Nonlinear and chaotic dynamics of a Bose-Einstein condensate in an optical cavity

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We present a theoretical analysis of a Bose-Einstein condensate (BEC) enclosed in an optical cavity driven by an external laser beam. In the limit where the cavity field is adiabatically eliminated, the interaction between the condensate and the cavity displays a range of nonlinear dynamical behaviours, such as multi-stability, quasi-periodicity, and conservative chaotic oscillations. We show that chaotic oscillations are ubiquitous when repulsive atom-atom interactions are included in the model. Despite the complexity of the full coupled BEC-cavity system, the essential features of its nonlinear *and chaotic* behaviour are low-dimensional and are well described by considering the evolution of only a few motional states of the BEC.

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1. INTRODUCTION

The interaction between Bose-Einstein condensates (BECs) and optical fields has been a subject of intense activity and interest since the first realisation of BECs in the mid-1990s [1, 2]. In particular, optical lattices have been a valuable tool in BEC experiments, allowing the mimicking of atomic lattices in condensed matter systems and consequently the investigation of fundamental problems such as quantum phase transitions [3]. Initially, optical lattices used in cold atoms and BEC experiments were static entities, but experiments have since demonstrated BECs enclosed in a high-quality optical cavity [4], where a standing-wave cavity mode plays the role of an optical lattice that is dynamic in character and that evolves self-consistently with the BEC [5]. This results in nonlinear behaviour such as optical bistability, which can be described via a simplified model of the BEC-cavity system using only two motional modes of the condensate coupled with a single optical cavity mode [6, 7]. When the position of one of the cavity mirrors is allowed to oscillate, it has been shown that the behaviour of the BEC-cavity system can be chaotic in nature [8]. Theoretical studies have in the main neglected the atom-atom interactions, but recent work has shown that the inclusion of (repulsive) interactions between condensate atoms has a strong effect on the system's nonlinearity [9, 10].

In this paper we consider the full model of a BEC-cavity system that solves the Gross-Pitaevskii equation coupled to the cavity field and driven by an external laser. As well as considering the limiting case where atom-atom interactions are neglected, we also include repulsive atom-atom interactions for two specific BEC configurations corresponding to transverse and longitudinal BEC alignments with the cavity. Using this model, we show

that even when atom-atom interactions are neglected, the system displays a variety of behaviours, specifically quasi-periodic and chaotic oscillations, which are an intrinsic feature of the coupling between the cavity field and the BEC. With the addition of repulsive atom-atom interactions, we further show that the region of optical bistability is shifted to higher input intensities (in agreement with [10]), and that chaotic oscillations are, beyond a critical pump intensity, ubiquitous.

The results from the full model suggest that some observed behaviours of the BEC-cavity system, which have been attributed to stochastic effects [6], could actually be deterministic, chaotic phenomena arising from the nonlinear nature of the cavity-BEC interaction. Further, although the behaviour of the system is rather complex, its dynamics are low-dimensional, and consequently it should be possible to describe its main features by using a reduced model involving a limited number of BEC motional states. We demonstrate a strong correspondence between the full numerical simulation and a restricted model incorporating a reduced number of motional states. We determine the minimum number of modes necessary to describe the nonlinear and chaotic features of the BEC-cavity interaction correctly.

2. BEC IN A CAVITY DRIVEN BY A LASER

The system described by our model is shown schematically in Fig. 1, and consists of a BEC coupled to a single mode of a Fabry-Pérot optical cavity, which is pumped by a blue-detuned laser. The pump field, with frequency ω_p , is dispersively coupled with the atomic resonance ω_a , with the corresponding detuning given by $\Delta_a = \omega_p - \omega_a$, and the detuning between the pump and cavity field at ω_c is given by $\Delta_c = \omega_p - \omega_c$. The potential depth of the lattice formed by the standing-wave cavity mode is characterised by $\hbar U_0$, where $U_0 = g_0^2/\Delta_a$ is the frequency shift associated with the dispersive effect of a single atom in the cavity and g_0 is the maximum coupling strength

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FIG. 1: (Colour online.) A sketch of the BEC-cavity system, where the single-mode cavity field interacts with a cigar-shaped condensate that is aligned (a) transverse to the cavity axis, and (b) along the cavity axis. In both cases, the cavity is driven by a pump laser η ; losses are given by κ . In an experimental setup, the addition of a detector can be used to record the cavity's transmitted light, which can provide a measure of, e.g., the atom-atom interaction [9].

between an atom and a cavity photon. The pump-atom detuning Δ_a is assumed to be sufficiently large such that the N atoms in the cavity can be assumed to remain in their ground state with negligible spontaneous emission. The cavity photon number $|\alpha|^2$ is affected by the input pump beam, cavity losses, and by the interaction of the BEC with the cavity mode.

