

# Entropic uncertainty minimum for angle and angular momentum

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## Abstract

Uncertainty relations are key components in the understanding of the nature of quantum mechanics. In particular, entropic relations are preferred in the study of angular position and angular momentum states. We propose a new form of angle–angular momentum state that provides, for all practical purposes, a lower bound on the entropic uncertainty relation,  $H_\varphi + H_m$ , for any given angular uncertainty, thus improving upon previous bounds. We establish this by comparing this sum with the absolute minimum value determined by a global numerical search. These states are convenient to work with both analytically and experimentally, which suggests that they may be of use for quantum information purposes.

Keywords: entropic uncertainty, orbital angular momentum, quantum information

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(Some figures may appear in colour only in the online journal)

## 1. Introduction

Current demands for secure, high-bandwidth communications have resulted in a lot of interest in new protocols in quantum information based on spatial modes carrying orbital angular momentum (OAM) [1–3]. These are defined within an infinite-dimensional, discrete Hilbert space and are therefore of considerable interest for quantum information purposes as they have the potential to generate multi-dimensional entangled states. The potential for each photon to carry more than one bit of information [4] means that in applications such as quantum key distribution (QKD) not only is the data transfer rate increased in proportion to the number of bits of information carried by each photon but the security of the protocol is also increased.

Uncertainty relations are of particular importance in the study of quantum-limited measurements as they impose a fundamental limit upon the precision with which conjugate pairs of physical properties of a system can be measured. In other words, the more precisely one property can be

determined, the less well known will be its counterpart. Indeed, the complementarity of incompatible observables is one of the fundamentally distinctive properties of quantum theory [5]. This feature is given quantitative form in the Heisenberg uncertainty principle [6] and its generalization to any pair of observables [7], which place bounds on the product of the statistical uncertainties or standard deviations for the pair. The uncertainty relation for angular momentum and angular position [8, 9] is much less well-known than the Heisenberg uncertainty relationship for linear position and momentum. It is also a little more complicated as the periodic nature of angular position imposes a finite range of  $2\pi$  on the angle observable and the resultant uncertainty relation has a state-dependent lower bound [7]. In the commonly-chosen range  $-\pi \leq \varphi < \pi$  it can be expressed as [8–11]

$$\Delta m \Delta \varphi \geq \frac{1}{2} |1 - 2\pi P(-\pi)|, \quad (1)$$

where  $P(-\pi)$  is the angular probability density at the boundary of the chosen range.

In this case there are strong reasons for considering other measures of uncertainty. In particular, *entropic* uncertainties are particularly well suited to states with a dependence on angular position as they avoid the issue of the angle periodicity [12]. Among the other advantages of entropic

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uncertainties is that they distinguish between broad, single-peaked distributions and distributions with two, or more, narrow but widely separated peaks. While both distributions can have a large uncertainty, the latter will have a much smaller entropy. Moreover, entropic uncertainties are well-adapted for quantum communications theory. Indeed, security proofs for QKD are expressed in terms of information or entropy bounds [13]. This suggests that the entropic uncertainty relation for angle and angular momentum may find application in secure communications based on OAM [14].

Entropic uncertainty relations place a lower bound on the sum of the entropies associated with the probability distributions for a pair of observables [12, 15–19]. An important feature of this lower bound is that it is a constant [16], which is in contrast to the lower bound for the product of standard deviations which is, in general, state-dependent [7]. For linear position and momentum and for angular momentum and angle, for example, the entropic uncertainty relations are [15, 17, 18]

$$H_x + H_p \geq \log(e\pi), \quad (2)$$

$$H_m + H_\varphi \geq \log(2\pi), \quad (3)$$

where the entropies are derived from the corresponding wavefunctions:

$$\begin{aligned} H_x &= -\int_{-\infty}^{\infty} dx |\psi(x)|^2 \log |\psi(x)|^2, \\ H_p &= -\int_{-\infty}^{\infty} dp |\psi(p)|^2 \log |\psi(p)|^2, \\ H_m &= -\sum_{m=-\infty}^{\infty} |\psi(m)|^2 \log |\psi(m)|^2, \\ H_\varphi &= -\int_{-\pi}^{\pi} d\varphi |\psi(\varphi)|^2 \log |\psi(\varphi)|^2. \end{aligned} \quad (4)$$

We note that in the last of these, any  $2\pi$ -integration range may be chosen as  $\psi(\varphi)$  is  $2\pi$ -periodic. The explicit values of these entropies depend, of course, on the base of the logarithm chosen. Here, for definiteness, we select the natural base so that log should be interpreted as  $\log_e$ .

