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ALMOST SURE EXPONENTIAL STABILITY IN THE NUMERICAL SIMULATION OF STOCHASTIC DIFFERENTIAL EQUATIONS*

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Abstract. This paper is mainly concerned with whether the almost sure exponential stability of stochastic differential equations (SDEs) is shared with that of a numerical method. Under the global Lipschitz condition, we first show that the SDE is pth moment exponentially stable (for $p \in (0, 1)$) if and only if the stochastic theta method is pth moment exponentially stable for a sufficiently small step size. We then show that the pth moment exponential stability of the SDE or the stochastic theta method implies the almost sure exponential stability of the SDE or the stochastic theta method, respectively. Hence, our new theory enables us to study the almost sure exponential stability of the SDEs using the stochastic theta method, instead of the method of the Lyapunov functions. That is, we can now carry out careful numerical simulations using the stochastic theta method with a sufficiently small step size Δt. If the stochastic theta method is pth moment exponentially stable for a sufficiently small $p \in (0, 1)$, we can then infer that the underlying SDE is almost surely exponentially stable. Our new theory also enables us to show the ability of the stochastic theta method to reproduce the almost sure exponential stability of the SDEs. In particular, we give positive answers to two open problems, (P1) and (P2) listed in section 1.

Key words. almost sure exponential stability, moment exponential stability, stochastic theta method, Lipschitz condition, linear growth condition

AMS subject classifications. 65C30, 60H35

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1. Introduction.

Stochastic differential equations (SDEs) have been widely used in many branches of science and industry with the emphasis being on stability analysis [1, 6, 11, 13, 16]. One of the most powerful techniques in the study of stochastic stability (e.g., the moment exponential stability or almost sure exponential stability) is the method of Lyapunov functions. Suppose that we are required to find out whether an SDE is stochastically stable. In the absence of an appropriate Lyapunov function, we may carry out careful numerical simulations using a numerical method with a "small" step size Δt. We are then left with two key questions:

(Q1) *If the SDE is stochastically stable (e.g., mean square exponentially stable or almost surely exponentially stable), will the numerical method be stochastically stable?*

(Q2) *If the numerical method is stochastically stable for small Δt, can we infer that the underlying SDE is stochastically stable?*

There are various types of stochastic stabilities, including the exponential stability in mean square, almost sure exponential stability, and asymptotic stability in probability [11, 14, 15]. In any case, both questions (Q1) and (Q2) deal with asymptotic ($t \to \infty$) properties and hence cannot be answered directly by applying traditional finite-time convergence results.

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In the case where stochastic stability means exponential stability in mean square, results that answer (Q1) and (Q2) for scalar, linear systems can be found in [7, 20, 21]. Baker and Buckwar [3] consider the stability of numerical methods for scalar constant delay SDEs under the assumptions of the global Lipschitz coefficients and the existence of a Lyapunov function. Schurz [21] also has results for nonlinear SDEs. In particular, for nonlinear SDEs under the global Lipschitz condition, Higham, Mao, and Stuart [9] show that the exponential stability in mean square for the SDE is equivalent to the exponential stability in mean square of the numerical method (e.g., the Euler–Maruyama and the stochastic theta method) for sufficiently small step sizes. For further developments in this area, we refer the reader to [5, 8, 19, 22, 24, 27], for example, and the references therein.

In the case where stochastic stability means almost sure exponential stability, so far most results answer (Q1), but few address (Q2). In fact, unlike in the mean square case, the first paper that addressed these questions on a reasonable class of SDEs was Higham, Mao, and Stuart [10] in 2007. In their paper, they answered (Q1) and (Q2) for the linear scalar SDE using the Euler–Maruyama method. For the nonlinear SDE, they answered (Q1) using the Euler–Maruyama method for the SDE under the linear growth condition plus the additional condition which guaranteed the almost sure exponential stability of the SDE. For the nonlinear SDE without the linear growth condition, they answered (Q1) using the backward Euler method. The research in this area has since then been developed by many authors, e.g., [4, 17, 25, 26], but all of these authors have addressed (Q1). There are many open problems in this direction. Two of them are stated below:

(P1) If the multidimensional linear SDE

$$dy(t) = A_0 y(t)dt + \sum_{i=1}^{m} A_i y(t) dw_i(t) \tag{1.1}$$

is almost surely exponentially stable, will a numerical method be almost surely exponentially stable for all sufficiently small step sizes?

(P2) Under the conditions in Theorem 4.5 below, the nonlinear SDE (2.1) is almost surely exponentially stable. This theorem is one of the most useful criteria on almost sure exponential stability. However, will a numerical method be almost surely exponentially stable for all sufficiently small step sizes?

Should we have some answers to (Q1) and (Q2) in the case where stochastic stability means almost sure exponential stability, these problems would be solved to a certain degree. In this paper we will address (Q1) and (Q2) in the sense of almost sure exponential stability in two steps. Under the global Lipschitz condition, we first show that the SDE is pth moment exponentially stable (for $p \in (0, 1)$) if and only if the stochastic theta method is pth moment exponentially stable for a sufficiently small step size. We will then show that the pth moment exponential stability of the SDE or the stochastic theta method implies the almost sure exponential stability of the SDE or the stochastic theta method, respectively. Hence, our new theory enables us to study the almost sure exponential stability of the SDEs using the stochastic theta method instead of the method of the Lyapunov functions. That is, we can now carry out careful numerical simulations using the stochastic theta method with a sufficiently small step size Δt. If the stochastic theta method is pth moment exponentially stable for a sufficiently small $p \in (0, 1)$, we can then infer that the underlying SDE is almost surely exponentially stable. Our new theory also enables us to show the ability of the stochastic theta method to reproduce the almost sure exponential stability of the

SDEs. In particular, we will be able to solve problems (P1) and (P2). In fact, it is known (see, e.g., [2]) that the linear SDE (1.1) is almost surely exponentially stable if and only if it is pth moment exponentially stable for a sufficiently small $p \in (0, 1)$. Moreover, under the conditions of Theorem 4.5, we will show that the nonlinear SDE (2.1) is pth moment exponentially stable for a sufficiently small $p \in (0, 1)$. Applying our new results, we will hence have the positive results on both (P1) and (P2).

Before we proceed to establish our new theory, we should point out that the way we establish our new results on (Q1) and (Q2) in the sense of the pth moment exponential stability for sufficiently small $p \in (0, 1)$ is motivated by our earlier paper [9] that deals with the 2nd moment exponential stability. However, there are at least two significant differences which we highlight below:

- It was assumed in [9] that a numerical method is available which, given a step size $\Delta t > 0$, computes discrete approximations $x_k \approx y(k\Delta t)$, with $x_0 = y_0$. It was also assumed that there is a well-defined interpolation process that extends the discrete approximation $\{x_k\}_{k\in\mathbb{Z}^+}$ to a continuous time approximation $\{x(t)\}_{t\in\mathbb{R}^+}$, with $x(k\Delta t) = x_k$. However, in general, only the discrete approximations x_k are computable but not the continuous-time approximation $\{x(t)\}_{t\in\mathbb{R}^+}$. It is therefore more useful in practice if the theory is only based on the discrete approximations x_k, and this is what we will establish in this paper. In other words, in this paper, we only need a numerical method which computes the discrete approximations $x_k \approx y(k\Delta t)$, and we do not require the continuous-time approximations.
- Mathematically speaking, many inequalities used in [9] for the 2nd moment do not work for the pth moment when $p \in (0, 1)$. We therefore have to develop new techniques to handle the pth moment. Moreover, our new theory is based on the discrete approximations x_k, so we have to apply the discrete-time analysis in many proofs in this paper, while [9] used the continuous-time analysis essentially.

2. *p*th moment exponential stability.

2.1. Stochastic differential equations. Throughout this paper, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $w(t) = (w_1(t), \ldots, w_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $|\cdot|$ denote both the Euclidean norm in $\mathbb{R}^n$ and the trace norm in $\mathbb{R}^{n\times m}$. For $a, b \in \mathbb{R}$, we use $a \vee b$ and $a \wedge b$ for $\max\{a, b\}$ and $\min\{a, b\}$, respectively.

