Robustly exponential stabilization of hybrid uncertain systems by feedback controls based on discrete-time observations

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Abstract

This paper deals with the problem of stabilizing an hybrid stochastic system with norm bounded uncertainties. State-feedback controls based on discrete-time observations are designed in the drift and diffusion parts of the system. The controlled system will be robustly exponentially stable in mean-square. Applying linear matrix inequality techniques, criteria to determine controllers and time lags are developed. One numerical example is given to verify our techniques.

Key words: Brownian motion, hybrid uncertain systems, robust stabilization, feedback control, discrete-time observations.

1. Introduction

Some stochastic systems may experience abrupt changes in their structures and parameters because of environment changes, random failures of components, etc. Hybrid stochastic systems with continuous-time Markov chains have been used to model such systems. An important class of hybrid systems is the hybrid stochastic differential equation, or stochastic differential equation with Markovian switching. One of the important issues in the study of hybrid SDEs is stability analysis arising from automatic control. There are many papers in this area and we mention, for example, Ji and Chizeck (1990), Basak et al. (1996), Mao (1999), Mao et al. (2000), Mao (2002), Wang et al. (2002), Mao and Yuan (2006), Mao (2007). In particular, Mao (1999) and Mao (2002) are two of most cited papers while Mao and Yuan (2006) is the first book in this area. Applying different methods and techniques, such as Lyapunov functions and functionals, M-matrices and LMIs, various of criteria on stability of hybrid SDEs or SDDEs could be found in these references.

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In practice, when we estimate parameters such as matrices in linear parts, there exist uncertainties which lead to uncertain systems. There are various types of uncertainties under discussion, such as time varying structured uncertainty(see for example, Shamma (1994), Mao et al. (1998), Moon et al. (2001), Lu et al. (2003), Chen et al (2005), Yue and Han (2005), Huang and Mao (2009), Wang and Bai (2012), Kuang and Deng (2012), Hartung et al. (2013), Zhu et al. (2014a)), polytopic-type uncertainty(Peaucelle et al. (2000), Shaked (2001), Xia and Jia (2002), He et al. (2005), Li et al. (2008), Li et al. (2009)), and interval uncertainty(Mao and Selfridge (2001), Mao (2002), Mao and Yuan (2006), Udom (2012)). Robust stability of a uncertain system means that the system will be always stable for any possible quantities of uncertainties. Ichikawa (1982) studied the robust stability for linear and semilinear uncertain systems. Robust stability of SDDEs with uncertainties or perturbations was investigated in Mao (1996) and Mao et al. (1998). In Chen et al (2005), an LMI approach was applied to get robust exponential stability in mean square for uncertain stochastic systems with multiple delays, where a Lyapunov-Krasovskii functional had been used for discussion. Kuang and Deng (2012) investigated exponential stability for a class of uncertain stochastic systems with multiple delays and nonlinear perturbations, while Zhu et al. (2014b) analyzed the robustness of globally stable stochastic delayed systems when there were uncertain perturbations in parameters.

When a system is unstable, some useful controllers have been designed to stabilize original system. The common used controllers are feedback controllers with or without delays. To uncertain systems, the problem has been called robust stabilization. We need to design a controller such that an unstable uncertain system becomes stable robustly. In Wang et al. (2002), the problem of stabilizing bilinear uncertain time-delay stochastic systems with markovian jumping parameters had been discussed. A state feedback controller had been designed such that the controlled system was stable in mean square. Lu et al. (2003) designed robust feedback stabilization controller for uncertain stochastic systems with time-varying delays. Toward almost sure exponential stabilization of uncertain stochastic systems, Hu and Mao (2008) designed state-feedback controllers. Huang and Mao (2009) proposed a robust delayed-state-feedback controller to exponentially stabilize uncertain stochastic systems based on delay dependent stability criteria. Wang and Shen (2012) had dealt with robust stochastic stabilization and H_∞ control of uncertain stochastic systems with time-varying delay and nonlinear perturbation. And Zhu et al. (2014a) discussed robust stabilization problem for a class of linear uncertain stochastic systems with Markovian switching. A robust state-feedback controller was designed for exponential stabilization.

Recently, Mao (2013) and Mao et al. (2014) proposed a new feedback controller based on discrete-time state observations. Regular feedback controls require continuous observations of the system state, while this new feedback controller only needs discrete state observations, which is more realistic and cost less in practice. Although similar problems in deterministic differential systems have been studied such as in Allwright et al. (2005) and Ebihara et al. (2011), Mao (2013) is the first paper in the area of SDEs. The aim of this paper is to design a feedback controller for a hybrid stochastic system with norm bounded uncertainties based on discrete state observations. Moreover, controllers will be put not only in drift part, but also in diffusion part of the system. The rest of this paper is arranged as follows. In Section 2, some notations, definitions and lemmas are recalled. Main results will be stated in Section 3. A numerical example is covered in Section 4.

2. Problem statement

Throughout this paper, we use following notations. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous with $\mathcal{F}_0$ containing all P-null sets). Let $B(t) = (B_1(t), \cdots, B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. For $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm. $\|A\| = \max\{|Ax| : |x| = 1\}$ means the operator norm of a matrix A. If A is a vector or matrix, its transpose is denoted by A^{T}. For two symmetric matrices A and B, $A > (<, \geq, \leq)B$ means that $A - B$ is positive definite(negative definite, positive semidefinite, negative semidefinite). For a symmetric matrix A, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ mean the smallest and largest eigenvalues of A, respectively. The integer part of a real number x will be denoted as $[x]$.

