Stochastic Dynamics of SIRS Epidemic Models with Random Perturbation *

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Abstract: In this paper, we consider a stochastic SIRS model with parameter perturbation, which is a standard technique in modeling population dynamics. In our model, the disease transmission coefficient and the removal rates are all affected by noise. We show that the stochastic model has a unique positive solution as it is essential in any population model. Then we establish conditions for extinction or persistence of the infectious disease. When the infective part is forced to expire, the susceptible part converges weakly to an inverse-gamma distribution with explicit shape and scale parameters. In case of persistence, by new stochastic Lyapunov functions, we show the ergodic property and positive recurrence of the stochastic model. We also derive an estimate for the mean of the stationary distribution. The analytical results are all verified by computer simulations, including examples based on experiments in laboratory populations of mice.

Key words: Ergodic property; extinction; positive recurrence; stochastic Lyapunov functions.

1 Introduction

Numerous health agencies frequently use mathematical models to analyze the spread and the control of infectious diseases in host populations. There is an intensive literature on the mathematical epidemiology, for examples, [9, 10, 11, 12, 14, 17, 18, 20, 22, 23, 25, 28, 29, 31, 32, 40, 41, 42, 43, 44, 45, 46, 48] and the references therein. In particular, [5, 6, 38] are excellent books in this area.

One of classic epidemic models is the SIR model, which subdivides a homogeneous host population into three epidemiologically distinct types of individuals, the susceptible, the infective, and the removed, with their population sizes denoted by S, I and R, respectively. It is suitable for some infectious diseases of permanent or long immunity, such as chickenpox, smallpox, measles, etc. For some diseases, see e.g. influenza and sexual diseases, the removed or recovered individuals finally go back to the susceptible state, called the SIRS model, which can be characterized by the following differential equations

$$\begin{cases} dS = (\lambda - \beta SI - d_S S + \gamma R)dt, \\ dI = (\beta SI - (d_I + \upsilon)I)dt, \\ dR = (\upsilon I - (d_R + \gamma)R)dt. \end{cases} \tag{1.1}$$

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Recall that the parameter $\lambda > 0$ is the rate of susceptible individuals recruited into the population (either by birth or immigration) per unit time; $\beta > 0$ is some transmission coefficient and it is assumed that the rate at which the susceptible individuals acquire the infection is proportional to the number of encounters between the susceptible and infective individuals per unit time, being βSI; $d_S > 0$ is the natural mortality rate or the removal rate of the susceptible individual; $d_I > 0$ is the removal rate of infectious individual and usually be the plus of the natural mortality rate and the mortality rate caused by the disease; $\upsilon > 0$ is the recovery rate of infective individual; $d_R > 0$ is the removal rate of the recovered individual and $\gamma > 0$ is the rate at which the recovered individual loses immunity. It is well known that if the reproductive number $R_0 = \frac{\lambda\beta}{d_S(d_I+\upsilon)} \leq 1$([13], [2], [6], [26], etc), Eq. (1.1) has a globally stable disease free equilibrium $E_0 = (\frac{\lambda}{d_S}, 0, 0)$ whilst $R_0 > 1$ there exists a unique globally stable endemic equilibrium $E^* = (S^*, I^*, R^*)$.

There is a lot of variability in the spread of the disease and this is incorporated in a model via assumptions about stochasticity in the transmission coefficient β and the removal rates d_S, d_I and d_R, which is one of standard ways in modeling stochastic population systems (see e.g. [19], [30], [24], [47]). By the stochastic Lyapunov functions, we obtain some analytical results for stochastic model posed in this paper. In particular we establish conditions for extinction or persistence of the infective population. In case of persistence, we show the existence and the uniqueness of the stationary distribution. Furthermore, we derive an estimate for the mean of the stationary distribution.

Throughout this paper, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets), and let $B(t) = (B_1(t), \cdots, B_N(t))$ be an N-dimensional standard Brownain motion (i.e. the N components $B_i(t), 1 \leq i \leq N$ are independent scalar Brownian motions). In practice we usually estimate a parameter by an average value plus an error term. In this case, the parameters β, d_S, d_I and d_R in Eq. (1.1) change to random variables $\tilde{\beta}, \widetilde{d_S}, \widetilde{d_I}$ and $\widetilde{d_R}$ respectively such that

$$\tilde{\beta} = \beta + \text{error}_0, \widetilde{d_S} = d_S + \text{error}_1, \widetilde{d_I} = d_I + \text{error}_2, \widetilde{d_R} = d_R + \text{error}_3.$$

Accordingly, Eq. (1.1) becomes

$$\begin{cases} dS = (\lambda - \beta SI - d_S S + \gamma R)dt - SI\text{error}_0 dt - S\text{error}_1 dt, \\ dI = (\beta SI - (d_I + \upsilon)I)dt + SI\text{error}_0 dt - I\text{error}_2 dt, \\ dR = (\upsilon I - (d_R + \gamma)R)dt - R\text{error}_3 dt. \end{cases} \tag{1.2}$$

By the central limit theorem, the error terms $\text{error}_i dt, 0 \leq i \leq 3$ may be approximated by a normal distribution with zero mean and variances $\sigma_i^2 dt, 0 \leq i \leq 3$, respectively. That is, $\text{error}_i dt \sim N(0, \sigma_i^2 dt)$. Since these $\text{error}_i dt, 0 \leq i \leq 3$ may correlate to each other, we represent them by the N-dimensional Brownian motion $B(t) = (B_1(t), \cdots, B_N(t))$ as follows

$$\text{error}_i dt = \sum_{j=1}^{N} \sigma_{ij} dB_j(t), \quad 0 \leq i \leq 3,$$

where $dB_j(t) = B_j(t + dt) - B_j(t)$, σ_{ij} are all real numbers such that

$$\sigma_i^2 := \sum_{j=1}^{N} \sigma_{ij}^2, \quad 0 \leq i \leq 3, \quad \sigma^2 := \sum_{i=0}^{3} \sigma_i^2.$$

Thus Eq. (1.2) is characterized by the following Itô SDE

$$\begin{cases} dS = (\lambda - \beta SI - d_S S + \gamma R)dt - SI\sum_{j=1}^{N}\sigma_{0j}dB_j(t) - S\sum_{j=1}^{N}\sigma_{1j}dB_j(t), \\ dI = (\beta SI - (d_I + \upsilon)I)dt + SI\sum_{j=1}^{N}\sigma_{0j}dB_j(t) - I\sum_{j=1}^{N}\sigma_{2j}dB_j(t), \\ dR = (\upsilon I - (d_R + \gamma)R)dt - R\sum_{j=1}^{N}\sigma_{3j}dB_j(t). \end{cases} \tag{1.3}$$

Obviously when $\sigma_1^2+\sigma_2^2+\sigma_3^2=0$, $d(S+I+R)=(\lambda-(d_S S+d_I I+d_R R))dt$ and thus $S+I+R$ is bounded. There are lots of papers in such a case. For example, Tornatore, Buccellato and Vetro [47] discuss the asymptotic stability of the disease free equilibrium of the SDE SIR model; Chen and Li [15] study another SDE version of the SIR model both with and without delay where they introduce stochastic noise in a way different from ours and that of Tornatore, Buccellato and Vetro [47]; Lu [30] extends the results of [47] by including the possibility of temporary immunity and improving the analytical bound on the sufficient condition of the stability of the disease free equilibrium. Recently, Gray et. al [19] establish the conditions of extinction and persistence of the SDE SIS model. But there are few papers considered the stochastic perturbations on both disease transmission coefficient and removal rates, which may happen in the real world. In our model, $S+I+R$ is not bounded in case of $\sigma_1^2+\sigma_2^2+\sigma_3^2\neq 0$ even if it is sufficiently small. Here we use the limit results of Chow [16] and moments methods to derive conditions on the asymptotic stability of the disease free equilibrium. We also note that the role of the influx parameter λ is different between the deterministic and stochastic model (see Remark 4.1). In the case of persistence, by new stochastic Lyapunov functions to counteract high-order terms, we establish conditions for the persistence of $I(t)$, where we show the existence of its stationary distribution and ergodic property. We also give an estimate of the mean by the ergodic theory.

It should be pointed out that only some of the parameters are varied but not all in this paper. One reason is to avoid the paper becoming too complicated. Another reason is because the technique of parameter perturbations used in this paper is not appropriate for some parameters. For example, if λ were perturbed by a Brownian motion, the value of S might become negative.