Consider an n-dimensional Itô SDE

$$dy(t) = f(y(t))dt + g(y(t))dw(t) \tag{2.1}$$

on $t \geq 0$ with initial value $y(0) = y_0 \in \mathbb{R}^n$, where

$$f : \mathbb{R}^n \to \mathbb{R}^n \quad \text{and} \quad g : \mathbb{R}^n \to \mathbb{R}^{n\times m}.$$

Throughout this paper, unless otherwise specified, we impose the following assumption as a standing hypothesis.

ASSUMPTION 2.1. *Both f and g satisfy the global Lipschitz condition. That is, there is a positive constant K such that*

$$|f(x) - f(y)| \vee |g(x) - g(y)| \leq K|x - y| \quad \forall x, y \in \mathbb{R}^n. \tag{2.2}$$

Moreover, $f(0) = 0$ *and* $g(0) = 0$.

We should point out that the reason we assume that $f(0) = 0$ and $g(0) = 0$ is because this paper is concerned with the stochastic stability of the trivial solution $y(t) \equiv 0$. It is also easy to see that this assumption implies the linear growth condition

$$|f(y)| \vee |g(y)| \le K|y| \quad \forall y \in \mathbb{R}^n. \tag{2.3}$$

Moreover, it is well known (see, e.g., [1, 6, 16]) that under Assumption 2.1, the SDE (2.1) has a unique global solution $y(t)$ on $t \ge 0$ and the solution satisfies

$$\mathbb{E}|y(t)|^p \le H(t,p,K)|y_0|^p, \quad t \ge 0, \tag{2.4}$$

for any $p > 0$, where $H(t,p,K)$ is a positive constant dependent on t,p,K only. In this paper, it is useful to have $H(t,p,K)$ explicitly for $p \in (0,1)$. It is straightforward to show that

$$\mathbb{E}|y(t)|^2 \le 3|y_0|^2 + 3K^2(t+1)\int_0^t \mathbb{E}|y(s)|^2 ds.$$

This, by the Gronwall inequality, yields

$$\mathbb{E}|y(t)|^2 \le 3|y_0|^2 e^{3K^2 t(t+1)}. \tag{2.5}$$

Hence, when $p \in (0,1)$,

$$\mathbb{E}|y(t)|^p \le (\mathbb{E}|y(t)|^2)^{p/2} \le 3^{p/2} e^{1.5pK^2 t(t+1)} |y_0|^p.$$

In other words, we have

$$H(t,p,K) = 3^{p/2} e^{1.5pK^2 t(t+1)} \tag{2.6}$$

for $p \in (0,1)$.

Of course, we may consider a more general case, for example, where the SDE has its random initial data $y(0) = \xi$ which is an $\mathcal{F}_0$-measurable $\mathbb{R}^n$-valued random variable such that $\mathbb{E}|\xi|^p < \infty$ $\forall p > 0$. In this case, by the Markov property of the solution, we can easily see that the solution satisfies

$$\mathbb{E}|y(t)|^p = \mathbb{E}(\mathbb{E}(|y(t)|^p|\xi)) \le \mathbb{E}(H(t,p,K)|\xi|^p) = H(t,p,K)\mathbb{E}|\xi|^p, \quad t \ge 0, \tag{2.7}$$

$\forall p > 0$. It is therefore clear why it is enough to consider only the deterministic initial value $y(0) = y_0 \in \mathbb{R}^n$. Moreover, for any $t_0 \ge 0$, we can regard $y(t)$ on $t \ge t_0$ as the solution of the SDE (2.1) on $t \ge t_0$ with initial data $y(t_0)$ at $t = t_0$. As the SDE is time-homogeneous, we hence see from (2.7) that

$$\mathbb{E}|y(t)|^p \le H(t-t_0,p,K)\mathbb{E}|y(t_0)|^p, \quad t \ge t_0, \tag{2.8}$$

for any $p > 0$. In this section we consider the pth moment exponential stability of the origin, which we define as follows (see, e.g., [6, 11, 14, 15]).

DEFINITION 2.2. *Let* $p > 0$. *The SDE* (2.1) *is said to be* exponentially stable in the pth moment *if there is a pair of positive constants* λ *and* M *such that, for every initial value* $y_0 \in \mathbb{R}^n$,

$$\mathbb{E}|y(t)|^p \le M|y_0|^p e^{-\lambda t} \quad \forall t \ge 0. \tag{2.9}$$

We refer to λ *as a* rate constant *and* M *as a* growth constant.

As in the explanation of (2.8) we see that (2.9) is equivalent to the following more general form:

$$\mathbb{E}|y(t)|^p \le M\mathbb{E}|y(t_0)|^p e^{-\lambda(t-t_0)} \quad \forall t \ge t_0 \ge 0. \tag{2.10}$$

2.2. Numerical solutions. We suppose that a numerical method is available which, given a step size $\Delta t > 0$, computes discrete approximations $x_k \approx y(k\Delta t)$ for $k \in \mathbb{Z}^+$ with $x_0 = y_0$. We also require that the process defined by the numerical method possess the following Markov property:

- Given $x_{\bar{k}}$ for some $\bar{k} \in \mathbb{Z}^+$, the process $\{x_k\}_{k \geq \bar{k}}$ can be regarded as the process which is produced by the numerical method applied to the SDE (2.1) on $t \geq \bar{k}\Delta t$ with the initial data $y(\bar{k}\Delta t) = x_{\bar{k}}$. Hence, the probability distributions of the process $\{x_k\}_{k \geq \bar{k}}$ are fully determined, given $x_{\bar{k}}$, but how the process has reached $x_{\bar{k}}$ has no further use. In other words, the discrete-time process $\{x_k\}_{k \in \mathbb{Z}^+}$ is a Markov process. Moreover, by the time-homogeneity of the SDE, $\{x_k\}_{k \in \mathbb{Z}^+}$ is of course time-homogeneous.

Such a discrete-time process will be illustrated for the class of stochastic theta methods in the next section. Following Definition 2.2, we now define the pth moment exponential stability for the discrete-time approximate solutions $\{x_k\}_{k \in \mathbb{Z}^+}$.

Definition 2.3. *Let $p > 0$. For a given step size $\Delta t > 0$, a numerical method is said to be* exponentially stable in the pth moment *on the SDE* (2.1) *if there is a pair of positive constants γ and N such that with initial value $y_0 \in \mathbb{R}^n$*

$$\mathbb{E}|x_k|^p \leq N|y_0|^p e^{-\gamma k \Delta t} \quad \forall k \in \mathbb{Z}^+. \tag{2.11}$$

By the time-homogeneous Markov property, we see that (2.11) is equivalent to the following more general form:

$$\mathbb{E}|x_k|^p \leq N\mathbb{E}|x_{\bar{k}}|^p e^{-\gamma(k-\bar{k})\Delta t} \quad \forall k \geq \bar{k} \geq 0. \tag{2.12}$$

2.3. Assumptions and results. From now on we will let $p \in (0,1)$. We wish to know whether the numerical method shares the pth moment exponential stability with the SDE. That is, we wish to address both (Q1) and (Q2) in the sense of the pth moment exponential stability. For this purpose, we impose a natural finite pth moment condition on the numerical methods.

Assumption 2.4. *Let $p \in (0,1)$. For all sufficiently small Δt the numerical method applied to* (2.1) *with initial condition $x_0 = y_0 \in \mathbb{R}^n$ satisfies*

$$\sup_{0 \leq k\Delta t \leq T} \mathbb{E}|x_k|^p \leq \bar{H}(T,p,K)|y_0|^p \quad \forall T \geq 0, \tag{2.13}$$

where $\bar{H}(T,p,K)$ is a positive constant dependent on T,p,K only, but independent of the initial value y_0 and the step size Δt.

By the time-homogeneous Markov property of the numerical method, we see easily from this assumption that

$$\mathbb{E}|x_k|^p \leq \bar{H}(k-\bar{k},p,K)\mathbb{E}|x_{\bar{k}}|^p, \quad k \geq \bar{k} \geq 0. \tag{2.14}$$

We also impose a natural finite-time convergence condition on the numerical methods.

Assumption 2.5. *Let $p \in (0,1)$. For all sufficiently small Δt the numerical method applied to* (2.1) *with initial condition $x_0 = y_0 \in \mathbb{R}^n$ satisfies*

$$\sup_{0 \leq k\Delta t \leq T} \mathbb{E}|x_k - y(k\Delta t)|^p \leq C_T|y_0|^p h(\Delta t) \quad \forall T \geq 0, \tag{2.15}$$

where C_T depends on T but not on y_0 and Δt, and $h : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing continuous function with $h(0) = 0$.