Let $r(t), t \geq 0$ be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j \end{cases}$$

where $\Delta > 0$ and $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$, while $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$. Assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$.

Denote by $C(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$ the family of continuous functions from $\mathbb{R}^n \times \mathbb{R}_+$ to $\mathbb{R}_+$, also by $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times S; \mathbb{R}_+)$ the family of continuous functions $V(x, t, i)$ from $\mathbb{R}^n \times \mathbb{R}_+ \times S$ to $\mathbb{R}_+$ such that for each $i \in S$, $V(x, t, i)$ is continuously twice differentiable in x and once in t.

Consider following controlled hybrid uncertain stochastic system on $t \geq 0$

$$\begin{aligned} dx(t) \quad = \quad & [(A(r(t)) + \Delta A(t, r(t)))x(t) + H^0(r(t))u(x(\delta(t)), r(t))]dt \\ & + \sum_{k=1}^{m} [(B^k(r(t)) + \Delta B^k(t, r(t)))x(t) + H^k(r(t))u(x(\delta(t)), r(t))]dw_k(t), \end{aligned} \tag{1}$$

with initial data

$$x(0) = x_0 \in L^2_{\mathcal{F}_0}(\mathbb{R}^n), r(0) = r_0 \in S, \tag{2}$$

where for any $i \in S, k = 1, 2, \cdots, m$, $A(i) = A_i$ and $B^k(i) = B^k_i$ are known matrices, while $\Delta A(t, i)$, $\Delta B^k(t, i)$ are assumed to be norm bounded, i.e.,

$$\Delta A(t, i) = L_A F_A(t) N_i, \Delta B^k(t, i) = L_B F_B(t) E^k_i, \tag{3}$$

with known constant matrices L_A, N_i, L_B, E_i^k and matrix-valued functions $F_A(t)$ and $F_B(t)$ having Lebesgue-measurable elements and satisfying

$$F_A^{\mathrm{T}}(t)F_A(t) \le I, F_B^{\mathrm{T}}(t)F_B(t) \le I \tag{4}$$

for any $t \in \mathbb{R}_+$. The control terms $H^0(r(t))u(x(\delta(t)), r(t))$ and $H^k(r(t))u(x(\delta(t)), r(t))$ take the form of

$$u(x(\delta(t)), r(t)) = K(r(t))x(\delta(t)) \tag{5}$$

with $\delta(t) = [t/\tau]\tau$ for $t \ge 0$, while $H^k(i) = H_i^k \in \mathbb{R}^{n\times q}, k = 0, 1, \cdots, m$ as $r(t) = i$ are given matrices.

Any uncertainties $\Delta A(t, i)$ and $\Delta B^k(t, i)$ satisfying equations (3) and (4) are said to be admissible. The controlled system (1) is a special hybrid uncertain system with a bounded variable delay with coefficients satisfying the local Lipschitz condition and the linear growth condition with respect to $x(t)$ and $x(\delta(t))$. Using the existence-uniqueness theorem on hybrid delayed SDEs(see Mao and Yuan (2006)), there exists a unique solution $x(t)$ to (1) under initial conditions (2). Moreover, the solution satisfies $E|x(t)|^2 < \infty$ for $t \ge 0$.

Definition 2.1. *The controlled hybrid uncertain stochastic system* (1) *with initial conditions* (2) *is said to be robustly exponentially stable in mean square, if there is a positive constant* $\lambda > 0$*, such that for any admissible uncertainties* $\Delta A(t, r(t))$ *and* $\Delta B^k(t, r(t))$*, the solution* $x(t)$ *satisfies*

$$\limsup_{t\to\infty} \frac{1}{t} \log(E|x(t)|^2) \le -\lambda. \tag{6}$$

To other similar systems, Lu et al. (2003) used the control $u(x(t)) = Kx(t)$ to stabilize uncertain stochastic systems with time-varying delay. The same control was applied in Hu and Mao (2008) for stabilizing an uncertain system in the sense of almost surely exponential stability, while Zhu et al. (2014a) used such control to stabilize an uncertain hybrid system with uncertain transition rates. Another delay feedback control of the form $Kx(t - \tau)$ was used in Huang and Mao (2009) for robust stabilization of an uncertain system with the same delay. In this paper, we aim to design feedback controls in both drift and diffusion parts based on the discrete-time state observations such that the controlled system (1) is robustly exponentially stable in mean square.

Following two lemmas will be useful for further discussion, which can be referred in Moon et al. (2001) and Xu et al (2006).

Lemma 2.2. *For any vectors* $u \in \mathbb{R}^q, v \in \mathbb{R}^l$ *and a matrix* $M \in \mathbb{R}^{q\times l}$*, the inequality*

$$2u^{\mathrm{T}}Mv \le ru^{\mathrm{T}}MGM^{\mathrm{T}}u + \frac{1}{r}v^{\mathrm{T}}G^{-1}v \tag{7}$$

holds for any symmetric positive definite matrix $G \in \mathbb{R}^{l\times l}$ *and number* $r > 0$.