The paper is organized as follows. In Section 2, we introduce some preliminaries to be used in the later sections. In Section 3, we prove the positivity of the solution which is essential in stochastic population dynamics. In Section 4, we establish the conditions for the extinction of infectious diseases whist the susceptible population converges weakly to an inverse-gamma distribution with explicit shape and scale parameters, where the mean and the variance of the susceptible are also expressed explicitly. In Section 5, we discuss the ergodicity of the SDE model under mild conditions. The mean of the stationary distribution is also estimated by the ergodic theory. In Section 6, we make a concluding remark to complete the paper.

2 Preliminary

In this paper we will use the law of large numbers for martingales and some criteria on ergodicity of the SDEs, so let us recall some classic results on them.

Assume that $(s_n, \mathcal{F}_n, n \geq 1)$ is a martingale with $E|s_n| < \infty$, and $x_1 = s_1$, $x_n = s_n - s_{n-1}$ for $n \geq 2$. The following result is given by Chow (Theorem 5(a) in [16]), and we introduce it as a lemma.

Lemma 2.1. *Let $(y_n, \mathcal{F}_{n-1}, n \geq 2)$ be strictly positive stochastic sequence such that $Ex_n y_n^{-1} < \infty$. If $1 \leq p \leq 2$, then*

$$\lim_{n\to\infty} s_n y_n^{-1} = 0$$

a.e. where

$$\sum_{n=2}^{\infty} E(|x_n|^p|\mathcal{F}_{n-1}) y_n^{-p} < \infty, \quad y_n \uparrow \infty.$$

Next, we give some criteria on the ergodic property. Denote

$$R_+^l = \{x \in R^l : x_i > 0 \text{ for all } 1 \leq i \leq l\}.$$

In general, let X be a regular temporally homogeneous Markov process in $E_l \subset R_+^l$ described by the SDE

$$dX(t) = b\left(X(t)\right) dt + \sum_{r=1}^{d} \sigma_r \left(X(t)\right) dB_r(t), \tag{2.1}$$

with initial value $X(t_0) = x_0 \in E_l$ and $B_r(t), 1 \leq r \leq d$, are standard Brownian motions defined on the above probability space. The diffusion matrix is defined as follows

$$A(x) = \left(A_{ij}(x)\right)_{1\leq i,j\leq l}, \quad A_{ij}(x) = \sum_{r=1}^{d} \sigma_r^i(x)\sigma_r^j(x).$$

Define the differential operator L associated with the equation (2.1) by

$$L = \sum_{i=1}^{l} b_i(x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{l} A_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}.$$

If L acts on a function $V \in C^{2,1}(E_l \times R_+; R)$, then

$$LV(x) = \sum_{i=1}^{l} b_i(x)\frac{\partial V}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{l} A_{ij}(x)\frac{\partial^2 V}{\partial x_i \partial x_j},$$

where $V_x = (\frac{\partial V}{\partial x_1}, \cdots, \frac{\partial V}{\partial x_l})$ and $V_{xx} = \left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)_{l\times l}$. By Itô's formula, we have

$$dV(X(t)) = LV(X(t))dt + \sum_{r=1}^{d} V_x(X(t))\sigma_r\left(X(t)\right) dB_r(t).$$

Lemma 2.2. *([21]) We assume that there exists a bounded domain $U \subset E_l$ with regular boundary, having the following properties:*

(B.1) In the domain U and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix $A(x)$ is bounded away from zero.

(B.2) If $x \in E_l \backslash U$, the mean time τ at which a path issuing from x reaches the set U is finite, and $\sup_{x\in K} E_x\tau < \infty$ for every compact subset $K \subset E_l$.

Then, the Markov process $X(t)$ has a stationary distribution $\upsilon(\cdot)$ with density in E_l such that for any Borel set $B \subset E_l$

$$\lim_{t\to\infty} P(t, x, B) = \upsilon(B),$$

and

$$P_x\left\{\lim_{T\to\infty} \frac{1}{T}\int_0^T f\big(x(t)\big)dt = \int_{E_l} f(x)\upsilon(dx)\right\} = 1,$$

for all $x \in E_l$ and $f(x)$ being a function integrable with respect to the probability measure υ.

Remark 2.1. (i) The existence of the stationary distribution with density is shown by Theorem 4.1 on page 119 and Lemma 9.4 on page 138 in [21] while the ergodicity and the weak convergence are shown by Theorem 5.1 on page 121 and Theorem 7.1 on page 130 in [21].

(ii) To verify Assumptions (B.1) and (B.2), it suffices to show that there exists a bounded domain U with regular boundary and a non-negative C^2-function V such that $A(x)$ is uniformly elliptical in U and for any $x \in E_l\backslash U$, $LV(x) \le -C$ for some $C > 0$ (See e.g. [49], page 1163).

3 Existence and uniqueness of positive solution

In order for the SDE SIRS model to make sense, we must show that this model has a unique global positive solution. Since the SDE SIRS model is a special SDE, the existing general existence-uniqueness theorem on SDEs (see e.g., [33], [34], [36]) is not applicable. We need to establish the following theorem.

Theorem 3.1. *For any given initial value $(S(0), I(0), R(0)) \in R^3_+$, the SDE (1.3) has a unique global positive solution with probability one, namely,*

$$\mathbb{P}\{(S(t), I(t), R(t)) \in R^3_+, \forall t \ge 0\} = 1.$$

Proof. Note that the coefficients of SDE (1.3) are locally Lipschitz continuous. By well known results (see e.g., [33], [34], [36]), there is a unique local solution on $[0, \tau_e)$, where τ_e is the explosion time.

Let $m_0 \ge 0$ be sufficiently large such that $S(0), I(0), R(0)$, all lie in the interval $[m_0^{-1}, m_0]$. For each integer $m \ge m_0$, define the stopping time

$$\tau_m = \inf\{t \in [0, \tau_e) : \min\{S(t), I(t), R(t)\} \le m^{-1} \text{ or } \max\{S(t), I(t), R(t)\} \ge m\}.$$

As usual, we set $\inf \emptyset = \infty$. Clearly, τ_m's are increasing. Set $\tau_\infty = \lim\limits_{m\to\infty} \tau_m$, where $0 \le \tau_\infty \le \tau_e$ a.e. If we show that $\tau_\infty = \infty$ a.e., then $\tau_e = \infty$ and the solution remains in R^3_+ for all $t \ge 0$, a.e. If this statement is false, then there is a pair of constants $T > 0$ and $\epsilon \in (0, 1)$ such that

$$P\{\tau_\infty \le T\} > \epsilon.$$

Hence there is an integer $m_1 \ge m_0$ such that

$$P\{\tau_m \le T\} \ge \epsilon \text{ for all } m \ge m_1. \tag{3.1}$$

Let $x = (S, I, R)$, and define the C^3-function $V_1 : R^3_+ \to R_+$ by

$$V_1(x) = (S - a - a\log\frac{S}{a}) + (I - 1 - \log I) + (R - 1 - \log R),$$

where a is a positive constant to be determined later.
By Itô's formula, we see

$$\begin{aligned} LV_1(x) = &\lambda - d_S S - d_I I - d_R R + a\left(\beta I + d_S - \frac{\lambda}{S} - \frac{\gamma R}{S}\right) - \beta S - \frac{\upsilon I}{R} + d_I + \upsilon + d_R + \gamma \\ &+ a\sum_{j=1}^{N}(I\sigma_{0j} + \sigma_{1j})^2 + \sum_{j=1}^{N}(S\sigma_{0j} - \sigma_{2j})^2 + \sigma_3^2. \end{aligned} \tag{3.2}$$

Let $a\beta \leq d_I$, then there exists a constant C_1 such that

$$LV_1(x) \leq C_1 + a\sum_{j=1}^{N}(I\sigma_{0j} + \sigma_{1j})^2 + \sum_{j=1}^{N}(S\sigma_{0j} - \sigma_{2j})^2.$$

Define a function $V_2 : R_+^3 \to R_+$ by

$$V_2(x) = (S + I + R)^2, \quad x = (S, I, R).$$

Then

$$\begin{aligned} LV_2(x) &= 2(S + I + R)(\lambda - d_S S - d_I I - d_R R) + \sum_{j=1}^{N}(S\sigma_{1j} + I\sigma_{2j} + R\sigma_{3j})^2 \\ &\leq (S + I + R)^2 + \lambda^2 + (\sigma_1^2 + \sigma_2^2 + \sigma_3^2)(S + I + R)^2 \\ &\leq C_2 + C_3(S + I + R)^2, \end{aligned} \tag{3.3}$$

where C_2, C_3 are another positive constants.