An important point to note is that for linear position and momentum, the lower bound on the entropy sum (2), is satisfied for the Gaussian wavefunctions that also satisfy the equality in the more familiar Heisenberg uncertainty principle. These states are referred to as the *intelligent states* [12, 20] and, indeed, have been demonstrated experimentally [10]. For angular momentum and angle the intelligent states that satisfy the equality in (3) have the form of a truncated Gaussian centred on  $\varphi = 0$  [10]:

$$\psi_{\text{int}} = (\sqrt{2\pi b} \operatorname{erf}(\pi/\sqrt{2b}))^{-\frac{1}{2}} \exp\left(-\frac{\varphi^2}{4b}\right), \quad (5)$$

where  $b = 1/(2\lambda)$  and  $\operatorname{erf}(x) = (2/\sqrt{\pi}) \int_0^x e^{-t^2} dt$  is the error function [21], as can be seen from the angular probability distribution in figure 1. At difference with position and linear momentum, in the case of angle and angular momentum it is possible, for a given  $\Delta\varphi$ , to find states for which the uncertainty product in (1) is *lower* than that for the intelligent states

[11, 22–24]. Such *minimum uncertainty states* [22], which may be expressed in terms of confluent hypergeometric functions [11] or Mathieu wave functions [23], are possible only because the state-dependent lower bound in (1) takes a yet lower value than the uncertainty product. For the entropic uncertainty (3), in contrast to the linear position and momentum states, *only* the angular-momentum *eigenstates*, for which  $H_m = 0$ , satisfy the equality. Any state for which the angular distribution is *peaked* necessarily leads to an entropic sum,  $H_m + H_\varphi$ , in excess of  $\log(2\pi)$ . Our aim is then to obtain a stronger lower bound on the sum  $H_m + H_\varphi$  for application when we do not have a flat angular probability distribution,  $P(\varphi) = |\psi(\varphi)|^2$ .

## 2. Analysis

One motivation for pursuing this specific problem is the existence of quantum communications protocols based on angle and angular-momentum complementarity [25]. It is known that uncertainty principles, both variance based and entropic, impose fundamental limits on quantum information tasks. For example, the ability to perform quantum steering is related to bounds in certain entropic uncertainty principles [26]. Similarly, one can relate the Tsilson bound to the uncertainty principle [27]. Furthermore, many estimates of the secret key rate for QKD depend on bounds for an entropic uncertainty principle [28, 29]. Understanding the minimum uncertainties for angle and OAM is thus important for exploiting OAM for quantum communication. Even approximate results are useful, as they can give a strong indication of the optimal performance of tasks such as QKD.

The simplest approach to our problem is to find a bound given any particular value for the angular standard deviation,  $\Delta\varphi$ . This has the added advantage that in fixing  $\Delta\varphi$  we can make a natural connection with existing work on the uncertainty principle for angular momentum and angle [10–12].

To obtain a stronger lower bound we require the lowest possible entropic sum,  $H_m + H_\varphi$ , for a given angular uncertainty,  $\Delta\varphi$ , and to find the form of the states which achieve this bound. Note, the fact that entropy is concave implies that the minimum uncertainty state will be a pure state [30]. We have not been able to find an analytical form for the lower bound and so have had to rely on a numerical optimisation to find this. There is, however, an excellent analytical approximation to the states that reach this lower bound. The states we have found have Gaussian amplitudes in the angular momentum basis:

$$\psi_{\text{OAM}} = \frac{1}{\sqrt{\vartheta_3(0, e^{-2a})}} \sum_{m=-\infty}^{\infty} e^{-am^2}, \quad (6)$$

where  $a$  is a real and positive parameter,  $1/\sqrt{\vartheta_3(0, e^{-2a})}$  is the normalisation factor and  $\vartheta_3$  is an elliptic theta function of the third kind, defined as  $\vartheta_3(u, q) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2inu}$  [31].

