AB_j} - \mathbb{E}_{z} \rho_A^z \otimes \rho_{B_j}^z \right) \right\|_1 \leq \sqrt{2 \ln(2)} \frac{\log d_A}{k}.
\]

Now, by Lemma 1, we have.

\[
\left\| \rho_{AB_j} - \mathbb{E}_{z} \rho_A^z \otimes \rho_{B_j}^z \right\|_1 \leq (d_A)^2 \max_{M_j} \left\| \text{id}_A \otimes M_j \left( \rho_{AB_j} - \mathbb{E}_{z} \rho_A^z \otimes \rho_{B_j}^z \right) \right\|_1,
\]

and so

\[
\mathbb{E} \left\| \rho_{AB_j} - \mathbb{E}_{z} \rho_A^z \otimes \rho_{B_j}^z \right\|_1 \leq \sqrt{2 \ln(2)} \frac{(d_A)^4 \log d_A}{k}.
\]

Note that \( \mathbb{E}_{z} \rho_A^z \otimes \rho_{B_j}^z = \sum_z p(z) \rho_A^z \otimes \rho_{B_j}^z \) is the Choi-Jamiolkowski state of a measure-and-prepare channel \( \mathcal{E}_j \) [3], since \( \mathbb{E}_{z} \rho_A^z = \rho_A = 1/d_A \). It is explicitly given by

\[
\mathcal{E}_j(X) := d_A \mathbb{E}_{z} \text{tr}((\rho_A^z)^T X) \rho_{B_j}^z.
\]

Note that the POVM \( \{d_A p(z) \rho_A^z \} \) is independent of \( j \).

Thanks to Lemma 2, we can now bound the distance of two maps by the distance of their Choi-Jamiolkowski states

\[
\| \text{tr}_{B_j} \circ \Lambda - \mathcal{E}_j \|_\diamond \leq d_A \| \rho_{AB_j} - \mathbb{E}_{z} \rho_A^z \otimes \rho_{B_j}^z \|_1,
\]

to find

\[
\mathbb{E} \left\| \text{tr}_{B_j} \circ \Lambda - \mathcal{E}_j \right\|_\diamond \leq \sqrt{2 \ln(2)} \frac{(d_A)^6 \log d_A}{k}.
\]
Then
\[
\mathbb{E}_j \|\text{tr}_{\Lambda_j} \circ \Lambda - \mathcal{E}_j\|_\diamond \leq \mathbb{E}_j \|\text{tr}_{\Lambda_j} \circ \Lambda - \mathcal{E}_j\|_\diamond + \frac{k}{n} \mathbb{E}_j \|\text{tr}_{\Lambda_j} \circ \Lambda - \mathcal{E}_j\|_\diamond \leq \sqrt{\frac{2 \ln(2) (d_A)^6 \log(d_A)}{k}} + \frac{2k}{n},
\]
(22)
where we used that the diamond norm between two ctp maps is upper-bounded by 2.

Choosing \( k \) to minimize the latter bound we obtain\(^1\)
\[
\mathbb{E}_j \|\text{tr}_{\Lambda_j} \circ \Lambda - \mathcal{E}_j\|_\diamond \leq \left( \frac{27 \ln(2) (d_A)^6 \log(d_A)}{n} \right)^{1/3}.
\]
(23)

Finally applying Markov’s inequality,
\[
\Pr_i \left( \|\text{tr}_{\Lambda_j} \circ \Lambda - \mathcal{E}_j\|_\diamond \geq \frac{1}{\delta} \left( \frac{27 \ln(2) (d_A)^6 \log(d_A)}{n} \right)^{1/3} \right) \leq \delta.
\]
(24)
\[
\Box
\]

**B. Supplementary Note 2 - Proof of Theorem 2**

The proof of Theorem 2 follows along the same lines as Theorem 1:

**Theorem 2 (restatement).** Let \( \Lambda : \mathcal{D}(A) \rightarrow \mathcal{D}(B_1 \otimes \ldots \otimes B_n) \) be a ctp map. For any subset \( S_t \subseteq [n] \) of \( t \) elements, define \( \Lambda_{S_t} := \text{tr}_{\bigcup_{j \in S_t} B_j} \circ \Lambda \) as the effective channel from \( \mathcal{D}(A) \) to \( \mathcal{D}(\bigotimes_{l \in S_t} B_l) \). Then for every \( 1 > \delta > 0 \) there exists a measurement \( \{M_k\}_k \) (where \( M_k \geq 0 \), \( \sum_k M_k = I \) such that for more than a \((1 - \delta)\) fraction of the subsets \( S_t \subseteq [n] \),
\[
\|\Lambda_{S_t} - \mathcal{E}_{S_t}\|_\diamond \leq \left( \frac{27 \ln(2) (d_A)^6 \log(d_A) t}{n \delta^3} \right)^{1/3},
\]
(25)

with
\[
\mathcal{E}_{S_t}(X) := \sum_k \text{tr}(M_k X) \sigma_{S_t,k},
\]
(26)

for states \( \sigma_{S_t,k} \in \mathcal{D}(\bigotimes_{l \in S_t} B_l) \).