Let $V(x) = V_1(x) + V_2(x)$. By (3.2) and (3.3), there exists a positive constant C such that

$$LV(x) \leq C + CV(x).$$

Let $\widetilde{V}(x,t) = e^{-Ct}(1 + V(x))$, then

$$L\widetilde{V}(x) = -Ce^{-Ct}(1 + V(x)) + e^{-Ct}LV(x) \leq 0.$$

Let $x(t) = (S(t), I(t), R(t)), t \geq 0$, by Itô's formula, we have for any $m \geq m_1$

$$E\widetilde{V}(x(t \wedge \tau_m), t \wedge \tau_m) = \widetilde{V}(x(0)) + E\int_0^{t\wedge\tau_m} L\widetilde{V}(x(u), u)ds \leq \widetilde{V}(x(0)). \tag{3.4}$$

Set $\Omega_m = \{\tau_m \leq T\}$ and by (3.1), $\mathbb{P}\{\Omega_m\} \geq \varepsilon$. Note that for every $\omega \in \Omega_m$, $V(x(\tau_m, \omega)) \geq b_m :=$ $\min\{V(y)|$y has a individual as m^{-1} or $m\} \to \infty$, as $m \to \infty$. It then follows from (3.4) that

$$\varepsilon b_m \leq E[V(x(\tau_m, \omega)I_{\Omega_m})] \leq e^{CT}\widetilde{V}(x(0)).$$

Letting $m \to \infty$ leads to the contradiction

$$\infty > e^{CT}\widetilde{V}(x(0)) \geq \infty.$$

Therefore $\tau_\infty = \infty$, a.e., whence the proof is complete.

□

4 Disease Extinction

In an epidemic model, the disease extinction or persistence are two of the most interesting and important issues. In this section, we will establish conditions for the disease extinction in the SDE SIRS model (1.3) and discuss the disease persistence in the next section.

Lemma 4.1. *Let $N(t) = S(t)+I(t)+R(t)$, then there exists some $p_0 > 1$ such that for any $p \in (1, p_0)$,*

$$\sup_n E\left(\max_{t\in[n,n+1]} N^p(t)\right) < +\infty. \tag{4.1}$$

Proof. Obviously, there exists some $p_0 > 1$ such that for any $p \in (1, p_0)$,

$$C_p = p\check{d} - \frac{p(p-1)\sigma^2}{2} > 0.$$

Note that

$$dN(t) = (\lambda - (d_S S(t) + d_I I(t) + d_R R(t)))dt - \sum_{j=1}^{N}(S\sigma_{1j} + I\sigma_{2j} + R\sigma_{3j})dB_j(t).$$

For any $p > 1$, we have

$$\begin{aligned} d\left(e^{0.5C_pt}N^p(t)\right) &= e^{0.5C_pt}\Big[\lambda pN^{p-1}(t) - pN^{p-1}(t)(d_S S(t) + d_I I(t) + d_R R(t)) \\ &+ \frac{p(p-1)}{2}N^{p-2}(t)\sum_{j=1}^{N}(S\sigma_{1j} + I\sigma_{2j} + R\sigma_{3j})^2 + 0.5C_pN^p(u)\Big]\, dt + dM(t), \end{aligned} \tag{4.2}$$

where $M(t) = -\int_0^t pN^{p-1}(u)e^{0.5C_pu}\sum_{j=1}^{N}(S\sigma_{1j} + I\sigma_{2j} + R\sigma_{3j})dB_j(t)$, $t \geq 0$, is a local martingale.

Denote $\max\{d_S, d_I, d_R\}$ and $\min\{d_S, d_I, d_R\}$ by $\hat{d}$ and $\check{d}$, respectively. Then

$$\begin{aligned} E\left(e^{0.5C_pt}N^p(t)\right) &\leq N^p(0) + E\int_0^t e^{0.5C_pu}\left[\lambda pN^{p-1}(u) - p\check{d}N^p(u) + \frac{p(p-1)\sigma^2}{2}N^p(u) + 0.5C_pN^p(u)\right] du \\ &\leq N^p(0) + E\int_0^t e^{0.5C_pu}\left[\lambda pN^{p-1}(u) - 0.5C_pN^p(u)\right] dt. \end{aligned}$$

Hence,

$$e^{0.5C_pt}EN^p(t) \leq N^p(0) + \frac{K}{0.5C_p}e^{0.5C_pt},$$

where $K = \sup_{x>0}(\lambda px^{p-1} - 0.5C_px^p)$. This implies that for any $p \in (1, p_0)$,

$$\sup_{t\geq 0} EN^p(t) \leq N^p(0) + \frac{K}{0.5C_p} < +\infty. \tag{4.3}$$

By (4.2), we have

$$\max_{t\in[n,n+1]} N^p(t) \leq N^p(n) + \int_n^{n+1}\left[\lambda pN^{p-1}(u) - C_pN^p(u)\right] dt + \max_{t\in[n,n+1]} |\widetilde{M}(t)|,$$

where $\widetilde{M}(t)=\int_n^t pN^{p-1}(u)\sum_{j=1}^N(S\sigma_{1j}+I\sigma_{2j}+R\sigma_{3j})dB_j(t)$, $t\in[n,n+1]$.
Therefore,

$$E\left(\max_{t\in[n,n+1]}N^p(t)\right)\le EN^p(n)+\int_n^{n+1}\left[\lambda pEN^{p-1}(u)+C_pEN^p(u)\right]dt+E\left(\max_{t\in[n,n+1]}|\widetilde{M}(t)|\right).$$

By (4.3), there exists a positive constant C_1 independent of n such that

$$E\left(\max_{t\in[n,n+1]}N^p(t)\right)\le C_1+E\left(\max_{t\in[n,n+1]}|\widetilde{M}(t)|\right). \tag{4.4}$$

Since

$$\begin{aligned}\widetilde{M}(t)=N^p(t)-N^p(n)-\int_n^t&\Big[\lambda pN^{p-1}(u)-pN^{p-1}(u)(d_SS(u)+d_II(u)+d_RR(u))\\&+\frac{p(p-1)}{2}N^{p-2}(u)\sum_{j=1}^N(S\sigma_{1j}+I\sigma_{2j}+R\sigma_{3j})^2\Big]du.\end{aligned}$$

Hence for some $C_2>0$, we have

$$|\widetilde{M}(t)|\le N^p(t)+N^p(n)+C_2\int_n^t\left[N^{p-1}(u)+N^p(u)\right]dt. \tag{4.5}$$

For any $p\in(1,p_0)$, there exists $p'>1$ such that $pp'\in(1,p_0)$ and the constants C_3,C_4 independent of n, we have

$$\begin{aligned}E\left(\max_{t\in[n,n+1]}|M(t)|^{p'}\right)&\le C_3\max_{t\in[n,n+1]}E|M(t)|^{p'}\\&\le C_3E\left[N^{pp'}(t)+N^{pp'}(n)+\int_n^{n+1}\left[N^{(p-1)p'}(u)+N^{pp'}(u)\right]dt\right]\\&\le C_4,\end{aligned}$$

where the first inequality is derived from the maximal inequality for martingales, the second by (4.5) and Jensen's inequality, and the last by (4.3).
By (4.4), we have

$$E\left(\max_{t\in[n,n+1]}N^p(t)\right)\le C_1+\left(E\left(\max_{t\in[n,n+1]}|M(t)|^{p'}\right)\right)^{1/p'}<+\infty,$$

whence the proof is complete. □

Lemma 4.2. *For any $1\le j\le N$,*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T S(t)dB_j(t)=0,\quad \lim_{T\to\infty}\frac{1}{T}\int_0^T I(t)dB_j(t)=0,\quad \lim_{T\to\infty}\frac{1}{T}\int_0^T R(t)dB_j(t)=0,\quad a.e.$$

Proof. Note that for any $T \in [n, n+1)$,

$$\frac{1}{T}\int_0^T S(t)dB_j(t) = \frac{n}{T}\cdot\frac{1}{n}\int_0^n S(t)dB_j(t) + \frac{1}{T}\int_n^T S(t)dB_j(t).$$

Let $x_i = \int_{i-1}^i S(t)dB_j(t)$, $1 \le i \le n$. For some $p \in (1, p_0)$, by the B-D-G inequality, there exists some $C < +\infty$ such that

$$E|x_i|^p \le CE\left(\int_{i-1}^i S^2(t)dt\right)^{p/2} \le CE\left(\max_{t\in[i-1,i]} S^p(t)\right) \le C\sup_n E\left(\max_{t\in[n,n+1]} N^p(t)\right) < +\infty,$$

where the last inequality is derived from Lemma 4.1. Then

$$E\left(\sum_{i=1}^\infty \frac{(|x_i|^p|\mathcal{F}_{i-1})}{i^p}\right) = \sum_{i=1}^\infty \frac{E|x_i|^p}{i^p} < \infty,$$

which implies

$$\sum_{i=1}^\infty \frac{(|x_i|^p|\mathcal{F}_{i-1})}{i^p} < \infty. \quad a.e.,$$

