FRAME PATTERNS IN n-CYCLES

MILES JONES, SERGEY KITAEV, AND JEFFREY REMMEL

ABSTRACT. In this paper, we study the distribution of the number occurrences of the simplest frame pattern, called the μ pattern, in n-cycles. Given an n-cycle C, we say that a pair $\langle i,j\rangle$ matches the μ pattern if $i<j$ and as we traverse around C in a clockwise direction starting at i and ending at j, we never encounter a k with $i<k<j$. We say that $\langle i,j\rangle$ is a nontrivial μ-match if $i+1<j$. We say that an n-cycle C is incontractible if there is no i such that $i+1$ immediately follows i in C. We show that number of incontractible n-cycles in the symmetric group S_n is D_{n-1} where D_n is the number of derangements in S_n. We show that number of n-cycles in S_n with exactly k μ-matches can be expressed as a linear combination of binomial coefficients of the form $\binom{n-1}{i}$ where $i \leq 2k+1$. We also show that the generating function $NTI_{n,\mu}(q)$ of q raised to the number of nontrivial μ-matches in C over all incontractible n-cycles in S_n is a new q-analogue of D_{n-1} which is different from the q-analogues of the derangement numbers that have been studied by Garsia and Remmel and by Wachs. We will show that there is a rather surprising connection between the charge statistic on permutations due to Lascoux and Schüzenberger and our polynomials in that the coefficient of the smallest power of q in $NTI_{2k+1,\mu}(q)$ is the number of permutations in S_{2k+1} whose charge path is a Dyck path. Finally, we show that $NTI_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}}$ and $NT_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}}$ are the number of partitions of k for sufficiently large n.

1. INTRODUCTION

Mesh patterns were introduced in [2] by Brändén and Claesson, and they were studied in a series of papers (e.g. see [6] by Kitaev and Liese, and references therein). A particular class of mesh patterns is *boxed patterns* introduced in [1] by Avgustinovich et al., who later suggested to call this type of patterns *frame patterns*. The simplest frame pattern which is called the μ pattern is defined as follows. Let S_n denote the set of all permutations of $\{1,\ldots,n\}$. Given $\sigma=\sigma_1\sigma_2\ldots\sigma_n \in S_n$, we say that a pair $\langle \sigma_i,\sigma_j\rangle$ is an occurrence of the μ pattern in σ if $i<j$, $\sigma_i<\sigma_j$, and there is no $i<k<j$ such that $\sigma_i<\sigma_k<\sigma_j$ (the last condition is indicated by the shaded area in Figure 2). The μ pattern is shown in Figure 1 using the notation in [2].

Instituto de Matemática,Universidad de Talca, Camino Lircay S/N Talca, Chile `mjones@inst-mat.utalca.cl` Supported by FONDECYT (Fondo Nacional de Desarrollo Científico y Tecnológico de Chile) postdoctoral grant #3130631.

Department of Computer and Information Sciences, University of Strathclyde, Glasgow G1 1XH, UK. Email: `sergey.kitaev@cis.strath.ac.uk`.

Department of Mathematics, University of California, San Diego, La Jolla, CA 92093-0112. USA. Email: `jremmel@ucsd.edu`.

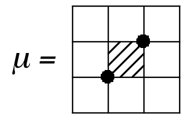

FIGURE 1. The mesh pattern μ.

Similarly, we say that the pair $\langle \sigma_i, \sigma_j \rangle$ is an occurrence of the μ' pattern in σ if $i < j$, $\sigma_i > \sigma_j$, and there is no $i < k < j$ such that $\sigma_i > \sigma_k > \sigma_j$. For example, if $\sigma = 4\,8\,2\,6\,1\,3\,5\,7$, then the occurrences of μ in σ are

$$\langle 4,8\rangle, \langle 4,6\rangle, \langle 4,5\rangle, \langle 2,6\rangle, \langle 2,3\rangle, \langle 6,7\rangle, \langle 1,3\rangle, \langle 3,5\rangle, \langle 5,7\rangle$$

and the occurrences of μ' in σ are

$$\langle 4,2\rangle, \langle 4,3\rangle, \langle 8,2\rangle, \langle 8,6\rangle, \langle 8,7\rangle, \langle 2,1\rangle, \langle 6,1\rangle, \langle 6,3\rangle, \langle 6,5\rangle.$$

We let $N_\mu(\sigma)$ (resp., $N_{\mu'}(\sigma)$) denote the number of occurrences of the μ (resp., μ') in σ. The *reverse* of $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, σ^r, is the permutation $\sigma_n \sigma_{n-1} \ldots \sigma_1$, and the *complement* of σ, σ^c, is the permutation $(n+1-\sigma_1)(n+1-\sigma_2)\ldots(n+1-\sigma_n)$. It is easy to see that $N_\mu(\sigma) = N_{\mu'}(\sigma^r) = N_{\mu'}(\sigma^c)$ and thus, since the reverse and complement are trivial bijections from S_n to itself, studying the distribution of μ-matches in S_n is equivalent to studying the distribution of μ'-matches in S_n.

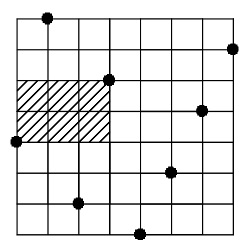

FIGURE 2. The graph of the permutation 48261357 with the occurrence $\langle 4,6\rangle$ highlighted.

If we graph a given permutation σ as dots on a grid as in Figure 2, then one can see that each occurrence of μ is a pair of increasing dots such that there are no dots within the rectangle created by the original dots. The occurrence $\langle 4,6\rangle$ is highlighted in Figure 2; in particular, there are no dots within the shaded rectangle.

Jones and Remmel studied the distribution of cycle-occurrences of classical consecutive patterns in [4]. See [5] for a comprehensive introduction to the theory of permutation patterns. In this paper, we shall study the distribution of the *cycle-occurrences* of μ in the cycle structure of permutations in the symmetric group S_n. That is, suppose that we are given a k-cycle C in a permutation σ in the symmetric group S_n. Then we will always write $C = (c_0, \ldots, c_{k-1})$ where c_0 is the smallest element of the cycle. We will always draw such a cycle with c_0 at the top and assume that we traverse around the cycle in a clockwise direction. For example, if $C = (1, 4, 6, 2, 7, 5, 8, 3)$, then we would picture C as in Figure 3. We say that c_s is *cyclically between* c_i and c_j in C, if starting at c_i, we encounter c_s before we encounter c_j as we traverse around the cycle in a clockwise direction. Alternatively, we can identify C with the permutation $c_0c_1 \ldots c_{k-1}$. In this notation, c_s is cyclically between c_i and c_j if either $i < s < j$, or $j < i < s$, or $s < j < i$. For example, in the cycle $C = (1, 4, 6, 2, 7, 5, 8, 3)$, 8 is cyclically between 2 and 3. Then we say that the pair $\langle c_i, c_j \rangle$ is a *cycle-occurrence of* μ *in* C if $c_i < c_j$ and there is no c_s such that $c_i < c_s < c_j$ and c_s is cyclically between c_i and c_j. Similarly, we say that the pair $\langle c_i, c_j \rangle$ is a cycle-occurrence of μ' in C, if $c_i > c_j$ and there is no c_s such that $c_i > c_s > c_j$ and c_s is cyclically between c_i and c_j. As is the case with permutations, the study of the number of cycle-occurrences of μ in the cycle structures of permutations is equivalent to the study of the number of cycle-occurrences of μ' in the cycle structures of permutations. That is, given the cycle structure $C_1, \ldots, C_k$ of a permutation $\sigma \in S_n$, the cycle complement of σ, $\sigma^{\text{cyc-c}}$, is the permutation whose cycle structure arises from the cycle structure of σ by replacing each number i by $n + 1 - i$. For example, if σ consists of the cycles $(1, 4, 2, 6), (3, 8), (5, 9, 7)$, then $\sigma^{\text{cyc-c}}$ consists of the cycles $(9, 6, 8, 4), (7, 2), (5, 1, 3)$. It is then easy to see that for all $\sigma \in S_n$, σ has k cycle-occurrences of μ if and only if $\sigma^{\text{cyc-c}}$ has k cycle-occurrences of μ'.

Let $\mathcal{C}_n$ be the set of n-cycles in S_n. If $C = (1, c_1, \ldots, c_{n-1}) \in \mathcal{C}_n$, then it is clear that $\langle i, i+1 \rangle$ is always a cycle-occurrence of μ in C. We shall call such cycle-occurrences *trivial occurrences of* μ or *trivial* μ*-matches* in C and all other cycle-occurrences of μ in C will be called *nontrivial occurrences of* μ or *nontrivial* μ*-matches* in C. We let $N_\mu(C)$ denote the number of occurrences of μ in C and $NT_\mu(C)$ denote the number of nontrivial occurrences of μ in C. For example, if $C = (1, 4, 6, 2, 7, 5, 8, 3)$, the nontrivial occurrences of μ in C are the pairs $\langle 1, 4 \rangle$, $\langle 2, 7 \rangle$, $\langle 2, 5 \rangle$, $\langle 4, 6 \rangle$, and $\langle 5, 8 \rangle$, so that $NT_\mu(C) = 5$. Clearly, if $C = (1, c_1, \ldots, c_{n-1})$ is an n-cycle in $\mathcal{C}_n$, then $N_\mu(C) = (n-1) + NT_\mu(C)$, since for all $1 \le i \le n-1$, $\langle i, i+1 \rangle$ will be a trivial occurrence of μ in C. If C is a 1-cycle, then $NT_\mu(C) = N_\mu(C) = 0$.

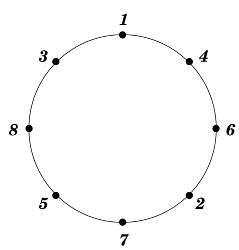

FIGURE 3. The cycle $C = (1, 4, 6, 2, 7, 5, 8, 3)$.

If $\sigma_1 \ldots \sigma_n$ is any sequence of distinct integers, we let $\text{red}(\sigma)$ denote the permutation of S_n that is obtained by replacing the ith smallest element of $\{\sigma_1, \ldots, \sigma_n\}$ by i for $i = 1, \ldots n$. For example, $\text{red}(3\ 7\ 10\ 5\ 2) = 2\ 4\ 5\ 3\ 1$. Similarly, if $C = (c_0, \ldots, c_{k-1})$ is a k-cycle in some permutation $\sigma \in S_n$, we let $\text{red}(C)$ be the k-cycle in S_k which is obtained by replacing the ith largest element of C by i. For example, if $C = (2,6,7,3,9)$, then $\mathrm{red}(C) = (1,3,4,2,5)$. In such a situation, we let $NT_\mu(C) = NT_\mu(\mathrm{red}(C))$. Finally, if σ consists of cycles $C^{(1)},\ldots,C^{(\ell)}$, then we let $CNT_\mu(\sigma) = \sum_{i=1}^{\ell} NT_\mu(C^{(i)})$.

We note that if one wishes to study the generating function

$$CNT_\mu(q,x,t) = 1 + \sum_{n\geq 1} \frac{t^n}{n!} \sum_{\sigma\in S_n} q^{CNT_\mu(\sigma)} x^{\mathrm{cyc}(\sigma)}, \tag{1}$$

where $\mathrm{cyc}(\sigma)$ denotes the number of cycles in σ, then it is enough to study the generating function

$$NT_\mu(q,t) = \sum_{n\geq 1} \frac{t^n}{n!} NT_{n,\mu}(q) \tag{2}$$

where $NT_{n,\mu}(q) = \sum_{C\in\mathcal{C}_n} q^{NT_\mu(C)}$. That is, it easily follows from the exponential formula that if w is a weight function on n-cycles $C \in \mathcal{C}_n$ and for any permutation σ whose cycle structure is $C_1,\ldots,C_k$, we define $w(\sigma) = \prod_{i=1}^{k} w(\mathrm{red}(C_i))$, then

$$1 + \sum_{n\geq 1} \frac{t^n}{n!} \sum_{\sigma\in S_n} w(\sigma) x^{\mathrm{cyc}(\sigma)} = e^{x\sum_{n\geq 1}\frac{t^n}{n!}\sum_{C\in\mathcal{C}_n} w(C)}.$$

Hence

$$CNT_\mu(q,x,t) = e^{xNT_\mu(q,t)}. \tag{3}$$

The main focus of this paper is to study the polynomials $NT_{n,\mu}(q)$.

In Table 1 of the appendix, we give a table of $NT_{n,\mu}(q)$ for $n \leq 10$ which was calculated using Mathematica. Given a polynomial $f(x) = \sum_{i=0}^{n} c_i x^i$, we let $f(x)|_{x^i} = c_i$ denote the coefficient of x^i in $f(x)$. We shall show that for any fixed $k \geq 0$, there are constants $c_0, c_1, \ldots, c_{2k+1}$ such that $NT_{n,\mu}(q)|_{q^k} = \sum_{i=1}^{2k+1} c_i \binom{n-1}{i}$ for all $n \geq 2k+2$. To prove this result, we need to study what we call the contraction of a cycle. Define a *bond* of an n-cycle $C \in \mathcal{C}_n$ to be a pair of consecutive integers $(a, a+1)$ such that a is immediately followed by $a+1$ in the cycle C. For example, the bonds of the cycle $(1,4,5,6,2,7,8,3)$ are the pairs $(4,5)$, $(5,6)$, and $(7,8)$. If $C = (c_1, c_2, \ldots, c_n)$ is an n-cycle where $c_1 = 1$, define the sets $R_1, R_2, \ldots, R_k$ recursively as follows. First, let $1 = c_1 \in R_1$. Then inductively, if $c_i \in R_j$, then $c_{i+1} \in R_j$ if $c_{i+1} = c_i + 1$, and $c_{i+1} \in R_{j+1}$ otherwise. For example, for the cycle $C = (1,2,4,6,7,8,3,5)$ these sets are $R_1 = \{1,2\}, R_2 = \{4\}, R_3 = \{6,7,8\}, R_4 = \{3\}, R_5 = \{5\}$. We will call these sets the *consecutive runs of* C. Define the *contraction* of a cycle C with consecutive runs $R_1, R_2, \ldots, R_k$ to be

$$\mathrm{cont}(C) = \mathrm{red}(\max(R_1), \max(R_2), \ldots, \max(R_k)),$$

where $\max(R_i)$ denotes the maximum element in a run R_i. For example, the contraction of $C = (1,2,4,6,7,8,3,5)$ is

$$\mathrm{cont}(1,2,4,6,7,8,3,5) = \mathrm{red}(2,4,8,3,5) = (1,3,5,2,4).$$

We say that C *contracts* to A if $\mathrm{cont}(C) = A$. We say that a cycle C is *incontractible* if $\mathrm{cont}(C) = C$. Thus an incontractible n-cycle C is an n-cycle such that there are no integers

i such that $i+1$ immediately follows i in C. We let $\mathcal{IC}_n$ denote the set of incontractible n-cycles in $\mathcal{C}_n$. We will show that $|\mathcal{IC}_n|$ is D_{n-1} where D_n is the number of derangements of S_n, i.e. the number of $\sigma \in S_n$ such that σ has no fixed points. We will also study the polynomials

$$NTI_{n,\mu}(q) = \sum_{C \in \mathcal{IC}_n} q^{NT_\mu(C)}.$$

Table 2 of the appendix gives the polynomials $NTI_{n,\mu}(q)$ for $n \leq 10$.