**Proof.** Since the proof is very similar to the proof of Theorem 1, we will only point out the differences.

Let \( \rho_{A_1,\ldots,A_n} := \text{id}_A \otimes \Lambda(\Phi) \) be the Choi-Jamiolkowski state of \( \Lambda \) and \( C = \{C_1, \ldots, C_{n/t}\} \) be a partition of \([n]\) into \( n/t \) sets of \( t \) elements each. Define \( \pi_C := \text{id}_A \otimes M_1 \otimes \ldots \otimes M_{n/t}(\rho) \), for quantum-classical channels \( M_1, \ldots, M_{n/t} \) defined as \( M_i(X) := \sum_l \text{tr}(N_{i,l} X) |l\rangle \langle l| \) for a POVM \( \{N_{i,l}\}_l \) with \( M_i \) acting on \( \bigcup_{j \in C_i} B_j \).

As in the proof of Theorem 1, by the chain rule,
\[
\log d_A \geq \max_{C_{j_1},\ldots,C_{j_k},M_{j_1},\ldots,M_{j_k}} I(A : B_{C_{j_1}}, \ldots, B_{C_{j_k}}|\pi_C)
\]
\[
= \max_{C_{j_1},\ldots,C_{j_k},M_{j_1},\ldots,M_{j_k}} \left( I(A : B_{C_{j_1}}|\pi_C) + \ldots + I(A : B_{C_{j_k}}|B_{C_{j_1}}, \ldots, B_{C_{j_{k-1}}}|\pi_C) \right) =: f(t),
\]
(27)

\(^1\) The expression \( a/\sqrt{k} + bk \) is minimal for \( k = (\frac{a}{2b})^{2/3} \). We further use that for \( b = 2/n < 1 \) it holds \( b^{1/3} \geq b^{5/6} \).
where the expectation is taken uniformly over the choice of non-overlapping sets \(C_{j_1}, \ldots, C_{j_k} \in [n]^t\). We have

\[
f(t) = \mathbb{E}_{C_{j_1}, \ldots, C_{j_k}} \max_{M_{j_1}, \ldots, M_{j_k-1}} \left( I(A : B_{C_{j_1}}) \pi_C + \ldots + \max_{M_{j_k}} I(A : B_{C_{j_k}} | B_{C_{j_1}}, \ldots, B_{C_{j_{k-1}}}) \pi_C \right)
\]

(28)

\[
\geq \mathbb{E}_{C_{j_1}, \ldots, C_{j_k-1}} \max_{M_{j_1}, \ldots, M_{j_k-1}} \mathbb{E}_{C_{j_k}} \left( I(A : B_{C_{j_1}}) \pi_C + \ldots + \max_{M_{j_k}} I(A : B_{C_{j_k}} | B_{C_{j_1}}, \ldots, B_{C_{j_{k-1}}}) \pi_C \right)
\]

(29)

\[
\geq \mathbb{E}_{C_{j_1}, \ldots, C_{j_k-1}} \max_{M_{j_1}, \ldots, M_{j_k-1}} \left( I(A : B_{C_{j_1}}) \pi_C + \ldots + \max_{C_{j_k}} I(A : B_{C_{j_k}} | B_{C_{j_1}}, \ldots, B_{C_{j_{k-1}}}) \pi_C \right)
\]

(30)

\[
\geq \mathbb{E}_{C_{j_1}, \ldots, C_{j_k-1}} \max_{M_{j_1}, \ldots, M_{j_k-1}} \left( I(A : B_{C_{j_1}}) \pi_C + \ldots + I(A : B_{C_{j_{k-1}}} | B_{C_{j_1}}, \ldots, B_{C_{j_{k-2}}}) \pi_C \right)
\]

(31)

