A Note on Exponential Almost Sure Stability of Stochastic Differential Equation

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Abstract

Our goal is to relax a sufficient condition for the exponential almost sure stability of a certain class of stochastic differential equations. Compare to the existing theory, we prove the almost sure stability, replacing Lipschitz continuity and linear growth conditions by the existence of a strong solution of the underlying stochastic differential equation. This result is extendable for the regime-switching system. An explicit example is provided for the illustration purpose.

Keywords. Almost sure stability, Stochastic differential equation, Regime-switching, Besel squared process.
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1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})$ be a filtered probability space satisfying general conditions, on which $B(t) \in \mathbb{R}^m$ is a standard m-dimensional Brownian motion. Consider a stochastic differential equation (SDE) of

$$dX(t) = f(X(t), t)dt + g(X(t), t)dB(t), \ X(t_0) = x_0, \tag{1}$$

where $X(t) \in \mathbb{R}^d$, $f : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^d$, $g : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^{d \times m}$. It is well known that SDE (1) has a unique strong solution under the following assumptions,

Assumption 1 *f and g obey the local Lipschitz continuity and Linear growth condition.*

In the past a few decades, the stability of such a system has been well studied as an important feature, see more details in [1, 3] and the references therein. Our interest of this note is to relax a sufficient condition for the almost sure exponential stability based on the existing result of Theorem 3.3 of the book [3].

For convenience, we define an operator L by, for all $\varphi \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R})$.

$$L\varphi(x, t) = \varphi_t(x, t) + \varphi_x(x, t) \cdot f(x, t) + \frac{1}{2} Trace \left(\varphi_{xx}(x, t)(gg^T)(x, t) \right).$$

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Theorem 3.3 of the book [3] provides a sufficient condition for the exponential almost sure stability under the following assumptions, and its proof is referred to [3].

Theorem 1 (Theorem 3.3 of [3]) *Suppose SDE* (1) *satisfies Assumption* 1 *and*

$$f(0,t) = g(0,t) = 0, \ \forall t \in [t_0, \infty). \tag{2}$$

Also, assume that there exists a function $V \in C^{2,1}(\mathbb{R}^d \times [t_0, \infty); \mathbb{R}_+)$, *and constants* $p > 0, c_1 > 0, c_2 \in \mathbb{R}, c_3 \geq 0$, *such that for all* $x \neq 0$ *and* $t \geq t_0$,

(H1) $c_1 |x|^p \leq V(x,t)$,

(H2) $LV(x,t) \leq c_2 V(x,t)$,

(H3) $|V_x(x,t) \cdot g(x,t)|^2 \geq c_3 V^2(x,t)$.

Then,

$$\limsup_{t \to \infty} \frac{1}{t} \log |X(t; t_0, x_0)| \leq -\frac{c_3 - 2c_2}{2p} \qquad a.s.$$

In particular, SDE (1) *is exponentially stable in almost sure sense, if* $c_3 > 2c_2$.

Together with Assumption 1, the equation (2) implies zero is an absorbing state for $X(t)$. Moreover, one can also conclude $X(t) > 0$ almost surely if the initial x_0 is given strictly positive.

In this below, we will present a stability result with weaker sufficient condition. In particular, we will remove Assumption 1 and (2). Instead, we impose the existence of the strong solution as follows.

Assumption 2 *SDE* (1) *has a strong solution for any initial* x_0.

The proof of the next result is relegated to the next section for better illustration.

Theorem 2 (Main stability result) *If SDE* (1) *satisfies Assumption* 2, *and assume that there exists a function* $V \in C^{2,1}(\mathbb{R}^d \times [t_0, \infty); \mathbb{R}_+)$, *and constants* $p > 0, c_1 > 0, c_2 \in \mathbb{R}, c_3 \geq 0$, *such that for* $t \geq t_0$, (H1)-(H3) *satisfy. Then*

$$\limsup_{t \to \infty} \frac{1}{t} \log |X(t; t_0, x_0)| \leq -\frac{c_3 - 2c_2}{2p} \qquad a.s. \tag{3}$$

In the above, we extend the definition to $\log 0 = -\infty$.

It is well known that Assumption 1 is only a sufficient condition for the existence of a strong solution, but not a necessary condition, see more discussions in Chapter 5 of [2]. In the next section after the proof of Theorem 2, we will provide an example, which can be verified to be almost surely exponential stable by Theorem 2, but not by Theorem 1. Section 3 includes an extension of the result to regime-switching diffusion and other possible generalizations.

2   A Proof of the Main Result and an Example

In this section, we will provide a proof of Theorem 2 and an Example for the illustration purpose.