We shall show that for any n-cycle C, $NT_\mu(C) = NT_\mu(cont(C))$ and that for any incontractible k-cycle A, there are precisely $\binom{n-1}{k-1}$ n-cycles C such that $cont(C) = A$. It follows that

$$NT_{n,\mu}(q)|_{q^k} = \sum_{s=1}^{n} \sum_{\substack{A \in \mathcal{IC}_s, \\ NT_\mu(A)=k}} \binom{n-1}{s-1}.$$

Then we shall show that the lowest power of q that appears in either $NTI_{2n,\mu}(q)$ or $NTI_{2n+1,\mu}(q)$ is q^n. It will then follow that

$$NT_{n,\mu}(q)|_{q^k} = \sum_{s=1}^{2k+1} \sum_{\substack{A \in \mathcal{IC}_s, \\ NT_\mu(A)=k}} \binom{n-1}{s-1}.$$

For example, it will then follow from our tables of the $NTI_{n,\mu}(q)$s that $NT_{n,\mu}(q)|_q = \binom{n-1}{2}$ for $n \geq 3$ and $NT_{n,\mu}(q)|_{q^2} = \binom{n-1}{2} + 2\binom{n-1}{4}$ for $n \geq 5$. Clearly, for $n \geq 1$, $NT_{n,\mu}(q)|_{q^0} = 1$ because $(1, 2, \ldots, n)$ is the only n-cycle that has no nontrivial occurrences of μ and $NT_{n,\mu}(q)|_{q^{\binom{n-1}{2}}} = 1$ since the only n-cycle with $\binom{n-1}{2}$ nontrivial occurrences of μ is $(1, n, n-1, n-2, \ldots, 2)$.

We computed that the sequence $(NTI_{2k+1,\mu}(q)|_{q^k})_{k \geq 3}$ starts out $1, 2, 5, 14, \ldots$ which suggests that $NTI_{2k+1,\mu}(q)|_{q^k} = C_k = \frac{1}{n+1}\binom{2n}{n}$ where C_k is the kth *Catalan number*. Our proof of this fact lead us to a surprising connection between the *charge statistic on permutations* as defined by Lascoux and Schützenberger [7] and our problem. That is, given a permutation $\sigma = \sigma_1 \ldots \sigma_n$, one defines the index, $\mathrm{ind}_\sigma(\sigma_i)$, of σ_i in σ as follows. First, $\mathrm{ind}_\sigma(1) = 0$. Then, inductively, $\mathrm{ind}_\sigma(i+1) = \mathrm{ind}_\sigma(i)$ if $i+1$ appears to the right of i in σ and $\mathrm{ind}_\sigma(i+1) = 1 + \mathrm{ind}_\sigma(i)$ if $i+1$ appears to the left i in σ. For example, if $\sigma = 1\ 4\ 8\ 5\ 9\ 6\ 2\ 7\ 3$ and we use subscripts to indicate the index of i in σ, then we see that the indices associated with σ are

$$\sigma = 1_0\ 4_1\ 8_2\ 5_1\ 9_2\ 6_1\ 2_0\ 7_1\ 3_0.$$

The charge of σ, $\mathrm{ch}(\sigma)$, is equal to $\sum_{i=1}^n \mathrm{ind}_\sigma(i)$. Thus, for example, if $\sigma = 1\ 4\ 8\ 5\ 9\ 6\ 2\ 7\ 3$, then $\mathrm{ch}(\sigma) = 8$. We will associate a path which we call the *charge path* of σ whose vertices are elements $(i, \mathrm{ind}_\sigma(\sigma_i))$ and whose edges are $\{(i, \mathrm{ind}_\sigma(\sigma_i)), (i+1, \mathrm{ind}_\sigma(\sigma_{i+1}))\}$ for $i = 1, \ldots, n$. For example, if $\sigma = 1\ 4\ 8\ 5\ 9\ 6\ 2\ 7\ 3$ as above, then charge path of σ, which we denote by $\mathrm{chpath}(\sigma)$, is pictured in Figure 4. In our particular example, the charge graph of σ is a *Dyck path* (a lattice path with steps (1,1) and (1,-1) from the origin

(0,0) to $(2n, 0)$ that never goes below the x-axis) and if $C = (1,4,8,5,9,6,2,7,3)$ is the n-cycle induced by σ, then $NT_\mu(C) = 4$. This is no accident. That is, we shall show that if $C = (1, \sigma_2, \ldots, \sigma_{2n+1})$ is $2n+1$-cycle in $\mathcal{IC}_{2n+1}$, then $NT_\mu(C) = n$ if and only if the charge path of $\sigma = 1\sigma_2 \ldots \sigma_{2n+1}$ is a Dyck path of length $2n$.

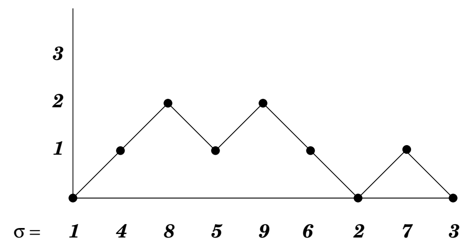

FIGURE 4. The charge path of $\sigma = 1\ 4\ 8\ 5\ 9\ 6\ 2\ 7\ 3$.

We also observed that $NTI_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}}$ and $NT_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}}$ is the number of partitions of k for sufficiently large n. Here a sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition of n if $\lambda_1 \geq \cdots \geq \lambda_k$ and $\sum_{i=1}^k \lambda_i = n$. The *Ferrers diagram* F_λ of a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ is the set of left justified rows of squares with λ_i squares in the i row from above. For example, $F_{(3,2,2,1)}$ is pictured in Figure 5.

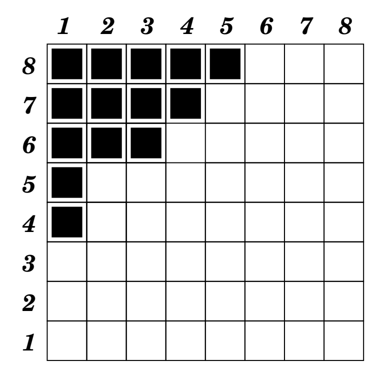

FIGURE 5. The Ferrers diagram of the partition $(3, 2, 2, 1)$.

In fact, we will show that for $k \leq n-2$, $NT_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}}$ equals the number of partitions of k. We show this by plotting the *non-μ-matches* of an n-cycle $C \in \mathcal{C}_n$, i.e. the pairs of integers $\langle i, j \rangle$ with $i < j$ which are not occurrences of the μ pattern in C, on an $n \times n$ grid and shading a cell in the jth row and ith column if and only if $\langle i, j \rangle$ is a non-μ-match of C. For example, the non-μ-matches of the cycle $C = (1,8,7,6,5,11,4,3,10,2,9)$ are $\langle 1,9 \rangle, \langle 1,10 \rangle, \langle 1,11 \rangle, \langle 2,10 \rangle, \langle 2,11 \rangle, \langle 3,11 \rangle$ and $\langle 4,11 \rangle$. These pairs can be plotted on the 11×11 grid as shown in Figure 6. The shaded plot in Figure 6 is of the form of the Ferrers diagram of the integer partition $\lambda = (4,2,1)$. We will show that if $C \in \mathcal{C}_n$ and there are

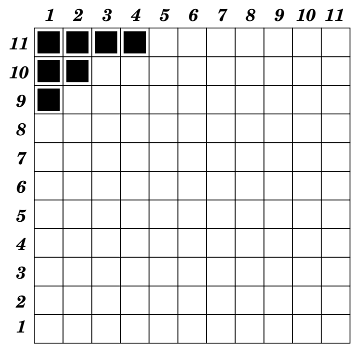

FIGURE 6. The non-matches of $(1,8,7,6,5,11,4,3,10,2,9)$ plotted on an 11×11 grid

fewer than $n-2$ non-matches in C, then shaded squares in the plot of the non-μ-matches in C will be of the form of a Ferrers diagram of an integer partition. Moreover, we shall show that if $C \in \mathcal{C}_n$ has fewer than $n-2$ non-matches, then C must be incontractible. Hence, it follows that $NTI_{n,\mu}(q)\big|_{q^{\binom{n-1}{2}-k}}$ and $NT_{n,\mu}(q)\big|_{q^{\binom{n-1}{2}-k}}$ are the number of partitions of k for sufficiently large n.

The outline of this paper is as follows. In Section 2, we shall study the properties of contractions of n-cycles and show that for all $n \geq 1$, there are positive integers $c_{1,k}, \ldots, c_{2k+1,k}$ such that $NT_{n,\mu}(q)|_{q^k} = \sum_{s=\left\lfloor \frac{3+\sqrt{1+8k}}{2} \right\rfloor}^{2k+1} c_{s,k}\binom{n-1}{s-1}$. In Section 3, we shall study incontractible cycles and the polynomials $NTI_{n,\mu}(q)$. We shall show that for all $n \geq 1$, $NTI_{2n+1,\mu}(q)|_{q^n} = C_n$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number. In Section 4, we study the non-μ-matches in n-cycles of $\mathcal{C}_n$ and use them to show that $NTI_{n,\mu}(q)\big|_{q^{\binom{n-1}{2}-k}}$ and $NT_{n,\mu}(q)\big|_{q^{\binom{n-1}{2}-k}}$ are the number of partitions of k for sufficiently large n. In Section 5, we will state our conclusions and some open problems.

2. The contraction of a cycle.

In this section, we establish some basic properties about the contraction of n-cycles.

Proposition 1. *For any n-cycle $C \in \mathcal{C}_n$, $NT_\mu(C) = NT_\mu(\mathrm{cont}(C))$.*

Proof. We proceed by induction on n. The theorem is clearly true for $n = 1$. Now, suppose that we are given an n-cycle $C \in \mathcal{C}_n$ and $R_1, \ldots, R_k$ are the consecutive runs. If $k = n$, then, clearly, C is incontractible. Otherwise, suppose that s is the least i such that $|R_i| \geq 2$. Let $R_s = \{i, i+1, \ldots, j\}$. Then it is easy to see that in C, there are no pairs $\langle a, t\rangle$ which are nontrivial occurrences of μ in C for $i \leq a \leq j-1$ since $a+1$ immediately follows a in C. Similarly, there are no pairs $\langle s, b\rangle$ which are nontrivial occurrences of μ in C for $i+1 \leq b \leq j$ since $b-1$ immediately precedes b in C. Now, suppose that C' arises from C by removing $i+1, \ldots, j$ and replacing each $s > j$ by $s-(j-i)$. Then it is easy to see that

(1) if $s < t \leq i$, $\langle s,t\rangle$ is a nontrivial occurrence of μ in C if and only if $\langle s,t\rangle$ is a nontrivial occurrence of μ in C',
(2) if $s \leq i < j < t$, then $\langle s,t\rangle$ is a nontrivial occurrence of μ in C if and only if $\langle s, t-(j-i)\rangle$ is a nontrivial occurrence of μ in C', and
(3) if $j \leq s < t$, then $\langle s,t\rangle$ is a nontrivial occurrence of μ in C if and only if $\langle s-(j-i), t-(j-i)\rangle$ is a nontrivial occurrence of μ in C'.

It follows that $NT_\mu(C) = NT_\mu(C')$. Moreover, it is easy to see that $\mathrm{cont}(C) = \mathrm{cont}(C')$. By induction $NT_\mu(C') = NT_\mu(\mathrm{cont}(C'))$. Hence, $NT_\mu(C) = NT_\mu(C') = NT_\mu(\mathrm{cont}(C')) = NT_\mu(\mathrm{cont}(C))$. □

It is easy to count the number of n-cycles $C \in \mathcal{C}_n$ for which $\mathrm{cont}(C) = A$. That is, we have the following proposition.

Proposition 2. *Let $A \in \mathcal{IC}_\ell$ be an incontractible cycle of length ℓ. The number of n-cycles $C \in \mathcal{C}_n$ such that* $\mathrm{cont}(C) = A$ *is* $\binom{n-1}{\ell-1}$.

Proof. If $\mathrm{cont}(C) = A$, then there are ℓ consecutive runs of C, namely $R_1, \ldots, R_\ell$ such that $\mathrm{red}(\max(R_1), \ldots, \max(R_\ell)) = A$. Since the maximum of one of the consecutive runs must be n, there are $\binom{n-1}{\ell-1}$ ways to choose the rest of the maxima. This will determine the consecutive runs and then we order them so that $\mathrm{red}(\max(R_1), \ldots, \max(R_\ell)) = A$.