From Eqs. (27) and (28), we obtain

\[
\log d_A \geq \sum_{q=1}^{k} \mathbb{E}_{C_{j_1}, \ldots, C_{j_{q-1}}} \min_{M_{j_1}, \ldots, M_{j_{q-1}}} \mathbb{E}_{C_{j_q}} \max_{M_{j_q}} I(A : B_{C_{j_q}} | B_{C_{j_1}}, \ldots, B_{C_{j_{q-1}}}) \pi \geq \frac{\log d_A}{t},
\]

(32)

and so there exists a \(q \leq k\) such that

\[
\mathbb{E}_{C_{j_1}, \ldots, C_{j_{q-1}}} \min_{M_{j_1}, \ldots, M_{j_{q-1}}} \mathbb{E}_{C_{j_q}} \max_{M_{j_q}} I(A : B_{C_{j_q}} | B_{C_{j_1}}, \ldots, B_{C_{j_{q-1}}}) \pi \leq \frac{\log d_A}{t},
\]

(33)

We can follow the proof of Theorem 1 without any modifications to obtain that

\[
\mathbb{E}_{C_{j} \notin C} \left\| \text{tr}_{C_{j}} \circ \Lambda - \mathcal{E}_{C_{j}} \right\|_{\diamond} \leq \sqrt{2 \ln(2)} \frac{(d_A)^6 \log d_A}{k},
\]

(34)

Then

\[
\mathbb{E}_{C_{j}} \left\| \text{tr}_{C_{j}} \circ \Lambda - \mathcal{E}_{C_{j}} \right\|_{\diamond} \leq \sqrt{2 \ln(2)} \frac{(d_A)^6 \log d_A}{k} + \frac{2kt}{n}.
\]

Choosing \(k\) to minimize the right-hand side as done in the proof of Theorem 1 and applying Markov’s inequality, we obtain the result. 

\[\Box\]

C. Supplementary Note 3 - Proof of Proposition 3

We will make use the following well-known lemma:
Lemma 3. (Gentle Measurement [4]) Let $\rho$ be a density matrix and $N$ an operator such that $0 \leq N \leq I$ and $\text{tr}(N\rho) \geq 1 - \delta$. Then
\[
\|\rho - \sqrt{N}\rho\sqrt{N}\|_1 \leq 2\sqrt{\delta}.
\] (34)

Proposition 3 (restatement). Let $\mathcal{E}$ be the channel given by Eq. (26). Suppose that for every $i = \{1, \ldots, t\}$ and $1 > \delta > 0$,
\[
\min_{\rho \in \mathcal{D}(A)} p_{\text{guess}}(\{\text{tr}(M_k\rho), \sigma_{B_i,k}\}) \geq 1 - \delta.
\] (35)

Then there exists POVMs $\{N_{B_1,k}\}, \ldots, \{N_{B_2,k}\}$ such that
\[
\min_{\rho} \sum_k \text{tr}(M_k\rho) \text{tr} \left( \bigotimes_i N_{B_i,k} \sigma_{B_i \ldots B_i,k} \right) \geq 1 - 6t\delta^{1/4}.
\] (36)

Proof. For simplicity we will prove the claim for $t = 2$. The general case follows by a similar argument.

Since for $j = \{1, 2\}$, $\min_{\rho \in \mathcal{D}(A)} p_{\text{guess}}(\{\text{tr}(M_k\rho), \sigma_{B_i,k}\}) \geq 1 - \delta$, by the minimax theorem [5] it follows that there exists POVMs $\{N_{B_1,k}\}, \{N_{B_2,k}\}$ on $B_1$ and $B_2$, respectively, such that for $j \in \{1, 2\}$ and all $\rho \in \mathcal{D}(A)$,
\[
\sum_k \text{tr}(M_k\rho) \text{tr}(N_{B_j,k}\sigma_{B_j,k}) \geq 1 - \delta.
\] (37)

Fix $\rho$ and let $X_j := \{k : \text{tr}(N_{B_j,k}\sigma_{B_j,k}) \leq 1 - \sqrt{\delta}\}$ for $j = \{1, 2\}$. Then from Eq. (37),
\[
\sum_{k \in X_j} \text{tr}(\rho M_k) \leq \sqrt{\delta}.
\] (38)