Thus, by Lemma 2.1,

$$\lim_{n\to\infty}\frac{1}{n}\int_0^n S(t)dB_j(t) = 0, \quad a.e. \tag{4.6}$$

On the other hand, by Lemma 4.1 again, we have

$$\begin{aligned}
&\sum_{n=1}^\infty \mathbb{P}\left\{\max_{T\in[n,n+1]}\frac{1}{T}\left|\int_n^T S(t)dB_j(t)\right| > \varepsilon\right\} \\
&\le \sum_{n=1}^\infty \frac{1}{\varepsilon^p n^p} E\left|\max_{T\in[n,n+1]}\int_n^T S(t)dB_j(t)\right|^p \\
&\le C\sum_{n=1}^\infty \frac{1}{n^p} E\left(\max_{t\in[n,n+1]} S^p(t)\right) \\
&\le C\left(\sum_{n=1}^\infty \frac{1}{n^p}\right)\sup_n E\left(\max_{t\in[n,n+1]} N^p(t)\right) < +\infty.
\end{aligned}$$

Applying the well-known Borel-Cantelli lemma, we see

$$\mathbb{P}\left\{\lim_{n\to\infty}\max_{T\in[n,n+1]}\frac{1}{T}\left|\int_n^T S(t)dB_j(t)\right| = 0\right\} = 1. \tag{4.7}$$

Taking (4.6) and (4.7) into account, we get

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T S(t)dB_j(t) = 0, \quad a.e.$$

In the same way, we can prove the other assertions and hence the proof is complete. $\square$

Lemma 4.3.

$$\lim_{T\to\infty} \frac{N(T)}{T} = 0, \quad a.e.$$

Proof. For any $\varepsilon > 0$, we have

$$\sum_{n=1}^{\infty} \mathbb{P}\left\{\max_{T\in[n,n+1]} \frac{N(T)}{T} > \varepsilon\right\} \le \sum_{n=1}^{\infty} \frac{1}{\varepsilon^p n^p} E\left(\max_{t\in[n,n+1]} N^p(t)\right) < +\infty,$$

where the last inequality is derived from Lemma 4.1. Applying the Borel-Cantelli lemma, we see

$$\mathbb{P}\left\{\lim_{n\to\infty} \max_{T\in[n,n+1]} \frac{N(T)}{T} = 0\right\} = 1.$$

The proof is hence complete. □

Theorem 4.1. *If*

$$\frac{\lambda\left(\beta + \sum_{j=1}^{n} \sigma_{0j}\sigma_{2j}\right)}{d_S\left(d_I + \upsilon + \frac{\sigma_2^2}{2}\right)} - \frac{\lambda^2\sigma_0^2}{2d_S^2\left(d_I + \upsilon + \frac{\sigma_2^2}{2}\right)} < 1 \quad \textit{and} \quad \sigma_0^2 \le \frac{d_S}{\lambda}\left(\beta + \sum_{j=1}^{n} \sigma_{0j}\sigma_{2j}\right),$$

then for any initial value, the SDE (1.3) obeys

$$\lim_{T\to\infty} \frac{\log I(T)}{T} \le \frac{\lambda}{d_S}\left(\beta + \sum_{j=1}^{n} \sigma_{0j}\sigma_{2j}\right) - \left(d_I + \upsilon + \frac{\sigma_2^2}{2}\right) - \frac{\lambda^2\sigma_0^2}{2d_S^2} < 0, \quad a.e. \tag{4.8}$$

In other words, the disease decays exponentially with probability one.

Proof. By Itô's formula, we have

$$\log(I(T)) = \log(I(0)) + \int_0^T f(x(t))dt + \sum_{j=1}^{N} \sigma_{0j} \int_0^T S(t)dB_j(t) - \sum_{j=1}^{N} \sigma_{2j} B_j(T), \tag{4.9}$$

where $f(x) = \beta S - (d_I + \upsilon) - \frac{1}{2}\sum_{j=1}^{N}(S\sigma_{0j} - \sigma_{2j})^2$ for $x = (S, I) \in R_+^2$.

$$\widetilde{N}(t) = S(t) + \frac{d_I}{d_S}I(t) + \frac{d_R}{d_S}R(t), \quad x(t) = (S(t), I(t)) \text{ and } y(t) = \frac{d_I}{d_S}I(t) + \frac{d_R}{d_S}R(t).$$

Then

$$
\begin{aligned}
f(x(t)) &= \beta S(t) - (d_I + \upsilon) - \frac{1}{2}\sigma_0^2 S^2(t) + \sum_{j=1}^{N} \sigma_{0j}\sigma_{2j} S(t) - \frac{\sigma_2^2}{2} \\
&= \beta(\widetilde{N}(t) - y(t)) - \left(d_I + \upsilon + \frac{\sigma_2^2}{2}\right) - \frac{1}{2}\sigma_0^2(\widetilde{N}(t) - y(t))^2 + \sum_{j=1}^{N} \sigma_{0j}\sigma_{2j}(\widetilde{N}(t) - y(t)) \\
&= \left(\beta + \sum_{j=1}^{N} \sigma_{0j}\sigma_{2j}\right)\widetilde{N}(t) - \left(\beta + \sum_{j=1}^{N} \sigma_{0j}\sigma_{2j}\right) y(t) - \left(d_I + \upsilon + \frac{\sigma_2^2}{2}\right) \\
&\qquad - \frac{\sigma_0^2}{2}\widetilde{N}^2(t) + \sigma_0^2 \widetilde{N}(t) y(t) - \frac{\sigma_0^2}{2} y^2(t) \\
&= \left(\beta + \sum_{j=1}^{N} \sigma_{0j}\sigma_{2j}\right)\widetilde{N}(t) - \left(\beta + \sum_{j=1}^{N} \sigma_{0j}\sigma_{2j} - \varepsilon\sigma_0^2\right) y(t) - \left(d_I + \upsilon + \frac{\sigma_2^2}{2}\right) \\
&\qquad - \frac{\sigma_0^2}{2}\widetilde{N}^2(t) + \sigma_0^2(\widetilde{N}(t) - \varepsilon) y(t) - \frac{\sigma_0^2}{2} y^2(t),
\end{aligned}
$$

where $\varepsilon > 0$ is a constant sufficiently small for $\varepsilon\sigma_0^2 \leq \beta + \sum_{j=1}^{N} \sigma_{0j}\sigma_{2j}$. Then

$$
\begin{aligned}
f(x(t)) &\leq \left(\beta + \sum_{j=1}^{N} \sigma_{0j}\sigma_{2j}\right)\widetilde{N}(t) - \left(d_I + \upsilon + \frac{\sigma_2^2}{2}\right) - \frac{\sigma_0^2}{2}\widetilde{N}^2(t) \\
&\qquad + \frac{\sigma_0^2}{2}(\widetilde{N}(t) - \varepsilon)^2 - \frac{\sigma_0^2}{2}(\widetilde{N}(t) - \varepsilon - y(t))^2 \\
&\leq \left(\beta + \sum_{j=1}^{N} \sigma_{0j}\sigma_{2j} - \sigma_0^2\varepsilon\right)\widetilde{N}(t) - \left(d_I + \upsilon + \frac{\sigma_2^2}{2}\right) + \frac{\varepsilon^2\sigma_0^2}{2}.
\end{aligned}
\tag{4.10}
$$

Recall that $N(t) = S(t) + I(t) + R(t)$, then

$$
dN(t) = (\lambda - d_S\widetilde{N}(t))dt + \sum_{j=1}^{N}(S\sigma_{1j} + I\sigma_{2j} + R\sigma_{3j})dB_j(t).
$$

Thus, we see

$$
\frac{N(T) - N(0)}{T} = \lambda - \frac{d_S}{T}\int_0^T \widetilde{N}(t)dt + \frac{1}{T}\sum_{j=1}^{N}\int_0^T (S\sigma_{1j} + I\sigma_{2j} + R\sigma_{3j})dB_j(t).
$$