Our notation emphasizes the dependence of C upon T as this is important in the subsequent analysis. It is also easy to see that C_T is nondecreasing in T. Moreover, for any $\bar{k} \in \mathbb{Z}^+$, if we let $\hat{y}(t)$ be the solution of the SDE (2.1) on $t \geq \bar{k}\Delta t$ with initial data $\hat{y}(\bar{k}\Delta t) = x_{\bar{k}}$, then, by the time-homogeneity of the SDE (2.1), condition (2.15) implies

$$\sup_{\bar{k}\Delta t \leq k\Delta t \leq \bar{k}\Delta t + T} \mathbb{E}|x_k - \hat{y}(t)|^p \leq C_T \mathbb{E}|x_{\bar{k}}|^p h(\Delta t). \tag{2.16}$$

These assumptions will be illustrated for the class of stochastic theta methods in the next section. The following lemma gives a positive answer to question (Q1) from section 1.

Lemma 2.6. *Assume that the SDE* (2.1) *is pth moment exponentially stable and satisfies* (2.9). *Under Assumptions* 2.4 *and* 2.5 *there exists a* $\Delta t^\star > 0$ *such that for every* $0 < \Delta t \leq \Delta t^\star$ *the numerical method is pth moment exponentially stable on the SDE* (2.1) *with rate constant* $\gamma = \frac{1}{2}\lambda$ *and growth constant* $N = \bar{H}(T+1, p, K)e^{\frac{1}{2}\lambda(T+1)}$, *where* $T = 3 + (4\log(2^p M))/\lambda$ *and* $\bar{H}(T+1, p, K)$ *is given by* (2.13). *(Please note that both* γ *and* N *are independent of* Δt.*)*

Proof. Without loss of any generality, we let $\Delta t < 1$. We divide the whole proof into 2 steps.

Step 1. By the definition of T, we observe that

$$2^p M e^{-\lambda T} = e^{-\frac{3}{4}\lambda(T+1)}. \tag{2.17}$$

By the elementary inequality

$$(a+b)^p \leq (2(a \vee b))^p \leq 2^p(a^p \vee b^p) \leq 2^p(a^p + b^p) \quad \forall a, b \geq 0, \tag{2.18}$$

we have

$$\mathbb{E}|x_k|^p \leq 2^p(\mathbb{E}|x_k - y(k\Delta t)|^p + \mathbb{E}|y(k\Delta t)|^p) \quad \forall k \in \mathbb{Z}^+. \tag{2.19}$$

In particular, for $k\Delta t \in [T, T+1]$ (such k exists as $\Delta t < 1$), using conditions (2.15) and (2.9), we then have

$$\mathbb{E}|x_k|^p \leq |y_0|^p 2^p \Big(C_{T+1} h(\Delta t) + M e^{-\lambda T} \Big). \tag{2.20}$$

This, together with (2.17), yields

$$\mathbb{E}|x_k|^p \leq |y_0|^p \Big(2^p C_{T+1} h(\Delta t) + e^{-\frac{3}{4}\lambda(T+1)} \Big), \quad k\Delta t \in [T, T+1].$$

Choose $\Delta t^\star \in (0,1)$ sufficiently small for

$$2^p C_{T+1} h(\Delta t^\star) + e^{-\frac{3}{4}\lambda(T+1)} \leq e^{-\frac{1}{2}\lambda(T+1)}.$$

Then, for every $0 < \Delta t \leq \Delta t^\star$,

$$\mathbb{E}|x_k|^p \leq |y_0|^p e^{-\frac{1}{2}\lambda(T+1)}, \quad k\Delta t \in [T, T+1]. \tag{2.21}$$

Let us now fix $\Delta t \in (0, \Delta t^\star]$ arbitrarily. Choose a positive integer $\bar{k}$ such that $\bar{k}\Delta t \in [T, T+1]$. It then follows from (2.21) that

$$\mathbb{E}|x_{\bar{k}}|^p \leq |y_0|^p e^{-\frac{1}{2}\lambda \bar{k}\Delta t}. \tag{2.22}$$

Moreover, by condition (2.13),

$$\mathbb{E}|x_k|^p \le \bar{H}(T+1,p,K)|y_0|^p, \quad 0 \le k \le \bar{k}.$$

Hence

$$\mathbb{E}|x_k|^p \le N|y_0|^p e^{-\frac{1}{2}\lambda k\Delta t}, \quad 0 \le k \le \bar{k}, \tag{2.23}$$

where $N = \bar{H}(T+1,p,K)e^{\frac{1}{2}\lambda(T+1)}$.

Step 2. Let us now consider the approximate solutions x_k for $k \ge \bar{k}$. By the Markov property described earlier in section 2, the process $\{x_k\}_{k\ge\bar{k}}$ can be regarded as the process which is produced by the numerical method applied to the SDE (2.1) on $t \ge \bar{k}\Delta t$ with the initial data $y(\bar{k}\Delta t) = x_{\bar{k}}$. On the other hand, let $\bar{y}(t)$ on $t \ge \bar{k}\Delta t$ be the unique solution of the SDE (2.1) with the initial data $\bar{y}(\bar{k}\Delta t) = x_{\bar{k}}$. By the Markov properties of the numerical solution and the true solution as well as the time-homogeneity of the SDE (2.1), condition (2.15) implies (alternatively, please recall (2.16)) that

$$\sup_{\bar{k}\Delta t \le k\Delta t \le \bar{k}\Delta t + T+1} \mathbb{E}|x_k - \bar{y}(t)|^p \le C_{T+1}\mathbb{E}|x_{\bar{k}}|^p h(\Delta t). \tag{2.24}$$

Moreover, by (2.9) (more precisely, by its equivalent form (2.10)), we have

$$\mathbb{E}|\bar{y}(t)|^p \le M\mathbb{E}|x_{\bar{k}}|^p e^{-\lambda(t-\bar{k}\Delta t)} \quad \forall t \ge \bar{k}\Delta t. \tag{2.25}$$

Using (2.24) and (2.25), we can show, in the same way as we did in Step 1, that

$$\mathbb{E}|x_{2\bar{k}}|^p \le \mathbb{E}|x_{\bar{k}}|^p e^{-\frac{1}{2}\lambda\bar{k}\Delta t} \tag{2.26}$$

and

$$\mathbb{E}|x_k|^p \le N\mathbb{E}|x_{\bar{k}}|^p e^{-\frac{1}{2}\lambda(k-\bar{k})\Delta t}, \quad \bar{k} \le k \le 2\bar{k}. \tag{2.27}$$

Repeating this procedure, we can show that for any nonnegative integer i,

$$\mathbb{E}|x_{(i+1)\bar{k}}|^p \le \mathbb{E}|x_{i\bar{k}}|^p e^{-\frac{1}{2}\lambda\bar{k}\Delta t} \tag{2.28}$$

and

$$\mathbb{E}|x_k|^p \le N\mathbb{E}|x_{i\bar{k}}|^p e^{-\frac{1}{2}\lambda(k-i\bar{k})\Delta t}, \quad i\bar{k} \le k \le (i+1)\bar{k}. \tag{2.29}$$

Consequently,

$$\mathbb{E}|x_{i\bar{k}}|^p \le \mathbb{E}|x_{(i-1)\bar{k}}|^p e^{-\frac{1}{2}\lambda\bar{k}\Delta t} \le \cdots \le |y_0|^p e^{-\frac{1}{2}\lambda i\bar{k}\Delta t},$$

and then

$$\mathbb{E}|x_k|^p \le N|y_0|^p e^{-\frac{1}{2}\lambda k\Delta t}, \quad i\bar{k} \le k \le (i+1)\bar{k}, \ i \ge 0.$$

That is, the numerical method is exponentially stable in the pth moment on the SDE (2.1) with rate constant $\gamma = \frac{1}{2}\lambda$ and growth constant $N = \bar{H}(T+1,p,K)e^{\frac{1}{2}\lambda(T+1)}$. The proof is hence complete. ∎

The next lemma gives a positive answer to question (Q2) from section 1.