Lemma 2.3. *Let A, B, D, F, W be matrices with suitable dimensions. If $W > 0$, $F^{\mathrm{T}}F \leq I$, then for any number $\varepsilon > 0$ such that $W^{-1} - \varepsilon DD^{\mathrm{T}} > 0$, it holds that*

$$(A + DFB)^{\mathrm{T}}W(A + DFB) \leq A^{\mathrm{T}}(W^{-1} - \varepsilon DD^{\mathrm{T}})^{-1}A + \varepsilon^{-1}B^{\mathrm{T}}B. \tag{8}$$

3. Main results

Denote $D_i^k = H_i^k K_i, i \in S, k = 0, 1, \cdots, m$ in system (1) with controller (5). We first discuss robustly exponential stability of uncertain hybrid stochastic system

$$\begin{aligned} dx(t) &= [(A(r(t)) + \Delta A(t, r(t)))x(t) + D^0_{r(t)}x(\delta(t))]dt \\ &\quad + \sum_{k=1}^{m}[(B^k(r(t)) + \Delta B^k(t, r(t)))x(t) + D^k_{r(t)}x(\delta(t))]dw_k(t). \end{aligned} \tag{9}$$

Following lemma estimates the difference between $x(t)$ and $x(\delta(t))$, which is useful for development of our design.

Lemma 3.1. *Let $x(t)$ be the solution of system* (9). *Set*

$$M_A = 2\max_{i \in S}(\|A_i\|^2 + \|L_A\|^2\|N_i\|^2), \qquad M_D^0 = \max_{i \in S}\|D_i\|^2,$$

$$M_B = 2\max_{i \in S}\sum_{k=1}^{m}\left(\|B_i^k\|^2 + \|L_B\|^2\|E_i^k\|^2\right), \qquad M_D^1 = \max_{i \in S}\sum_{k=1}^{m}\|D_i^k\|^2,$$

and define

$$K(\tau) = (8\tau^2 M_A + 8\tau M_B + 4\tau^2 M_D^0 + 4\tau M_D^1)e^{8\tau^2 M_A + 8\tau M_B} \tag{10}$$

for $\tau > 0$. If τ is small enough for $K(\tau) < 1/2$, then for any $t \geq 0$,

$$E|x(t) - x(\delta(t))|^2 \leq \frac{2K(\tau)}{1 - 2K(\tau)}E|x(t)|^2. \tag{11}$$

Proof. Fix an integer $l \geq 0$, for any $t \in [l\tau, (l+1)\tau)$,

$$\begin{aligned} x(t) - x(\delta(t)) &= x(t) - x(l\tau) \\ &= \textstyle\int_{l\tau}^{t}\big(A(r(s)) + \Delta A(s, r(s))x(s) + D^0_{r(s)}x(l\tau)\big)ds \\ &\quad + \sum_{k=1}^{m}\textstyle\int_{l\tau}^{t}\big[\big(B^k(r(s)) + \Delta B^k(s, r(s))\big)x(s) + D^k_{r(s)}x(l\tau)\big]dw_k(s) \end{aligned}$$

Using Hölder inequality and the Doob's martingale inequality, we can derive

$$\begin{aligned} &E|x(t) - x(l\tau)|^2 \\ \leq\ & 4E|\textstyle\int_{l\tau}^{t}\big(A(r(s)) + \Delta A(s, r(s))x(s)\big)ds|^2 + 4E|\int_{l\tau}^{t}D^0_{r(s)}x(l\tau)ds|^2 \\ &+4E|\sum_{k=1}^{m}\textstyle\int_{l\tau}^{t}\big(B^k(r(s)) + \Delta B^k(s, r(s))\big)x(s)dw_k(s)|^2 + 4E|\sum_{k=1}^{m}\int_{l\tau}^{t}\big(D^k_{r(s)}x(l\tau)\big)dw_k(s)|^2 \\ \leq\ & 4\tau\textstyle\int_{l\tau}^{t}E\big(\|A(r(s)) + \Delta A(s, r(s))\|^2|x(s)|^2\big)ds + 4\tau\int_{l\tau}^{t}E\big(\|D^0_{r(s)}\|^2|x(l\tau)|^2\big)ds \\ &+4\sum_{k=1}^{m}\textstyle\int_{l\tau}^{t}E\big(\|B^k(r(s)) + \Delta B^k(s, r(s))\|^2|x(s)|^2\big)ds + 4\sum_{k=1}^{m}\int_{l\tau}^{t}\big(E\|D^k_{r(s)}\|^2|x(l\tau)|^2\big)ds. \end{aligned} \tag{12}$$

Obviously, $\|F_A(t)\| \leq 1$ holds for any $t \geq 0$ from (4). Consequently, for any $i \in S$,

$$\|A_i + \Delta A(t,i)\|^2 \leq 2(\|A_i\|^2 + \|L_A F_A(t) N_i\|^2) \leq M_A. \tag{13}$$