Angular momentum and angle are related by a discrete Fourier transform, which leads to the angular wavefunction

$$\begin{aligned} \psi(\varphi) &= \frac{1}{\sqrt{2\pi\vartheta_3(0, e^{-2a})}} \sum_{m=-\infty}^{\infty} e^{-am^2} e^{im\varphi}, \\ &= \frac{\vartheta_3\left(\frac{\varphi}{2}, e^{-a}\right)}{\sqrt{2\pi\vartheta_3(0, e^{-2a})}}, \end{aligned} \quad (7)$$

where we have re-expressed the sum in terms of its corresponding theta function. A straightforward application of the Poisson sum-rule [31] leads to a, perhaps, more physically appealing representation in the form

$$\psi(\varphi) = \frac{1}{\sqrt{2a\vartheta_3(0, e^{-2a})}} \sum_{n=-\infty}^{\infty} \exp\left[-\frac{1}{4a}(\varphi - 2n\pi)^2\right]. \quad (8)$$

We note the similarity between (8) and equation (2.18) of [32].

Our Gaussian distribution of angular momentum states has the form, in the angular representation, of a sequence of overlapping equal Gaussians, peaked at  $2n\pi$ . Figure 1 shows the corresponding angular probability distribution in the angular range  $-\pi$  to  $\pi$  and the comparison with the intelligent states. In this form it is clear that the parameter  $a$  determines the width of the Gaussians. To be precise,  $a$  is proportional to the square root of the width of the Gaussians such that the full width at half maximum (FWHM) of the Gaussians is  $4\sqrt{\ln 2a}$ . As the parameter  $a$  is increased, the width of the Gaussians is increased and, as the angle state is related to the OAM state by (discrete) Fourier transform, the number of states in the OAM distribution decreases accordingly.

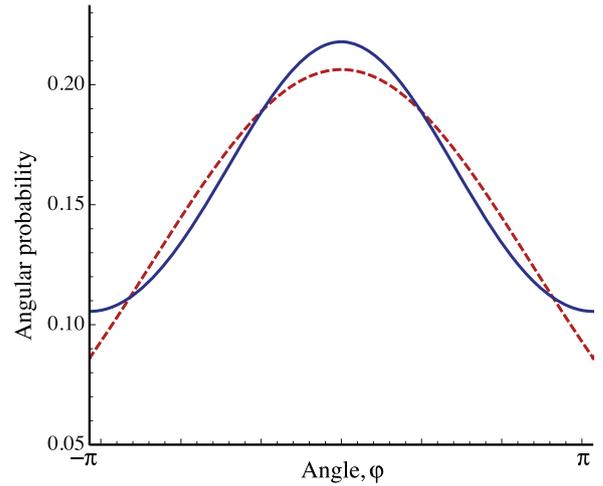
The angular uncertainty,  $\Delta\varphi$ , is defined via

$$\Delta\varphi^2 = \int_{-\pi}^{\pi} d\varphi \varphi^2 P(\varphi), \quad (9)$$

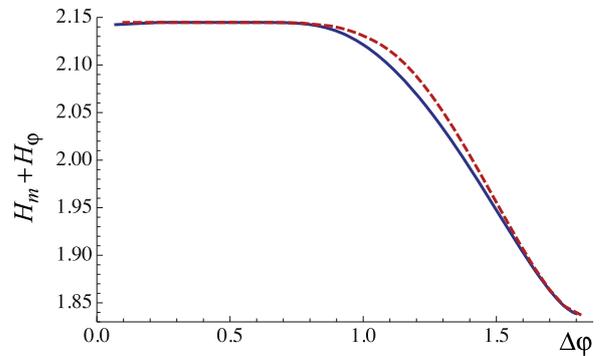
where the angular probability distribution is  $P(\varphi) = |\psi(\varphi)|^2$ . For the states (7) we calculate this to be

$$\begin{aligned} \Delta\varphi^2 &= \frac{\pi^2}{3} + \frac{2}{\vartheta_3(0, e^{-2a})} \\ &\times \sum_{\substack{m, m'=-\infty \\ m \neq m'}}^{\infty} \frac{(-1)^{m-m'}}{(m-m')^2} e^{-a(m^2+m'^2)}. \end{aligned} \quad (10)$$

Equation (10) depends solely on the parameter  $a$  and so we can calculate the entropic sum (3) as a function of angular uncertainty simply by varying  $a$ . Note that as  $a$  is increased from 0 towards  $\infty$ , the angular uncertainty increases from 0 to its limiting value of  $\pi/\sqrt{3}$ . The corresponding OAM distribution varies between an almost flat, continuous distribution and a single OAM eigenstate with variance  $\Delta m = 0$ .