LEMMA 2.7. *Assume that Assumptions* 2.4 *and* 2.5 *hold. Assume also that for a step size* $\Delta t > 0$*, the numerical method is pth moment exponentially stable with rate constant* γ *and growth constant* N*. If* Δt *satisfies*

$$2^p C_T h(\Delta t) + e^{-\frac{3}{4}\gamma T} \le e^{-\frac{1}{2}\gamma T}, \tag{2.30}$$

where $T = \bar{k}\Delta t$ *and* $\bar{k}$ *is the smallest integer which is no less than* $4\log(2^p N)/(\gamma\Delta t)$*, then the SDE* (2.1) *is pth moment exponentially stable with rate constant* $\lambda = \frac{1}{2}\gamma$ *and growth constant* $M = H(T,p,K)e^{\frac{1}{2}\gamma T}$*, where* $H(T,p,K)$ *is given in* (2.4).

Proof. It is easy to see from $4\log(2^p N)/(\gamma\Delta t) \le \bar{k}$ that

$$2^p N e^{-\gamma\bar{k}\Delta t} \le e^{-\frac{3}{4}\gamma\bar{k}\Delta t},$$

namely,

$$2^p N e^{-\gamma T} \le e^{-\frac{3}{4}\gamma T} \tag{2.31}$$

as $T = \bar{k}\Delta t$. By the elementary inequality (2.18)

$$\mathbb{E}|y(T)|^p \le 2^p(\mathbb{E}|x_k - y(T)|^p + \mathbb{E}|x_k|^p). \tag{2.32}$$

Using Assumptions 2.4 and 2.5 and (2.11), we obtain

$$\mathbb{E}|y(T)|^p \le 2^p|y_0|^p(C_T h(\Delta t) + Ne^{-\gamma T}).$$

This, together with (2.31) and (2.30), yields

$$\mathbb{E}|y(T)|^p \le |y_0|^p e^{-\frac{1}{2}\gamma T}. \tag{2.33}$$

Moreover, it follows from (2.4) that

$$\mathbb{E}|y(t)|^p \le H(T,p,K)|y_0|^p, \quad t \in [0,T].$$

Hence

$$\mathbb{E}|y(t)|^p \le M|y_0|^p e^{-\frac{1}{2}\gamma t}, \quad t \in [0,T], \tag{2.34}$$

where $M = H(T,p,K)e^{\frac{1}{2}\gamma T}$.

Let us now consider the solution $y(t)$ on $t \ge T$. As explained before, this can be regarded as the solution of the SDE (2.1) with the initial data $y(T)$ at time $t = T$. Moreover, let $\{\bar{x}_k\}_{k\ge\bar{k}}$ be the process which is produced by the numerical method applied to the SDE (2.1) on $t \ge T$ with the initial data $\bar{x}_{\bar{k}} = y(T)$. By the time-homogeneity of the SDE (2.1) as well as the Markov property of the true and numerical solutions, condition (2.15) implies

$$\sup_{\bar{k}\le k\le 2\bar{k}} \mathbb{E}|\bar{x}_k - y(k\Delta t)|^p \le C_T\mathbb{E}|y(T)|^p h(\Delta t). \tag{2.35}$$

Also, (2.11) implies

$$\mathbb{E}|\bar{x}_k|^p \le M\mathbb{E}|y(T)|^p e^{-\lambda(k-\bar{k})\Delta t} \quad \forall k \ge \bar{k}. \tag{2.36}$$

Using (2.35) and (2.36), we can show, in the same way that (2.33) and (2.34) were obtained, that

$$\mathbb{E}|y(2T)|^p \le \mathbb{E}|y(T)|^p e^{-\frac{1}{2}\gamma T} \tag{2.37}$$

and

$$\mathbb{E}|y(t)|^p \le M\mathbb{E}|y(T)|^p e^{-\frac{1}{2}\lambda(t-T)}, \quad t \in [T, 2T]. \tag{2.38}$$

Repeating this procedure, we can show that for any nonnegative integer i,

$$\mathbb{E}|y((i+1)T)|^p \le \mathbb{E}|y(iT)|^p e^{-\frac{1}{2}\gamma T} \tag{2.39}$$

and

$$\mathbb{E}|y(t)|^p \le M\mathbb{E}|y(iT)|^p e^{-\frac{1}{2}\gamma(t-iT)}, \quad t \in [iT, (i+1)T]. \tag{2.40}$$

Consequently,

$$\mathbb{E}|y(iT)|^p \le \mathbb{E}|y((i-1)T)|^p e^{-\frac{1}{2}\gamma T} \le \cdots \le |y_0|^p e^{-\frac{1}{2}\gamma iT}$$

and then

$$\mathbb{E}|y(t)|^p \le M|y_0|^p e^{-\frac{1}{2}\gamma t}, \quad t \in [iT, (i+1)T].$$

That is, the SDE (2.1) is pth moment exponentially stable with rate constant $\lambda = \frac{1}{2}\gamma$ and growth constant $M = H(T, p, K)e^{\frac{1}{2}\gamma T}$. The proof is hence complete. ∎

Lemmas 2.6 and 2.7 lead to the following theorem.

THEOREM 2.8. *Suppose that a numerical method satisfies Assumptions* 2.4 *and* 2.5. *Then the SDE is exponentially stable in the pth moment if and only if there exists a* $\Delta t > 0$ *such that the numerical method is exponentially stable in the pth moment with rate constant* γ, *growth constant* N, *step size* Δt, *and global error constant* C_T *for* $T = \bar{k}\Delta t$ *satisfying* (2.30), *where* $\bar{k}$ *is the smallest integer which is no less than* $4\log(2^p N)/(\gamma\Delta t)$.

Proof. The "if" part of the theorem follows from Lemma 2.7 directly. To prove the "only if" part, suppose the SDE is exponentially stable in the pth moment with rate constant λ and growth constant M. Lemma 2.6 shows that there is a $\Delta t^\star > 0$ such that for any step size $0 < \Delta t \le \Delta t^\star$, the numerical method is exponentially stable in the pth moment with rate constant $\gamma = \frac{1}{2}\lambda$ and growth constant $N = 2^p(C_{T+1} + M)e^{\frac{1}{2}\lambda(T+1)}$, where $T = 3 + (4\log(2^p M))/\lambda$. Noting that both of these constants are independent of Δt, it follows that we may reduce Δt if necessary until (2.30) becomes satisfied. ∎

We emphasize that Theorem 2.8 is an "if and only if" result, which shows that, under Assumptions 2.4 and 2.5 and for sufficiently small Δt, the pth moment exponential stability of the method is equivalent to the pth moment exponential stability of the SDE. Thus it is feasible to investigate exponential stability of the SDE from careful numerical simulations.

2.4. Improved results. In Lemmas 2.6 and 2.7, we found new rate constants that were within a factor $\frac{1}{2}$ of the given ones. We can of course make the new rate constants as close as possible to the given ones, say a factor of $1 - \epsilon$ for any $\epsilon \in (0, 1)$. The price we paid for this was an increase in the growth constants and a decrease in the step sizes. The following lemmas describe this situation more precisely.

LEMMA 2.9. *Assume that the SDE* (2.1) *is pth moment exponentially stable and satisfies* (2.9). *Let* $\epsilon \in (0, 1)$. *Under Assumptions* 2.4 *and* 2.5 *there exists a*

$\Delta t^\star > 0$ such that for every $0 < \Delta t \le \Delta t^\star$ the numerical method is pth moment exponentially stable on the SDE (2.1) with rate constant $\gamma = (1-\epsilon)\lambda$ and growth constant $N = 2^p(C_{T+1} + M)e^{(1-\epsilon)\lambda(T+1)}$, where $T = 2\log(2^pM)/(\epsilon\lambda) + (2-\epsilon)/\epsilon$. (Please note once again that both γ and N are independent of Δt.)