Similarly, for any $i \in S$, $\|B_i^k + \Delta B^k(t,i)\|^2 \leq 2(\|B_i^k\|^2 + \|L_B\|^2 \|E_i^k\|^2)$, and then

$$\sum_{k=1}^{m} \|B_i^k + \Delta B^k(t,i)\|^2 \leq M_B. \tag{14}$$

Combining (13) and (14) into (12), it will be held that

$$\begin{aligned} E|x(t) - x(l\tau)|^2 &\leq 4(M_A\tau + M_B) \int_{l\tau}^{t} E|x(s)|^2 ds + 4\tau(M_D^0 \tau + M_D^1) E|x(l\tau)|^2 \\ &\leq 8(\tau M_A + M_B) \int_{l\tau}^{t} E|x(s) - x(l\tau)|^2 ds \\ &\quad + (8\tau^2 M_A + 8\tau M_B + 4\tau^2 M_D^0 + 4\tau M_D^1) E|x(l\tau)|^2. \end{aligned} \tag{15}$$

By Gronwall's inequality, we have

$$E|x(t) - x(l\tau)|^2 \leq K(\tau) E|x(l\tau)|^2 \leq 2K(\tau)\big(E|x(t) - x(l\tau)|^2 + E|x(t)|^2\big), \tag{16}$$

which implies $E|x(t) - x(l\tau)|^2 \leq \frac{2K(\tau)}{1-2K(\tau)} E|x(t)|^2$ as required. □

We now state the main theorem on robustly exponential stability of system (9).

Theorem 3.2. *If there exist positive definite matrices Q_i and positive numbers $\mu_i, \varepsilon_i, \delta_{ik}, k = 1, 2, \cdots, m, i \in S$, such that for any $i \in S$,*

$$Q_i^{-1} - \varepsilon_i L_B L_B^T > 0 \tag{17}$$

and

$$\begin{aligned} \bar{Q}_i \;:=\; & Q_i(A_i + D_i^0) + (A_i + D_i^0)^{\mathrm{T}} Q_i + \mu_i Q_i L_A L_A^T Q_i + \mu_i^{-1} N_i^{\mathrm{T}} N_i + \sum_{j=1}^{N} \gamma_{ij} Q_j \\ & + \sum_{k=1}^{m} (B_i^k + D_i^k)^{\mathrm{T}} \big(Q_i^{-1} - \varepsilon_i L_B L_B^{\mathrm{T}}\big)^{-1} (B_i^k + D_i^k) + \varepsilon_i^{-1} \sum_{k=1}^{m} (E_i^k)^{\mathrm{T}} E_i^k + \sum_{k=1}^{m} \delta_{ik}^{-1} (E_i^k)^{\mathrm{T}} E_i^k \end{aligned} \tag{18}$$

are all negative definite matrices. Set

$$\lambda_m = \min_{i \in S} \lambda_{\min}(Q_i), \lambda_M = \max_{i \in S} \lambda_{\max}(Q_i), \lambda = \max_{i \in S} \lambda_{\max}(\bar{Q}_i), \tag{19}$$

$$\rho_i = \lambda_{\max}\Big(\sum_{k=1}^{m} (D_i^k)^{\mathrm{T}} Q_i D_i^k + \delta_{ik} (D_i^k)^{\mathrm{T}} Q_i L_B L_B^T Q_i D_i^k\Big), \rho = \max_{i \in S} \rho_i, \tag{20}$$

$$\bar{M}_i = \|Q_i D_i^0\| + \sum_{k=1}^{m} \|(B_i^k + D_i^k)^{\mathrm{T}} Q_i D_i^k\|, \bar{M} = \max_{i \in S} \bar{M}_i, \tag{21}$$

$$\lambda_\tau = \rho \frac{2K(\tau)}{1 - 2K(\tau)} + 2\sqrt{\frac{2\bar{M} K(\tau)}{1 - 2K(\tau)}} + \lambda. \tag{22}$$

If τ is sufficiently small for

$$K(\tau) < \frac{(\sqrt{\bar{M} - \lambda\rho} - \sqrt{\bar{M}})^2}{2\rho^2 + 2(\sqrt{\bar{M} - \lambda\rho} - \sqrt{\bar{M}})^2}, \tag{23}$$

then the solution satisfies

$$\limsup_{t \to \infty} \frac{1}{t} \log(E|x(t)|^2) \leq \frac{\lambda_\tau}{\lambda_M}.$$

Proof. First we note that $\lambda < 0$ and $\rho > 0$ from their definitions, so that $\sqrt{\bar{M} - \lambda\rho} > \sqrt{\bar{M}}$ holds. It can also be verified that condition (23) makes $K(\tau) < 1/2$ and $\lambda_\tau < 0$.

We use Lyapunov function $V(x(t), r(t)) = x^{\mathrm{T}}(t)Q(r(t))x(t)$ for further discussion, where $Q(i) = Q_i$ as $r(t) = i$. Applying the generalized Itô formulae to V, we have

$$dV(x(t), r(t)) = LV(x(t), r(t))dt + dM_1(t),$$

where $M_1(t)$ is a martingale with $M_1(0) = 0$ and

$$\begin{aligned} LV(x(t), i) \quad = \quad & 2x^{\mathrm{T}}(t)Q_i\Big((A_i + \Delta A(t,i))x(t) + D_i^0 x(\delta(t))\Big) + \sum_{j=1}^{N} \gamma_{ij} x^{\mathrm{T}}(t)Q_j x(t) \\ & + \sum_{k=1}^{m} \Big((B_i^k + \Delta B^k(t,i))x(t) + D_i^k x(\delta(t))\Big)^{\mathrm{T}} Q_i \Big((B_i^k + \Delta B^k(t,i))x(t) + D_i^k x(\delta(t))\Big) \end{aligned} \tag{24}$$