For example, suppose $A = (1,3,5,2,4)$ and $n = 8$. The 8-cycle that contracts to A corresponding to the choice $\{1,3,4,7\}$ out of the $\binom{7}{4}$ choices is obtained by letting the maxima of the consecutive runs be $\{1,3,4,7,8\}$. Therefore, the consecutive runs are $\{1\}, \{2,3\}, \{4\}, \{5,6,7\}, \{8\}$. Then we arrange them so that $\mathrm{red}(\max(R_1), \ldots, \max(R_\ell)) = A$ and you get $\{1\}, \{4\}, \{8\}, \{2,3\}, \{5,6,7\}$, and so the cycle is $(1,4,8,2,3,5,6,7)$. □

By partitioning the n-cycles $C \in \mathcal{C}_n$ such that $NT_\mu(C) = k$ by their contractions and applying Propositions 1 and 2, we see that for any $k \geq 0$ and $n \geq 1$,

$$NT_{n,\mu}(q)|_{q^k} = \sum_{s=1}^{n} \sum_{\substack{A \in \mathcal{IC}_s,\\ NT_\mu(A)=k}} \binom{n-1}{s-1}. \tag{4}$$

We shall prove in the next section that the smallest power of q that occurs in either $NTI_{2n,\mu}(q)$ or $NTI_{2n+1,\mu}(q)$ is q^n. Hence it follows that

$$\sum_{s=1}^{n} \sum_{\substack{A \in \mathcal{IC}_s,\\ NT_\mu(A)=k}} \binom{n-1}{s-1} = \sum_{s=1}^{2k+1} \sum_{\substack{A \in \mathcal{IC}_s,\\ NT_\mu(A)=k}} \binom{n-1}{s-1}.$$

We observed in the introduction the maximum number of nontrivial occurrences of μ that one can have for s-cycle is $\binom{s-1}{2}$. This means that for a cycle of length s to have k nontrivial occurrences, we must have that $\frac{(s-1)(s-2)}{2} \geq k$ or, equivalently, $s^2 - 3s + 2 - 2k \geq 0$. It follows that it must be the case that $s \geq \frac{3+\sqrt{1+8k}}{2}$. Hence,

$$\sum_{s=1}^{2k+1} \sum_{\substack{A \in \mathcal{IC}_s, \\ NT_\mu(A)=k}} \binom{n-1}{s-1} = \sum_{s=\left\lfloor \frac{3+\sqrt{1+8k}}{2} \right\rfloor}^{2k+1} \sum_{\substack{A \in \mathcal{IC}_s, \\ NT_\mu(A)=k}} \binom{n-1}{s-1}. \tag{5}$$

Thus we have the following theorem.

Theorem 3. *For any $k \geq 0$ and $n \geq 1$,*

$$NT_{n,\mu}(q)|_{q^k} = \sum_{s=\left\lfloor \frac{3+\sqrt{1+8k}}{2} \right\rfloor}^{2k+1} c_{s,k} \binom{n-1}{s-1} \tag{6}$$

where $c_{s,k} = NTI_{s,\mu}(q)|_{q^k}$.

It follows from our tables for $NTI_{n,\mu}(q)$ and the fact that $NTI_{11,\mu}(q)|_{q^5} = C_5 = 42$ which we will prove in Corollary 7 that for all $n \geq 1$

$$\begin{aligned}
NT_{n,\mu}(q)|_{q} &= \binom{n-1}{2}, \\
NT_{n,\mu}(q)|_{q^2} &= \binom{n-1}{3} + 2\binom{n-1}{4}, \\
NT_{n,\mu}(q)|_{q^3} &= \binom{n-1}{3} + 3\binom{n-1}{4} + 6\binom{n-1}{5} + 5\binom{n-1}{6}, \\
NT_{n,\mu}(q)|_{q^4} &= 2\binom{n-1}{4} + 13\binom{n-1}{5} + 27\binom{n-1}{6} + 29\binom{n-1}{7} + 14\binom{n-1}{8}, \text{ and} \\
NT_{n,\mu}(q)|_{q^5} &= \binom{n-1}{4} + 10\binom{n-1}{5} + 51\binom{n-1}{6} + 134\binom{n-1}{7} + \\
&\quad 181\binom{n-1}{8} + 130\binom{n-1}{9} + 42\binom{n-1}{10}.
\end{aligned}$$

3. Incontractible n-cycles

Let IC_n denote the number of incontractible n-cycles in $\mathcal{C}_n$. Clearly, there is only 1 incontractible 3-cycle, namely $(1,3,2)$, and there are only 2 incontractible 4-cycles, namely $(1,4,3,2)$ and $(1,3,2,4)$. Let D_n denote the number of permutations of length n with no fixed points. For example, $D_1 = 0$, $D_2 = 1$, and $D_3 = 2$. We call D_n the n-derangement number. It is well known that $D_n = (n-1)D_{n-1} + (n-1)D_{n-2}$ for $n \geq 3$. We will show that $IC_n = (n-2)IC_{n-1} + (n-2)IC_{n-2}$ for $n \geq 5$. Our proof of this recursion will allows us to prove the following theorem.

Theorem 4.

1. *For all $n \geq 3$, $IC_n = D_{n-1}$.*
2. *For all $n \geq 2$, the lowest power of q that appears in $NTI_{2n}(q)$ is q^n and the lowest power of q that occurs in $NTI_{2n+1}(q)$ is q^n.*

Proof. For part (1), we have computed that $IC_n = D_{n-1}$ for $n \in \{3, 4\}$. Thus we need only prove that for $n \geq 5$,

$$IC_n = (n-2)IC_{n-1} + (n-2)IC_{n-2}. \tag{7}$$

Let $[n] = \{1, \ldots, n\}$. Suppose that $n \geq 5$ and $C = (1, c_2, \ldots, c_n)$ is an n-cycle in $\mathcal{IC}_n$. Let $C \upharpoonright_{[n-1]} = (1, c'_2, \ldots, c'_{n-1})$ be the cycle obtained from C by removing n from C. For example if $C = (1, 3, 6, 2, 4, 5)$ then $C \upharpoonright_{[5]} = (1, 3, 2, 4, 5)$. We then have two cases depending on whether $C \upharpoonright_{[n-1]}$ is incontractible or not.

Case 1. $C \upharpoonright_{[n-1]} \in \mathcal{IC}_{n-1}$.
In this situation, it is easy to see that if $D = C \upharpoonright_{[n-1]}$, there are exactly $n-2$ cycles $C' \in \mathcal{IC}_n$ such that $D = C' \upharpoonright_{[n-1]}$. These cycles are the result of inserting n immediately after i in D for $i = 1, \ldots, n-2$. That is, let $D^{(i)}$ be the result of inserting n immediately after i in the cycle structure of D where $1 \leq i \leq n-2$. Then it is easy to see that $D^{(i)} \in \mathcal{IC}_n$ and $D = D^{(i)} \upharpoonright_{[n-1]}$. For example, if $D = (1, 3, 5, 4, 2)$ is the 5-cycle pictured at the top of Figure 7, then $D^{(1)}, \ldots, D^{(4)}$ are pictured on the second row of Figure 7, reading from left to right. Thus, there are $(n-2)$ IC_{n-1} n-cycles $C \in \mathcal{IC}_n$ such that $C \upharpoonright_{[n-1]}$ is an element of $\mathcal{IC}_{n-1}$. (Note that $D^{(n-1)}$ is not incontractible because it is obtained by inserting n directly after $n-1$.)

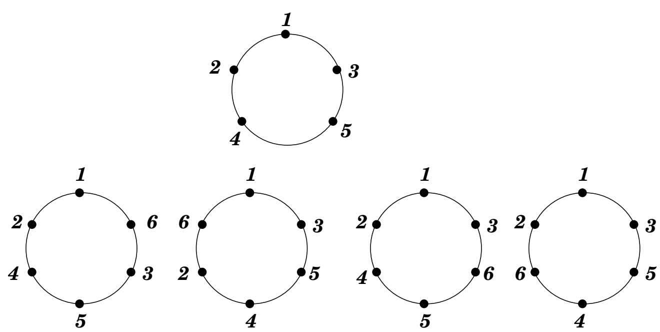

FIGURE 7. The four elements of $\mathcal{IC}_6$ that arise by inserting 6 into $C = (1, 3, 5, 4, 2)$.

We note that if $D = (1 = d_1, \ldots, d_{n-1})$ is an $(n-1)$-cycle in $\mathcal{IC}_{n-1}$, then

$$NT_\mu(D^{(i)}) \geq 1 + NT_\mu(D) \text{ for } i = 1, \ldots, n-2.$$

That is, if $\langle d_s, d_t \rangle$ is a nontrivial occurrence of μ in D, then the insertion of n does not effect whether $\langle d_s, d_t \rangle$ is a nontrivial occurrence of μ in $D^{(i)}$. Moreover, we are always guaranteed that $\langle i, n \rangle$ is a nontrivial occurrence of μ in $D^{(i)}$ since $i \leq n-2$. It is possible that $NT_\mu(D^{(i)}) - NT_\mu(D)$ is greater than or equal to 1. For example, in Figure 7, it is easy to see that if $D = (1, 3, 5, 4, 2)$, then there is only two pairs which are nontrivial occurrences of μ in C, namely $\langle 1, 3 \rangle$ and $\langle 3, 5 \rangle$, while in $D^{(1)}$, the insertion of 6 after 1 created three new nontrivial occurrences of μ in $D^{(1)}$ namely, the pairs $\langle 1, 6 \rangle$, $\langle 2, 6 \rangle$, and $\langle 4, 6 \rangle$. Thus,

$NT_\mu(D^{(1)}) - NT_\mu(D) = 5 - 2 = 3$. On the other hand, it is easy to see that if $D \in \mathcal{IC}_{n-1}$, then

$$NT_\mu(D^{(n-2)}) = 1 + NT_\mu(D). \tag{8}$$

That is, if we insert n immediately after $n-2$ in D, then for $i = 1, \ldots, n-3$, $\langle i, n\rangle$ cannot be a nontrivial occurrence of μ in $D^{(n-2)}$ because $n-2$ will be between i and n in D. Thus, we will create exactly one more pair which is a nontrivial occurrence of μ in $D^{(n-2)}$ that was not in D, namely, $\langle n-2, n\rangle$.

Case 2 $C \restriction_{[n-1]} \notin \mathcal{IC}_n$.
In this case, it must be that in C, n is the only element between i and $i+1$ in C for some i so that in $C \restriction_{[n-1]}$, $i+1$ immediately follows i. Clearly, i is the only j in $C \restriction_{[n-1]}$ such that $j+1$ immediately follows j. Hence, we can construct an element of $D \in \mathcal{IC}_{n-2}$ from $C \restriction_{[n-1]}$ by removing $i+1$ and then replacing j by $j-1$ for $i+1 < j \leq n-1$. Vice versa, given $D \in \mathcal{IC}_{n-2}$ and $1 \leq i \leq n-2$, let $D^{[i]}$ be the cycle that results from D by first replacing elements $j \geq i+1$, by $j+1$, then replacing i by a pair i immediately followed by $i+1$, and finally inserting n between i and $i+1$. This process is pictured in Figure 8 for the cycle $D = (1, 3, 2, 4)$. In such a situation, we shall say that $D^{[i]}$ arises by expanding D at i.

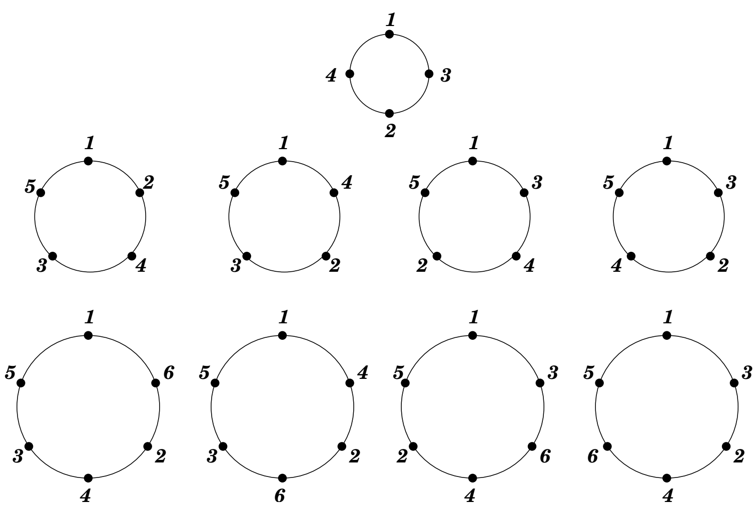

FIGURE 8. The four elements of $\mathcal{IC}_6$ that arise by expanding $D = (1, 3, 2, 4)$.

We should note that if $D = (1 = d_1, \ldots, d_{n-2})$ is an $(n-2)$-cycle in $\mathcal{IC}_{n-2}$, then

$$NT_\mu(D^{[i]}) \geq 1 + NT_\mu(D) \text{ for } i = 1, \ldots, n-2.$$

In the first step of the expansion of D at i, we replace each $j \geq i+1$ by $j+1$ and then replace i by a consecutive pair i followed by $i+1$ to get a cycle D_i. It is clear that this keeps the number of nontrivial occurrences of μ the same. That is, it is easy to see that there will be no pair $\langle i,t\rangle$ which is a nontrivial occurrence of μ in D_i since i is immediately followed by $i+1$ in D_i. Moreover, it is easy to check that

(1) if $s<t\leq i$, then $\langle s,t\rangle$ is a nontrivial occurrence of μ in D if and only if $\langle s,t\rangle$ is a nontrivial occurrence of μ in D_i,
(2) if $s<i<t$, then $\langle s,t\rangle$ is a nontrivial occurrence of μ in D if and only if $\langle s,t+1\rangle$ is a nontrivial occurrence of μ in D_i.
(3) $\langle i,t\rangle$ is a nontrivial occurrence of μ in D if and only if $\langle i+1,t+1\rangle$ is a nontrivial occurrence of μ in D_i, and
(4) if $i<s<t$, then $\langle s,t\rangle$ is a nontrivial occurrence of μ in D if and only if $\langle s+1,t+1\rangle$ is a nontrivial occurrence of μ in D_i.

Then for every pair $\langle r,s\rangle$ which is a nontrivial occurrence of μ in D_i, there exists a nontrivial occurrence of μ in $D^{[i]}$ after inserting n between i and $i+1$. Finally we will create at least one new nontrivial occurrence of μ in D, namely $\langle i,n\rangle$.