Starting with the three-dimensional Gross-Pitaevskii equation, and following the derivation described in e.g. [11, 12], we arrive at a mean-field, one-dimensional description of the condensate dynamics, where the equation of motion for the coupled BEC-cavity system is (cf. [13])

$$i\hbar\dot{\Psi}(X, T) = \left(\frac{-\hbar^2}{2m} \frac{\partial^2}{\partial X^2} + |\alpha(T)|^2 \hbar U_0 \sin^2(kX) + g_{1D} |\Psi|^2 \right) \Psi(X, T), \quad (1)$$

where the cavity photon number is given by

$$|\alpha|^2 = \frac{\eta^2}{\kappa^2 + (\Delta_c - U_0 \mathcal{O})^2}. \quad (2)$$

The 1-D atom-atom interaction parameter is given by $g_{1D} = g_{3D}/(\pi\sigma_y\sigma_z) = 2\hbar\omega_\perp a$, where $g_{3D} = 4\pi\hbar^2 a/m$ is the 3D atom-atom interaction parameter, a is the scattering length of the BEC, and m is the atomic mass. $\sigma_{y,z}$ are the dimensions of the trap in the y and z directions.

The transverse trap frequency is $\omega_\perp = \sqrt{\omega_y\omega_z}$, where $\omega_{y,z} = 2\hbar/(m\sigma_{y,z}^2)$ are the trap frequencies in the (y, z) directions. The overlap integral $\mathcal{O} = \langle \Psi | \sin^2(kX) | \Psi \rangle$ describes the driving of the cavity mode by the condensate, where $k = 2\pi/\lambda$ and λ are, respectively, the wavenumber and wavelength of the optical field, κ is the cavity decay rate, and η is the cavity pumping rate. The BEC wave function Ψ is normalised such that $\langle \Psi | \Psi \rangle = N$.

In this paper we consider a condensate formed by ^{87}Rb atoms, with $m = 1.44 \times 10^{-25}$ kg, and with parameters that correspond to the experimental setup described in [4], so that $\Delta_a = 2\pi \times 58$ GHz, $g_0 = 2\pi \times 14.1$ MHz, $\kappa = 2\pi \times 1.3$ MHz, $\lambda = 780$ nm, and the number of atoms $N = 10^5$. The results are however applicable to wide ranges of the parameter values.

In Eq. (2), it has been assumed that the optical cavity field, α , decays significantly faster than the timescale of the BEC dynamics in the optical lattice produced by the cavity mode; i.e., the cavity decay rate κ is significantly greater than the recoil frequency $\omega_r = \hbar k^2/2m$. This assumption is well justified in many BEC-cavity experiments, such as in the case of [4], where κ exceeds ω_r by a factor of ~ 100 . For the purposes of this paper we first concentrate on coupling between atoms solely via the cavity field. We initially assume close proximity to a Feshbach resonance, allowing us to set the scattering length to zero (and thus $g_{1D} = 0$) in order to neglect atom-atom interactions. Later we consider two specific instances where $g_{1D} > 0$, corresponding to two distinct experimental setups. The first instance, shown in Fig. 1a, considers the situation where the BEC is aligned transversely to the cavity axis to simulate a configuration similar to that used at ETH [4, 6, 14, 15] and designed to minimise the effect of interactions between the atoms. The second instance, shown in Fig. 1b, involves a more common cigar-shaped BEC aligned in the direction of the cavity axis (as in, e.g., [16, 17]).