The first term can be treated by Lemma 2.2 as

$$\begin{aligned} & 2x^{\mathrm{T}}(t)Q_i\Big((A_i + \Delta A(t,i))x(t) + D_i^0 x(\delta(t))\Big) \\ = \;& x^{\mathrm{T}}(t)\big(Q_i(A_i + D_i^0) + (A_i + D_i^0)^{\mathrm{T}} Q_i\big)x(t) + 2x^{\mathrm{T}}(t)Q_i L_A F_A(t) N_i x(t) - 2x^{\mathrm{T}}(t)Q_i D_i^0\big(x(t) - x(\delta(t))\big) \\ \leq \;& x^{\mathrm{T}}(t)\big(Q_i(A_i + D_i^0) + (A_i + D_i^0)^{\mathrm{T}} Q_i\big)x(t) + \mu_i x^{\mathrm{T}}(t)(Q_i L_A L_A^{\mathrm{T}} Q_i)x(t) + \mu_i^{-1} x^{\mathrm{T}}(t)(N_i^{\mathrm{T}} N_i)x(t) \\ & -2x^{\mathrm{T}}(t)Q_i D_i^0\big(x(t) - x(\delta(t))\big). \end{aligned} \tag{25}$$

For the last term in (24), we can rewrite it as

$$\begin{aligned} & \sum_{k=1}^{m} \Big((B_i^k + \Delta B^k(t,i))x(t) + D_i^k x(\delta(t))\Big)^{\mathrm{T}} Q_i \Big((B_i^k + \Delta B^k(t,i))x(t) + D_i^k x(\delta(t))\Big) \\ = \;& \sum_{k=1}^{m} x^{\mathrm{T}}(t)(B_i^k + D_i^k + L_B F_B(t) E_i^k)^{\mathrm{T}} Q_i (B_i^k + D_i^k + L_B F_B(t) E_i^k)x(t) \\ & + \sum_{k=1}^{m} (x(t) - x(\delta(t)))^{\mathrm{T}} (D_i^k)^{\mathrm{T}} Q_i D_i^k (x(t) - x(\delta(t))) \\ & -2\sum_{k=1}^{m} x^{\mathrm{T}}(t)(B_i^k + D_i^k)^{\mathrm{T}} Q_i D_i^k (x(t) - x(\delta(t))) - 2\sum_{k=1}^{m} x^{\mathrm{T}}(t)(L_B F_B(t) E_i^k)^{\mathrm{T}} Q_i D_i^k (x(t) - x(\delta(t))). \end{aligned} \tag{26}$$

By Lemma 2.2 and 2.3, we have for any k,

$$\begin{aligned} & x^{\mathrm{T}}(t)\big(B_i^k + D_i^k + L_B F_B(t) E_i^k\big)^{\mathrm{T}} Q_i \big(B_i^k + D_i^k + L_B F_B(t) E_i^k\big)x(t) \\ \leq \;& x^{\mathrm{T}}(t)\Big((B_i^k + D_i^k)^{\mathrm{T}} \big(Q_i^{-1} - \varepsilon_i L_B L_B^{\mathrm{T}}\big)^{-1} (B_i^k + D_i^k) + \varepsilon_i^{-1} (E_i^k)^{\mathrm{T}} E_i^k\Big)x(t), \end{aligned} \tag{27}$$

and

$$
\begin{array}{ll}
& -2\sum_{k=1}^{m} x^{\mathrm{T}}(t)\big(L_B F_B(t) E_i^k\big)^{\mathrm{T}} Q_i D_i^k \big(x(t)-x(\delta(t))\big) \\
\leq & \sum_{k=1}^{m} \delta_{ik}^{-1} x^{\mathrm{T}}(t)(E_i^k)^{\mathrm{T}} E_i^k x(t) + \sum_{k=1}^{m} \delta_{ik}\big(x(t)-x(\delta(t))\big)^{\mathrm{T}}\big((D_i^k)^{\mathrm{T}} Q_i L_B L_B^T Q_i D_i^k\big)\big(x(t)-x(\delta(t))\big).
\end{array}
\tag{28}
$$

Combining (25)-(28) together into (24), and by the definitions of $\bar{M}_i$ and ρ_i, we will have for any $i \in S$,

$$
\begin{array}{rcl}
LV(x(t),i) & \leq & x^{\mathrm{T}}(t)\bar{Q}_i x(t) - 2x^{\mathrm{T}}(t)\Big(Q_i D_i^0 + \sum_{k=1}^{m}(B_i^k + D_i^k)^{\mathrm{T}} Q_i D_i^k\Big)\big(x(t)-x(\delta(t))\big) \\
& & +\big(x(t)-x(\delta(t))\big)^{\mathrm{T}} \sum_{k=1}^{m}\Big((D_i^k)^{\mathrm{T}} Q_i D_i^k + \delta_{ik}^{-1}(D_i^k)^{\mathrm{T}} Q_i L_B L_B^{\mathrm{T}} Q_i D_i^k\Big)\big(x(t)-x(\delta(t))\big) \\
& \leq & \lambda_{\max}(\bar{Q}_i)|x(t)|^2 + 2\bar{M}_i|x(t)||x(t)-x(\delta(t))| + \rho_i|x(t)-x(\delta(t))|^2.
\end{array}
\tag{29}
$$