Again it is possible that $NT_\mu(D^{[i]})-NT_\mu(D)$ is greater than or equal to 1. For example, in Figure 8, it is easy to see that if $D=(1,3,5,4,2)$, there are only two pairs that are nontrivial occurrences of μ in D, namely $\langle 1,3\rangle$ and $\langle 2,4\rangle$, which correspond to the pairs $\langle 1,4\rangle$ and $\langle 3,5\rangle$ that are nontrivial occurrences of μ in $D^{[2]}$. However, the insertion of 6 between 2 and 3 in D_2 created two new nontrivial occurrences of μ in $D^{[2]}$, namely, $\langle 2,6\rangle$ and $\langle 4,6\rangle$. Thus, $NT_\mu(D^{[2]})-NT_\mu(D)=4-2=2$. On the other hand, it is easy to see that if $D\in\mathcal{IC}_{n-2}$, then

$$NT_\mu(D^{[n-2]})=1+NT_\mu(D). \tag{9}$$

That is, if we insert n immediately after $n-2$ in D_{n-2}, then for $i=1,\ldots,n-3$, $\langle i,n\rangle$ cannot be a nontrivial occurrences of μ in $D^{[n-2]}$ because $n-2$ will be between i and n in $D^{[n-2]}$. Thus, we will create exactly one occurrence of μ in $D^{[n-2]}$ that was not in D_{n-2}, namely, $\langle n-2,n\rangle$.

For part (2), note that we have shown by direct calculation that the lowest power of q appearing in $NTI_{4,\mu}(q)$ and $NTI_{5,\mu}(q)$ is q^2. Thus, the part (2) holds for $n=2$. Now, assume part (2) holds for $n\geq 2$. Then we shall show the corollary holds for $n+1$. We have shown that each $(2n+2)$-cycle C in $\mathcal{IC}_{2n+2}$ is either of the form $D^{(i)}$ for some $D\in\mathcal{IC}_{2n+1}$ and $1\leq i\leq 2n$ in which case $NT_\mu(D^{(i)})\geq 1+NT_\mu(D)$, or of the form $E^{[i]}$ for some $E\in\mathcal{IC}_{2n}$ and $1\leq i\leq 2n$ in which case $NT_\mu(E^{[i]})\geq 1+NT_\mu(E)$. But by induction, we know that $NT_\mu(D)\geq n$ and $NT_\mu(E)\geq n$ so that $NT_\mu(C)\geq n+1$. Thus, the smallest possible power of q that can appear in $NTI_{2n+2,\mu}(q)$ is $n+1$. On the other hand, we can assume by induction that there is a $D\in\mathcal{IC}_{2n+1}$ such that $NT_\mu(D)=n$ in which case we know that $NT_\mu(D^{(n-2)})=1+NT_\mu(D)=n+1$. Hence, the coefficient of q^{n+1} in $NT_{2n+2,\mu}(q)$ is non-zero.

We have shown that each $(2n+3)$-cycle C in $\mathcal{IC}_{2n+3}$ is either of the form $D^{(i)}$ for some $D\in\mathcal{IC}_{2n+2}$ and $1\leq i\leq 2n+1$ in which case $NT_\mu(D^{(i)})\geq 1+NT_\mu(D)$, or of the form $E^{[i]}$ for some $E\in\mathcal{IC}_{2n+1}$ and $1\leq i\leq 2n+1$ in which case $NT_\mu(E^{[i]})\geq 1+NT_\mu(E)$. But by induction, we know that $NT_\mu(D)\geq n+1$ and $NT_\mu(E)\geq n$ so that $NT_\mu(C)\geq n+1$.

Thus, the smallest possible power of q that can appear in $NTI_{2n+3,\mu}(q)$ is $n+1$. On the other hand, we can assume by induction that there is a $D \in \mathcal{IC}_{2n+1}$ such that $NT_\mu(D) = n$ in which case we know that $NT_\mu(D^{[n-2]}) = 1 + NT_\mu(D) = n+1$. Hence, the coefficient of q^{n+1} in $NT_{2n+3,\mu}(q)$ is non-zero. □

We have not been able to find a recursion for the polynomials $NTI_{n,\mu}(q)$. The problem with the recursion implicit in the proof of Theorem 4 is that for n-cycles $D \in \mathcal{IC}_n$, the contributions of $\sum_{i\geq 1} q^{NT_\mu(D^{(i)})}$ are not uniform. For example, if $D = (1,3,5,4,2)$, then $NT_\mu(D) = 2$, $NT_\mu(D^{(1)}) = 5$, $NT_\mu(D^{(2)}) = 4$, $NT_\mu(D^{(3)}) = 4$, and $NT_\mu(D^{(4)}) = 3$ so that $\sum_{i=1}^4 q^{NT_\mu(D^{(i)})} = (q + 2q + q^3)q^{NT_\mu(D)}$. However, $D = (1,5,4,3,2)$, then $NT_\mu(D) = 6$, $NT_\mu(D^{(1)}) = 10$, $NT_\mu(D^{(2)}) = 9$, $NT_\mu(D^{(3)}) = 8$, and $NT_\mu(D^{(4)}) = 7$ so that $\sum_{i=1}^4 q^{NT_\mu(D^{(i)})} = (q + q^2 + q^3 + q^4)q^{NT_\mu(D)}$. A similar phenomenon occurs for $\sum_{i\geq 1} NT_\mu(D^{[i]})$. For example, it is easy to see from Figure 6 that if $D = (1,3,2,4)$, then $NT_\mu(D) = 2$, $NT_\mu(D^{[1]}) = 3$, $NT_\mu(D^{[2]}) = 4$, $NT_\mu(D^{[3]}) = 3$, and $NT_\mu(D^{[4]}) = 3$, so that $\sum_{i=1}^4 q^{NT_\mu(D^{[i]})} = (3q + q^2)q^{NT_\mu(D)}$. However, as one can see from Figure 9 below that if $D = (1,4,3,2)$, then $NT_\mu(D) = 3$, $NT_\mu(D^{[1]}) = 6$, $NT_\mu(D^{[2]}) = 5$, $NT_\mu(D^{[3]}) = 4$, and $NT_\mu(D^{[4]}) = 4$, so that $\sum_{i=1}^4 q^{NT_\mu(D^{[i]})} = (2q + q^2 + q^3)q^{NT_\mu(D)}$.

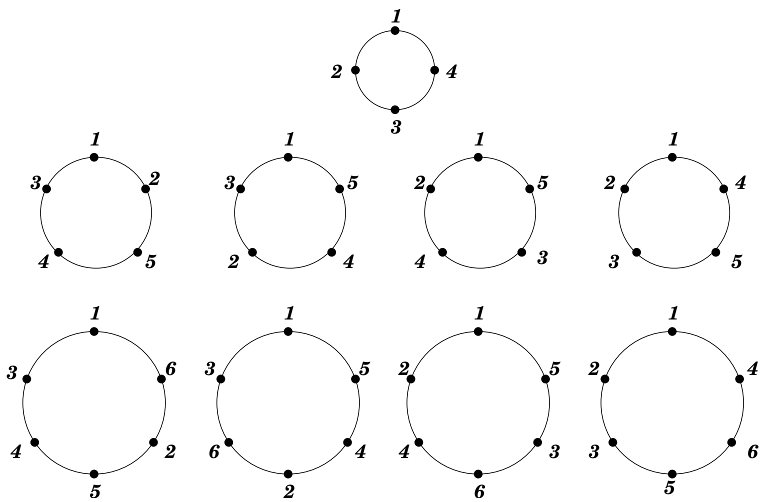

FIGURE 9. The four elements of $\mathcal{IC}_6$ that arise by expanding $D = (1,4,3,2)$.

Our next goal is to show that $NTI_{2n+1,\mu}(q)|_{q^n} = C_n$ where C_n is the nth Catalan number.

Theorem 5. *Suppose that $\sigma = \sigma_1 \ldots \sigma_{2n+1} \in S_{2n+1}$ and* $\mathrm{chpath}(\sigma) = P$ *is a Dyck path of length $2n$. Then the n-cycle $C_\sigma = (\sigma_1, \ldots, \sigma_{2n+1})$ is incontractible and $NT_\mu(C) = n$.*

Proof. Since the path P starts at $(0,0)$, it follows that $\mathrm{ind}_\sigma(\sigma_1) = 0$. This can only happen if $\sigma_1 = 1$. Thus, $C_\sigma = (1, \sigma_2, \ldots, \sigma_{2n+1})$ has the standard form of an n-cycle. Note that the only way $i+1$ immediately follows i in C_σ is if there is a j such that $\sigma_j = i$ and $\sigma_{j+1} = i+1$. But then $ind_\sigma(\sigma_j) = ind_\sigma(\sigma_{j+1})$ which would imply that the charge path of σ has a level which means that there is an edge drawn horizontally. Since we are assuming that the charge path of C_σ is a Dyck path, there can be no such i and, hence, C_σ is incontractible.

Next we claim that if the ith step of P is an up-step, then $\langle \sigma_i, \sigma_{i+1}\rangle$ is a nontrivial occurrence of μ in C_σ. That is, if the ith step of P is an up-step, then $\mathrm{ind}_\sigma(\sigma_i) = k$ and $\mathrm{ind}_\sigma(\sigma_{i+1}) = k+1$ for some $k \geq 0$. Since P is a Dyck path, there must be some $j > i+1$ such that $\mathrm{ind}_\sigma(\sigma_j) = k$ because we must pass through level k from the point $(i+1, k+1)$ in $\mathrm{chpath}(\sigma)$ to get back to the point $(2n, 0)$ which is the end point of the path. Let i_j denote the smallest $j > i$ such that $\mathrm{ind}_\sigma(\sigma_j) = k$. It follows from the definition of the function ind_σ that it must be the case that $\sigma_i = \ell$ and $\sigma_{i_j} = \ell+1$. Moreover, it must be the case that $\sigma_{i+1} > \ell+1$ so that $\langle \sigma_i, \sigma_{i+1}\rangle$ is a nontrivial occurrence of μ in C_σ. We claim that there can be no $t \neq \sigma_{i+1}$ such that $\langle \sigma_i, t\rangle$ is a nontrivial occurrence of μ in C_σ. That is, since P is a Dyck path, we know that all the vertices on P between (i, k) and (i_j, k) lie on levels strictly greater than k. It is easy to see from our inductive definition of ind_σ that it must be the case that σ_{i+1} is the least element of $\sigma_{i+1}, \sigma_{i+2}, \ldots, \sigma_{i_j-1}$ and hence the fact that σ_{i+1} immediately follows σ_i in C_σ implies that none of the pairs $\langle \sigma_i, \sigma_s\rangle$ is a nontrivial occurrence of μ in C_σ for $i+1 < s < i_j$. But then as we traverse around the cycle C_σ, the fact that $\sigma_i = \ell$ and $\sigma_{i_j} = \ell+1$ implies that none of the pairs $\langle \sigma_i, \sigma_s\rangle$ where either $i_j < s \leq 2n+1$ or $1 \leq s < i$ are nontrivial occurrences of μ in C_σ. Thus, if the ith step of P is an up-step, then $\langle \sigma_i, t\rangle$ is an occurrence of μ in C_σ for exactly one t, namely $t = \sigma_{i+1}$.

Next suppose that the ith step of P is a down-step. Then, we claim that there is no t such that $\langle \sigma_i, t\rangle$ is a nontrivial occurrence of μ in C_σ. We then have two cases.

Case 1. $\mathrm{ind}_\sigma(\sigma_i) = k$ and there is a $j > i$ such that $\mathrm{ind}_\sigma(\sigma_j) = k$.
In this case, let i_j be the least $j > i$ such that $\mathrm{ind}_\sigma(\sigma_j) = k$. It then follows that if $\sigma_i = \ell$, then $\sigma_{i_j} = \ell+1$. Because P is a Dyck path, it must be the case that $\mathrm{ind}_\sigma(\sigma_s) < k$ for all $i < s < i_j$ so that $\sigma_s < \ell$ for all $i < s < i_j$. But this means that as we traverse around the cycle C_σ, the first element that we encounter after $\sigma_i = \ell$ which is bigger than ℓ is $\sigma_{i_j} = \ell+1$ and hence there is no $t > \sigma_i$ such that $\langle \sigma_i, t\rangle$ is a nontrivial occurrence of μ in C_σ.

Case 2. $\mathrm{ind}_\sigma(\sigma_i) = k$ and there is no $j > i$ such that $\mathrm{ind}_\sigma(\sigma_j) = k$.
In this case since P is a Dyck path, all the elements σ_s such that $s > i$ must have $\mathrm{ind}_\sigma(\sigma_s) < k$ and hence $\sigma_s < \sigma_i$ since, in a charge graph, all the elements whose index in σ is less than k are smaller than all the elements whose index is equal to k in σ. We now have two subcases.

Subcase 2.1. There is an $s < i$ such that $\mathrm{ind}_\sigma(\sigma_s) = k+1$.
Let s_j be the smallest $s < i$ such that $\mathrm{ind}_\sigma(\sigma_s) = k+1$. Since σ_i was the rightmost element whose index relative to σ equals k, it follows that if $\sigma_i = \ell$, then $\sigma_{s_j} = \ell+1$ since it is the leftmost element whose index relative to σ equals $k+1$. Moreover, for all $t < s_j$, either

σ_t has index less than k in σ or σ_t has index k in σ. In either case, our definition of ind_σ ensures that $\sigma_t < \ell$. But this means that as we traverse around the cycle C_σ, the first element that we encounter after $\sigma_i = \ell$ which is bigger than ℓ is $\sigma_{s_j} = \ell + 1$ and hence there is no $t > \sigma_i$ such that $\langle \sigma_i, t\rangle$ is a nontrivial occurrence of μ in C_σ.

Subcase 2.2. There is no $s < i$ such that $\mathrm{ind}_\sigma(\sigma_s) = k + 1$.
In this case, σ_i has the highest possible index in σ and it is the rightmost element whose index in σ is k which means that $\sigma_i = 2n + 1$. Hence, there is no $t > \sigma_i$ such that $\langle \sigma_i, t\rangle$ is an occurrence of μ in C_σ.