Let $T \to \infty$, Lemma 4.2 and Lemma 4.3 yields

$$
\frac{1}{T}\int_0^T \widetilde{N}(t)dt = \frac{\lambda}{d_S}, \quad a.e.
$$

Since Lemma 4.2 and Lemma 4.3 implies that $\lim_{T\to\infty} \frac{1}{T}\int_0^T S(t)dB_j(t) = \lim_{T\to\infty} \frac{B_j(T)}{T} = 0$, a.e., by (4.9), (4.10) and let $\varepsilon = \frac{\lambda}{d_S}$, we have

$$
\lim_{T\to\infty} \frac{\log I(T)}{T} \leq \frac{\lambda}{d_S}\left(\beta + \sum_{j=1}^{N} \sigma_{0j}\sigma_{2j}\right) - \left(d_I + \upsilon + \frac{\sigma_2^2}{2}\right) - \frac{\lambda^2\sigma_0^2}{2d_S^2} < 0.
$$

The proof is hence complete. $\square$

Remark 4.1. In the deterministic model (1.1), λ is the rate of susceptible individuals recruited into the population (either by birth or immigration) per unit time; β is some transmission coefficient and it is assumed that the rate at which the susceptible individuals acquire the infection is proportional to the number of encounters between the susceptible and infective individuals per unit time, being βSI; d_S is the natural mortality rate or the removal rate of the susceptible individual; d_I is the removal rate of infectious individual and may be the sum of the natural mortality rate and the disease mortality rate; υ is the recovery rate of infective individual; d_R is the removal rate of the recovered individual and γ is the rate at which the recovered individual loses immunity. Thus, from the biological point of view, $\frac{1}{d_S}$ is the average death age of the susceptible individual, that is, the average lifespan of the susceptible individual; $\frac{\lambda}{d_S}$ is the number of the susceptible population without infection during the life span; $\frac{1}{d_I+\upsilon}$ is the mean infective period, or the mean course of infection. It is well-known that the basic reproductive number R_0 of the deterministic model (1.1) is just the product of the transmission coefficient β per unit time, the number $\frac{\lambda}{d_S}$ of when all of the individuals in the population are initially susceptible and the mean infective period $\frac{1}{d_I+\upsilon}$, which is actually the average number of secondary infections produced by one infected individual during the mean course of infection in a completely susceptible population and thus determines whether a disease persists or goes extinct.

Because of the existence of random fluctuations in the environment, we consider the stochastic model (1.3) to investigate how the randomness affects the behavior of the disease transmission. By Theorem 4.1, under some mild condition and

$$\begin{aligned}
R_{0;\sigma_{ij}} &:= \frac{\lambda}{d_S} \cdot \left(\beta + \sum_{j=1}^{N} \sigma_{0j}\sigma_{2j} - \frac{\lambda \sigma_0^2}{2d_S} \right) \cdot \frac{1}{d_I + \upsilon + \frac{\sigma_2^2}{2}} \\
&= \frac{\lambda \left(\beta + \sum_{j=1}^{N} \sigma_{0j}\sigma_{2j} \right)}{d_S \left(d_I + \upsilon + \frac{\sigma_2^2}{2} \right)} - \frac{\lambda^2 \sigma_0^2}{2d_S^2 \left(d_I + \upsilon + \frac{\sigma_2^2}{2} \right)} \\
&< 1,
\end{aligned}$$

we obtain

$$\begin{aligned}
\lim_{T\to\infty} \frac{\log I(T)}{T} &\le \frac{\lambda}{d_S} \left(\beta + \sum_{j=1}^{N} \sigma_{0j}\sigma_{2j} \right) - \left(d_I + \upsilon + \frac{\sigma_2^2}{2} \right) - \frac{\sigma_0^2}{2} \left(\frac{\lambda}{d_S} \right)^2 \\
&= \left(d_I + \upsilon + \frac{\sigma_2^2}{2} \right) \left(R_{0;\sigma_{ij}} - 1 \right) \\
&< 0, \quad a.e.
\end{aligned}$$

That is, in the stochastic model (1.3), the random fluctuations have their effect on the transmission of infectious diseases. Note also that under the condition of Theorem 4.1, the average number of secondary infections in the stochastic model is less than 1 which implies that the disease dies out eventually.

Theorem 4.2. *Under the conditions of Theorem 4.1, the susceptible class $S(t)$ of the SDE (1.3) converges weakly to an inverse-gamma distribution ν, which is the distribution of the reciprocal of a* *gamma distribution with shape parameter* $\frac{2d_S}{\sigma_1^2}+1$ *and scale parameter* $\frac{\sigma_1^2}{2\lambda}$*. Moreover, we have*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T S(t)dt=\frac{\lambda}{d_S},\quad a.e.$$

Furthermore, if $d_S>\frac{\sigma_1^2}{2}$*, then*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\left(S(t)-\frac{\lambda}{d_S}\right)^2 dt=\frac{\sigma_1^2}{2d_S-\sigma_1^2}\left(\frac{\lambda}{d_S}\right)^2,\quad a.e.$$

Proof. In the same way as Theorem 4.1 was proved, we can show that

$$\limsup_{T\to\infty}\frac{1}{T}\log R(T)\leq-\left(d_R+\gamma+\frac{\sigma_3^2}{2}\right)<0,\quad a.e. \tag{4.11}$$

Recall that $N(t)=S(t)+I(t)+R(t)$, $t\geq 0$ and

$$dN(t)=(\lambda-d_SN(t)+f(t))\,dt+\sum_{j=1}^N(\sigma_{1j}N(t)+g_j(t))dB_j(t),$$

where $f(t)=(d_S-d_I)I(t)+(d_S-d_R)R(t)$ and $g_j(t)=(\sigma_{2j}-\sigma_{1j})I(t)+(\sigma_{3j}-\sigma_{1j})R(t)$ for any $t\geq 0, 1\leq j\leq N$. Applying Theorem 3.1 in [35] again,

$$N(T)=\widetilde{\Phi}(T)\left(N(0)+\int_0^T\widetilde{\Phi}^{-1}(t)\left(\lambda+f(t)-\sum_{j=1}^N\sigma_{1j}g_j(t)\right)dt+\sum_{j=1}^N\int_0^T\widetilde{\Phi}^{-1}(t)g_j(t)dB_j(t)\right),$$

where $\widetilde{\Phi}(t)=\exp\left\{-\left(d_S+\frac{\sigma_1^2}{2}\right)t-\sum_{j=1}^N\sigma_{1j}B_j(t)\right\}$, $t\geq 0$.
Using (4.8) and (4.11), it is easy to show

$$\lim_{T\to\infty}\widetilde{\Phi}(T)\int_0^T\widetilde{\Phi}^{-1}(t)\left(f(t)-\sum_{j=1}^N\sigma_{1j}g_j(t)\right)dt=0.$$

On the other hand, note that $\lim_{T\to\infty}\widetilde{\Phi}^{-1}(T)=\infty$ and by (4.8), (4.11) again,

$$\int_0^\infty\frac{(\widetilde{\Phi}^{-1}(t))^2g_j^2(t)}{(1+\widetilde{\Phi}^{-1}(t))^2}dt<\infty,\quad a.e.,\qquad 1\leq j\leq N.$$

Thus Theorem 3.4 in [35] yields

$$\lim_{T\to\infty}\widetilde{\Phi}(T)\left(\sum_{j=1}^N\int_0^T\widetilde{\Phi}^{-1}(t)g_j(t)dB_j(t)\right)=0,\quad a.e.$$

Therefore

$$N(T)=\widetilde{\Phi}(T)\left(N(0)+\int_0^T\lambda\widetilde{\Phi}^{-1}(t)dt\right)+o(1),\quad a.e.,$$

where $o(1) \to 0$, a.e. as $T \to \infty$. On the other hand, since $\lim_{T\to\infty} I(t) = \lim_{T\to\infty} R(t) = 0$, a.e.,

$$S(T) = \widetilde{\Phi}(T)\left(N(0) + \int_0^T \lambda\widetilde{\Phi}^{-1}(t)dt\right) + o(1), \quad a.e. \tag{4.12}$$

Let $X(t)$ be the solution of the following lienar SDE

$$dX(t) = (\lambda - d_S X(t))\, dt + X(t)\sum_{j=1}^N \sigma_{1j} dB_j(t), \quad X(0) = N(0). \tag{4.13}$$

By Theorem 3.1 in [35], $X(t)$ can be expressed as

$$X(t) = \widetilde{\Phi}(T)\left(N(0) + \int_0^T \lambda\widetilde{\Phi}^{-1}(t)dt\right).$$