Consequently, we obtain for any $t \geq 0$,

$$
LV(x(t),r(t)) \leq \lambda|x(t)|^2 + 2\bar{M}|x(t)||x(t)-x(\delta(t))| + \rho|x(t)-x(\delta(t))|^2. \tag{30}
$$

Now applying the generalized Itô formula to $e^{\theta t}V(x(t),r(t))$ with $\theta := -\lambda_\tau/\lambda_M > 0$, we obtain for any $t \geq 0$,

$$
e^{\theta t}x^{\mathrm{T}}(t)Q(r(t))x(t) = x^{\mathrm{T}}(0)Q(r(0))x(0) + \int_0^t e^{\theta s}[\theta x^{\mathrm{T}}(s)Q(r(s))x(s) + LV(x(s),r(s))]ds + M_2(t),
$$

where $M_2(t)$ is a continuous martingale with $M_2(0) = 0$. Taking expectation on both sides, and using (30) and Fubini's theorem, we get

$$
\begin{array}{c}
e^{\theta t}E\Big(x^{\mathrm{T}}(t)Q(r(t))x(t)\Big) \leq \lambda_M E|x(0)|^2 \\
+\int_0^t e^{\theta s}\big[(\theta\lambda_M + \lambda)E|x(s)|^2 + 2\bar{M}E(|x(s)||x(s)-x(\delta(s))|) + \rho E|x(s)-x(\delta(s))|^2\big]ds.
\end{array}
\tag{31}
$$

Setting $a = \sqrt{\frac{2\bar{M}K(\tau)}{1-2K(\tau)}} > 0$ and applying Lemma 3.1, it is true that

$$
2\bar{M}E(|x(s)||x(s)-x(\delta(s))|) \leq aE|x(t)|^2 + \frac{\bar{M}}{a}E|x(t)-x(\delta(t))|^2 \leq 2aE|x(t)|^2. \tag{32}
$$

Substituting (32) into (31), and using Lemma 3.1 again, we can obtain that for any $t \geq 0$,

$$
\lambda_m e^{\theta t}E|x(t)|^2 \leq \lambda_M E|x(0)|^2 + \int_0^t e^{\theta s}(\theta\lambda_M + \lambda_\tau)E|x(s)|^2 ds = \lambda_M E|x_0|^2,
$$

which is just equivalent to

$$
\limsup_{t\to\infty}\frac{1}{t}\log E|x(t)|^2 \leq -\theta
$$

as required.

□

From sufficient conditions in Theorem 3.2, we need to find Q_i and other positive constants, such that for any i, $\bar{Q}_i < 0$. Fortunately, we can convert requirements (17) and (18) into LMIs, which are easier to be checked. To see this, set $P_i = Q_i^{-1}$ for any $i \in S$ and multiply $\bar{Q}_i$ by P_i from both left and right. It is easy to see

$$
\begin{aligned}
P_i \bar{Q}_i P_i \;\; = \;\; & P_i(A_i + D_i^0)^{\mathrm{T}} + (A_i + D_i^0)P_i + \mu_i L_A L_A^T + \mu_i^{-1} P_i N_i^{\mathrm{T}} N_i P_i + \gamma_{ii} P_i \\
& + \sum_{k=1}^{m} P_i (B_i^k + D_i^k)^{\mathrm{T}} (P_i - \varepsilon_i L_B L_B^{\mathrm{T}})^{-1} (B_i^k + D_i^k) P_i \\
& + \varepsilon_i^{-1} \sum_{k=1}^{m} P_i (E_i^k)^{\mathrm{T}} E_i^k P_i + \sum_{k=1}^{m} \delta_{ik}^{-1} P_i (E_i^k)^{\mathrm{T}} E_i^k P_i \\
& + \sum_{i \neq j} (\sqrt{\gamma_{ij}} P_i) P_j^{-1} (\sqrt{\gamma_{ij}} P_i).
\end{aligned} \tag{33}
$$

Now for $i \in S$, set

$$
\Pi_i = \begin{pmatrix}
\Pi_{11i} & P_i N_i^{\mathrm{T}} & \Pi_{31i}^{\mathrm{T}} & \Pi_{41i}^{\mathrm{T}} & \Pi_{51i}^{\mathrm{T}} & \Pi_{61i}^{\mathrm{T}} \\
N_i P_i & -\mu_i I & 0 & 0 & 0 & 0 \\
\Pi_{31i} & 0 & \Pi_{33i} & 0 & 0 & 0 \\
\Pi_{41i} & 0 & 0 & \Pi_{44i} & 0 & 0 \\
\Pi_{51i} & 0 & 0 & 0 & \Pi_{55i} & 0 \\
\Pi_{61i} & 0 & 0 & 0 & 0 & \Pi_{66i}
\end{pmatrix} \tag{34}
$$

where blocks are defined as

$$\Pi_{11i} = P_i(A_i + D_i^0)^{\mathrm{T}} + (A_i + D_i^0)P_i + \gamma_{ii} P_i + \mu_i L_A L_A^T;$$