The only other possibility is that we have not considered is that $i = 2n + 1$. Since P is a Dyck path, we know that $\mathrm{ind}_\sigma(\sigma_{2n+1}) = 0$, $\mathrm{ind}_\sigma(\sigma_1) = 0$, and $\mathrm{ind}_\sigma(\sigma_2) = 1$. But then it follows that if $\sigma_{2n+1} = \ell$, then $\sigma_1 < \ell$ and $\sigma_2 = \ell + 1$. But this means that as we traverse around the cycle C_σ, the first element that we encounter after $\sigma_{2n+1} = \ell$ which is bigger than ℓ is $\sigma_2 = \ell + 1$ and hence there is no $t > \sigma_i$ such that $\langle \sigma_{2n+1}, t\rangle$ is a nontrivial occurrence of μ in C_σ.

Thus, we have shown that the nontrivial occurrences of μ in C_σ correspond to the up-steps of $\mathrm{chpath}(\sigma)$ and hence $NT_\mu(C_\sigma) = n$. ☐

Our next theorem will show that if $C = (1, \sigma_2, \dots, \sigma_{2n+1})$ is a $(2n + 1)$-cycle in $\mathcal{IC}_{2n+1}$ where $NT_\mu(C) = n$, then the charge path of $\sigma_C = 1\sigma_2 \dots \sigma_{2n+1}$ must be a Dyck path.

Theorem 6. *Let $C = (1, \sigma_2, \dots, \sigma_{2n+1}) \in \mathcal{IC}_{2n+1}$ and $\sigma_C = 1\sigma_2 \dots \sigma_{2n+1}$. Then, if $NT_\mu(C) = n$, the charge path of σ_C must be a Dyck path.*

Proof. Our proof proceeds by induction on n. For $n = 1$, the only element of $\mathcal{IC}_3$ is $C = (1, 3, 2)$ and it is easy to see that $\mathrm{chpath}(132)$ is a Dyck path.

Now assume by induction that if $D = (1, \sigma_2, \dots, \sigma_{2n+1})$ is a $(2n+1)$-cycle in $\mathcal{IC}_{2n+1}$ such that $NT_\mu(D) = n$, then the charge path of $\sigma_D = 1\sigma_2 \dots \sigma_{2n+1}$ is a Dyck path. Now suppose that $C = (1, \sigma_2, \dots, \sigma_{2n+3})$ is a $(2n + 3)$-cycle in $\mathcal{IC}_{2n+3}$ such that $NT_\mu(C) = n + 1$. Let $\sigma_C = \sigma = 1\sigma_2 \dots \sigma_{2n+3}$. Then, we know by our proof of Theorem 4 that either $C = D^{(j)}$ for some $D \in \mathcal{IC}_{2n+2}$ or $C = D^{[j]}$ for some $D \in \mathcal{IC}_{2n+1}$. We claim that C cannot be of the form $D^{(j)}$ for some $D \in \mathcal{IC}_{2n+2}$. That is, we know that for all $D \in \mathcal{IC}_{2n+2}$ we have $NT_\mu(D) \geq n+1$ and, hence, $NT_\mu(D^{(j)}) \geq 1 + NT_\mu(D) \geq n+2$. Thus, there must be some $D \in \mathcal{IC}_{2n+1}$ such that $C = D^{[j]}$ for some j. But then we know that $P_D = \mathrm{chpath}(D)$ is a Dyck path. Let $D = (1, \tau_2, \dots, \tau_{2n+1})$ and $\sigma_D = \tau = \tau_1 \dots \tau_{2n+1}$. Note that it follows from our arguments in Theorem 4 that $NT_\mu(D^{[j]}) - NT_\mu(D)$ is equal to the number of pairs $\langle s, 2n + 3\rangle$ which are nontrivial occurrences of μ in $D^{[j]}$. It is also the case that the charge path of σ is easily constructed from the charge path of τ. That is, suppose that $\tau_i = j$. Then $1, \dots, j$ are in the same order in σ and τ so that $\mathrm{ind}_\sigma(s) = \mathrm{ind}_\tau(s)$ for $1 \leq s \leq j$. In going from τ to σ, we first increased the value of any $t > j$ by one and then inserted $j + 1$ immediately after j to get a permutation α. Thus $\mathrm{ind}_\alpha(j) = \mathrm{ind}_\alpha(j + 1)$. Moreover, $j+1, \dots, 2n+2$ are in the same relative order in α as $j, \dots, 2n+1$ in τ so that for $j+1 \leq t \leq 2n+1$, $\mathrm{ind}_\tau(t) = \mathrm{ind}_\alpha(t+1)$. Finally, inserting $2n+3$ in α between j and $j+1$

has no effect on the indices assigned to $1,\dots,2n+2$. Thus for $1 \leq s \leq j$, $\text{ind}_\sigma(j) = \text{ind}_\tau(j)$ and for $j+1 \leq t \leq 2n+1$, $\text{ind}_\sigma(t+1) = \text{ind}_\tau(t)$. This means that one can construct the charge path of σ by essentially starting with the charge graph of τ, then replacing the vertex $(i, \text{ind}_\tau(\tau_i))$ by a horizontal edge, and finally replacing that horizontal edge by a pair of edges $\{(i, \text{ind}_\sigma(\sigma_i)), (i+1, \text{ind}_\sigma(2n+3)\}$ and $\{(i+1, \text{ind}_\sigma(2n+3)), ((i+2), \text{ind}_\sigma(\sigma_{i+2})\}$. In particular, this means the charge paths from 1 up to i are identical for both σ and τ and the charge path from $i+2$ to $2n+3$ in σ is identical to the charge path from i to $2n+1$ in τ.

We now consider several cases depending on which i is such that $\tau_i = j$ and the value $\text{ind}_\tau(\tau_i)$ in the Dyck path P_D.

Case 1. $j = \tau_i$ where $\text{ind}_\tau(\tau_i) < \text{ind}_\tau(\tau_{i+1})$.
In this case, we will have $\sigma_i = \tau_i$, $\sigma_{i+1} = 2n+3$, $\sigma_{i+2} = 1+\tau_i$, and $\sigma_{i+3} = 1+\tau_{i+1}$. Since P_D is a Dyck path, there will be some k such that $\text{ind}_\tau(\tau_i) = k$, and $\text{ind}_\tau(\tau_{i+1}) = k+1$ so that we will be in the situation pictured in Figure 10. Because $\text{ind}_\sigma(\sigma_{i+3}) = k+1$, it must be the case that $\text{ind}_\sigma(2n+3) > k+1$. It is easy to see that $\langle \sigma_i, 2n+3\rangle$ is an occurrence of μ in C. We then have two subcases.

Subcase 1A. There is an $s < i$ such that $\text{ind}_\sigma(\sigma_s) = k+1$.
In this case, let s_i be the largest s such that $s < i$ and $\text{ind}_\sigma(\sigma_s) = k+1$. Because P_D is a Dyck path, it must be the case that for all $s_i < t < i$, $\text{ind}_\sigma(\sigma_{s_i}) \leq k$ because on the path from $(s, k+1)$ to (i, k), we cannot have an $s < t < i$ such that $\text{ind}_\sigma(\sigma_t) > k+1$ without having some $t < u < i$ such that $\text{ind}_\sigma(\sigma_u) = k+1$ which would violate our choice of s_i. But this means that $\sigma_t < \sigma_i$ for all $s_i < t < \sigma_i$ and hence, $\langle \sigma_{s_i}, 2n+3\rangle$ is a nontrivial occurrence of μ in C. Thus, $NT_\mu(C) \geq NT_\mu(D) + 2 = n+2$.

Subcase 1B. There is no $s < i$ such that $\text{ind}_\sigma(\sigma_s) = k+1$.
In this case, let s_i be the largest s such that $i < s$ and $\text{ind}_\sigma(\sigma_s) = k$. Because P_D is a Dyck path such an s must exist since $\text{ind}_\sigma(\sigma_{i+3}) = k+1$ and a Dyck path that reaches level $k+1$ must subsequently descend to level k. But this means that for all $t < i$, $\text{ind}_\sigma(\sigma_t) \leq k$ and hence, $\sigma_t \leq \sigma_i$. Moreover, for all $r > s_i$, $\text{ind}_\sigma(\sigma_r) < k$ and hence, $\sigma_r \leq \sigma_i$. But it then follows that since $\sigma_{s_i} > \sigma_i$, $\langle \sigma_{s_i}, 2n+3\rangle$ is a nontrivial occurrence of μ in C. Thus, $NT_\mu(C) \geq NT_\mu(D) + 2 = n+2$.

Case 2. $j = \tau_{2n+1}$.
In this case, we will have $\sigma_{2n+1} = \tau_{2n+1}$, $\sigma_{2n+2} = 2n+3$ and $\sigma_{2n+3} = \tau_{2n+1}+1$. Thus, the charge path of $D^{[\tau_{2n+1}]}$ looks like one of the two situations pictured in Figure 11. That is, we have two subcases.

Subcase 2A. $\text{ind}_\sigma(\sigma_{2n+2}) = 1$.
In this case, $\text{chpath}(\sigma)$ will be a Dyck path. This case can only happen if $\text{ind}_\tau(\tau_i) \leq 1$ for all $1 \leq i \leq 2n+1$.

Subcase 2B. $\text{ind}_\sigma(\sigma_{2n+2}) \geq 2$.

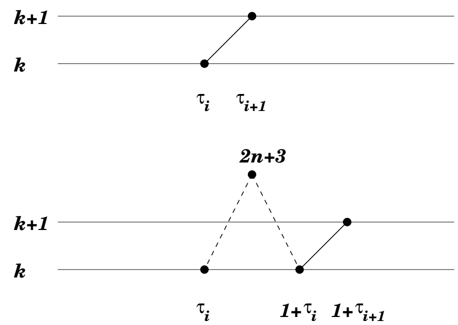

FIGURE 10. The charge path of $D^{[\tau_{2n+1}]}$ in Case 1.

In this case, chpath(σ) will not be a Dyck path. This case can only happen if there is an s such that $\mathrm{ind}_\tau(\tau_s) = \mathrm{ind}_\sigma(\tau_i + s) \geq 2$. But then since $\mathrm{ind}_\sigma(\sigma_{2n}) = 1$, it must be the case that $\sigma_{2n} < \tau_s + 1 < 2n + 3$. Hence both $\langle \sigma_{2n+1}, 2n+3 \rangle$ and $\langle \sigma_{2n}, 2n+3 \rangle$ are nontrivial occurrences of μ in C so that $NT_\mu(C) \geq NT_\mu(D) + 2 = n + 2$.

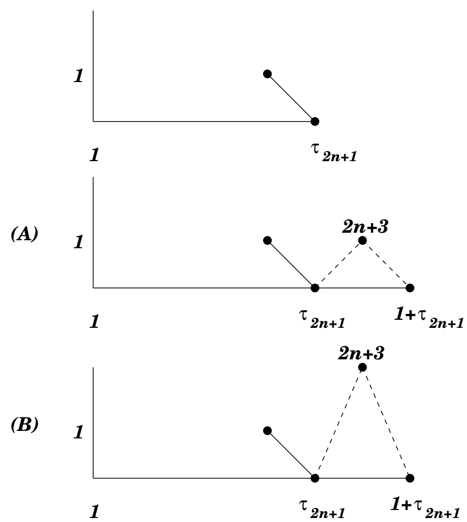

FIGURE 11. The charge path of $D^{[\tau_{2n+1}]}$ in Case 2.

Case 3. $j = \tau_i$ where $\mathrm{ind}_\tau(\tau_{i-1}) > \mathrm{ind}_\tau(\tau_i) > \mathrm{ind}_\tau(\tau_{i+1})$.
In this case, we will have $\sigma_{i-1} = 1 + \tau_{i-1}$, $\sigma_i = \tau_i$, $\sigma_{i+1} = 2n+3$, $\sigma_{i+2} = 1 + \tau_i$, and $\sigma_{i+3} = \tau_{i+1}$. There are now two cases.

Case 3A. $\tau_{i-1} < 2n + 1$.
In this case, we will have the situation pictured in Figure 12. That is, since $\sigma_{i-1} = 1 + \tau_{i-1} < 2n + 2$, it must be the case the either $2n+2$ is to the left of σ_{i-1} in which case

$\mathrm{ind}_\sigma(2n+2) > k+1$ or $2n+2$ is the right of σ_{i+3} in which case $\mathrm{ind}_\sigma(2n+2) \geq k+1$ and $2n+3$ is to the left of $2n+2$ in σ. In either case, it must be that $\mathrm{ind}_\sigma(2n+3) > k+1$. It is then easy to see that both $\langle \sigma_{i-1}, 2n+3\rangle$ and $\langle \sigma_i, 2n+3\rangle$ are nontrivial occurrences of μ in C so that $NT_\mu(C) \geq NT_\mu(D)+2 = n+2$.

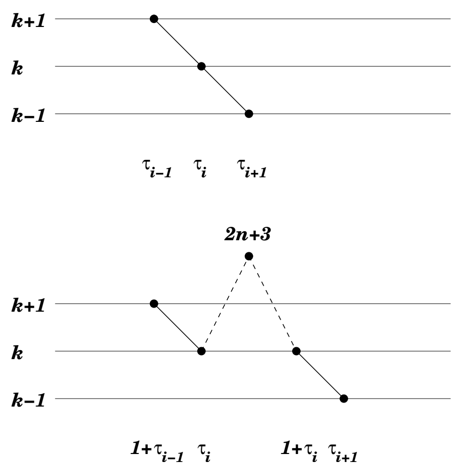

FIGURE 12. The charge path of $D^{[\tau_i]}$ in Case 3A.

Case 3B. $\tau_{i-1} = 2n+1$.
In this case, we will have the situation pictured in Figure 13. In this situation, the charge path of $D^{[\tau_i]}$ will be a Dyck path.

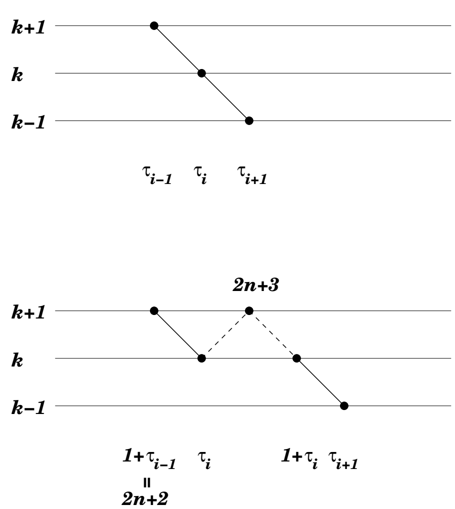

FIGURE 13. The charge path of $D^{[\tau_i]}$ in Case 3B.