Proof. The proof is very similar to that of Lemma 2.6. In fact, by the definition of T, we have

$$2^pMe^{-\lambda T} = e^{-(1-0.5\epsilon)\lambda(T+1)}. \tag{2.41}$$

For $k\Delta t \in [T, T+1]$, it then follows from (2.20) that

$$\mathbb{E}|x_k|^p \le |y_0|^p\Big(2^pC_{T+1}h(\Delta t) + e^{-(1-0.5\epsilon)\lambda(T+1)}\Big).$$

Choose $\Delta t^\star \in (0,1)$ sufficiently small for

$$2^pC_{T+1}h(\Delta t^\star) + e^{-(1-0.5\epsilon)\lambda(T+1)} \le e^{-(1-\epsilon)\lambda(T+1)}.$$

Then, for every $0 < \Delta t \le \Delta t^\star$,

$$\mathbb{E}|x_k|^p \le |y_0|^pe^{-(1-\epsilon)\lambda(T+1)}, \quad k\Delta t \in [T, T+1]. \tag{2.42}$$

Let us now fix $\Delta t \in (0, \Delta t^\star]$ arbitrarily. Choose a positive integer $\bar{k}$ such that $\bar{k}\Delta t \in [T, T+1]$. It then follows from (2.42) that

$$\mathbb{E}|x_{\bar{k}}|^p \le |y_0|^pe^{-(1-\epsilon)\lambda\bar{k}\Delta t}. \tag{2.43}$$

Moreover, it follows from (2.19) and Assumptions 2.4 and 2.5 and (2.9) that

$$\mathbb{E}|x_k|^p \le |y_0|^p2^p(C_{T+1} + M), \quad 0 \le k \le \bar{k}.$$

Hence

$$\mathbb{E}|x_k|^p \le N|y_0|^pe^{-(1-\epsilon)\lambda k\Delta t}, \quad 0 \le k \le \bar{k}, \tag{2.44}$$

where $N = 2^p(C_{T+1} + M)e^{(1-\epsilon)\lambda(T+1)}$. The remaining proof is almost the same as Step 2 in the proof of Lemma 2.6. ◻

LEMMA 2.10. *Let $\epsilon \in (0,1)$ and assume that Assumptions 2.4 and 2.5 hold. Assume also that for a step size $\Delta t > 0$, the numerical method is pth moment exponentially stable with rate constant γ and growth constant N. If Δt satisfies*

$$2^pC_Th(\Delta t) + e^{-(1-0.5\epsilon)\gamma T} \le e^{-(1-\epsilon)\gamma T}, \tag{2.45}$$

where $T = \bar{k}\Delta t$ and $\bar{k}$ is the smallest integer which is not less than $2\log(2^pN)/(\epsilon\gamma\Delta t)$, then the SDE (2.1) is pth moment exponentially stable with rate constant $\lambda = (1-\epsilon)\gamma$ and growth constant $M = H(T,p,K)e^{(1-\epsilon)\gamma T}$, where $H(T,p,K)$ is given in (2.4).

The proof is similar to that of Lemma 2.7 and so is omitted.

3. The stochastic theta method. The theory established in the previous section requires that the numerical solutions not only have the Markov property described in section 2 but also that they satisfy Assumptions 2.4 and 2.5. The question is, do such numerical solutions exist? We will give a positive answer here by considering the class of stochastic theta methods. Given a free parameter $\theta \in [0,1]$, the numerical solutions by the stochastic theta method are defined by

$$x_{k+1} = x_k + (1-\theta)f(x_k)\Delta t + \theta f(x_{k+1})\Delta t + g(x_k)\Delta w_k, \quad k \in \mathbb{Z}^+, \tag{3.1}$$

with the initial value $x_0 = y_0$, where $\Delta w_k = w((k+1)\Delta t) - w(k\Delta t)$. When $\theta = 0$, (3.1) is the widely used Euler–Maruyama method (see, e.g., [12, 16]). In this case, (3.1) is an explicit equation that defines x_{k+1}. But when $\theta \neq 0$, (3.1) represents a nonlinear system that is to be solved for x_{k+1}. By the classical Banach fixed-point theorem, it is easy to show the following (see, e.g., [9, 23]):

- Under the global Lipschitz condition (2.1), if $K\theta\Delta t < 1$, then (3.1) can be solved uniquely for x_{k+1}, with probability 1.

From now on we always assume that the step size $\Delta t < 1/(K\theta)$, so that the stochastic theta method (3.1) is well-defined. (We will in fact require $\Delta t < 1/(\sqrt{10}K\theta)$ later.) In other words, we can compute the discrete approximations $x_k \approx x(k\Delta t)$, with $x_0 = y_0$.

For our theory to work, we first need to show that the stochastic theta method possesses the Markov property, namely, the discrete process $\{x_k\}_{k\in\mathbb{Z}^+}$ is a time-homogeneous Markov process. This can been seen easily because, given $x_{\bar{k}}$ for some $\bar{k} \in \mathbb{Z}^+$, the process $\{x_k\}_{k\geq\bar{k}}$ can be fully determined by (3.1), but how the process has reached $x_{\bar{k}}$ has no further use. The following theorem shows that the stochastic theta method satisfies Assumption 2.4.

THEOREM 3.1. *Let Assumption* 2.1 *hold. Let* $p \in (0,1)$ *and let* Δt *be sufficiently small for* $\sqrt{10}K\theta\Delta t < 1$. *Then the discrete process* $\{x_k\}_{k\in\mathbb{Z}^+}$ *defined by the stochastic theta method* (3.1) *satisfies*

$$\sup_{0\leq k\Delta t\leq T} \mathbb{E}|x_k|^p \leq \bar{H}(T,p,K)|y_0|^p \quad \forall T \geq 0, \tag{3.2}$$

where $\bar{H}(T,p,K) = (11)^{p/2}e^{5pTK^2(T+1)}$.

Proof. To prove the lemma, let us introduce two continuous-time step processes,

$$z_1(t) = \sum_{k=0}^{\infty} x_k \mathbf{1}_{[k\Delta t,(k+1)\Delta t)}(t), \quad z_2(t) = \sum_{k=0}^{\infty} x_{k+1} \mathbf{1}_{[k\Delta t,(k+1)\Delta t)}(t),$$

with $\mathbf{1}_G$ denoting the indicator function for the set G. It is easy to see that $z_1(k\Delta t) = z_2((k-1)\Delta t) = x_k$. For convenience, we will let $t_k = k\Delta t$ for $k \in \mathbb{Z}^+$ from now on. It is easily shown that

$$x_{k+1} = y_0 + \int_0^{t_{k+1}} [(1-\theta)f(z_1(s)) + \theta f(z_2(s))]ds + \int_0^{t_{k+1}} g(z_1(s))dw(s). \tag{3.3}$$

Noting that

$$\begin{aligned}\int_0^{t_{k+1}} f(z_2(s))ds &= \int_0^{t_k} f(z_2(s))ds + \int_{t_k}^{t_{k+1}} f(z_2(s))ds \\ &= \int_{t_1}^{t_{k+1}} f(z_1(s))ds + f(x_{k+1})\Delta t,\end{aligned}$$

we get

$$x_{k+1} = y_0 - \theta f(y_0)\Delta t + \theta f(x_{k+1})\Delta t + \int_0^{t_{k+1}} f(z_1(s))ds + \int_0^{t_{k+1}} g(z_1(s))dw(s). \tag{3.4}$$

Hence

$$\begin{aligned}|x_{k+1}|^2 \leq 5\bigg\{&|y_0|^2 + (\theta\Delta t)^2|f(y_0)|^2 + (\theta\Delta t)^2|f(x_{k+1})|^2 \\ &+ \Big|\int_0^{t_{k+1}} f(z_1(s))ds\Big|^2 + \Big|\int_0^{t_{k+1}} g(z_1(s))dw(s)\Big|^2\bigg\}.\end{aligned}$$

By the linear growth condition (2.3) (followed from Assumption 2.1) as well as the Hölder inequality and the property of the Itô integral, we can show

$$\mathbb{E}|x_{k+1}|^2 \le 5\bigg\{|y_0|^2 + (K\theta\Delta t)^2|y_0|^2 + (K\theta\Delta t)^2\mathbb{E}|x_{k+1}|^2 \\ + K^2(t_{k+1}+1)\int_0^{t_{k+1}} \mathbb{E}|z_1(s))|^2 ds\bigg\}.$$

This, together with the condition $K\theta\Delta t < 1/\sqrt{10}$, yields

$$\begin{aligned} \mathbb{E}|x_{k+1}|^2 &\le 11|y_0|^2 + 10K^2(t_{k+1}+1)\int_0^{t_{k+1}} \mathbb{E}|z_1(s))|^2 ds \\ &= 11|y_0|^2 + 10K^2(t_{k+1}+1)\Delta t \sum_{j=0}^{k} \mathbb{E}|x_j|^2. \end{aligned} \tag{3.5}$$

Consequently, $\forall k \in \mathbb{Z}^+$ such that $t_{k+1} \le T$,

$$\mathbb{E}|x_{k+1}|^2 \le 11|y_0|^2 + 10K^2(T+1)\Delta t \sum_{j=0}^{k} \mathbb{E}|x_j|^2. \tag{3.6}$$

By the discrete Gronwall inequality (see, e.g., [14, 15]), we hence obtain

$$\sup_{0\le t_{k+1}\le T} \mathbb{E}|x_{k+1}|^2 \le 11|y_0|^2 e^{10TK^2(T+1)}.$$

Finally, we have

$$\sup_{0\le t_{k+1}\le T} \mathbb{E}|x_{k+1}|^p \le \Big(\sup_{0\le t_{k+1}\le T} \mathbb{E}|x_{k+1}|^2\Big)^{p/2} \le \bar{H}(T,p,K)|y_0|^p,$$

as required. □

Let us now proceed to show that the stochastic theta method satisfies Assumption 2.5. We need a lemma.