For the purposes of our analysis, we rewrite Eq. (1) in terms of dimensionless variables and parameters using the transformation $t = T/T_0$, where $T_0 = mL^2/4\hbar$, with $L = \lambda/2$ being the lattice period, $x = 2X/L$, and $\psi = \Psi\sqrt{L/N}$, giving

$$i\dot{\psi}(x, t) = \frac{i}{2} \frac{\partial^2 \psi}{\partial x^2} - i|\alpha|^2 V_0 \sin^2\left(\frac{\pi x}{2}\right) \psi - i\Lambda |\psi|^2 \psi, \quad (3)$$

where $\Lambda = g_{1D}NT_0/(\hbar L)$ and $V_0 = U_0T_0$ are dimensionless parameters that characterise the scattering length and BEC-lattice coupling strength, respectively. Note that the dimensionless BEC wave function ψ is normalised such that $\langle \psi | \psi \rangle = 1$, and thus

$$|\alpha|^2 = \frac{\eta^2}{\kappa^2} \left(1 + \left(\frac{U_0 N \mathcal{O}'}{2\kappa} - \frac{\Delta_c}{\kappa} \right)^2 \right)^{-1}, \quad (4)$$

where the overlap integral $\mathcal{O}' = \langle \psi | \sin^2(\pi x/2) | \psi \rangle$.

In order to understand the dynamical behavior of Eqs. (3)-(4), it is important to determine

if the flow is conservative or dissipative. When the cavity-field dynamics are incorporated, the system is clearly dissipative because of the losses thorough the cavity mirror and relaxes to a sub-space where such dissipations are balanced by the cavity pumping rate. When the cavity field dynamics is adiabatically eliminated as done in [8] and in Eq. (4), the dynamics starts and evolve on such sub-space. By evaluating the Jacobian J of Eqs. (3)-(4) and their complex conjugate, it is possible to demonstrate that

$$SJ + J^T S = 0 \quad (5)$$

where the skew-symmetric matrix S is of the form

$$S = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \quad (6)$$

with I being the identity matrix [18]. This implies that the flow (3)-(4) is symplectic and it conserves the volume in phase space. It is important to stress that the volume is conserved on the sub-space of the full phase space. This is further confirmed by a zero divergence of the flow and the conservation in time of the total atomic density. We note that a symplectic flow may not be Hamiltonian in character although Hamiltonian flows are always symplectic [19]. In our case, the full Hamiltonian character of the interaction mediated through the integral $\mathcal{O}' = \langle \psi | \sin^2(\pi x/2) | \psi \rangle$ cannot, at present, be established. We are however in the presence of a conservative dynamical system.

3. BISTABILITY, NONLINEAR DYNAMICS, AND CHAOS IN A NON-INTERACTING CONDENSATE

3.1. Bistability

Eqs. (3) and (4) have steady-state solutions, depicted in Fig. 2 for the case where atomic collisions are neglected ($\Lambda = 0$), that show nonlinear resonance and bistability when the pump intensity exceeds a critical value [4, 6]. In order to investigate the nonlinear dynamics of the system, we solve the full Eqs. (3) and (4) using a split-step FFT method for a fixed value of the pump-cavity detuning, $\Delta_c - NU_0/2 = -5\kappa$. We use the pump rate η as a controllable parameter for investigation of the dynamics of the bistable region of the resonance curves. Note that, although we use a fixed value for the detuning here, the system behaviour described is generic, and that similar results are obtained for a wide range of values of Δ_c if the pump intensity is also modified accordingly.

The system conserves the total number of atoms N during the dynamical evolution. For this reason, it is important to specify the initial condition corresponding to particular experimental conditions. In this paper

FIG. 2: (Colour online.) Resonance curves of increasing size for $\eta/\kappa = 0.32$ (green, smallest), 0.64 (blue, dotted), 1.16 (black, solid), and 1.40 (red, dashed-dotted, largest). The vertical dashed line (brown), at $\Delta_c - U_0 N/2 = -5\kappa$, indicates the detuning used to investigate the behaviour of the system for increasing pump intensities, and the inset shows the region near this detuning in more detail, for the values $\eta/\kappa = 1.40$ (red, leftmost dashed-dotted curve), 1.16 (black, middle curve), and 0.64 (blue, rightmost dotted curve). The y -axis is normalised by $|\alpha|_{cr}^2$, the critical value beyond which bistability occurs. (Note that the scattering length here is set to zero.) Quantities are dimensionless.

we consider the BEC initially at rest in a homogeneous state where the number of atoms is uniformly distributed across the optical cavity. We shall see that the choice of this initial condition corresponds to the dynamics on invariant manifolds when considering modal expansion of the BEC wave function Ψ in Sec. 5.