**Figure 1.** The angular probability distribution,  $P(\varphi) = |\psi(\varphi)|^2$ , in the range  $-\pi$  to  $\pi$  for the overlapping Gaussian states (blue, solid) compared to the truncated Gaussian intelligent states (red, dashed), corresponding to angular variance  $\Delta\varphi = 1.61$ .



**Figure 2.** Entropic uncertainty relation,  $H_m + H_\varphi$ , as a function of  $\Delta\varphi$  for the intelligent states (red, dashed) and our Gaussian distribution states (blue, solid). Both tend to the limits  $\log(e\pi) = 2.14$  and  $\log(2\pi) = 1.84$  as  $\Delta\varphi \rightarrow 0$  and  $\Delta\varphi \rightarrow \infty$ , respectively.

It is straightforward to obtain the angular momentum entropy for our Gaussian state:

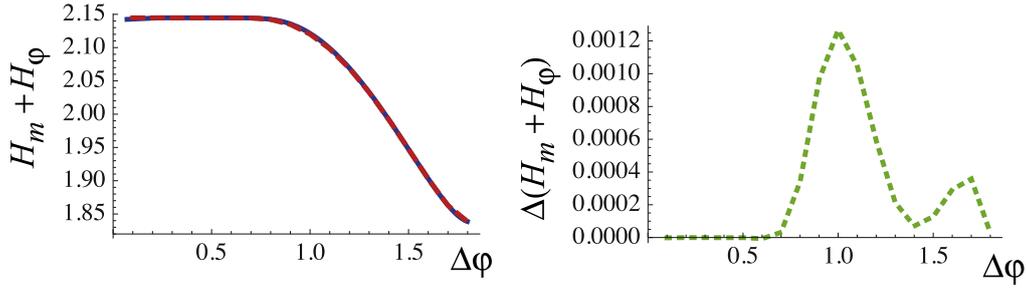
$$H_m = \log \vartheta_3(0, e^{-2a}) + 2a\Delta m^2, \quad (11)$$

where  $\Delta m^2$  is the angular-momentum variance given by

$$\Delta m^2 = -\frac{1}{2\vartheta_3(0, e^{-2a})} \frac{d}{da} \vartheta_3(0, e^{-2a}). \quad (12)$$

Evaluating the angular entropy presents more of a challenge. We do not have an expression for this in closed form but do have a very good upper bound on its value, which we present in the appendix. It is straightforward, however, to determine the value of  $H_\varphi$  numerically. The result, shown in figure 2, clearly shows a lower entropic uncertainty sum for the Gaussian states than for the intelligent states over a large range of  $\Delta\varphi$ .

Although we do not have an exact analytical expression for the entropic uncertainty over the entire range of angular



**Figure 3.** Left: The entropic uncertainty of our new states (red, dashed) is almost identical to the numerically minimized sum (blue, solid); right: the *maximum* difference between them is of the order of 0.1%.

uncertainty, it is straightforward to obtain limiting forms for large and small values of  $\Delta\varphi$ . In the limit  $\Delta\varphi$  tends to its maximum value,  $\pi/\sqrt{3}$ , our state becomes the angular momentum eigenstate with  $m = 0$ . For this state we have  $H_m = 0$  and  $H_\varphi = \log(2\pi)$  and we reach the global minimum value given in (2). At the other extreme, for small values of  $\Delta\varphi$ , our Gaussian peaks become well separated and it suffices to consider only the peak centred at  $\varphi = 0$ :

$$\psi(\varphi) \approx (2a\pi)^{-\frac{1}{4}} \exp\left(-\frac{\varphi^2}{4a}\right). \quad (13)$$

In this limit our angle entropy is

$$H_\varphi = \frac{1}{2} \log(2ea\pi). \quad (14)$$

For the angular momentum variance in this limit we find  $\Delta m^2 = 1/(4a)$  and our angular momentum entropy is

$$H_m = \frac{1}{2} \log\left(\frac{e\pi}{2a}\right). \quad (15)$$

Thus our entropy sum tends to the value  $\log(e\pi)$ , which is the value found for the unbounded and continuous observables,  $x$  and  $p$ . The reasons for this are, of course, that in this limit the angular momentum distribution is broad and slowly varying and so approaches a continuous distribution, and the angle distribution is sharply peaked around  $\varphi = 0$  and so is insensitive to the ends of the angle range at  $\varphi = \pm\pi$ .