2.1   Proof of Theorem 2

In the above Assumption 2 we do not exclude possibly zero solutions. Therefore, although $\log V(x(t), t)$ is well defined for all $t \in (0, \infty)$ under extended logarithm, it is not directly applicable to Ito's formula to calculate $d \log V(x(t), t)$ due to possible value $\log 0$. For this reason, we define a new strictly positive function

$$V^\eta(x, t) = V(x, t) + e^{-\eta t},$$

for some constant $\eta > 0$ to be given by (9). By Ito's formula on V^η, we obtain

$$dV^\eta(X(t), t) = (LV(X(t), t) - \eta e^{-\eta t})dt + (V_x \cdot g)(X(t), t)dB_t.$$

For the strictly positive function V^η, we can apply Ito's formula on $\log V^\eta(X(t), t)$, obtain

$$
\begin{aligned}
d \log V^\eta(X(t), t) &= \frac{1}{V^\eta(X(t), t)} dV^\eta(X(t), t) - \frac{1}{2} \frac{1}{V^\eta(X(t), t)^2} d\langle V^\eta, V^\eta \rangle(X(t), t) \\
&= \left(\frac{LV}{V^\eta}(X(t), t) - \frac{1}{2} \left| \frac{V_x \cdot g}{V^\eta} \right|^2 (X(t), t) - \frac{\eta e^{-\eta t}}{V^\eta}(X(t), t) \right) dt \qquad (4) \\
&\qquad + \frac{V_x \cdot g}{V^\eta}(X(t), t)dB_t.
\end{aligned}
$$

Rewriting it into integral form

$$
\begin{aligned}
\log V^\eta(X(t), t) &= \log V^\eta(x_0, t_0) \\
&\quad + \int_{t_0}^t \left(\frac{LV}{V^\eta}(X(s), s) - \frac{1}{2} \left| \frac{V_x \cdot g}{V^\eta} \right|^2 (X(s), s) - \frac{\eta e^{-\eta s}}{V^\eta}(X(s), s) \right) ds \\
&\qquad\qquad\qquad\qquad + \int_{t_0}^t \frac{V_x \cdot g}{V^\eta}(X(s), s)dB_s.
\end{aligned}
$$
$$(5)$$

Denote the last term by

$$M(t) = \int_{t_0}^t \frac{V_x \cdot g}{V^\eta}(X(s), s)dB_s.$$

Note that $M(t)$ is a continuous martingale with initial value $M(t_0) = 0$. Assign $\epsilon \in (0, 1)$, by exponential martingale inequality ([5, 7])

$$\mathbb{P}\left(\sup_{0 \le t \le n} [M(t) - \frac{\epsilon}{2} \int_{t_0}^t \frac{|V_x \cdot g|^2}{|V^\eta|^2}(X(s), s)ds] > \frac{2}{\epsilon} \log n \right) \le \frac{1}{n^2}$$

Applying Borel-Cantelli lemma, we will see that for almost all $\omega \in \Omega$, there is an integer $n_0 = n_0(\omega)$ such that when $n \ge n_0$

$$M(t) \le \frac{2}{\epsilon} \log n + \frac{\epsilon}{2} \int_{t_0}^t \frac{|V_x \cdot g|^2}{|V^\eta|^2}(X(s), s)ds \qquad (6)$$

for all $0 \le t \le n$.
Using (H2) and (H3) on (5) and (6), it leads to the inequality

$$\log V^\eta(X(t), t) \le \log V^\eta(x_0, t_0) + \int_{t_0}^t I(s)ds + \frac{2}{\epsilon} \log n. \qquad (7)$$

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where
$$I(s) = \frac{c_2 V}{V^\eta}(X(s), s) - \frac{\frac{1}{2}(1-\epsilon)c_3|V|^2}{|V^\eta|^2}(X(s), s) - \frac{\eta e^{-\eta s}}{V^\eta}(X(s), s).$$

Define $G : \mathbb{R}_+ \to \mathbb{R}$ by
$$G(y) = \frac{c_2 y}{y+1} - \frac{(1-\epsilon)c_3 y^2}{2(y+1)^2} - \frac{\eta}{y+1}.$$

In terms of function $G(\cdot)$, $I(s)$ has representation of

$$I(s) = G\Big(V(X(s), s) \cdot e^{\eta s}\Big). \tag{8}$$

Now, by taking special η of

$$\eta := (1-\epsilon)c_3 - c_2. \tag{9}$$

One can observe that
$$G'(y) = \frac{(1-\epsilon)c_3}{(y+1)^3} \geq 0, \ \forall y \in (0, \infty)$$

which means $G(\cdot)$ is an non-decreasing function on $(0, \infty)$. Hence, we can find upper bound of G by calculating its limit at infinity, that is

$$G(y) \leq c_2 - \frac{1}{2}(1-\epsilon)c_3 = \lim_{y \to \infty} G(y), \ \forall y \in \mathbb{R}_+.$$