Thus (4.12) yields

$$\lim_{T\to\infty} (S(T) - X(T)) = 0, \quad a.e. \tag{4.14}$$

Assume that $\{B(t), t \geq 0\}$ is a standard Brownian motion. Since the processes $\{\sum_{j=1}^N \sigma_{1j} B_j(t), t \geq 0\}$ and $\{\sigma_1 B(t), t \geq 0\}$ are equivalent in distribution, we may replace $\sum_{j=1}^N \sigma_{1j} dB_j(t)$ by $\sigma_1 dB(t)$ in the SDE (4.13). Let $Y(t) = X(t) - \frac{\lambda}{d_S}$, then $Y(t)$ satisfies

$$dY = -d_S Y dt + \sigma_1\left(Y + \frac{\lambda}{d_S}\right) dB(t). \tag{4.15}$$

Theorem 2.1 (a) in [7] with $C = 1$ implies that $Y(t)$ is stable in distribution, so does $X(t)$. Let $q(x) = \exp\left(-2\int_1^x \frac{\lambda - d_S y}{\sigma_1^2 y^2} dy\right)$. We have

$$\int_1^\infty q(x)dx = \infty, \quad \int_0^1 q(x)dx = \infty, \quad \int_0^\infty \frac{dx}{\sigma_1^2 q(x) x^2} < \infty.$$

So $X(t)$ is ergodic (Theorem 1.16 in [27]), and its unique invariant distribution ν has density $(Mx^2 p(x))^{-1}$, where $p(x) = \exp\left(\frac{2\lambda}{\sigma_1^2 x} + \frac{2d_S}{\sigma_1^2}\ln x\right), x > 0$, M is a normal constant. By computation, ν is an inverse-gamma distribution, which is the reciprocal of a gamma distribution with shape parameter $\frac{2d_S}{\sigma_1^2} + 1$ and scale parameter $\frac{\sigma_1^2}{2\lambda}$. It is clear that the stability in distribution implies that the limiting distribution is just the invariant distribution. Therefore, $X(t)$ converges weakly to ν as $t \to \infty$. By (4.14), we conclude that $S(t)$ converges weakly to ν, too.

Note that

$$\begin{aligned}\frac{N(T)}{T} = \frac{N(0)}{T} + \lambda - \frac{d_S}{T}\int_0^T S(t)dt - \frac{d_I}{T}\int_0^T I(t)dt - \frac{d_R}{T}\int_0^T R(t)dt \\ - \sum_{j=1}^N \int_0^T (\sigma_{ij} S(t) + \sigma_{2j} I(t) + \sigma_{3j} R(t)) dB_j(t).\end{aligned}$$

Under the conditions of Theorem 4.1, we get

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T I(t)dt = 0, \quad \lim_{T\to\infty} \frac{1}{T}\int_0^T R(t)dt = 0, \quad a.e.,$$

which, together with Lemma 4.2 and Lemma 4.3, yields

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T S(t)dt=\frac{\lambda}{d_S},\quad a.e.$$

Write

$$\frac{1}{T}\int_0^T S^2(t)dt=\frac{1}{T}\int_0^T(S(t)-X(t))^2dt+\frac{2}{T}\int_0^T(S(t)-X(t))X(t)dt+\frac{1}{T}\int_0^T X^2(t)dt.$$

By (4.14) and $\lim_{T\to\infty}\frac{1}{T}\int_0^T X(t)dt=\frac{\lambda}{d_S}$, we have

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T(S(t)-X(t))^2dt=\lim_{T\to\infty}\frac{1}{T}\int_0^T(S(t)-X(t))X(t)dt=0,\quad a.e.,$$

which implies

$$\lim_{T\to\infty}\left(\frac{1}{T}\int_0^T S^2(t)dt-\frac{1}{T}\int_0^T X^2(t)dt\right)=0,\quad a.e.$$

Therefore,

$$\lim_{T\to\infty}\left(\frac{1}{T}\int_0^T\left(S(t)-\frac{\lambda}{d_S}\right)^2dt-\frac{1}{T}\int_0^T\left(X(t)-\frac{\lambda}{d_S}\right)^2dt\right)=0,\quad a.e.$$

Applying Itô's formula to $\left(X(t)-\frac{\lambda}{d_S}\right)^2$, we get

$$\begin{aligned}&\frac{1}{2T}\left(X(T)-\frac{\lambda}{d_S}\right)^2=\frac{\sigma_1^2}{2}\left(\frac{\lambda}{d_S}\right)^2+\frac{\lambda\sigma_1^2}{d_ST}\int_0^T\left(X(t)-\frac{\lambda}{d_S}\right)dt\\&-\left(d_S-\frac{\sigma_1^2}{2}\right)\frac{1}{T}\int_0^T\left(X(t)-\frac{\lambda}{d_S}\right)^2dt+\frac{\sigma_1}{T}\int_0^T X(t)\left(X(t)-\frac{\lambda}{d_S}\right)dB(t),\end{aligned}\tag{4.16}$$

If $d_S>\frac{\sigma_1^2}{2}$, in the same way as Lemmas 4.1, 4.2 and 4.3 were proved, we can show

$$\lim_{T\to\infty}\frac{1}{T}\left(X(T)-\frac{\lambda}{d_S}\right)^2=0,\quad\lim_{T\to\infty}\frac{1}{T}\int_0^T X(t)\left(X(t)-\frac{\lambda}{d_S}\right)dB(t)=0,\quad a.e.$$

Let $T\to\infty$ in (4.16), we get

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\left(X(t)-\frac{\lambda}{d_S}\right)^2dt=\frac{\sigma_1^2}{2d_S-\sigma_1^2}\left(\frac{\lambda}{d_S}\right)^2,\quad a.e.$$

This completes the proof. □

Example 4.1. As a slightly more realistic example to illustrate our analytical results, we introduce an SIRS model which is used to investigate the dynamics of *Pasteurella muris* in colonies of laboratory mice (see [4]). We choose parameters from p362 and p363 in [4]. That is, $\lambda=0.33$, and the other parameters are:

Parameter	β	d_S	d_R	d_I	ν	γ
Value/day	0.0056	0.006	0.006	0.0066	0.04	0.0021

Fig.1

Fig.2

To see the effect of random fluctuations,we consider the following diffusion coefficient matrix

$$\sigma = \begin{pmatrix} 0.01 & 0 & 0 & 0 \\ 0 & 0.001 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.1 \end{pmatrix}.$$

Note that in the deterministic model, $R_0 = 2.9057$. Hence, the infectious disease will persist in the deterministic model. For the stochastic model, it is easy to check that the conditions of Theorem 4.1 are satisfied and the infectious disease will eventually become extinctive due to the effect of random fluctuations. In Fig.1, the red, blue and green lines represent the susceptible, the infective and the recovered individuals, respectively. It is seen that the infective and the recovered individuals are forced to expire. The following table shows sample means and variances at different time which are very close to our theoretical results in Theorem 4.2.

Time point	20	40	60	80	100
Sample mean	57.1999	56.1885	55.1778	55.2005	55.1588
Sample variance	0.0356	0.0318	0.0257	0.0285	0.0233

To verify the density function of the stationary distribution ν, we use software R to get an estimate for the kernel densities of ν and $S(t)$, respectively. In Fig.2 and Fig.3, these histograms of kernel densities look alike, and thus confirm our analytical results.

5 Ergodicity

In this section, we discuss the persistence of the SDE (1.3) by the ergodic property of Markov processes.

Theorem 5.1. *If the matrix $(\sigma_{kj})_{0\le k\le 3, 1\le j\le N}$ has its full row rank, $R_0 > 1$, $\check{d} := \min\{d_S, d_I, d_R\} > 6\sigma^2$ and*

$$\begin{aligned} &\left(d_S + \frac{\gamma d_S}{d_S + d_R} - C_1\right) S^{*2} \bigwedge \left(\frac{\gamma d_I}{d_S + d_R} - C_2\right) I^{*2} \bigwedge \left(\frac{\gamma(d_I + d_R)(d_R + \gamma)}{\upsilon(d_S + d_R)} + \frac{\gamma d_R}{d_S + d_R} - C_3\right) R^{*2} \\ &> \left(\frac{\gamma(d_S + d_I)}{\beta(d_S + d_R)} + \frac{d_I + \upsilon}{\beta}\right) \sigma_2^2 I^* + C_1 S^{*2} + C_2 I^{*2} + C_3 R^{*2} + \frac{\lambda^4 \sigma_0^2}{\check{d}^3(\check{d} - 6\sigma^2)}, \end{aligned} \tag{5.1}$$

where (S^*, I^*, R^*) is the endemic equilibrium of (1.1), $C_1 = 2\left(\frac{\gamma(d_S+d_I)}{\beta(d_S+d_R)} + \frac{d_I+\upsilon}{\beta}\right) I^* \sigma_0^2 + \frac{\gamma\sigma^2}{d_S+d_R} + 2\sigma_1^2$, $C_2 = \frac{\gamma\sigma^2}{d_S+d_R}$, $C_3 = \frac{\gamma\sigma^2}{d_S+d_R} + \frac{\gamma(d_I+d_R)}{\upsilon(d_S+d_R)}\sigma_3^2$, then the solution of the SDE (1.3) is an ergodic and positive recurrent Markov process.