LEMMA 3.2. *Let Assumption* 2.1 *hold. Let* Δt *be sufficiently small for* $2K^2\Delta t < 1$. *Then the solution of the SDE* (2.1) *has the property*

$$\mathbb{E}|y(t) - y(t_k)|^2 \vee \mathbb{E}|y(t) - y(t_{k+1})|^2 \le \bar{C}_T \Delta t |y_0|^2 \tag{3.7}$$

$\forall 0 \le t_k \le t \le t_{k+1} \le T$, *where* $\bar{C}_T = 3(1+2K^2)e^{3K^2T(T+1)}$.

Proof. Noting

$$y(t) - y(t_k) = \int_{t_k}^{t} f(y(s))ds + \int_{t_k}^{t} g(y(s))dw(s),$$

we can show easily that

$$\mathbb{E}|y(t) - y(t_k)|^2 \le 2K^2(\Delta t + 1)\int_{t_k}^{t} \mathbb{E}|y(s)|^2 ds \le (1+2K^2)\int_{t_k}^{t} \mathbb{E}|y(s)|^2 ds.$$

By (2.5), we then have

$$\mathbb{E}|y(t) - y(t_k)|^2 \le 3(1+2K^2)e^{3K^2T(T+1)}\Delta t|y_0|^2.$$

Similarly, we can show

$$\mathbb{E}|y(t) - y(t_{k+1})|^2 \le 3(1 + 2K^2)e^{3K^2T(T+1)}\Delta t|y_0|^2.$$

The proof is therefore complete. □

THEOREM 3.3. *Let Assumption* 2.1 *hold. Let* $p \in (0, 1)$ *and let* Δt *be sufficiently small for*

$$(6 \vee 2K^2)\Delta t < 1. \tag{3.8}$$

Then the stochastic theta method solution (3.1) *and the true solution of the SDE* (2.1) *satisfy*

$$\sup_{0\le t_k\le T} \mathbb{E}|x_k - y(t_k)|^p \le C_T(\Delta t)^{p/2}|y_0|^p \quad \forall T > 0, \tag{3.9}$$

where

$$C_T = [36K^2T(2T + 1)(1 + 2K^2)]^{p/2}e^{7.5pK^2T(T+1)},$$

which is independent of Δt *and* y_0.

Proof. It follows from (2.1) and (3.3) that for any $0 \le t_{k+1} \le T$,

$$x_{k+1} - y(t_{k+1}) = \int_0^{t_{k+1}} \big((1 - \theta)[f(z_1(s)) - f(y(s))] + \theta[f(z_2(s)) - f(y(s))]\big)\, ds + \int_0^{t_{k+1}} [g(z_1(s)) - g(y(s))]\, dw(s).$$

Define

$$y_1(t) = \sum_{k=0}^{\infty} y(t_k)\mathbf{1}_{[k\Delta t,(k+1)\Delta t)}(t), \quad y_2(t) = \sum_{k=0}^{\infty} y(t_{k+1})\mathbf{1}_{[k\Delta t,(k+1)\Delta t)}(t).$$

Then

$$\begin{aligned} x_{k+1} - y(t_{k+1}) = &\int_0^{t_{k+1}} \big((1 - \theta)[f(z_1(s)) - f(y_1(s))] + \theta[f(z_2(s)) - f(y_2(s))]\big)ds \\ &+ \int_0^{t_{k+1}} \big((1 - \theta)[f(y_1(s)) - f(y(s))] + \theta[f(y_2(s)) - f(y(s))]\big)ds \\ &+ \int_0^{t_{k+1}} \big([g(z_1(s)) - g(y_1(s))] + [g(y_1(s)) - g(y(s))]\big)dw(s). \end{aligned}$$

But

$$\int_0^{t_{k+1}} [f(z_2(s)) - f(y_2(s))]ds = \int_{t_1}^{t_{k+1}} [f(z_1(s)) - f(y_1(s))]ds + [f(x_{k+1}) - f(y(t_{k+1}))]\Delta t.$$

Hence

$$\begin{aligned} x_{k+1} - y(t_{k+1}) = &\,\theta[f(x_{k+1}) - f(y(t_{k+1}))]\Delta t + \int_0^{t_{k+1}} [f(z_1(s)) - f(y_1(s))]ds \\ &+ \int_0^{t_{k+1}} \big((1 - \theta)[f(y_1(s)) - f(y(s))] + \theta[f(y_2(s)) - f(y(s))]\big)ds \\ &+ \int_0^{t_{k+1}} \big([g(z_1(s)) - g(y_1(s))] + [g(y_1(s)) - g(y(s))]\big)dw(s). \end{aligned}$$

This, together with Lemma 3.2, implies

$$\begin{aligned}
&\mathbb{E}|x_{k+1}-y(t_{k+1})|^2\\
&\le 6(K\theta\Delta t)^2\mathbb{E}|x_{k+1}-y(t_{k+1})|^2+6K^2(T+1)\int_0^{t_{k+1}}\mathbb{E}|z_1(s)-y_1(s)|^2ds\\
&+6K^2(T+1)\int_0^{t_{k+1}}\mathbb{E}|y_1(s)-y(s)|^2ds+6K^2T\int_0^{t_{k+1}}\mathbb{E}|y_2(s)-y(s)|^2ds\\
&\le 6(K\theta\Delta t)^2\mathbb{E}|x_{k+1}-y(t_{k+1})|^2+6K^2(T+1)\Delta t\sum_{j=0}^{k}\mathbb{E}|x_j-y(t_j)|^2\\
&+6K^2T(2T+1)\bar{C}_T\Delta t|y_0|^2.
\end{aligned}$$

However, by (3.8),

$$6(K\theta\Delta t)^2\le 3(2K^2\Delta t)\Delta t\le 3\Delta t\le 0.5.$$

So

$$\begin{aligned}
\mathbb{E}|x_{k+1}-y(t_{k+1})|^2\le 12K^2(T+1)\Delta t\sum_{j=0}^{k}\mathbb{E}|x_j-y(t_j)|^2&\\
+12K^2T(2T+1)\bar{C}_T\Delta t|y_0|^2.&
\end{aligned}$$

This, by the discrete Gronwall inequality, yields

$$\mathbb{E}|x_{k+1}-y(t_{k+1})|^2\le 12K^2T(2T+1)\bar{C}_T\Delta t|y_0|^2e^{12K^2T(T+1)}$$

$\forall 0\le t_{k+1}\le T$. Finally, the required assertion (3.9) follows as

$$\mathbb{E}|x_{k+1}-y(t_{k+1})|^p\le\left(\mathbb{E}|x_{k+1}-y(t_{k+1})|^2\right)^{p/2}.$$

The proof is hence complete. ∎

Remark 3.4. It is useful to point out that condition (3.8) implies $\sqrt{10}K\theta\Delta t<1$, which is the condition required for Theorem 3.1. In fact, if $6\ge 2K^2$, namely, $\sqrt{3}\ge K$, then (3.8) means $6\Delta t<1$. Hence

$$\sqrt{10}K\theta\Delta t\le\sqrt{30}\Delta t<1.$$

But if $\sqrt{3}<K$, then (3.8) means $2K^2\Delta t<1$. Thus

$$\sqrt{10}K\theta\Delta t<2\sqrt{3}K\Delta t<2K^2\Delta t<1.$$

That is, we always have $\sqrt{10}K\theta\Delta t<1$ if (3.8) holds. We therefore see from Theorems 3.1 and 3.3 that under our standing Assumption 2.1, the stochastic theta method satisfies Assumptions 2.4 and 2.5 as long as the step size Δt is sufficiently small for (3.8) to hold. By our theory established in section 2, we can further conclude that for the nonlinear SDEs under the global Lipschitz condition, the pth moment exponential stability for the SDE is equivalent to the pth moment exponential stability of the stochastic theta method for sufficiently small step sizes.