Case 4. $j = \tau_i$ where $\text{ind}_\tau(\tau_{i-1}) < \text{ind}_\tau(\tau_i) > \text{ind}_\tau(\tau_{i+1})$.
In this case, we will have $\sigma_{i-1} = \tau_{i-1}$, $\sigma_i = \tau_i$, $\sigma_{i+1} = 2n+3$, $\sigma_{i+2} = 1+\tau_i$, and $\sigma_{i+3} = \tau_{i+1}$. There are now two subcases.

Subcase 4A. $\text{ind}_\sigma(2n+3) > k+1$.
In this case, we have the situation pictured in Figure 14. It will always be the case that $\langle \sigma_i, 2n+3\rangle$ is a nontrivial occurrence of μ in C. We then have two more subcases.

Subcase 4A1. There is an $s < i$ such that $\text{ind}_\sigma(\sigma_s) = k+1$.
In this case, let s_i be the largest s such that $s < i$ and $\text{ind}_\sigma(\sigma_s) = k+1$. Because P_D is a Dyck path, we can argue as in Case 1A that it must be the case that for all $s_i < t < i$, $\text{ind}_\sigma(\sigma_{s_t}) \leq k$. But this means that $\sigma_t < \sigma_i$ for all $s_i < t < \sigma_i$ and hence, $\langle \sigma_{s_i}, 2n+3\rangle$ is also a nontrivial occurrence of μ in C. Thus, $NT_\mu(C) \geq NT_\mu(D) + 2 = n+2$.

Subcase 4A2. There is no $s < i$ such that $\text{ind}_\sigma(\sigma_s) = k+1$.
In this case, let s_i be the largest s such that $i < s$ and $\text{ind}_\sigma(\sigma_s) = k$. Note that s_i exists since $\text{ind}_\sigma(\sigma_{i+2}) = k$. But this means that for all $t < i$, $\text{ind}_\sigma(\sigma_t) \leq k$ and, hence, $\sigma_t < \sigma_i$. Moreover, for all $r > s_i$, $\text{ind}_\sigma(\sigma_r) < k-1$ and hence, $\sigma_r < \sigma_i$. But it then follows that since $\sigma_{s_i} > \sigma_i$, $\langle \sigma_{s_i}, 2n+3\rangle$ is also a nontrivial occurrence of μ in C. Thus, $NT_\mu(C) \geq NT_\mu(D) + 2 = n+2$.

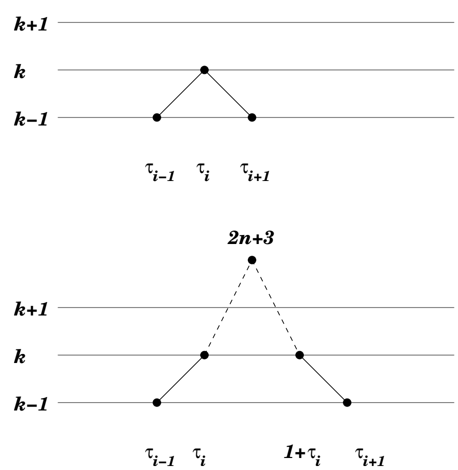

FIGURE 14. The charge path of $D^{[\tau_i]}$ in Case 4A.

Subcase 4B. $\text{ind}_\sigma(2n+3) = k+1$.
In this case, we have the situation pictured in Figure 15 and, hence, $\text{chpath}(\sigma)$ will be a Dyck path.

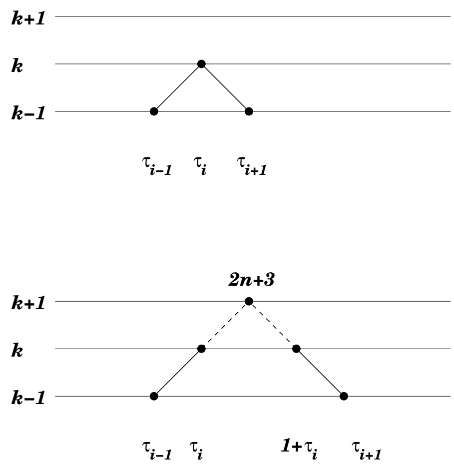

FIGURE 15. The charge path of $D^{[\tau_i]}$ in Case 4B.

Thus, we have shown that if $D \in \mathcal{IC}_{2n+1}$ is such that $NT_\mu(D) = n$ and $D^{[i]} = C$, then either chpath(σ) is a Dyck path or $NT_\mu(D^{[i]}) \geq n+2$. Hence, if $NT_\mu(C) = n+1$, then chpath(σ) is a Dyck path. □

Theorems 5 and 6 yield the following corollary.

Corollary 7. *$NTI_{2n+1,\mu}(q)|_{q^n} = C_n$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number.*

4. Plots and Integer Partitions

In this section we will show that $NTI_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}}$ and $NT_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}}$ are equal to the number of partitions of k for sufficiently large n by plotting the *non-μ-matches* on a grid and mapping them to Ferrers diagrams of partitions.

Given an n-cycle $C \in \mathcal{C}_n$, we say that a pair $\langle i,j\rangle$ with $i<j$ is a non-μ-match of C if $\langle i,j\rangle$ is not an occurrence of μ in C. In other words, $\langle i,j\rangle$ is a non-μ-match if there exists an integer x such that $i<x<j$ and x is cyclically between i and j in C. Furthermore, let $\mathcal{NM}_\mu(C)$ be the set of non-μ-matches in C and let $NM_\mu(C) = |\mathcal{NM}_\mu(C)|$. Let

$$NM_{n,\mu}(q) = \sum_{C \in \mathcal{C}_n} q^{NM_\mu(C)}. \tag{10}$$

Note that for any $C \in \mathcal{C}_n$, $NM_\mu(C) + NT_\mu(C) = \binom{n-1}{2}$ so that

$$NM_{n,\mu}(q)|_{q^k} = NT_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}}.$$

Table 3 of the appendix shows the polynomials $NM_{n,\mu}(q)$ for $1 \leq n \leq 10$. Our data suggests the following theorem.

Theorem 8. *For $k < n-2$, $NM_{n,\mu}(q)|_{q^k} = a(k)$ where $a(k)$ is the number of integer partitions of k.*

To prove Theorem 8, we need to define what we call the plot of non-μ-matches in $C \in \mathcal{C}_n$. We consider an $n \times n$ grid where the rows are labeled with $1, 2, \ldots, n$, reading from bottom to top, and the columns are labeled with $1, 2, \ldots, n$, reading from left to right. The cell (i, j) is the cell which lies in the ith row and jth column. Then, given a set S of ordered pairs $\langle i, j \rangle$ with $1 \leq i \leq n$, we let $plot_n(S)$ denote the diagram that arises by shading a cell (j, i) on the $n \times n$ grid if and only if $\langle i, j \rangle \in S$. Given an n-cycle $C \in \mathcal{C}_n$, we let $NMplot(C) = plot_n(\mathcal{NM}_\mu(C))$. For example, if $C = (1, 4, 5, 3, 8, 7, 2, 6)$, then

$$\mathcal{NM}_\mu(C) = \{\langle 1,5\rangle, \langle 1,6\rangle, \langle 1,7\rangle, \langle 1,8\rangle, \langle 2,5\rangle, \langle 2,7\rangle, \langle 2,8\rangle, \langle 3,5\rangle, \langle 4,6\rangle, \langle 4,7\rangle, \langle 4,8\rangle\}$$

and $NMplot(C)$ is pictured in Figure 16.

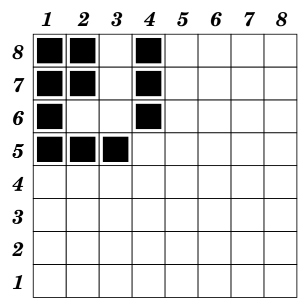

FIGURE 16. $NMplot((1, 4, 5, 3, 8, 7, 2, 6))$.

First, we observe that if $C \in \mathcal{C}_n$, then $\mathcal{NM}_\mu(C)$ completely determines C. That is, we have the following theorem.

Theorem 9. *If S is a set of ordered pairs such that $\mathcal{NM}_\mu(C) = S$ for some n-cycle $C \in \mathcal{C}_n$, then C is the only n-cycle such that $\mathcal{NM}_\mu(C) = S$.*

Proof. Our proof proceeds by induction on n. Clearly, the theorem holds for $n = 1$ and $n = 2$. Now, assume that the theorem holds for n. Let $C' \in \mathcal{C}_{n+1}$ and $S = \mathcal{NM}_\mu(C')$. Let $C = (1, c_2, \ldots, c_n)$ be the n-cycle that is obtained from C by removing $n+1$. Then it is easy to see that $\mathcal{NM}_\mu(C) = S - \{\langle i, n+1\rangle : \langle i, n+1\rangle \in S\}$. Hence, C is the unique n-cycle in $\mathcal{C}_n$ such that $\mathcal{NM}_\mu(C) = S - \{\langle i, n+1\rangle : \langle i, n+1\rangle \in S\}$. Let $C^{(i)}$ denote the cycle that results by inserting $n+1$ immediately after i in C. Since C is unique, it follows that C' must equal $C^{(i)}$ for some $1 \leq i \leq n$. However, it is easy to see that if $1 \leq i < n$, then $\langle j, n+1\rangle \in \mathcal{NM}_\mu(C^{(i)})$ for $1 \leq j < i$ and $\langle i, n+1\rangle \notin \mathcal{NM}_\mu(C^{(i)})$. If $i = n$, then $\langle j, n+1\rangle \in \mathcal{NM}_\mu(C^{(i)})$ for $1 \leq j < n$. Thus, it follows that $\mathcal{NM}_\mu(C^{(1)}), \ldots, \mathcal{NM}_\mu(C^{(n)})$ are pairwise distinct. Hence, there is exactly one cycle C' such that $\mathcal{NM}_\mu(C') = S$. $\square$

If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is a partition of n, the *Ferrers diagram* of a partition λ on the $m \times m$ grid, denoted $FD_m(\lambda)$, is the diagram that results by shading the squares $(1, m - i + 1), (2, m - i + 1), \ldots, (\lambda_i, m - i + 1)$ in the ith row from the top for $i = 1, \ldots, \ell$. For example, the Ferrers diagram of $\lambda = (5, 4, 3, 1, 1)$ on a 8×8 grid is pictured in Figure 17.

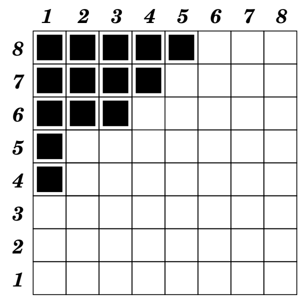

FIGURE 17. $FD_8(5,4,3,1,1)$

It is easy to see that the following three properties characterize the Ferrers diagrams of partitions λ in an $n \times n$ grid.

(1) If λ is not empty, then the square $(n,1)$ is shaded in $FD_m(\lambda)$.
(2) If $x \neq 1$ and the square (y,x) is shaded in $FD_m(\lambda)$, then the square $(y,x-1)$ is shaded in $FD_m(\lambda)$.
(3) If $y \neq n$ and the square (y,x) is shaded in $FD_m(\lambda)$, then the square $(y+1,x)$ is shaded in $FD_m(\lambda)$.

This means that if S is a set of pairs (i,j) such that $plot_n(S)$ is a Ferrers diagram, then it must be the case that

(1) if S is not empty, then the square $(1,n) \in S$,
(2) if $x \neq 1$ and the square $(x,y) \in S$, then $(x-1,y) \in S$, and
(3) if $y \neq n$ and the square $(x,y) \in S$, then $(x,y+1) \in S$.

Our next lemma and corollary will show that if $C \in \mathcal{C}_n$ and $NM_\mu(C) < n-2$, then $\mathcal{NM}_\mu(C)$ has the same three properties.

Lemma 10. *Assume that $C \in \mathcal{C}_n$.*

(1) *If $\langle 1,n\rangle \notin \mathcal{NM}_\mu(C)$ and $NM_\mu(C) \neq 0$, then $NM_\mu(C) \geq n-2$.*
(2) *If $x \neq 1$, $\langle x,y\rangle \in \mathcal{NM}_\mu(C)$, and $\langle x-1,y\rangle \notin \mathcal{NM}_\mu(C)$, then $NM_\mu(C) \geq n-2$.*
(3) *If $y \neq n$, $\langle x,y\rangle \in \mathcal{NM}_\mu(C)$, and $\langle x,y+1\rangle \notin \mathcal{NM}_\mu(C)$, then $NM_\mu(C) \geq n-2$.*

Proof. Let $C = (1,c_1,\ldots,c_{n-1}) \in \mathcal{C}_n$ be an n-cycle where $n \geq 4$.

For part (1), suppose that $\langle 1,n\rangle \notin \mathcal{NM}_\mu(C)$ and $NM_\mu(C) \neq 0$. Then, there are no integers that are cyclically in between 1 and n in C so that $c_1 = n$. Since $NM_\mu(C) \neq 0$, $C \neq (1,n,n-1,\ldots,3,2)$. Thus, C must be of the form

$$C = (1,n,n-1,\ldots,n-a,b,\underbrace{\ldots}_{A}b+1\underbrace{\ldots}_{B})$$

where $b+1 < n-a$. That is, the sequence $c_1 > c_2 > \cdots > c_{a+1}$ consists of the decreasing interval from n to $n-a$ and then $b \leq n-a-2$. It follows that the pairs $\langle 1,b+1\rangle, \langle b,n\rangle,\ldots,\langle b,n-a\rangle$ are all *non-μ-matches* in C which accounts for $a+2$ non-μ-matches in C. Let A be the set of integers cyclically between b and $b+1$ in C and let B be the set of integers cyclically between $b+1$ and 1 in C. Note that

(1) if $x \in A$ and $x < b$, then $\langle x, n\rangle \in \mathcal{NM}_\mu(C)$,
(2) if $x \in A$ and $x > b$, then $\langle 1, x\rangle \in \mathcal{NM}_\mu(C)$,
(3) if $x \in B$ and $x < b$, then $\langle x, b+1\rangle \in \mathcal{NM}_\mu(C)$, and
(4) if $x \in B$ and $x > b$, then $\langle 1, x\rangle \in \mathcal{NM}_\mu(C)$.