Fig.3

Proof. Note that the diffusion coefficient $g(x)$ of the SDE (1.3) is defined by

$$g(x) = (g_{ij}(x)), \quad 1 \le i \le 3, \quad 1 \le j \le N, \quad x = (S, I, R)^\tau \in R^3_+,$$

where $g_{1j}(x) = -SI\sigma_{0j} - S\sigma_{1j}$, $g_{2j}(x) = SI\sigma_{0j} - I\sigma_{2j}$, $g_{3j}(x) = -R\sigma_{3j}$, $1 \le j \le N$.
If the matrix $(\sigma_{kj})_{0\le k\le 3, 1\le j\le N}$ has the full row rank, then $\text{rank}(g(x)) = 3$ and thus $A(x) := g(x)g^\tau(x)$ is positive definite in R^3_+. Since $g(x)$ is continuous in x, $A(x)$ is uniformly elliptical in any compact set $K \subset R^3_+$.

By Lemma 2.2 and its remark, it suffices to find a positive Lyapunov function $V(x)$ and a compact set $K \subset R^3_+$ such that $LV(x) \le -C$ for some $C > 0$ and $x \in R^3_+/K$.
When $R_0 > 1$, there exists unique positive equilibrium (S^*, I^*, R^*) of (1.1) such that

$$\lambda + \gamma R^* = \beta S^* I^* + d_S S^*, \quad \beta S^* = d_I + \upsilon, \quad \upsilon I^* = (d_R + \gamma)R^*. \tag{5.2}$$

For $x = (S, I, R)^\tau \in R^3_+$, we define a Lyapunov function V_1 by

$$V_1(x) = I - I^* - I^* \log \frac{I}{I^*} = I^* \left(\frac{I}{I^*} - 1 - \log \frac{I}{I^*} \right) > 0.$$

Itô's formula and (5.2) yield

$$\begin{aligned} LV_1(x) &= (I - I^*)(\beta S - d_I - \upsilon) + \frac{I^*}{2} \sum_{j=1}^N (\sigma_{0j} S - \sigma_{2j})^2 \\ &= \beta(I - I^*)(S - S^*) + \frac{\sigma_0^2 I^*}{2} S^2 - SI^* \sum_{j=1}^N \sigma_{0j}\sigma_{2j} + \frac{\sigma_2^2 I^*}{2} \\ &\le \beta(I - I^*)(S - S^*) + \sigma_0^2 I^* S^2 + \sigma_2^2 I^*. \end{aligned}$$

Define V_2 by

$$V_2(x) = \frac{1}{2}(R - R^*)^2.$$

Then by (5.2) we compute

$$\begin{aligned} LV_2(x) &= (R - R^*)(\upsilon I - (d_R + \gamma)R) + \frac{\sigma_3^2}{2} R^2 \\ &= \upsilon(R - R^*)(I - I^*) - (d_R + \gamma)(R - R^*)^2 + \frac{\sigma_3^2}{2} R^2. \end{aligned}$$

Define V_3 by

$$V_3(x) = \frac{1}{2}(S - S^* + I - I^* + R - R^*)^2.$$

Then applying Itô's formula and (5.2) again, we get

$$\begin{aligned} LV_3(x) &= (S - S^* + I - I^* + R - R^*)(\lambda - d_S S - d_I I - d_R R) + \frac{1}{2} \sum_{j=1}^N (\sigma_{1j} S + \sigma_{2j} I + \sigma_{3j} R)^2 \\ &\le (S - S^* + I - I^* + R - R^*)(\lambda - d_S S - d_I I - d_R R) + \frac{1}{2}(S^2 + I^2 + R^2)(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \\ &\le -d_S(S - S^*)^2 - d_I(I - I^*)^2 - d_R(R - R^*)^2 + \frac{\sigma^2}{2}(S^2 + I^2 + R^2) \\ &\quad - (d_S + d_I)(S - S^*)(I - I^*) - (d_S + d_R)(S - S^*)(R - R^*) - (d_I + d_R)(I - I^*)(R - R^*). \end{aligned}$$

Define V_4 by

$$V_4(x) = \frac{1}{2}(S - S^*)^2.$$

Then

$$\begin{aligned}
LV_4(x) &= (S - S^*)(\lambda - \beta SI - d_S S + \gamma R) + \frac{1}{2}\sum_{j=1}^{N}(\sigma_{0j} SI + \sigma_{1j} S)^2 \\
&= -d_S(S - S^*)^2 - \beta(S - S^*)(SI - S^* I^*) + \gamma(S - S^*)(R - R^*) + \frac{1}{2}\sum_{j=1}^{N}(\sigma_{0j} SI + \sigma_{1j} S)^2 \\
&\leq -d_S(S - S^*)^2 - \beta S^*(S - S^*)(I - I^*) - \beta(S - S^*)^2 I + \gamma(S - S^*)(R - R^*) + \sigma_0^2 S^2 I^2 + \sigma_1^2 S^2 \\
&\leq -d_S(S - S^*)^2 - \beta S^*(S - S^*)(I - I^*) + \gamma(S - S^*)(R - R^*) + \sigma_0^2 S^2 I^2 + \sigma_1^2 S^2
\end{aligned}$$

Define V_5 by

$$V_5(x) = \frac{1}{4}(S + I + R)^4.$$

We compute

$$\begin{aligned}
LV_5(x) &= (S + I + R)^3(\lambda - d_S S - d_I I - d_R R) + \frac{3}{2}(S + I + R)^2\sum_{j=1}^{N}(\sigma_{1j} S + \sigma_{2j} I + \sigma_{3j} R)^2 \\
&\leq \lambda(S + I + R)^3 - \check{d}(S + I + R)^4 + \frac{3\sigma^2}{2}(S + I + R)^4,
\end{aligned}$$

where $\check{d} = \min\{d_S, d_I, d_R\}$. Applying Young's inequality, we get

$$\lambda(S + I + R)^3 \leq \frac{\lambda^4}{4\check{d}^3} + \frac{3\check{d}}{4}(S + I + R)^4,$$

which yields

$$LV_5(x) \leq \frac{\lambda^4}{4\check{d}^3} - \left(\frac{\check{d}}{4} - \frac{3\sigma^2}{2}\right)(S + I + R)^4.$$

At last, we consider

$$V(x) = \left(\frac{\gamma(d_S + d_I)}{\beta(d_S + d_R)} + \frac{d_I + \upsilon}{\beta}\right)V_1 + \frac{\gamma(d_I + d_R)}{\upsilon(d_S + d_R)}V_2 + \frac{\gamma}{d_S + d_R}V_3 + V_4 + \frac{4\sigma_0^2}{\check{d} - 6\sigma^2}V_5.$$

Applying Itô's formula and (5.2), we get

$$
\begin{aligned}
LV(x) \leq & -\left(d_S + \frac{\gamma d_S}{d_S + d_R}\right)(S - S^*)^2 - \frac{\gamma d_I}{d_S + d_R}(I - I^*)^2 - \left(\frac{\gamma(d_I + d_R)(d_R + \gamma)}{\upsilon(d_S + d_R)} + \frac{\gamma d_R}{d_S + d_R}\right)(R - R^*)^2 \\
& + \left(\frac{\gamma(d_S + d_I)}{\beta(d_S + d_R)} + \frac{d_I + \upsilon}{\beta}\right)\sigma_0^2 I^* S^2 + \left(\frac{\gamma(d_S + d_I)}{\beta(d_S + d_R)} + \frac{d_I + \upsilon}{\beta}\right)\sigma_2^2 I^* + \frac{\sigma_3^2}{2}\frac{\gamma(d_I + d_R)}{\upsilon(d_S + d_R)} R^2 \\
& + \frac{\gamma\sigma^2}{2(d_S + d_R)}(S^2 + I^2 + R^2) + \sigma_1^2 S^2 + \sigma_0^2 S^2 I^2 + \frac{\lambda^4\sigma_0^2}{\check{d}^3(\check{d} - 6\sigma^2)} - \sigma_0^2(S + I + R)^4 \\
\leq & -\left(d_S + \frac{\gamma d_S}{d_S + d_R} - C_1\right)(S - S^*)^2 - \left(\frac{\gamma d_I}{d_S + d_R} - C_2\right)(I - I^*)^2 \\
& - \left(\frac{\gamma(d_I + d_R)(d_R + \gamma)}{\upsilon(d_S + d_R)} + \frac{\gamma d_R}{d_S + d_R} - C_3\right)(R - R^*)^2 \\
& + \left(\frac{\gamma(d_S + d_I)}{\beta(d_S + d_R)} + \frac{d_I + \upsilon}{\beta}\right)\sigma_2^2 I^* + C_1 S^{*2} + C_2 I^{*2} + C_3 R^{*2} + \frac{\lambda^4\sigma_0^2}{\check{d}^3(\check{d} - 6\sigma^2)}.
\end{aligned}
$$