4. Almost sure exponential stability. Our paper is mainly concerned with the almost sure exponential stability of both true and numerical solutions with the objective of finding positive answers to problems (P1) and (P2) in section 1. It is therefore time to relate the pth moment exponential stability to the almost sure exponential stability. We first cite a theorem from [18, Theorem 5.9, p. 167] which shows that under our standing hypothesis, the pth moment exponential stability of the true solutions implies the almost sure exponential stability.

THEOREM 4.1. *Let Assumption* 2.1 *hold and let* $p \in (0,1)$*. Assume that the SDE* (2.1) *is pth moment exponentially stable and satisfies* (2.9)*. Then the solution of the SDE* (2.1) *satisfies*

$$\limsup_{t\to\infty} \frac{1}{t} \log(|y(t)|) \le -\frac{\lambda}{p} \quad a.s.$$

$\forall y_0 \in \mathbb{R}^n$*. That is, the SDE is also almost surely exponentially stable.*

The following theorem is an analogue for the numerical solutions.

THEOREM 4.2. *Assume that the numerical method is pth moment exponentially stable and satisfies* (2.11)*. Then the method satisfies*

$$\limsup_{k\to\infty} \frac{1}{k\Delta t} \log(|x_k|) \le -\frac{\gamma}{p} \quad a.s. \tag{4.1}$$

$\forall y_0 \in \mathbb{R}^n$*. That is, the method is also almost surely exponentially stable.*

Proof. Let $\epsilon \in (0,\gamma)$ be arbitrary. By the Chebyshev inequality,

$$\mathbb{P}\Big(|x_k| > e^{-(\gamma-\epsilon)k\Delta t/p}\Big) \le N|y_0|^p e^{-\epsilon k\Delta t}, \quad k \in \mathbb{Z}^+.$$

By the well-known Borel–Cantelli lemma, we see that for almost all $\omega \in \Omega$,

$$|x_k| \le e^{-(\gamma-\epsilon)k\Delta t/p} \tag{4.2}$$

holds for all but finitely many k. Hence, there exists a $k_0(\omega)$ $\forall \omega \in \Omega$ excluding a $\mathbb{P}$-null set, for which (4.2) holds whenever $k \ge k_0$. Consequently, for almost all $\omega \in \Omega$, if $k\Delta t \le t \le (k+1)\Delta t$ and $k \ge k_0$,

$$\frac{1}{k\Delta t} \log(|x_k|) \le -\frac{(\gamma-\epsilon)k\Delta t}{pt} \le -\frac{(\gamma-\epsilon)k}{p(k+1)}.$$

Hence

$$\limsup_{k\to\infty} \frac{1}{k\Delta t} \log(|x_k|) \le -\frac{\gamma-\epsilon}{p} \quad \text{a.s.}$$

and the required (4.1) follows by letting $\epsilon \to 0$. ∎

Before we proceed to give our positive answers to problems (P1) and (P2), let us highlight what our new theory established thus far enables us to do under the standing Assumption 2.1:

- Suppose that we are required to find out whether the SDE (2.1) is almost surely exponentially stable. In the absence of an appropriate Lyapunov function, we can now carry out careful numerical simulations using the stochastic theta method with a sufficiently small step size Δt. If the stochastic theta method is pth moment exponential stable for a sufficiently small $p \in (0,1)$, we can then infer that the underlying SDE is almost surely exponentially stable.

- If the SDE is pth moment exponentially stable for some $p \in (0,1)$, then the stochastic theta method is almost surely exponentially stable for all sufficiently small step sizes Δt.

In other words, we have obtained positive answers to both (Q1) and (Q2) in section 1. We are also in a position to give our positive answers to problems (P1) and (P2) of that section.

4.1. Answer to (P1). Consider the n-dimensional linear SDE

$$dy(t) = A_0 y(t)dt + \sum_{i=1}^{m} A_i y(t) dw_i(t) \tag{4.3}$$

on $t \geq 0$ with the initial value $y(0) = y_0 \in \mathbb{R}^n$, where $A_i \in \mathbb{R}^{n\times n}$ $\forall 0 \leq i \leq m$. Let us cite a theorem from Arnold, Kliemann, and Oeljeklaus [2].

THEOREM 4.3. *The linear SDE* (4.3) *is almost surely exponentially stable if and only if it is pth moment exponentially stable for a sufficiently small* $p \in (0,1)$.

From this theorem and our theory in sections 2 and 3, we obtain the following theorem, which gives a positive answer to problem (P1).

THEOREM 4.4. *If the linear SDE* (4.3) *is almost surely exponentially stable, then the stochastic theta method is almost surely exponentially stable for all sufficiently small step sizes.*

4.2. Answer to (P2). There are many results on the almost sure exponential stability of the nonlinear SDE (2.1) (see, e.g., [1, 6, 11, 13, 14, 15]). We cite one of the most useful criteria from Mao [16, Theorem 3.3, p. 121].

THEOREM 4.5. *Let Assumption* 2.1 *hold. Assume that there exists a function* $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+; \mathbb{R}^+)$ *and constants* $q > 0$, $\bar{c}_1 \geq c_1 > 0$, $c_2 \in \mathbb{R}$, $c_3 \geq 0$ *with* $c_3 > 2c_2$ *such that for all* $y \neq 0$ *and* $t \geq 0$,

$$c_1|y|^q \leq V(y,t) \leq \bar{c}_1|y|^q, \quad LV(y,t) \leq c_2 V(y,t),$$

$$|V_x(y,t)g(y,t)|^2 \geq c_3 (V(y,t))^2,$$

where the diffusion operator L acting on the $C^{2,1}$-functions is defined by

$$LV(y,t) = V_t(y,t) + V_y(y,t)f(x,t) + 0.5\mathrm{trace}(g^T(y,t)V_{yy}(y,t)g(y,t)),$$

in which

$$V_t(y,t) = \frac{\partial V(y,t)}{\partial t}, \quad V_y(y,t) = \Big(\frac{\partial V(y,t)}{\partial y_i}\Big)_{1\times n}, \quad V_{yy}(y,t) = \Big(\frac{\partial^2 V(x,t)}{\partial y_i \partial y_j}\Big)_{n\times n}.$$

Then the SDE (2.1) *is almost surely exponentially stable.*

The following lemma shows that under the same conditions as those of Theorem 4.5, the SDE (2.1) is pth moment exponentially stable for all sufficiently small p.

LEMMA 4.6. *Let the conditions of Theorem* 4.5 *hold. Let* $\epsilon \in (0,1)$ *be sufficiently small for*

$$p := \epsilon q < 1 \quad \text{and} \quad 0.5(1-\epsilon)c_3 > c_2. \tag{4.4}$$

Then the solution of the SDE (2.1) *satisfies*

$$\mathbb{E}|y(t)|^p \leq M|y_0|^p e^{-\lambda t} \quad \forall t \geq 0 \tag{4.5}$$

$\forall y_0 \in \mathbb{R}^n$, *where* $M = (\bar{c}_1/c_1)^\epsilon$ *and* $\lambda = \epsilon(0.5(1-\epsilon)c_3 - c_2)$.

Proof. Assertion (4.5) holds when $y_0 = 0$, so we need to show it for $y_0 \neq 0$. Fix any $y_0 \neq 0$. By Mao [16, Lemma 3.2, p. 120], we observe that $y(t) \neq 0$ $\forall t \geq 0$ almost surely. Let $U(y,t) = (V(y,t))^\epsilon$. Then it is easy to show that for $y \neq 0$ and $t \geq 0$, the diffusion operator L acting on $U(y,t)$ has the form

$$LU(y,t) = \epsilon(V(y,t))^{\epsilon-1}LV(y,t) - 0.5\epsilon(1-\epsilon)(V(y,t))^{\epsilon-2}|V_y(y,t)g(y,t)|^2.$$

By the conditions, we then see

$$LU(y,t) \leq -\lambda U(y,t).$$

Consequently

$$L(e^{\lambda t}U(y,t)) = \lambda e^{\lambda t}U(y,t) + e^{\lambda t}LU(y,t) \leq 0.$$

An application of the Itô formula implies

$$e^{\lambda t}\mathbb{E}U(y(t),t) \leq U(y_0,0) \quad \forall t \geq 0.$$

This yields the required assertion (4.5) by the fact $c_1^\epsilon |y|^p \leq U(y,t) \leq \bar{c}_1^\epsilon |y|^p$. ☐

From this lemma and our theory in sections 2 and 3, we obtain the following theorem, which gives a positive answer to problem (P2).