It follows that each element in A and B is part of at least one non-μ-match in C so that we know that $NM_\mu(C) \geq a + 2 + |A| + |B|$. Thus, since $|A| + |B| = n - a - 4$, we have $NM_\mu(C) \geq n - 2$.

For part (2), suppose that $x \neq 1$, $\langle x, y\rangle \in \mathcal{NM}_\mu(C)$, and $\langle x-1, y\rangle \notin \mathcal{NM}_\mu(C)$. Then x cannot be cyclically between $x-1$ and y in C. Also, there exists an integer z with $x < z < y$ such that z is cyclically between x and y in C, but z is not cyclically between $x-1$ and y in C. It follows that C is of the form

$$C = (\underbrace{1, \ldots}_{A_3}, x, \underbrace{\ldots, z, \ldots}_{A_1}, x-1, \underbrace{\ldots}_{A_2}, y, \underbrace{\ldots}_{A_3})$$

where A_1 is the set of integers cyclically between x and $x-1$ in C that are not equal to z, A_2 is the set of integers cyclically between $x-1$ and y in C, and A_3 is the set of integers between y and x in C. Note that $\langle x, y\rangle$ and $\langle x-1, z\rangle$ are in $\mathcal{NM}_\mu(C)$. Moreover,

(1) if $a \in A_1$ and $a < x-1$, then $\langle a, y\rangle \in \mathcal{NM}_\mu(C)$,
(2) if $a \in A_1$ and $a > x$, then $\langle x-1, a\rangle \in \mathcal{NM}_\mu(C)$,
(3) if $a \in A_2$ and $a < x-1$, then $\langle a, z\rangle \in \mathcal{NM}_\mu(C)$,
(4) there are no $a \in A_2$ with $x-1 < a < y$ since $\langle x-1, y\rangle \notin \mathcal{NM}_\mu(C)$,
(5) if $a \in A_2$ and $a > y$, then $\langle 1, a\rangle \in \mathcal{NM}_\mu(C)$,
(6) if $a \in A_3$ and $a < z$, then $\langle a, y\rangle \in \mathcal{NM}_\mu(C)$, and
(7) if $a \in A_3$ and $a > z$, then $\langle x, a\rangle \in \mathcal{NM}_\mu(C)$.

Therefore, any integer $a \in A_1 \cup A_2 \cup A_3$ is part of a distinct non-μ-match in C. Since $|A_1 \cup A_2 \cup A_3| = n - 4$, it follows that $NM_\mu(C) \geq 2 + n - 4 = n - 2$.

For part (3), suppose that $y \neq n$, $\langle x, y\rangle \in \mathcal{NM}_\mu(C)$, and $\langle x, y+1\rangle \notin \mathcal{NM}_\mu(C)$. Then y cannot be cyclically between x and $y+1$ in C. Also, there exists an integer z with $x < z < y$ such that z is cyclically between x and y in C, but z is not cyclically between x and $y+1$ in C. Thus, C must be of the form

$$C = (\underbrace{1, \ldots}_{B_3}, x, \underbrace{\ldots}_{B_1}, y+1, \underbrace{\ldots, z, \ldots}_{B_2}, y, \underbrace{\ldots}_{B_3})$$

where B_1 is the set of integers cyclically between x and $y+1$ in C, B_2 is the set of integers cyclically between $y+1$ and y in C that are not equal to z, and B_3 is the set of integers cyclically between y and x in C. Note that $\langle x, y\rangle$ and $\langle z, y+1\rangle$ are in $\mathcal{NM}_\mu(C)$. Moreover,

(1) if $b \in B_1$ and $b < x$, then $\langle b, y\rangle \in \mathcal{NM}_\mu(C)$,
(2) there is no $b \in B_1$ with $x < b < y+1$ since $\langle x, y+1\rangle \notin \mathcal{NM}_\mu(C)$,
(3) if $b \in B_1$ and $b > y+1$, then $\langle z, b\rangle \in \mathcal{NM}_\mu(C)$,
(4) if $b \in B_2$ and $b < y$, then $\langle b, y+1\rangle \in \mathcal{NM}_\mu(C)$,
(5) if $b \in B_2$ and $b > y+1$, then $\langle x, b\rangle \in \mathcal{NM}_\mu(C)$,

(6) if $b \in B_3$ and $b < z$, then $\langle b, y\rangle \in \mathcal{NM}_\mu(C)$, and
(7) if $b \in B_3$ and $b > z$ then $\langle x, b\rangle \in \mathcal{NM}_\mu(C)$.

Therefore, any integer $b \in B_1 \cup B_2 \cup B_3$ is part of a distinct non-μ-match in C. Since $|B_1 \cup B_2 \cup B_3| = n-4$, it follows that $NM_\mu(C) \geq 2+n-4 = n-2$. $\square$

Corollary 11. *If $NM_\mu(C) < n-2$, then $NMplot(C)$ is a Ferrers diagram.*

Proof. This corollary follows directly from Lemma 10. That is, suppose that $C \in \mathcal{C}_n$ and $NM_\mu(C) < n-2$. First, if $NM_\mu(C) = 0$, then $C = (1, n, n-1, \ldots, 2)$ in which case $FD_n(C)$ has no shaded squares which correspond to the empty partition. Thus, assume that $1 \leq NM_\mu(C) < n-2$. Then, it follows from part (1) of Lemma 10 that $\langle 1, n\rangle \in \mathcal{NM}_\mu(C)$. Next it follows from part (2) of Lemma 10 that if $x \neq 1$ and $\langle x, y\rangle \in \mathcal{NM}_\mu(C)$, then $\langle x-1, y\rangle \in \mathcal{NM}_\mu(C)$. Finally, it follows from part (3) of Lemma 10 that if $y \neq n$ and $\langle x, y\rangle \in \mathcal{NM}_\mu(C)$, then $\langle x, y+1\rangle \in \mathcal{NM}_\mu(C)$. Hence, the shaded cells $NMplot(C)$ must be a Ferrers diagram of a partition λ of $NM_\mu(C)$. $\square$

If $n \geq 3$, we let T_n be the plot of the Ferrers diagram in the $n \times n$ grid corresponding the to partition $\lambda = (n-2, n-3, \ldots, 1)$. Thus, for example, T_7 is pictured in Figure 18. One can see that the sets that T_n have the property that $|\{\lambda : \lambda \text{ is a partition and } FD_n(\lambda) \subseteq T_n\}| = C_n$ where C_n is the nth Catalan number since the lower boundaries of the plots of $FD(\lambda)$ for such λ correspond to Dyck paths.

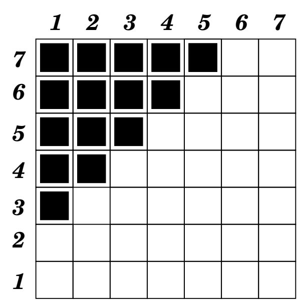

FIGURE 18. The Ferrers diagram T_7 on the 7×7 grid.

Our next theorem will show that for any $n \geq 3$ and any partition $\lambda \subseteq T_n$, we can construct an n-cycle $C \in \mathcal{C}_n$ such that $NMplot(C) = FD_n(\lambda)$.

Theorem 12. *Suppose that $n \geq 3$ and λ is a partition such that $\lambda \subseteq T_n$. Then, there is an n-cycle $C \in \mathcal{C}_n$ such that $NMplot(C) = FD_n(\lambda)$.*

Proof. We proceed by induction on n. For $n = 3$, it is easy to see that if $C^{(1)} = (1, 3, 2)$, the $NMplot(C^{(1)})$ is empty and if $C^{(2)} = (1, 2, 3)$, then $NMPlot(C^{(2)}) = T_3$. Note that $C^{(1)}$ and $C^{(2)}$ have the property that if the largest part of the corresponding partition is of size i, then 3 immediately follows $i+1$ in the cycle. Thus, our theorem holds for $n = 3$.

Now, suppose that $n > 3$ and $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition which is contained in T_n. It follows that $\lambda^- = (\lambda_2, \ldots, \lambda_k)$ is contained in T_{n-1}. Assume by induction that there is an $(n-1)$-cycle C' such that $NMplot(C') = FD_{n-1}(\lambda^-)$ and $n-1$ immediately follows $\lambda_2 + 1$ in C'. Thus, in C', the pairs $\langle \lambda_2 + 1, n-1\rangle, \langle \lambda_2 + 2, n-1\rangle, \ldots, \langle n-3, n-1\rangle$ must match μ in C'. This means that if $\lambda_2 + 1 \leq n-3$, then $n-3$ must lie between $n-2$ and $n-1$ in C'. Next if $\lambda_2 + 1 \leq n-4$, then $n-4$ must lie between $n-3$ and $n-1$ in C'. In general, if $\lambda_2 + 1 \leq n-k$, then $n-k$ must lie between $n-k+1$ and $n-1$ in C'. Thus, C' must be of the following form

$$C' = (\underbrace{\ldots}_{A_{n-1}} n-2 \underbrace{\ldots}_{A_{n-2}} n-3 \underbrace{\ldots}_{A_{n-3}} n-4 \underbrace{\ldots}_{A_{n-4}} \cdots (\lambda_2+2) \underbrace{\ldots}_{A_{\lambda_2+2}} (\lambda_2+1), (n-1) \underbrace{\ldots}_{A_{n-1}}).$$

where A_i is the set of elements cyclically between $n-i$ and $n-i-1$ in C' for $i = 1, \ldots, n-\lambda_2-2$. Now let C be the cycle that results from C' by inserting n immediately after $\lambda_1 + 1$ in C'. Inserting n into C' does not effect on whether pairs $\langle i, j\rangle$ with $1 \leq i < j \leq n-1$ are μ-matches in C. That is, for such pairs $\langle i, j\rangle \in \mathcal{NM}_\mu(C)$ if and only if $\langle i, j\rangle \in \mathcal{NM}_\mu(C')$. Thus, the diagram of $NMplot(C')$ and $NMplot(C)$ are the same up to row $n-1$. Now, in row n, we know that the cells $(n, 1), \ldots, (n, \lambda_1)$ are shaded since the fact that n immediately follows $\lambda_1 + 1$ in C means that $\langle i, n\rangle \in \mathcal{NM}_\mu(C)$ for $i = 1, \ldots, \lambda_1$. However, it is easy to see from the form of C' above that $\langle n-2, n\rangle, \langle n-3, n\rangle, \ldots, \langle \lambda_1 + 1, n\rangle$ are μ-matches in C since $n-2, n-3, \ldots, \lambda_1 + 1$ appear in decreasing order as we traverse clockwise around the cycle C. Thus, $NMplot(C) = FD_n(\lambda)$. □

Note the proof of Theorem 12 gives a simple algorithm to construct an n-cycle C_λ such that $NMplot(C_\lambda) = FD_n(\lambda)$ for any $\lambda \subseteq T_n$. For example, suppose that $n = 12$ and $\lambda = (3, 3, 2, 1)$ so that the Ferrers diagram in the 12×12 grid is pictured in Figure 19.

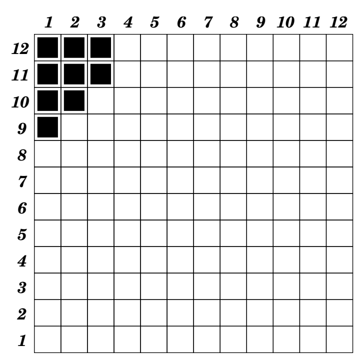

FIGURE 19. $sh_{12}(3, 3, 2, 1)$.

Since there are no non-μ-matches in the first eight rows, we must start with the cycle $C_8 = (1, 8, 7, 6, 5, 4, 3, 2)$. Then, our proof of Theorem 12 tells us that we should build up the cycle structure by first creating a cycle $C_9 \in \mathcal{C}_9$ by inserting 9 immediately after 2 in C_8, since the number of non-μ-matches in row 9 is 1. Then we create a cycle $C_{10} \in \mathcal{C}_{10}$

by inserting 10 immediately after 3 in C_9 since the number of non-μ-matches in row 10 is 2. Then, we create a cycle $C_{11} \in \mathcal{C}_{11}$ by inserting 11 immediately after 4 in C_{10} since the number of non-μ-matches in row 11 is 3. Finally, we create a cycle $C_{12} = C_\lambda \in \mathcal{C}_{12}$ by inserting 12 immediately after 4 in C_{11} since the number of non-μ-matches in row 12 is 3. Thus,

$$\begin{aligned} C_9 &= (1,8,7,6,5,4,3,2,9), \\ C_{10} &= (1,8,7,6,5,4,3,10,2,9), \\ C_{11} &= (1,8,7,6,5,4,11,3,10,2,9), \text{ and} \\ C_{12} &= (1,8,7,6,5,4,12,11,3,10,2,9). \end{aligned}$$

Then we have that

$$\begin{aligned} \mathcal{NM}_\mu((1,8,7,6,5,4,12,11,3,10,2,9)) = &\langle 1,12\rangle, \langle 2,12\rangle, \langle 3,12\rangle, \\ &\langle 1,11\rangle, \langle 2,11\rangle, \langle 3,11\rangle, \\ &\langle 1,10\rangle, \langle 2,10\rangle, \\ &\langle 1,9\rangle. \end{aligned}$$

Now we can prove Theorem 8.