### 3. Entropic minimum

It remains to determine how close the entropy sum for our state is to the elusive true minimum value. We have tested this numerically using standard search techniques. Note that we constrain the angular probability of the state to have minima at  $\varphi = \pm\pi$  in order to maximize the rhs of equation (1). We also assume that, like both the intelligent states (6) and our overlapping Gaussian states (8), the angular probability function is symmetric and increases monotonically from the minima to a maximum at  $\varphi = 0$ . This, and hence also the corresponding OAM state, are then numerically optimized to minimize the sum  $H_m + H_\varphi$  using an iterative optimisation process. This iteration is repeated several times with a

randomized optimization pathway to confirm that our optimization is global.

Comparing our numerical result for the entropic sum for our overlapping Gaussian state (6, 7) with this global minimum, figure 3, we find a very close agreement between the two with the greatest difference,  $\Delta(H_m + H_\varphi)$ , for  $\Delta\varphi \approx 1$ , amounting to only about 0.06%. This difference is unlikely to be significant for any but the most sensitive of measurements and suggests that the overlapping Gaussian states provide a practical approximation to the states that minimize the entropic sum for a given angular width.

### 4. Conclusion

In conclusion, we have presented a new form of angle and orbital angular momentum states which have a Gaussian distribution in angular momentum and consist of a sum of overlapping Gaussians in angle. These are an excellent analytical approximation to the states that minimize the entropic sum,  $H_m + H_\varphi$ , and thus provide, what is *for all practical purposes*, a state independent lower bound on  $H_\varphi + H_m$  for any angular width, thereby improving upon previous bounds. We suggest that these states are therefore ideal candidates for use in high-bandwidth, secure quantum information protocols.

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### Appendix A. Analytic approximation for $H_\varphi$

We seek an analytic approximation, preferably an upper bound, to  $H_\varphi$  for our overlapping Gaussian state (7). We find this by making use of the relative entropy, which in our case

takes the form [33]

$$H [P(\varphi)||Q(\varphi)] = \int_{-\pi}^{\pi} d\varphi P(\varphi) \log \frac{P(\varphi)}{Q(\varphi)}. \quad (\text{A.1})$$

This quantity has the merit of being always greater than or equal to zero, taking the value zero *only* if  $P(\varphi) = Q(\varphi)$ . Hence, if we set  $P(\varphi) = |\psi(\varphi)|^2$  then we find the bound

$$H_{\varphi} \leq - \int_{-\pi}^{\pi} d\varphi P(\varphi) \log Q(\varphi), \quad (\text{A.2})$$

for *any* probability distribution  $Q(\varphi)$ .

We find different upper bounds for different choices of  $Q(\varphi)$ . One suitable choice for the probability distribution  $Q(\varphi)$  is to use the truncated Gaussian state, which is also the intelligent state for angle and angular momentum [9, 10]. This gives:

$$Q(\varphi) = Z^{-1}(b) \exp\left(-\frac{\varphi^2}{2b}\right), \quad (\text{A.3})$$

where  $Z(b) = \sqrt{2\pi b} \operatorname{erf}(\pi/\sqrt{2b})$  and the positive parameter  $b$  is yet to be determined. A straightforward calculation then gives the bound

$$H_{\varphi} \leq \log Z(b) + \frac{\Delta\varphi_P^2}{2b}, \quad (\text{A.4})$$

where  $\Delta\varphi_P^2$  is the variance associated with the probability distribution  $P(\varphi)$ . To find the best bound we simply choose  $b$  so as to minimize our upper bound. To this end we differentiate our bound with respect to  $b$  to find the extremum:

$$\begin{aligned} \frac{d}{db} \left( \log Z(b) + \frac{\Delta\varphi_P^2}{2b} \right) &= 0 \\ \Rightarrow \frac{1}{Z} \frac{dZ}{db} - \frac{\Delta\varphi_P^2}{2b^2} &= 0. \end{aligned} \quad (\text{A.5})$$