3.2. Nonlinear dynamics and chaos

Examples of cavity photon-number evolution as a function of time are shown in Fig. 3 for three values of η ($\eta/\kappa = 0.99, 1.08$, and 1.40), corresponding to three distinct cases of crossing the bistable region of the nonlinear resonance curves indicated by the dotted line in Fig. 2. As was demonstrated in [4], the photon-number evolution displays nonlinear oscillations. In order to distinguish between the different kinds of nonlinear oscillation that occur, we study the frequency spectrum of the intracavity intensity/photon number and the evolution of the BEC for these different pumping rates. Fig. 4 shows the temporal spectra of the cavity photon number $|\alpha|^2$ for the same values of pump rate, η , as used in Fig. 3, i.e., $\eta/\kappa = 0.99, 1.08$, and 1.40 . It can be seen that the frequency spectra change qualitatively as η is increased: the spectrum is dominated by a single frequency indi-

cating regular oscillation when $\eta/\kappa = 0.99$ (see inset in Fig. 4), while there is a much broader spectrum suggestive of chaotic behaviour when $\eta/\kappa = 1.08$, and when $\eta/\kappa = 1.40$, the spectrum is composed of a number of distinct frequencies characteristic of quasi-periodic nonlinear oscillations.

In order to further investigate the nonlinear behaviour of the full system, we obtain phase-space plots using the modes

$$c_n(t) = \frac{1}{L_D} \int_{-L_D/2}^{L_D/2} \psi(x, t) e^{-i\pi n x} dx, \quad (7)$$

where L_D is the length of the BEC in units of x , i.e., $L_D = 2L_{BEC}/L$, where L_{BEC} is the length of the BEC. Using the conjugate pair $X = -L_D\sqrt{8N}\text{Re}(c_0(c_1^* + c_{-1}^*)/2)$ and $P = L_D\sqrt{8N}\text{Im}(c_0(c_1^* + c_{-1}^*)/2)$, we are then able to plot the system's evolution in phase space (Fig. 5). By constructing Poincaré sections of the phase-space orbits, we are then able to evaluate the true periodicity of the system. Poincaré sections are evaluated at the zero-crossing of the real part of the first motional state of the condensate, c_1 . The sections resulting from this approach corresponding to the orbits of Fig. 5 are provided in Fig. 6.

At very low pump intensities ($\eta/\kappa < 0.10$ for our parameters), the system displays periodic behaviour in both the photon number $|\alpha|^2$ and the BEC dynamics, as shown by the Poincaré sections (which correspond to a single point). For values of η/κ above 0.33, the system enters the nonlinear regime (consider, for example, the resonance curve given in Fig. 2 for $\eta/\kappa = 0.64$). For values of η/κ above this, but below $\eta/\kappa = 1.07$, quasi-periodic behaviour is apparent in the Poincaré sections (cf. the innermost orbit of Fig. 5 with the corresponding projection in Fig. 6). We note however that, even for values of η/κ as large as 0.99, this quasi-periodicity is barely visible in the oscillations of the **cavity-field** intensity, as demonstrated by the inset in Fig. 3a.

Just above $\eta/\kappa = 1.07$, the system evolution approaches the separatrix that separates the motion confined around a single stationary point to orbits that circle around all three stationary points. The unstable separatrix is embedded in the orbit projection displayed in Fig. 5 for $\eta/\kappa = 1.08$ where weakly chaotic motion is observed. The critical value of η/κ corresponding to the cross-over between confined and extended orbits in the (X, P) plane (i.e., when the initial condition is exactly located on the separatrix) is estimated from the condition of tangency of the resonance curves, with respect to the detuning Δ_c , to be at $\eta/\kappa \approx 1.16$ (see inset of Fig. 2). **From our numerical simulations, we observe a first interval of chaotic motion for η/κ in the range 1.07–1.21, which includes the critical value of tangency. Since Eqs. (3)–(4) conserve the volume in phase space (see Section 2), the chaotic motion is “conservative” in nature, in contrast to dissipative chaos.**