Together with (8), we have
$$I(s) \leq c_2 - \frac{1}{2}(1-\epsilon)c_3.$$

With this inequality, we rewrite (10) by

$$\log V^\eta(X(t), t) \ \leq \log V^\eta(x_0, t_0) + \Big(c_2 - \frac{1}{2}(1-\epsilon)c_3\Big)(t - t_0) + \frac{2}{\epsilon}\log n \tag{10}$$

for all $0 \leq t \leq n$. Therefore, if t is chosen as $n-1 \leq t \leq n$

$$\frac{1}{t}\log V^\eta(X(t), t) \leq \frac{1}{t}\log V^\eta(x_0, t_0) + \Big(c_2 - \frac{1}{2}(1-\epsilon)c_3\Big)\frac{t - t_0}{t} + \frac{2}{\epsilon}\frac{\log(t+1)}{t}.$$

Passing the limit $t \to \infty$,

$$\limsup_{t \to \infty} \frac{1}{t}\log V^\eta(X(t), t) \leq \Big(c_2 - \frac{1}{2}(1-\epsilon)c_3\Big).$$

Finally, we reach the desired result (3) as a consequence of the arbitrariness of $\epsilon \in (0, 1)$, (H1), and $V \leq V^\eta$.

2.2 Example

Consider SDE given by

$$dX(t) = -X(t)dt + 2e^{-t/2}X^{1/2}(t)dB(t), \ X(0) \geq 0. \tag{11}$$

SDE (11) violates Assumption 1 due to its coefficient of $dB(t)$. On the other hand, SDE (11) satisfies Assumption 2, since we can find a strong solution of the form $X(t) = e^{-t}Y(t)$, where $Y(t)$ is a squared Bessel process with dimension $\delta = 0$, namely $BESQ^0$, which satisfies

$$dY(t) = 2\sqrt{Y(t)}dB(t), \ Y_0 = x_0,$$

see Chapter XI of [6]. In fact, the above $X(t)$ is the unique solution of SDE (11) by Theorem IX.3.5 of [6].

Set Liapunov function $V(x)$ by $V(x) = x$ on $(x,t) \in \mathcal{O} := \mathbb{R}_+ \times \mathbb{R}_+$. We only define V on $\mathcal{O}$, since $X(t) > 0$ almost surely provided that $x_0 > 0$. Then, we can verify, V satisfies (H1)-(H3) with the constants $p = 1, c_1 = 1, c_2 = -1, c_3 = 0$. Thus, $X(t)$ is exponentially stable in the almost sure sense, with

$$\limsup_{t \to \infty} \frac{1}{t} \log |X(t)| \leq -1.$$

3 Further extensions

In this part, we will present two generalizations of the stability result. The proof can be completed using exactly the same methodologies discussed above.

3.1 Regime-Switching model

Let $\gamma(\cdot)$ is a continuous Markov chain taking value in a finite state space $\mathcal{M} = \{1, 2, \ldots, m\}$ with generator $Q(x) = q_{ij}(x)$ satisfying $(q_{ij}(x))_{i,j \in \mathcal{M}} \geq 0$ for $j \neq i$ and $\sum_{j \in \mathcal{M}} q_{ij}(x) = 0$ for all $x \in \mathbb{R}^d$ and $i \in \mathcal{M}$. Associated to the Markov chain γ, we consider following stochastic differential equation:

$$dX(t) = f(X(t), \gamma(t), t)dt + g(X(t), \gamma(t), t)dB(t), \ X(t_0) = x_0, \gamma(t) = \gamma_0. \tag{12}$$

The stochastic differential equation of the form (12) is often referred to regime-switching diffusion. The applications of such a system have been witnessed in various areas, such as signal processing, production planning, biological systems, and see more in [4, 8, 9]. The generator of (12) can be written by, for $\varphi \in C^{2,1}(\mathbb{R}^d \times \mathcal{M} \times \mathbb{R}_+; \mathbb{R})$

$$L\varphi(x,i,t) \ = \varphi_t(x,i,t) + \varphi_x(x,i,t) \cdot f(x,i,t) + \frac{1}{2}Trace\left(\varphi_{xx}(x,i,t)(gg^T)(x,i,t)\right) + \sum_{\mathcal{M} \ni j \neq i} q_{ij}(x)\Big(g(x,j,t) - g(x,i,t)\Big).$$

If we go through similar steps in Section 2, we obtain the following result on the stability, and the proof is omitted to keep this note brief.

