On the Representability of Line Graphs

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1 Introduction

A graph $G = (V, E)$ is representable if there exists a word W over the alphabet V such that letters x and y alternate in W if and only if $(x, y) \in E$ for each $x \neq y$. Such a W is called a *word-representant* of G. Note that in this paper we use the term graph to mean a finite, simple graph, even though the definition of representable is applicable to more general graphs.

The notion of a representable graph comes from algebra, where it was used by Kitaev and Seif to study the growth of the free spectrum of the well known *Perkins semigroup* [5]. There are also connections between representable graphs and robotic scheduling as described by Graham and Zang in [1]. Moreover, representable graphs are a generalization of *circle graphs*, which was shown by Halldórsson, Kitaev and Pyatkin in [2], and thus they are interesting from a graph theoretical point of view. Finally, representable graphs are interesting from a combinatorics on words point of view as they deal with the study of alternations in words.

Not all graphs are representable. Examples of minimal (with respect to the number of nodes) non-representable graphs given by Kitaev and Pyatkin in [4] are presented in Fig. 1.

It was remarked in [2] that very little is known about the effect of the line graph operation on the representability of a graph. We attempt to shed some light on this subject by showing that the line graph of the smallest known non-representable graph, the wheel on five vertices, W_5, is in fact non-representable. In fact we prove a stronger result, which is that $L(W_n)$ (where $L(G)$ denotes the line graph of G) is non-representable for $n \geqslant 4$. From the non-representability of $L(W_4)$ we are led to a more general theorem regarding line graphs. Our main result is that $L^k(G)$, where G is not a cycle, a path or the claw graph,

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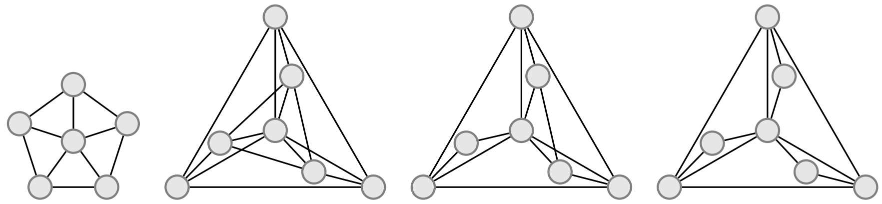

Fig. 1. Minimal non-representable graphs

is guaranteed to be non-representable for $k \geqslant 4$. However an important open question still remains: Is the line graph of a non-representable graph always non-representable.

2 Results

The *wheel graph*, denoted by W_n, is a graph we obtain from a cycle C_n by adding one external vertex adjacent to every other vertex.

A line graph $L(G)$ of a graph G is a graph on the set of edges of G such that in $L(G)$ there is an edge (a, b) if and only if edges a, b are adjacent in G.

Theorem 1. *The line graph $L(W_n)$ is not representable for each $n \geqslant 4$.*

Theorem 2. *The line graph $L(K_n)$ is not representable for each $n \geqslant 5$.*

It was shown by van Rooji and Wilf [6] that iterating the line graph operator on most graphs results in a sequence of graphs which grow without bound. This unbounded growth results in graphs that are non-representable after a small number of iterations of the line graph operator since they contain the line graph of a large enough clique.

Theorem 3. *If a connected graph G is not a path, a cycle, or the claw graph $K_{1,3}$, then $L^n(G)$ is not representable for $n \geqslant 4$.*

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