We are now able to explain the evolution of the photon number $|\alpha|^2$ with increasing values of η as displayed in

Fig. 3. For values of $\eta/\kappa < 1.07$, such as that of Fig. 3a, $|\alpha|^2$ displays a clear (quasi-)periodic evolution. Near the separatrix (e.g., at $\eta/\kappa = 1.08$, as in Fig. 3b), chaotic oscillations in the photon number become apparent; this behaviour is characteristic of conservative chaos. Finally for values beyond the separatrix region, the oscillation of the photon number is often quasi-periodic or locked between the two competing frequencies such as that used in Fig. 3c.

For a BEC-cavity system where the scattering length $a \approx 0$ (e.g., if operating close to a Feshbach resonance), we observe after the separatrix chaos an alternation of quasi-periodic and chaotic behaviours (see the Poincaré sections in Figs. 6a and 6b). The appearance and disappearance of chaotic dynamics is due to successive transitions from quasi-periodic motion to frequency lockings, and vice-versa, typical of the quasi-periodic route to chaos [20].

3.3. Spatial behaviour of the BEC

In addition to the (temporal) frequency spectra of the previous section, we also consider the time-averaged spatial spectra of the BEC wave function/order parameter, $\Psi(x, t)$, as shown in Fig. 7 for $\eta/\kappa = 0.99, 1.08$, and 1.40 corresponding to the phase-space plots of Fig. 5.

The spatial frequency components c_n with spatial periods $\lambda/2n$ are calculated from the integration of the full model equations using Eq. (7). The amplitude of the zero-frequency (spatially uniform) component of the BEC, c_0 , dominates the amplitudes of the non-zero spatial-frequency components for all the parameter ranges considered here (see for example the cases of Fig. 7). The significance of the additional spatial modes varies depending on the value of η . In Fig. 7a, where $\eta/\kappa = 0.99$, the dynamics is dominated by the first spatial mode/motional state of the BEC, i.e., the component of the BEC with spatial period equal to $\lambda/2$, with the second spatial harmonic having an amplitude that is smaller by three orders of magnitude. Higher harmonics have negligible amplitude. Fig. 7b, where $\eta/\kappa = 1.08$, shows the presence of additional spatial modes while the relative amplitude of each spatial frequency component to the next is reduced compared to Fig. 7a. This indicates that the spatial behaviour of the BEC is now more complex due to the chaotic motion. The distribution of excited spatial modes in Fig. 7c, where $\eta/\kappa = 1.40$, is similar to that of Fig. 7b, whose peaks are indicated on Fig. 7c for comparison. While the dominant spatial modes have now become even more significant, it should be noted that the highest frequency component visible in the spectrum of Fig. 7b at ± 12.5 is now absent, indicating a simpler dynamical regime than that of Fig. 7b.

There are two important facts about the dynamics of the modal amplitudes displayed in Fig. 7. First, we always observe symmetry between the averaged positive and negative n components with equal $|n|$. Second, al-

FIG. 3: (Colour online.) The temporal evolution of the photon number $|\alpha|^2$, for $\eta/\kappa = 0.99$ (Fig. (a), **where the upper panel is a magnification of the top part of the oscillatory trajectory**), 1.08 (Fig. (b)), and 1.40 (Fig. (c)), showing weakly quasi-periodic, chaotic, and quasi-periodic oscillations, respectively. **Note that all quantities are dimensionless (see text before Eq. (3)).**

though the dynamics for values of η/κ larger than one are quite complex, the exponential decrease of the mode amplitude with increasing $|n|$ demonstrates that the number of active modes is always finite and limited to less than nine.