Let $G = X_1^c \cap X_2^c$, with $X_j^c$ the complement of $X_j$. Then
\[
\sum_{k \in G} \text{tr}(M_k\rho) \text{tr} ((N_{B_1,k} \otimes N_{B_2,k}) \sigma_{B_1B_2,k}) \leq 1 - 2\sqrt{\delta} - 4\delta^{1/4}
\] (39)

where in the third line we used Lemma 3. In more detail, we have
\[
\text{tr} ((N_{B_1,k} \otimes N_{B_2,k}) \sigma_{B_1B_2,k}) = \text{tr}(N_{B_1,k}\sigma_{B_1,k}) \text{tr}(N_{B_2,k}\sigma'_{B_2,k}),
\] (40)

with $\sigma'_{B_2,k} := \text{tr}_{B_1}(N_{B_1,k}\sigma_{B_1,B_2,k})/\text{tr}(N_{B_1,k}\sigma_{B_1,k})$. Since $\text{tr}(N_{B_1,k}\sigma_{B_1,k}) \geq 1 - \delta^{1/2}$, Lemma 3 gives $\|\sigma'_{B_2,k} - \sigma_{B_2,k}\|_1 \leq 4\delta^{1/4}$. Then from Eq. (40),
\[
\text{tr} ((N_{B_1,k} \otimes N_{B_2,k}) \sigma_{B_1B_2,k}) \geq \text{tr}(N_{B_1,k}\sigma_{B_1,k}) \text{tr}(N_{B_2,k} \sigma_{B_2,k}) - 4\delta^{1/4}.
\] (41)

□
D. Supplementary Note 4 - Proof of Corollary 4

Corollary 4 will follow from Theorem 1 and the following well-known continuity relation for mutual information:

**Lemma 4.** (Alicki-Fannes Inequality [6]) For $\rho_{AB}$,

$$|H(A|B)_\rho - H(A|B)_{\sigma}| \leq 4\|\rho - \sigma\|_1 \log d_A + 2h_2(\|\rho - \sigma\|_1),$$

with $H(A|B) = S(AB) - S(B)$ and $h_2$ the binary entropy function.

If $S(A)_\rho = S(A)_\sigma$, then

$$|I(A : B)_\rho - I(A : B)_\sigma| \leq 4\|\rho - \sigma\|_1 \log d_A + 2h_2(\|\rho - \sigma\|_1).$$

**Corollary 4 (restatement).** Let $\Lambda : \mathcal{D}(B) \to \mathcal{D}(B_1 \otimes \ldots \otimes B_n)$ be a ctp map. Define $\Lambda_j := \text{tr}_{B_j} \circ \Lambda$ as the effective dynamics from $\mathcal{D}(B)$ to $\mathcal{D}(B_j)$. Then for every $1 > \delta > 0$ there exists a set $S \subseteq [n]$ with $|S| \geq (1 - \delta)n$ such that for all $j \in S$ and all states $\rho_{AB}$ it holds

$$I(A : B_j)_{\text{id}_A \otimes A_j, B(\rho_{AB})} \leq \max_{\Gamma_{QC} \in \mathcal{QC}} I(A : B)_{\text{id}_A \otimes \Gamma_{QC}^{\text{tr}}(\rho_{AB})} + 4\epsilon \log d_A + 2h_2(\epsilon),$$

where $\epsilon = \left( \frac{27 \ln(2)(d_B^6 \log(d_B))}{n^3} \right)^{1/3}$, $h_2(x) = -x \log x - (1 - x) \log(1 - x)$, and the maximum on the right-hand side is over quantum-classical channels $\Gamma_{QC}(X) = \sum_l \text{tr}(N_l X) ||l||$, with $\{N_l\}_l$ a POVM and $\{||l||\}_l$ a set of orthogonal states. As a consequence, for every $\rho_{AB}$,

$$\lim_{n \to \infty} \left( \max_{\Lambda_{\text{id}_A \otimes A_j, B(\rho_{AB})}} \mathbb{E} I(A : B_j)_{\text{id}_A \otimes A_j, B(\rho_{AB})} \right) = \max_{\Gamma_{QC} \in \mathcal{QC}} I(A : B)_{\text{id}_A \otimes \Gamma_{QC}^{\text{tr}}(\rho_{AB})},$$

with $\mathbb{E}_j X_j = \frac{1}{n} \sum_{i=1}^{n} X_j$, and the maximum on the left-hand side taken over any quantum operation $\Lambda : \mathcal{D}(B) \to \mathcal{D}(B_1 \otimes \ldots \otimes B_n)$.