Theorem 3 *Suppose SDE* (12) *has the unique strong solution, and there exists a function* $V \in C^{2,1}(\mathbb{R}^d \times \mathcal{M} \times [t_0, \infty); \mathbb{R}_+)$, *and constants* $p > 0, c_1 > 0, c_2 \in \mathbb{R}, c_3 \geq 0$, *such that for all* $x \neq 0$, $i \in \mathcal{M}$ *and* $t \geq t_0$,

(H1') $c_1|x|^p \leq V(x,i,t)$,

(H2') $LV(x,i,t) \leq c_2 V(x,i,t)$,

(H3') $|V_x(x,i,t) \cdot g(x,i,t)|^2 \geq c_3 V^2(x,i,t)$.

Then,

$$\limsup_{t\to\infty} \frac{1}{t} \log|X(t)| \leq -\frac{c_3 - 2c_2}{2p} \qquad a.s.$$

In particular, SDE (12) *is exponentially stable in almost sure sense, if* $c_3 > 2c_2$.

3.2 Delay systems

In this part, we consider stochastic differential delay equations, see [10] and the references therein.

$$dX(t) = f(X(t), X(t - \delta(t)), t)dt + g(X(t), X(t - \delta(t)), t)dB(t) \tag{13}$$

with initial data

$$X(t) = \psi(t), \quad \forall t \in [t_0 - \delta(t_0), t_0],$$

where $X(t) \in \mathbb{R}^d$, $f : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^d$, $g : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^{d \times m}$. we define an operator $L : C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}) \mapsto L^0(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}; \mathbb{R})$ by, for all $\varphi \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R})$.

$$L\varphi(x,y,t) = \varphi_t(x,t) + \varphi_x(x,t) \cdot f(x,y,t) + \frac{1}{2} Trace \left(\varphi_{xx}(x,t)(gg^T)(x,y,t) \right).$$

Assumption 3 *SDE* (13) *has a strong solution for any initial* x.

Theorem 4 *If SDE* (13) *satisfies Assumption* 3 *and*

$$f(0,y,t) = g(0,y,t) = 0, \forall (y,t) \in \mathbb{R}^d \times \mathbb{R}_+, \tag{14}$$

and assume that there exists a function $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}_+)$, *and constants* $p > 0, c_1 > 0, c_2 \in \mathbb{R}, c_3 \geq 0$, *such that for* $t \geq t_0$,

(H1") $c_1|x|^p \leq V(x,t)$,

(H2") $LV(x,y,t) \leq c_2 V(x,t)$,

(H3") $|V_x(x,t)g(x,y,t)|^2 \geq c_3 V^2(x,t)$.

Then

$$\limsup_{t\to\infty} \frac{1}{t} \log|x(t;t_0,x_0)| \leq -\frac{c_3 - 2c_2}{2p} \qquad a.s. \tag{15}$$

References

[1] R. Z. Has′minskiĭ. *Stochastic stability of differential equations*, volume 7 of *Monographs and Textbooks on Mechanics of Solids and Fluids: Mechanics and Analysis*. Sijthoff & Noordhoff, Alphen aan den Rijn, 1980. Translated from the Russian by D. Louvish.

[2] Ioannis Karatzas and Steven E. Shreve. *Methods of mathematical finance*, volume 39 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, 1998.

[3] Xuerong Mao. *Stochastic differential equations and their applications*. Horwood Publishing Series in Mathematics & Applications. Horwood Publishing Limited, Chichester, 1997.

[4] Xuerong Mao and Chenggui Yuan. *Stochastic differential equations with Markovian switching*. Imperial College Press, London, 2006.

[5] Michel Métivier. *Semimartingales*, volume 2 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1982. A course on stochastic processes.

[6] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

[7] Jia An Yan. *Yang yu suiji jifen yinlun*. Shanghai Science and Technology Publishing House, Shanghai, 1981.

[8] G. George Yin and Qing Zhang. *Continuous-time Markov chains and applications*, volume 37 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, 1998. A singular perturbation approach.

[9] G. George Yin and C. Zhu. *Hybrid switching diffusions: Properties and applications*, volume 63 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, 2010.

[10] Chenggui Yuan and Xuerong Mao. Robust stability and controllability of stochastic differential delay equations with Markovian switching. *Automatica J. IFAC*, 40(3):343–354, 2004.