FIG. 4: (Colour online.) Frequency spectra for $\eta/\kappa = 1.08$ (black, thick line) and 1.40 (blue, **dashed line**), with inset showing $\eta/\kappa = 0.99$ (red, thick line) alongside $\eta/\kappa = 1.40$ (**blue, dashed line**) in greater detail (note the leftward frequency shift for the higher value of η/κ). The spectra for $\eta/\kappa = 0.99$ and 1.40 both exhibit characteristic quasi-periodic behaviour, while the broadened spectrum for $\eta/\kappa = 1.08$ indicates chaos. **Quantities are dimensionless.**

FIG. 5: (Colour online.) Phase-space orbits for $\eta/\kappa = 0.99$ (blue, innermost orbit), 1.08 (black, **figure-of-eight** orbit), and 1.40 (red, outermost orbit), showing regular oscillations, (weakly) chaotic dynamics, and quasi-periodic behaviour, respectively. **Quantities are dimensionless.**

4. WIDESPREAD CHAOS IN A REPULSIVE BEC

In Section 3 it was shown that when atom-atom interactions/collisions were negligible the BEC-cavity mode interaction gave rise to chaotic dynamics for certain values of the pumping rate η/κ . Reducing the magnitude of the interaction term is possible experimentally by either moving closer to a Feshbach resonance or by manipulating the alignment of the condensate with respect to the cavity axis, thus reducing the cross-section of the condensate. Although this reduces the effect of atom-atom

FIG. 6: (Colour online.) (a) Poincaré sections constructed from the orbits shown in Fig. 5, for the values $\eta/\kappa = 0.99$ (blue crosses), 1.08 (black circles), and 1.40 (red squares). (b) Poincaré sections for $\eta/\kappa = 2.1$ (blue crosses), 2.4 (black circles), and 3.0 (red squares). **Quantities are dimensionless.**

interactions on the one-dimensional model, typical values of the scattering length in ^{87}Rb show that the magnitude of the interaction term can still be significant: for our parameters, $\Lambda = 85.5$ for a transversely aligned BEC, compared to $\Lambda = 204$ for a longitudinally aligned condensate.

The significance of atom-atom interactions is too great to be neglected (see, e.g., [10]); as mentioned in Sec. 1, for example, recent experiments have shown that the resonance of the BEC-cavity system is altered by the presence of atom-atom interactions. In this section, we therefore extend our analysis of the BEC-cavity's chaotic dynamics to include repulsively interacting condensates ($\Lambda > 0$) and show that chaotic dynamics becomes a ubiquitous feature of the BEC-cavity interaction. Poincaré sections for representative instances of repulsive condensates are used to indicate the presence of wide-ranging chaotic behaviour, while, for the purposes of quantifying the degree of chaos exhibited by each of the two alignments, we eval-

uate the widths along the X -axis of the orbits in the phase space (X, P) , such as those shown in Fig. 5. To do this, we record the thicknesses at the negative-to-positive zero crossing of P (this ensures that we record only the leftmost part of the phase-space orbit). As the measured widths of the plots increase, so too does the spread of the data points in the corresponding Poincaré sections. Transient behaviours have been discarded. In order to focus on the main features, we restrict our consideration to values of η/κ over the range 0.90–4.10; that is, only the quasi-periodic and chaotic domains are considered.

Fig. 6 shows the Poincaré sections for three different values of pump intensity η , exhibiting the variety of behaviours for the non-interaction model. Similarly, we now evaluate the sections for both the transverse ($\Lambda = 85.5$) and longitudinal ($\Lambda = 204$) BEC alignments. For $\Lambda > 0$, the onset of chaos is attained at larger values of η with respect to the non-interaction case: where previously the onset of chaotic dynamics was apparent from $\eta/\kappa \approx 1.07$, we find that it is now shifted upwards to $\eta/\kappa \approx 1.47$ ($\eta/\kappa \approx 1.88$) for $\Lambda = 85.5$ ($\Lambda = 204$). With η adjusted accordingly, we obtain Fig. 8 ($\Lambda = 85.5$) and Fig. 9 ($\Lambda = 204$).

The sections for the two alternative alignments both show that chaos is now a predominant feature of the interaction dynamics. As in Fig. 6, the section below the onset of chaotic dynamics displays quasi-periodicity. Beyond this however, we find that the behaviour is exclusively chaotic, with the two largest values of η/κ in each of Figs. 8 and 9 showing a spread in the distribution of points that increases in a roughly linear fashion with increasing η .