We can write the first part of this expression in terms of the variance for the probability distribution  $Q(\varphi)$ :

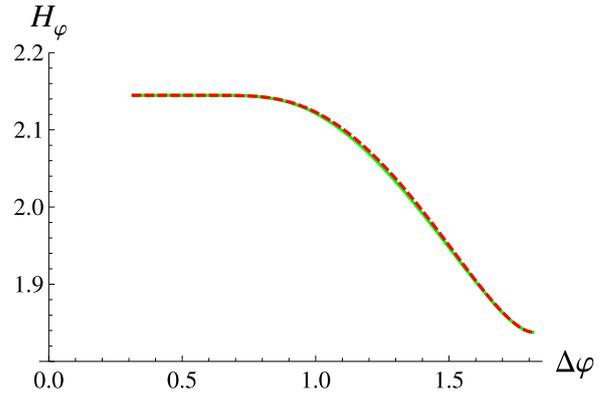
$$\begin{aligned} \frac{1}{Z} \frac{dZ}{db} &= \frac{1}{Z} \int_{-\pi}^{\pi} d\varphi \exp\left(-\frac{\varphi^2}{2b}\right) \\ &= \frac{1}{2b^2 Z} \int_{-\pi}^{\pi} d\varphi \varphi^2 \exp\left(-\frac{\varphi^2}{2b}\right) \\ &= \frac{\Delta\varphi_Q^2}{2b^2}. \end{aligned} \quad (\text{A.6})$$

It follows that the least upper bound occurs when we choose the two probability distributions,  $P(\varphi)$  and  $Q(\varphi)$  to have the same variance:  $\Delta\varphi_Q^2 = \Delta\varphi_P^2 = \Delta\varphi^2$ . Hence, our least upper bound for  $H_{\varphi}$ , which we denote  $\Gamma$ , has the form

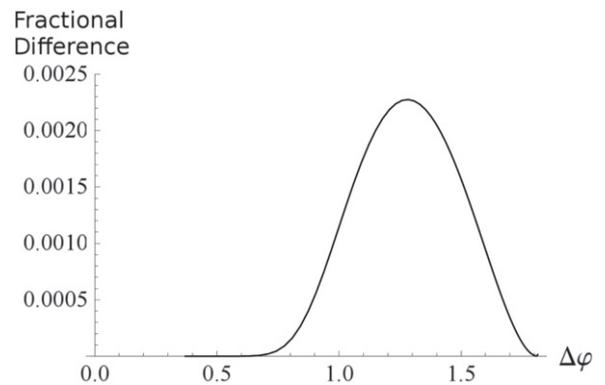
$$\Gamma = \log Z(b) + \frac{\Delta\varphi^2}{2b}, \quad (\text{A.7})$$

where  $b$  is chosen so that  $\Delta\varphi_Q^2 = \Delta\varphi^2$ .

We find that using  $\Gamma$  in place of  $H_{\varphi}$  gives an excellent approximation to the entropy sum evaluated numerically for



**Figure 4.** A plot comparing the exact numerically calculated value for  $H_m + H_{\varphi}$  with the minimum value for  $\Gamma$ . The solid green line is  $H_m + H_{\varphi}$ , while the dashed red line is for the upper bound  $\Gamma$ .



**Figure 5.** A plot of the fractional difference between  $\Gamma$  and  $H_m + H_{\varphi}$  against  $\Delta\varphi$ .

$\psi(\varphi)$ . In figure 4 we compare the exact numerical values for  $H_m + H_{\varphi}$ , plotted as a function of  $\Delta\phi$ , with  $H_m + \Gamma$ , where  $\Gamma$  is evaluated with  $\Delta\varphi_Q = \Delta\varphi$ . The solid green line is  $H_m + H_{\varphi}$ , while the dashed red line is  $H_m + \Gamma$ . Figure 4 shows that using  $\Gamma$  in place of  $H_{\varphi}$  is an excellent approximation. This can be shown further by plotting the fractional difference between the two curves plotted in figure 4, as shown in figure 5. We see that the fractional difference between the approximate and exact results is always less than 0.24%.

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