$$\Pi_{31i} = \Big((B_i^1 P_i + D_i^1 P_i)^{\mathrm{T}}, (B_i^2 P_i + D_i^2 P_i)^{\mathrm{T}}, \cdots, (B_i^m P_i + D_i^m P_i)^{\mathrm{T}} \Big)^{\mathrm{T}};$$

$$\Pi_{33i} = \mathrm{diag}\Big(\varepsilon_i L_B L_B^{\mathrm{T}} - P_i, \varepsilon_i L_B L_B^{\mathrm{T}} - P_i, \cdots, \varepsilon_i L_B L_B^{\mathrm{T}} - P_i \Big);$$

$$\Pi_{41i} = \Pi_{51} = \Big((E_i^1 P_i)^{\mathrm{T}}, (E_i^2 P_i)^{\mathrm{T}}, \cdots, (E_i^m P_i)^{\mathrm{T}} \Big)^{\mathrm{T}};$$

$$\Pi_{44i} = \mathrm{diag}\Big(-\varepsilon_i I, -\varepsilon_i I, \cdots, -\varepsilon_i I \Big);$$

$$\Pi_{55i} = \mathrm{diag}\Big(-\delta_{i1} I, -\delta_{i2} I, \cdots, -\delta_{im} I \Big);$$

$$\Pi_{61i} = \Big(\sqrt{\gamma_{i1}} P_i, \cdots, \sqrt{\gamma_{i,i-1}} P_i, \sqrt{\gamma_{i,i+1}} P_i, \cdots, \sqrt{\gamma_{iN}} P_i \Big)^{\mathrm{T}};$$

$$\Pi_{66i} = \mathrm{diag}\Big(-P_1, \cdots, -P_{i-1}, -P_{i+1}, \cdots, -P_N \Big).$$

By the well-known Schur complements (see Mao and Yuan (2006)), for any $i \in S$, LMIs $\Pi_i < 0$ are equivalent to $P_i \bar{Q}_i P_i < 0$ and then to $\bar{Q}_i < 0$. So if LMIs $\Pi_i < 0$ have solutions $P_i > 0, \mu_i > 0, \varepsilon_i > 0, \delta_{ik} > 0, i \in S, k = 1, 2, \cdots, m$, then $Q_i = P_i^{-1}$, $\mu_i, \varepsilon_i, \delta_{ki}$ are quantities required in Theorem 3.2.

Now we can design the robust controller in (5). Set $X_i = K_i P_i$ and note $D_i^k = H_i^k K_i, i \in S, k = 0, 1, \cdots, m$. According to Theorem 3.2, the criteria to guarantee exponential stability of controlled system (1) with controller (5) are to find matrices $Q_i > 0$ and K_i, positive numbers $\mu_i, \varepsilon_i, \delta_{ik}$ such that for any

$i \in S$,

$$\begin{aligned} &P_iA_i^{\mathrm{T}} + A_iP_i + X_i^{\mathrm{T}}(H_i^0)^{\mathrm{T}} + H_i^0X_i + \mu_iL_AL_A^T + \mu_i^{-1}P_iN_i^{\mathrm{T}}N_iP_i + \gamma_{ii}P_i \\ &\quad + \sum_{k=1}^m (B_i^kP_i + H_i^kX_i)^{\mathrm{T}}(P_i - \varepsilon_iL_BL_B^{\mathrm{T}})^{-1}(B_i^kP_i + H_i^kX_i) \\ &+\varepsilon_i^{-1}\sum_{k=1}^m P_i(E_i^k)^{\mathrm{T}}E_i^kP_i + \sum_{k=1}^m \delta_{ik}P_i(E_i^k)^{\mathrm{T}}E_i^kP_i + \sum_{i\neq j}(\sqrt{\gamma_{ij}}P_i)P_j^{-1}(\sqrt{\gamma_{ij}}P_i) < 0, \end{aligned} \tag{35}$$

which can then be transformed into LMIs:

$$\hat{\Pi}_i = \begin{pmatrix} \hat{\Pi}_{11i} & P_iN_i^{\mathrm{T}} & \hat{\Pi}_{31i}^{\mathrm{T}} & \Pi_{41i}^{\mathrm{T}} & \Pi_{51i}^{\mathrm{T}} & \Pi_{61i}^{\mathrm{T}} \\ N_iP_i & -\mu_iI & 0 & 0 & 0 & 0 \\ \hat{\Pi}_{31i} & 0 & \Pi_{33i} & 0 & 0 & 0 \\ \Pi_{41i} & 0 & 0 & \Pi_{44i} & 0 & 0 \\ \Pi_{51i} & 0 & 0 & 0 & \Pi_{55i} & 0 \\ \Pi_{61i} & 0 & 0 & 0 & 0 & \Pi_{66i} \end{pmatrix} \tag{36}$$

with

$$\hat{\Pi}_{11i} = P_iA_i^{\mathrm{T}} + X_i^{\mathrm{T}}(H_i^0)^{\mathrm{T}} + A_iP_i + H_i^0X_i + \gamma_{ii}P_i + \mu_iL_AL_A^T;$$
$$\hat{\Pi}_{31i} = \Big((B_i^1P_i + H_i^1X_i)^{\mathrm{T}}, (B_i^2P_i + H_i^2X_i)^{\mathrm{T}}, \cdots, (B_i^mP_i + H_i^mX_i)^{\mathrm{T}}\Big)^{\mathrm{T}};$$

and other same blocks as in Π.