Beyond a critical value of η , the widths of the phase plots, given in Figs. 10a–c, tend to increase in a roughly linear fashion with η for $\Lambda > 0$, while for the model neglecting atom-atom interactions ($\Lambda = 0$), the onset of chaotic behaviour is sporadic. It should be apparent that, for this system, chaotic motion cannot be neglected even when close to a Feshbach resonance.

As an additional measure of chaoticity, we also calculate the number of temporal frequency components of the photon-number evolution that exceed a fixed threshold value. For small values of η , where a minimal number of components stray above the cut-off point, there is good agreement between the plots of the spectral components and the orbit widths. In fact, the plot of the spectral components in the cavity field, for $\Lambda = 0$, is in good agreement with the orbit widths up to $\eta/\kappa \approx 3.2$. For $\Lambda > 0$, there are two main domains of behaviour, viz. before and after the respective critical points. From the onset of chaos at the critical points of $\eta/\kappa = 1.47$ (1.88) for $\Lambda = 85.5$ (204), a substantial increase in the number of high-magnitude components is apparent.

The widespread presence of nonlinear chaotic motion when $\Lambda > 0$ may appear at odds with recent predictions of decreasing regions of bistability when increasing atom-atom interaction strengths [10]. However, we also find that bistable regions shrink for $\Lambda > 0$ as depicted in

FIG. 7: (Colour online.) Averaged spatial spectra for $\eta/\kappa = 0.99, 1.08$, and 1.40 , **showing the amplitudes of the spatial frequency components c_n** . Fig. (a) indicates the two main modes that describe the BEC wave function. In (b), additional modes are apparent, and the prominence of the main mode is lessened as a consequence of the increased influence of the new modes. Once we reach $\eta/\kappa = 1.40$, we see that the highest-frequency mode apparent in (b) has disappeared. **Quantities are dimensionless.**

FIG. 8: (Colour online.) Phase-space orbits for a repulsive BEC ($\Lambda = 85.5$) when $\eta/\kappa = 1.43$ (blue crosses), 1.77 (black circles), and 2.22 (red squares), showing quasi-periodic oscillations and chaotic dynamics. **Quantities are dimensionless.**

FIG. 9: (Colour online.) Phase-space orbits for a repulsive BEC ($\Lambda = 204$) when $\eta/\kappa = 1.80$ (blue crosses), 2.27 (black circles), and 2.84 (red squares), showing quasi-periodic oscillations and chaotic dynamics. **Quantities are dimensionless.**

Figs. 11 and 12, in agreement with [10]. We have extrapolated that for high enough values of Λ , the bistability will disappear. As shown in Fig. 11, for a fixed value of η , the bistable region is decreased as Λ is increased; for an interaction strength of $\Lambda = 204$, corresponding to an axially aligned condensate, there is no bistability from $\eta/\kappa \approx 0.34$ (the bistability threshold for the $\Lambda = 0$ case) up to $\eta/\kappa \approx 0.63$. In an environment where the interaction strength can be tuned (e.g., by manipulating the scattering length near a Feshbach resonance), this provides the opportunity to use the BEC-cavity system as an **optical switch** [21]. This mechanism could potentially be used instead of, or in addition to, other switching

methods such as the introduction of a transverse beam above a threshold intensity [22] or the addition of a Kerr medium into the cavity [23].

5. REDUCED MODEL OF THE BEC-CAVITY NONLINEAR DYNAMICS

The results presented in Secs. 3 and 4 show that the dynamics of the BEC-cavity interaction as predicted by the full model represented by Eq. (3) is rich and complex. However, the spatial spectra shown in Fig. 7 suggest that even when the dynamics is highly nonlinear

FIG. 10: (Colour online.) Alternative approaches for quantifying the chaotic dynamics of the BEC-cavity system. The plots of (a)–(c) show the thickness of phase-space portraits, such as those from Fig. 5, for a range of values of η ; plots (d)–(f) show for the same range the number of components in the power spectra for $|\alpha|^2$ (normalised to unity) with a magnitude greater than 10^{-3} . Quantities are dimensionless.

FIG. 11: (Colour online.) The width of the region of bistability for both the non-interaction case (red dashed curve) and the two alternative alignments, for $\Lambda = 85.5$ (black middle curve) and $\Lambda = 204$ (blue dotted curve). Quantities are dimensionless.