Proof. Suppose that $k \leq n-2$. Let

$$\begin{aligned} FD_n(k) &= \{C \in \mathcal{C}_n : NMplot(C) = FD_n(\lambda) \text{ for some } \lambda \vdash k\} \text{ and} \\ NMp_n(k) &= \{C \in \mathcal{C}_n : NM_\mu(C) = k\}. \end{aligned}$$

Theorem 11 shows that $NMp_n(k) \subseteq FD_n(k)$ and Theorem 12 shows that $FD_n(k) \subseteq NMp_n(k)$. Thus, $NM_{n,\mu}(q)|_{q^k} = |NMp_n(k)| = |FD_n(k)|$. Hence, $NM_{n,\mu}(q)|_{q^k}$ equals the number of partitions of k. $\square$

Now, we will show that for $k < n-2$, $NTI_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}} = NT_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}} = a(k)$, where $a(k)$ is the number of partitions of k. First we shall show that if $C \in \mathcal{C}_n$ has fewer than $n-2$ non-matches, then C must be incontractible.

Lemma 13. *If $C \in \mathcal{C}_n$ has fewer than $n-2$ non-matches, then C must be incontractible.*

Proof. If $i+1$ immediately follows i in an n-cycle $C \in \mathcal{C}_n$, then, clearly, $\langle j, i+1\rangle$ are non-μ-matches for $1 \leq j < i$ and $\langle i,k\rangle$ is a non-μ-match for $i+2 \leq k \leq n$. This gives $n-2$ non-μ-matches. $\square$

Corollary 14. *For $k < n-2$, $NTI_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}} = NT_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}} = a(k)$ where $a(k)$ is the number of partitions of k.*

Proof. By definition, $NT_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}} = NM_{n,\mu}(q)|_{q^k}$. Thus, by Theorem 8, for $k < n-2$, $NT_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}} = a(k)$.

By Lemma 13, we have that if a cycle C has k non-μ-matches where $k < n-2$, then C is incontractible. It follows that if a cycle has $\binom{n-1}{2} - k$ non-trivial μ-matches, then it is incontractible. Thus, for $k < n-2$, $NTI_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}} = NT_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}}$. And so $NTI_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}} = a(k)$ for $k < n-2$. □

5. Conclusions and direction for further research

The main focus of this paper was to study the polynomials $NT_{n,\mu}(q) = \sum_{C \in \mathcal{C}_n} q^{NT_\mu(C)}$ and $NTI_{n,\mu}(q) = \sum_{C \in \mathcal{IC}_n} q^{NT_\mu(C)}$. We showed that $NTI_{n,\mu}(1)$ is the number of derangements of S_{n-1}. Thus, the polynomial $NTI_{n,\mu}(q)$ is a q-analogue of the derangement number D_{n-1}. There are several q-analogues of the derangement numbers that have been studied in the literature, see the papers by Garsia and Remmel [3] and Wachs[9]. Our q-analogue of D_{n-1} is different from either the Garsia-Remmel q-analogue or the Wachs q-analogue of the derangement numbers. Moreover, we proved that

$$NT_{n,\mu}(q)|_{q^k} = \sum_{s=1}^{2k+1} \sum_{\substack{A \in \mathcal{IC}_s,\\ NT_\mu(A)=k}} \binom{n-1}{s-1}$$

so that the coefficients of the polynomial $NT_{n,\mu}(q)$ can be expressed in terms of the coefficients of the polynomials $NTI_{j,\mu}(q)$. We also showed that $NTI_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}}$ equals the number of partition of k for $k < n-2$.

The main open question is to find some sort of recursion or generating function that would allow us to compute $NTI_{n,\mu}(q)$. Several of the sequences $(NT_{n,\mu}(q)|_{q^k})_{n \geq 2}$ appear in the OEIS [8]. For example, we showed that $NT_{n,\mu}(q)|_{q^2} = \binom{n-1}{3} + 2\binom{n-1}{4}$ so that the sequence $(NT_{n,\mu}(q)|_{q^2})_{n \geq 4}$ starts out $1, 6, 20, 50, 105, 196, 336, 540, 825, \ldots$. The nth term of this sequence has several combinatorial interpretations including being the number of $\sigma \in S_n$ which are 132-avoiding and have exactly two descents, the number of Dyck paths on length $2n+2$ with $n-1$ peaks, and the number of squares with corners on the $n \times n$ grid. It would be interesting to find bijections from such objects to our $(n+4)$-cycles $C \in \mathcal{C}_n$ such that $NT_\mu(C) = 2$. We proved that $NT_{n,\mu}(q)|_{q^3} = \binom{n-1}{3} + 3\binom{n-1}{4} + 6\binom{n-1}{5} + 5\binom{n-1}{6}$ so that the sequence $(NT_{n,\mu}(q)|_{q^3})_{n \geq 4}$ starts out

$$1, 7, 31, 102, 273, 630, 1302, 2472, 4389 \ldots.$$

This sequence does not appear in the OEIS. Similarly, the sequence $(NT_{n,\mu}(q)|_{q^4})_{n \geq 5}$ starts out $2, 23, 135, 561, 1870, 5328, 13476, \ldots$ and it does not appear in the OEIS.

Finally, it would be interesting to characterize the charge graphs of those cycles $C \in \mathcal{IC}_{2n}$ for which $NM_\mu(C) = n$. One can see from our tables that sequence $(NTI_{2n,\mu}(q)|_{q^n})_{n \geq 2}$ starts out $1, 6, 29, 130, \ldots$. Moreover, we have computed the $NTI_{12,\mu}(q)|_{q^6} = 562$. This suggests that this sequence is sequence A008549 in the OEIS. If so, this would mean that $NTI_{2n,\mu}(q)|_{q^n} = \sum_{i=0}^{n-2} \binom{2n-1}{i}$ for $n \geq 2$.

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Appendix: Tables of $NT_{n.\mu}(q)$, $NTI_{n,\mu}(q)$, and $NM_{n,\mu}(q)$.

Table 1. Polynomials $NT_{n,\mu}(q)$.

$NT_{1,\mu}(q)$	1
$NT_{2,\mu}(q)$	1
$NT_{3,\mu}(q)$	$1+q$
$NT_{4,\mu}(q)$	$1+3\,q+q^2+q^3$
$NT_{5,\mu}(q)$	$1+6\,q+6\,q^2+7\,q^3+2\,q^4+q^5+q^6$
$NT_{6,\mu}(q)$	$1+10\,q+20\,q^2+31\,q^3+23\,q^4+15\,q^5+13\,q^6+3\,q^7+2\,q^8+q^9+q^{10}$
$NT_{7,\mu}(q)$	$1+15\,q+50\,q^2+106\,q^3+135\,q^4+126\,q^5+119\,q^6+66\,q^7+46\,q^8+$ $25\,q^9+19\,q^{10}+5\,q^{11}+3\,q^{12}+2\,q^{13}+q^{14}+q^{15}$
$NT_{8,\mu}(q)$	$1+21\,q+105\,q^2+301\,q^3+561\,q^4+736\,q^5+850\,q^6+726\,q^7+603\,q^8+$ $418\,q^9+299\,q^{10}+174\,q^{11}+101\,q^{12}+65\,q^{13}+33\,q^{14}+27\,q^{15}+7\,q^{16}+$ $5\,q^{17}+3\,q^{18}+2\,q^{19}+q^{20}+q^{21}$
$NT_{9,\mu}(q)$	$1+28\,q+196\,q^2+742\,q^3+1870\,q^4+3311\,q^5+4820\,q^6+5541\,q^7+$ $5675\,q^8+5007\,q^9+4055\,q^{10}+3093\,q^{11}+2116\,q^{12}+1461\,q^{13}+888\,q^{14}+$ $646\,q^{15}+338\,q^{16}+217\,q^{17}+126\,q^{18}+80\,q^{19}+44\,q^{20}+35\,q^{21}+$ $11\,q^{22}+7\,q^{23}+5\,q^{24}+3\,q^{25}+2\,q^{26}+q^{27}+q^{28}$
$NT_{10,\mu}(q)$	$1+36\,q+336\,q^2+1638\,q^3+5328\,q^4+12253\,q^5+22392\,q^6+32864\,q^7+$ $41488\,q^8+45433\,q^9+44119\,q^{10}+40008\,q^{11}+32781\,q^{12}+25689\,q^{13}+$ $18551\,q^{14}+13710\,q^{15}+9137\,q^{16}+6179\,q^{17}+3971\,q^{18}+2568\,q^{19}+1640\,q^{20}+$ $1098\,q^{21}+640\,q^{22}+374\,q^{23}+251\,q^{24}+148\,q^{25}+100\,q^{26}+56\,q^{27}+46\,q^{28}+$ $15\,q^{29}+11\,q^{30}+7\,q^{31}+5\,q^{32}+3\,q^{33}+2\,q^{34}+q^{35}+q^{36}$

Table 2. Polynomials $NTI_{n,\mu}(q)$.

$NTI_{1,\mu}(q)$	1
$NTI_{2,\mu}(q)$	0
$NTI_{3,\mu}(q)$	q
$NTI_{4,\mu}(q)$	$q^2 + q^3$
$NTI_{5,\mu}(q)$	$2\,q^2 + 3\,q^3 + 2\,q^4 + q^5 + q^6$
$NTI_{6,\mu}(q)$	$6\,q^3 + 13\,q^4 + 10\,q^5 + 8\,q^6 + 3\,q^7 + 2\,q^8 + q^9 + q^{10}$
$NTI_{7,\mu}(q)$	$5\,q^3 + 27\,q^4 + 51\,q^5 + 56\,q^6 + 48\,q^7 + 34\,q^8 + 19\,q^9 + 13\,q^{10}$ $+5\,q^{11} + 3\,q^{12} + 2\,q^{13} + q^{14} + q^{15}$
$NTI_{8,\mu}(q)$	$29\,q^4 + 134\,q^5 + 255\,q^6 + 327\,q^7 + 323\,q^8 + 264\,q^9 + 187\,q^{10} + 139\,q^{11} +$ $80\,q^{12} + 51\,q^{13} + 26\,q^{14} + 20\,q^{15} + 7\,q^{16} + 5\,q^{17} + 3\,q^{18} + 2\,q^{19} + q^{20} + q^{21}$
$NTI_{9,\mu}(q)$	$14\,q^4 + 181\,q^5 + 694\,q^6 + 1413\,q^7 + 2027\,q^8 + 2307\,q^9 + 2139\,q^{10} + 1841\,q^{11} +$ $1392\,q^{12} + 997\,q^{13} + 652\,q^{14} + 458\,q^{15} + 282\,q^{16} + 177\,q^{17} + 102\,q^{18} + 64\,q^{19} +$ $36\,q^{20} + 27\,q^{21} + 11\,q^{22} + 7\,q^{23} + 5\,q^{24} + 3\,q^{25} + 2\,q^{26} + q^{27} + q^{28}$
$NTI_{10,\mu}(q)$	$130\,q^5 + 1128\,q^6 + 3965\,q^7 + 8509\,q^8 + 13444\,q^9 + 16918\,q^{10} + 18015\,q^{11} + 17121\,q^{12} +$ $14712\,q^{13} + 11663\,q^{14} + 8784\,q^{15} + 6347\,q^{16} + 4406\,q^{17} + 2945\,q^{18} + 1920\,q^{19} + 1280\,q^{20} +$ $819\,q^{21} + 541\,q^{22} + 311\,q^{23} + 206\,q^{24} + 121\,q^{25} + 82\,q^{26} + 47\,q^{27} + 37\,q^{28} + 15\,q^{29} +$ $11\,q^{30} + 7\,q^{31} + 5\,q^{32} + 3\,q^{33} + 2\,q^{34} + q^{35} + q^{36}$

Table 3. Polynomials $NM_{n,\mu}(q)$

$n = 1$	1
2	1
3	$1 + q$
4	$1 + q + 3\,q^2 + q^3$
5	$1 + q + 2\,q^2 + 7\,q^3 + 6\,q^4 + 6\,q^5 + q^6$
6	$1 + q + 2\,q^2 + 3\,q^3 + 13\,q^4 + 15\,q^5 + 23\,q^6 + 31\,q^7 + 20\,q^8 + 10\,q^9 + q^{10}$
7	$1 + q + 2\,q^2 + 3\,q^3 + 5\,q^4 + 19\,q^5 + 25\,q^6 + 46\,q^7 + 66\,q^8 + 119\,q^9 + 126\,q^{10} + 135\,q^{11} +$ $106\,q^{12} + 50\,q^{13} + 15\,q^{14} + q^{15}$
8	$1 + q + 2\,q^2 + 3\,q^3 + 5\,q^4 + 7\,q^5 + 27\,q^6 + 33\,q^7 + 65\,q^8 + 101\,q^9 + 174\,q^{10} + 299\,q^{11} + 418\,q^{12} +$ $603\,q^{13} + 726\,q^{14} + 850\,q^{15} + 736\,q^{16} + 561\,q^{17} + 301\,q^{18} + 105\,q^{19} + 21\,q^{20} + q^{21}$
9	$1 + q + 2\,q^2 + 3\,q^3 + 5\,q^4 + 7\,q^5 + 11\,q^6 + 35\,q^7 + 44\,q^8 + 80\,q^9 + 126\,q^{10} + 217\,q^{11} + 338\,q^{12} +$ $646\,q^{13} + 888\,q^{14} + 1461\,q^{15} + 2116\,q^{16} + 3093\,q^{17} + 4055\,q^{18} + 5007\,q^{19} + 5675\,q^{20} +$ $5541\,q^{21} + 4820\,q^{22} + 3311\,q^{23} + 1870\,q^{24} + 742\,q^{25} + 196\,q^{26} + 28\,q^{27} + q^{28}$
10	$1 + q + 2\,q^2 + 3\,q^3 + 5\,q^4 + 7\,q^5 + 11\,q^6 + 15\,q^7 + 46\,q^8 + 56\,q^9 + 100\,q^{10} + 148\,q^{11} + 251\,q^{12} +$ $374\,q^{13} + 640\,q^{14} + 1098\,q^{15} + 1640\,q^{16} + 2568\,q^{17} + 3971\,q^{18} + 6179\,q^{19} + 9137\,q^{20} +$ $13710\,q^{21} + 18551\,q^{22} + 25689\,q^{23} + 32781\,q^{24} + 40008\,q^{25} + 44119\,q^{26} + 45433\,q^{27} +$ $41488\,q^{28} + 32864\,q^{29} + 22392\,q^{30} + 12253\,q^{31} + 5328\,q^{32} + 1638\,q^{33} + 336\,q^{34} + 36\,q^{35} + q^{36}$