Theorem 3.3. *Assume that there exist matrices $P_i = P_i^{\mathrm{T}} > 0$, X_i, and positive numbers $\mu_i, \varepsilon_i, \delta_{ik}$, $i \in S, k = 1, 2, \cdots, m$, such that LMIs $\hat{\Pi}_i < 0$ hold for any $i \in S$. Set $Q_i = P_i^{-1}$, $K_i = X_iP_i^{-1}$. Then the controlled system* (1) *is robustly exponentially stable in mean square, if we set $u(x(t), r(t)) = K(r(t))x([t/\tau]\tau)$, where τ is small enough such that* (23) *holds.*

From above theorem, there are two steps to get the robust controller. In the first step, we need to find solutions for $\hat{\Pi}_i < 0, i \in S$. The second step is to calculate all quantities in (19), (20) and (21), and then to get small τ from condition (23).

4. An example

Consider a two-dimensional controlled uncertain hybrid system

$$\begin{aligned} dx(t) = \ & [(A(r(t)) + \Delta A(t, r(t)))x(t) + H^0(r(t))K(r(t))x(\delta(t))]dt \\ & +[(B(r(t)) + \Delta B(t, r(t)))x(t) + H^1(r(t))Kx(\delta(t))]dw(t), \end{aligned} \tag{37}$$

where $B(t)$ is a Brownian motion and $r(t)$ is a Markov chain taking values in $S = \{1, 2\}$ with transition matrix $\Gamma = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$. $\Delta A(t, r(t))$ and $\Delta B(t, r(t))$ are defined as in (3) and (4) and all coefficients are given by

$$A_1 = \begin{pmatrix} 0.5 & 0.2 \\ 0.3 & -0.5 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 0.1 \\ 0 & 1 \end{pmatrix}, B_1 = \begin{pmatrix} -1 & 0 \\ 0.5 & -1 \end{pmatrix}, B_2 = \begin{pmatrix} 1 & 0.1 \\ 0 & 1 \end{pmatrix};$$

$$L_A = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}, N_1 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix}, N_2 = \begin{pmatrix} -0.5 & 0 \\ 0 & -0.5 \end{pmatrix},$$

$$L_B = \begin{pmatrix} -0.1 & 0 \\ 0 & -0.1 \end{pmatrix}, E_1 = \begin{pmatrix} -0.2 & 0 \\ 0 & 0.2 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$H_1^0 = \begin{pmatrix} -8 & 0.1 \\ 0.5 & -3 \end{pmatrix}, H_2^0 = \begin{pmatrix} -6 & 0 \\ 0 & -5 \end{pmatrix}, H_1^1 = \begin{pmatrix} -1 & 0.5 \\ 2 & 1 \end{pmatrix}, H_2^1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Applying LMI package to (36), it can be found that

$$P_1 = \begin{pmatrix} 0.3609 & 0.0162 \\ 0.0162 & 0.4812 \end{pmatrix}, P_2 = \begin{pmatrix} 0.3866 & -0.0012 \\ -0.0012 & 0.4163 \end{pmatrix}, X_1 = \begin{pmatrix} 0.0891 & 0.1087 \\ -0.1649 & 0.1075 \end{pmatrix},$$

$$X_2 = \begin{pmatrix} 0.1659 & 0.0220 \\ -0.0167 & 0.2075 \end{pmatrix}, \mu_1 = 0.8197, \mu_2 = 0.8334, \delta_1 = 0.8201, \delta_2 = 0.8756, \varepsilon_1 = 0.8123, \varepsilon_2 = 0.8694$$

make $\hat{\Pi}_1 < 0$ and $\hat{\Pi}_2 < 0$.

Consequently, we can calculate $\lambda = -1.4963, \rho = 1.0712, \bar{M} = 7.4133$ and

$$K_1 = \begin{pmatrix} 0.2370 & 0.2178 \\ -0.4677 & 0.2393 \end{pmatrix}, K_2 = \begin{pmatrix} 0.4293 & 0.0542 \\ -0.0417 & 0.4984 \end{pmatrix}. \tag{38}$$

Also we should find small τ from condition (23). Direct calculation gives

$$K(\tau) = (47.6778\tau^2 + 28.1536\tau)\exp(17.7220\tau^2 + 26.2526\tau) \text{ and } \frac{(\sqrt{\bar{M} - \lambda\rho} - \sqrt{\bar{M}})^2}{2\rho^2 + 2(\sqrt{\bar{M} - \lambda\rho} - \sqrt{\bar{M}})^2} = 0.032.$$

We can see that (23) is satisfied as $\tau \le \tau_0 = 0.0011$. Now we can design the robust controller $u(x(\delta(t)), r(t)) = K(r(t))x([t/\tau_0]\tau_0)$ with $K(1) = K_1$ and $K(2) = K_2$ as in (38) such that (37) is robustly exponentially stable in mean square.

5. Conclusion

In this paper, a controller based on discrete time observations has been designed such that an uncertain hybrid stochastic system is robustly exponentially stable in mean square. The control is put not only in the drift term, but also in the diffusion term. Techniques have been developed to get the controller. Compared to other feedback controls, this kind of control is more realistic in practice.

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