THEOREM 4.7. *Under the conditions of Theorem* 4.5*, the stochastic theta method is almost surely exponentially stable for all sufficiently small step sizes.*

5. Examples. In this section we just discuss two examples to illustrate our positive answers to (P1) and (P2).

Example 5.1. Consider a two-dimensional linear SDE

$$dy(t) = A_0 y(t)dt + A_1 y(t)dw_1(t) + A_2 y(t)dw_2(t) \tag{5.1}$$

on $t \geq 0$ with initial value $y(0) = y_0 \in \mathbb{R}^2$, where

$$A_0 = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad A_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix},$$

in which a, σ_1, and σ_2 are real numbers. It is easy to see that the matrices A_0, A_1, and A_2 commute. It is known (see, e.g., [16, p. 100]) that the linear SDE (5.1) has the explicit solution

$$\begin{aligned} y(t) &= \exp\Big[\big(A_0 - 0.5(A_1^2 + A_2^2)\big)t + A_1 w_1(t) + A_2 w_2(t)\Big]\, y_0 \\ &= \exp\Big[\big(a - 0.5(\sigma_1^2 - \sigma_2^2)\big)I_2 t + A_1 w_1(t) + A_2 w_2(t)\Big]\, y_0, \end{aligned} \tag{5.2}$$

where I_2 is the 2×2 identity matrix. It is therefore easy to see that the linear SDE (5.1) is almost surely exponentially stable if and only if $a < 0.5(\sigma_1^2 - \sigma_2^2)$. By Theorem 4.4, we can then conclude that the stochastic theta method applied to the SDE (5.1) is almost surely exponentially stable for all sufficiently small step sizes, which is one of the key results of [4]. Figure 1 shows a computer simulation of the paths of $y_1(t)$ and $y_2(t)$ using the stochastic theta method with the parameter $\theta = 0.5$ and the step size $\Delta t = 0.001$ and the system parameters $a = 1$, $\sigma_1 = 2$, $\sigma_2 = 1$ as well as the initial values $y_1(0) = 1$, $y_2(0) = 2$. The computer simulation clearly supports our theoretical result.

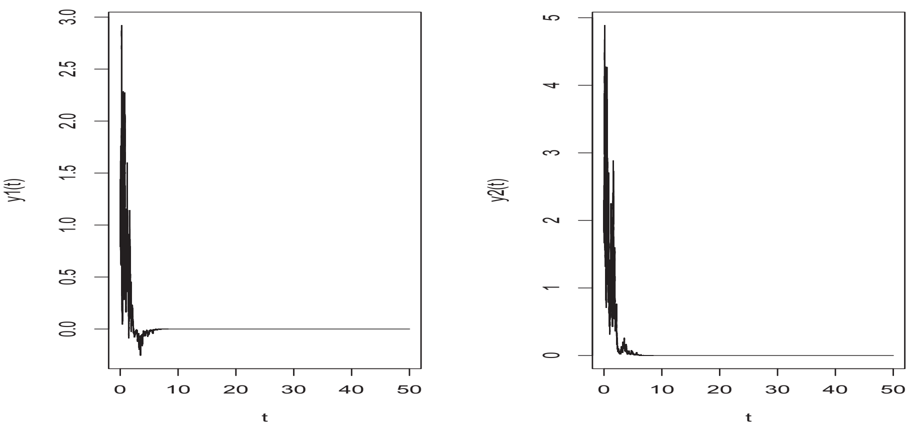

FIG. 1. *The sample paths of the solution to the SDE* (5.1).

Example 5.2. Consider the two-dimensional stochastic differential equation

$$dy(t) = f(y(t))dt + Gy(t)dw_1(t) \tag{5.3}$$

on $t \geq 0$ with initial value $y(0) = y_0 \in \mathbb{R}^2$,

$$f(y) = \begin{pmatrix} 2y_2 \cos y_1 + \sin y_1 \\ -y_1 + \sin y_2 \end{pmatrix}, \quad G = \begin{pmatrix} 3 & -0.3 \\ -0.3 & 3 \end{pmatrix}.$$

Let $V(y, t) = |y|^2$. It is easy to verify that

$$LV(y, t) = 2y_1(2y_2 \cos y_1 + \sin y_1) + 2y_2(-y_1 + \sin y_2) + |Gy|^2 \leq 13.89|x|^2$$

and

$$|V_x(x, t)Gx|^2 = |2x^T Gx|^2 \geq 29.16|x|^4.$$

By Theorem 4.5, the SDE (5.3) is almost surely exponentially stable, while by Theorem 4.7, the stochastic theta method applied to the SDE is also almost surely exponentially stable for all sufficiently small step sizes. Figure 2 is the computer simulation of the paths of $y_1(t)$ and $y_2(t)$ using the Euler–Maruyama method (i.e., the stochastic theta method with the parameter $\theta = 0$) and the step size $\Delta t = 0.001$ as well as the initial values $y_1(0) = 1$, $y_2(0) = 2$. Again, the computer simulation clearly supports our theoretical result.

6. Conclusions. In this paper, we have shown that, under the standing Assumption 2.1,

$$(A) \Leftarrow (B) \Leftrightarrow (C) \Rightarrow (D),$$

where

- (A) denotes the almost sure exponential stability of the SDE,
- (B) denotes the pth moment exponential stability of the SDE ($p \in (0, 1)$ is sufficiently small),

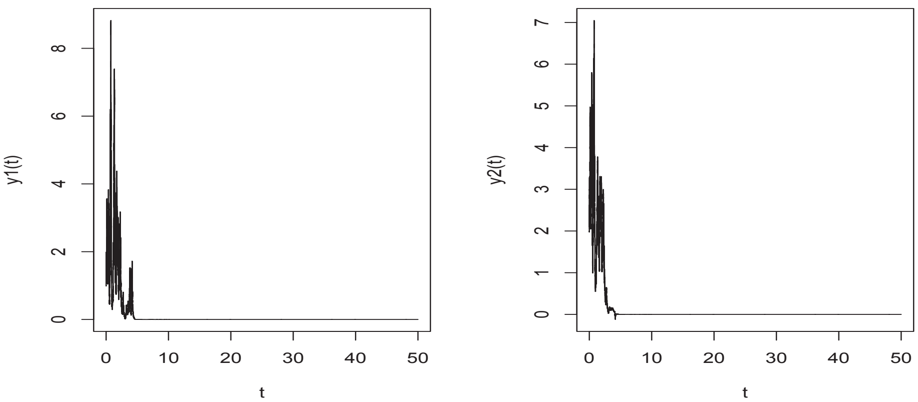

FIG. 2. *The sample paths of the solution to the SDE* (5.3).

(C) denotes the pth moment exponential stability of the stochastic theta method for a sufficiently small step size,

(D) denotes the almost sure exponential stability of the stochastic theta method for a sufficiently small step size.

In particular, our new theory enables us to study the almost sure exponential stability of SDEs using the stochastic theta method, instead of the method of the Lyapunov functions. That is, we can now carry out careful numerical simulations using the stochastic theta method with a sufficiently small step size Δt. If the stochastic theta method is pth moment exponentially stable for a sufficiently small $p \in (0, 1)$, we can then infer that the underlying SDE is almost surely exponentially stable.

Our new theory also enables us to show the ability of the stochastic theta method to reproduce the almost sure exponential stability of the SDEs. In the case when the underlying SDE is linear (namely, (4.3)), recalling the classical result $(A) \Leftrightarrow (B)$ established by Arnold, Kliemann, and Oeljeklaus [2], we have shown $(A) \Rightarrow (D)$. That is, we have obtained the positive answer to problem (P1). In the case when the underlying SDE is nonlinear, we know that (A) does not imply (B) in general. However, we have recalled one of the most useful criteria, Theorem 4.5, for almost sure exponential stability and shown that the SDE is pth moment exponentially stable for all sufficiently small p under the same conditions. Consequently, we have obtained a positive answer to problem (P2).

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