**Proof.** By definition, for all ctp maps $\Lambda$ and $\mathcal{E}$ acting on $B$, and for any state $\rho_{AB}$, it holds

$$\|\text{id}_A \otimes A_j(\rho) - \text{id}_A \otimes \mathcal{E}_B(\rho)\|_1 \leq \|\Lambda - \mathcal{E}\|_{\text{tr}}.$$

Combining Theorem 1 and Lemma 4 (specifically, Eq. (43)), we have that for every $1 > \delta > 0$ there exist a measurement $\{M_k\}_k$ and a set $S \subseteq [n]$ with $|S| \geq (1 - \delta)n$ such that for all $j \in S$ and all states $\rho_{A'B}$ it holds

$$I(A : B)_{\text{id}_A \otimes A_j(\rho_{AB})} \leq I(A : B)_{\text{id}_A \otimes \mathcal{E}_j(\rho_{AB})} + 4\epsilon \log d_A + 2h_2(\epsilon),$$

with

$$\mathcal{E}_j(X) = \sum_k \text{tr}(M_k X) |k\rangle \langle k|$$

and

$$\epsilon = \left( \frac{27 \ln(2)(d_B^6 \log(d_B))}{n^3} \right)^{1/3}.$$

The claim is then a simple consequence of substituting $\mathcal{E}_j$ with an optimal quantum-classical channel.
We now turn to the proof of Eq. (45). That the left-hand side of Eq. (45) is larger than the right-hand side is trivial. Indeed one can pick \( \Lambda = \Lambda_{B \rightarrow B_1 B_2 \ldots B_n} \) as the quantum-classical map that uses the POVM \( \{ N_l \} \) that achieves the accessible information \( I(A : B_c) = \max_{\Gamma \in \text{QC}} I(A : B_{id} \otimes \Gamma(\rho_{AB})) \) with measurement on \( B \) and stores the result in \( n \) classical registers, one for each \( B_i \): \( \Gamma(X) = \sum_l \text{tr}(N_l X) |l\rangle \langle l| \otimes^n \). To prove that the left-hand side of Eq. (45) is smaller than the right-hand side it is sufficient to use Eq. (46) for the choice \( \delta = n - \frac{1-n}{\eta} \), for any \( 0 < \eta < 1 \). Then one obtains,

\[
\frac{1}{n} \sum_{i=1}^{n} I(A : B_i) \leq \frac{1}{n} \left\{ (1 - \delta)n \left[ I(A : B_c) + 4\epsilon \log d_A + 2h_2(\epsilon) \right] + \delta n 2 \log d_A \right\}
\]

\[
= (1 - \delta) \left[ I(A : B_c) + 4\epsilon \log d_A + 2h_2(\epsilon) \right] + \delta 2 \log d_A \xrightarrow{n \to \infty} I(A : B_c)
\]

where we have used that

\[
\epsilon = \left( \frac{27 \ln(2) \log(d_B)^6}{n\delta^3} \right)^{1/3} \xrightarrow{n \to \infty} 0
\]

for our choice of \( \delta \), independently of the choice of \( \Lambda = \Lambda_{B \rightarrow B_1 B_2 \ldots B_n} \).

II. SUPPLEMENTARY METHODS

We make use of the following properties of the mutual information:

- Positivity of conditional mutual information:
  \[ I(A : B|C) := I(A : BC) - I(A : C) \geq 0. \] (51)

  This is equivalent to strong subadditivity and to monotonicity of mutual information under local operations [7].

- For a general state \( \rho_{AB} \) it holds [7]
  \[ I(A : B)_{\rho_{AB}} \leq 2 \min \{ \log d_A, \log d_B \}, \] (52)

  with the more stringent bound
  \[ I(A : B)_{\sigma_{sep}^{AB}} \leq \min \{ \log d_A, \log d_B \} \] (53)

  for a separable state \( \sigma_{sep}^{AB} \) [8].

- Chain rule [7]:
  \[ I(A : B_1 B_2 \ldots B_n) = I(A : B_1) + I(A : B_2|B_1) + I(A : B_3|B_1 B_2) + \ldots + I(A : B_n|B_1 B_2 \ldots B_{n-1}). \] (54)

- Pinsker’s inequality (for mutual information):
  \[ \frac{1}{2 \ln 2} \| \rho_{AB} - \rho_A \otimes \rho_B \|_1^2 \leq I(A : B)_{\rho_{AB}}. \] (55)

- Conditioning on classical information
  \[ I(A : B|Z)_{\rho} = \sum_z p(z) I(A : B)_{\rho_z} \] (56)

  for a state \( \rho_{ABZ} = \sum_z p(z) \rho_{z,AB} \otimes |z\rangle \langle z|_Z \), with \( \{|z\} \) an orthonormal set.
III. SUPPLEMENTARY REFERENCES