If (5.1) holds, then the episode

$$
\begin{aligned}
& \left(d_S + \frac{\gamma d_S}{d_S + d_R} - C_1\right)(S - S^*)^2 + \left(\frac{\gamma d_I}{d_S + d_R} - C_2\right)(I - I^*)^2 \\
& + \left(\frac{\gamma(d_I + d_R)(d_R + \gamma)}{\upsilon(d_S + d_R)} + \frac{\gamma d_R}{d_S + d_R} - C_3\right)(R - R^*)^2 \\
& = \left(\frac{\gamma(d_S + d_I)}{\beta(d_S + d_R)} + \frac{d_I + \upsilon}{\beta}\right)\sigma_2^2 I^* + C_1 S^{*2} + C_2 I^{*2} + C_3 R^{*2} + \frac{\lambda^4\sigma_0^2}{\check{d}^3(\check{d} - 6\sigma^2)}
\end{aligned}
$$

lies in the positive zone of R^3 and thus there exists a positive constant $\varepsilon > 0$ and a compact set K of R_+^3 such that for any $x \in R_+^3/K$,

$$
\begin{aligned}
& \left(d_S + \frac{\gamma d_S}{d_S + d_R} - C_1\right)(S - S^*)^2 + \left(\frac{\gamma d_I}{d_S + d_R} - C_2\right)(I - I^*)^2 \\
& + \left(\frac{\gamma(d_I + d_R)(d_R + \gamma)}{\upsilon(d_S + d_R)} + \frac{\gamma d_R}{d_S + d_R} - C_3\right)(R - R^*)^2 \\
& \geq \left(\frac{\gamma(d_S + d_I)}{\beta(d_S + d_R)} + \frac{d_I + \upsilon}{\beta}\right)\sigma_2^2 I^* + C_1 S^{*2} + C_2 I^{*2} + C_3 R^{*2} + \frac{\lambda^4\sigma_0^2}{\check{d}^3(\check{d} - 6\sigma^2)} + \varepsilon.
\end{aligned}
$$

Hence for any $x \in R_+^3/K$,

$$LV(x) \leq -\varepsilon < 0.$$

Applying Lemma 2.2, we prove the ergodic property and the positive persistence of the SDE (1.3) whence the proof is complete. □

Theorem 5.2. *Under the conditions of Theorem 5.1, let μ denote the stationary distribution of the*

SDE (1.3). *Then we have*

$$d_S \int_{R^3_+} x\mu(dx,dy,dz) + \left(d_I + \frac{\upsilon d_R}{d_R+\gamma}\right)\int_{R^3_+} y\mu(dx,dy,dz) = \lambda,$$
$$\upsilon \int_{R^3_+} y\mu(dx,dy,dz) = (d_R+\gamma)\int_{R^3_+} z\mu(dx,dy,dz),$$
$$\left(\beta + \sum_{j=1}^N \sigma_{0j}\sigma_{2j}\right)\int_{R^3_+} x\mu(dx,dy,dz) - \frac{\sigma_0^2}{2}\int_{R^3_+} x^2\mu(dx,dy,dz) = d_I + \upsilon + \frac{\sigma_2^2}{2}.$$

Proof. Consider

$$\begin{aligned}\frac{N(T)-N(0)}{T} &= \lambda - \frac{d_S}{T}\int_0^T S(t)dt - \frac{d_I}{T}\int_0^T I(t)dt - \frac{d_R}{T}\int_0^T R(t)dt \\ &\quad + \frac{1}{T}\sum_{j=1}^N \int_0^T (\sigma_{0j}S(t) + \sigma_{1j}I(t) + \sigma_{3j}R(t))dB_j(t).\end{aligned}$$

Let $T\to\infty$, by ergodic theorem and Lemma 4.2, 4.3, we have

$$d_S\int_{R^3_+} x\mu(dx,dy,dz) + d_I\int_{R^3_+} y\mu(dx,dy,dz) + d_R\int_{R^3_+} z\mu(dx,dy,dz) = \lambda. \tag{5.3}$$

Similarly, by

$$\frac{R(T)-R(0)}{T} = \frac{\upsilon}{T}\int_0^T I(t)dt - \frac{d_R+\gamma}{T}\int_0^T R(t)dt + \frac{1}{T}\sum_{j=1}^N \sigma_{3j}\int_0^T R(t)dB_j(t),$$

we have

$$\upsilon\int_{R^3_+} y\mu(dx,dy,dz) = (d_R+\gamma)\int_{R^3_+} z\mu(dx,dy,dz), \tag{5.4}$$

which, together with (5.3), yields,

$$d_S\int_{R^3_+} x\mu(dx,dy,dz) + \left(d_I + \frac{\upsilon d_R}{d_R+\gamma}\right)\int_{R^3_+} y\mu(dx,dy,dz) = \lambda.$$

Applying Itô's formula to $\log I(t)$, yields

$$\begin{aligned}\frac{\log I(T) - \log I(0)}{T} &= \left(\beta + \sum_{j=1}^N \sigma_{0j}\sigma_{2j}\right)\frac{1}{T}\int_0^T S(t)dt - \frac{\sigma_0^2}{2T}\int_0^T S^2(t)dt \\ &\quad - \left(d_I + \upsilon + \frac{\sigma_2^2}{2}\right) + \frac{1}{T}\sum_{j=1}^N\int_0^T (\sigma_{0j}S(t) - \sigma_{1j})dB_j(t).\end{aligned}$$

By the ergodic theorem again, we get $\lim_{T\to\infty}\frac{\log I(T)}{T}$ exists, a.e. Now, we claim that $\lim_{T\to\infty}\frac{\log I(T)}{T} = 0$, a.e. If this statement is false, then $\lim_{T\to\infty}\frac{\log I(T)}{T} > 0$ or $\lim_{T\to\infty}\frac{\log I(T)}{T} < 0$, a.e. This means that $\lim_{T\to\infty} I(T) = \infty$ or 0, a.e., which contradicts the conclusion of the weak convergence to the invariant distribution μ lying in R^3_+. Therefore, we have

$$\lim_{T\to\infty} \frac{\log I(T)}{T} = 0, \quad a.e.$$

and hence

$$\left(\beta + \sum_{j=1}^{N} \sigma_{0j}\sigma_{2j}\right) \int_{R^3_+} x\mu(dx, dy, dz) - \frac{\sigma_0^2}{2} \int_{R^3_+} x^2\mu(dx, dy, dz) = d_I + \upsilon + \frac{\sigma_2^2}{2},$$

whence the proof is complete. □

6 Conclusions

Stochastic epidemic models have been studied by many authors, see e.g., [1, 3, 8, 37, 39]. In this paper, we impose the stochasticity on the disease transmission coefficient β and the removal rates d_S, d_I, d_R of deterministic model (1.1). If only consider the perturbation of β, then $N(t) = S(t) + I(t) + R(t)$ is uniformly bounded, and some papers, see e.g. [15], [30], [47] and their references, mainly study its extinction, but there are few papers concerned on its ergodic property. If the removal rates are also effected by noise, which may happen in the real world, $N(t)$ is unbounded even if the noise is small. In such a case, we adapt Chow's approach ([16]) and the moment estimate to establish the conditions for extinction of infective population. Furthermore, we construct new stochastic Lyapunov functions to study the ergodic property of the SDE (1.3). When the deterministic model has endemic equilibrium, under some mild conditions, we prove that the SDE (1.3) is ergodic. In such a case, by ergodic theory, the average of solution converges to the mean of the stationary distribution as the time goes by.

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