FIG. 12: (Colour online.) The width of the bistable region for increasing Λ , for values of the parameter $\eta/\kappa = 0.44$ (bottom dotted curve and blue crosses), 0.72 (middle curve and black circles), and 1.00 (top dashed curve and red squares). Quantities are dimensionless.

and chaotic, the spatial dynamics of the BEC is relatively simple, being dominated by only a few spatial frequencies, which are integer multiples of the “lattice wavenumber” $k_L = 4\pi/\lambda$. In several previous studies (e.g., [6, 7, 16]), a highly simplified model involving only a spatially uniform BEC plus a component with spatial period $\lambda/2$ was used to describe the BEC-cavity dynamics. This model, although a dramatic simplification of the spatial description of the BEC, was still capable of describing some of the oscillatory and nonlinear features of the system behaviour, e.g., the bistability shown in Fig. 2 [4]. It should be noted, however, that it does not reproduce other important aspects of the system’s nonlinear dynamics, e.g., chaos.

The spatial spectra shown in Fig 7 suggest that the simplified model of [4, 6, 7] would be inadequate to describe the BEC-cavity system when, e.g., $\eta/\kappa = 1.08$, as higher-order spatial frequency components attain a sufficiently large amplitude to have a significant influence on the system dynamics. The presence of a limited number of spatial modes, however, suggests that the dynamics of the BEC remains confined to a small number of variables even when the system is chaotic. Here we derive a simplified model of the BEC-cavity interaction using a projection onto a number of spatial modes, which is capable not only of reducing to the model of [6], but also of describing chaos and including atom-atom interactions. Our model shows that, in contrast to the $\Lambda = 0$ case where the dynamics is mostly quasi-periodic, in cases where $\Lambda > 0$, chaos is ubiquitous beyond a critical input value. Furthermore, our model gives an alternative — deterministic — account for previously experimentally observed intermittent oscillations [6] attributed to stochastic detector noise.

5.1. Modal expansion

For our reduced model, we assume that the BEC wave function is periodic on the scale of the lattice period $\lambda/2$ and can therefore be well-described by a finite series of $N_{\max}$ spatial modes

$$\psi(x, t) = \frac{1}{\sqrt{n_L}} \sum_{n=-N_{\max}}^{N_{\max}} c_n(t) e^{in\pi x}, \quad (8)$$

where $n_L = L_{\text{BEC}}/(\lambda/2)$ is the number of lattice sites occupied by the BEC. Substituting $\psi(x, t)$ as defined in Eq. (8) into Eq. (3) we obtain

$$\begin{aligned} \frac{dc_n}{dt} = & -i \frac{n^2 \pi^2}{2} c_n - i \frac{|\alpha|^2 V_0}{2} \left(c_n - \frac{c_{n-1}}{2} - \frac{c_{n+1}}{2} \right) \\ & - i \Lambda' \sum_{k, m=-N_{\max}}^{N_{\max}} c_k c_m^* c_{n+m-k} \end{aligned} \quad (9)$$

where $\Lambda' = \Lambda/n_L$, while the overlap integral, $\mathcal{O}'$ in Eq. (4), becomes

$$\begin{aligned} \mathcal{O}' &= \left\langle \psi \left| \sin^2 \left(\frac{\pi x}{2} \right) \right| \psi \right\rangle \\ &= \frac{1}{2} \left[1 - \sum_{n=-N_{\max}}^{N_{\max}} c_n^* (c_{n-1} + c_{n+1}) \right]. \end{aligned} \quad (10)$$

In what follows, we investigate cases that involve truncating the modal expansion (i.e., reducing the value of $N_{\max}$), and comparing the predictions of these reduced models with the results from the full model presented above, in order to determine the level of model complexity and dimension required to capture the relevant features of the interaction dynamics.

The model of [6], which involves only a spatially uniform BEC component plus a component with spatial period $\lambda/2$, is recovered by imposing $N_{\max} = 1$ in Eqs. (8)–(10), $c_{-1} = c_1$ and $|c_0|^2 \gg |c_1|^2$. While this $N_{\max} = 1$ model is capable of describing some of the nonlinear features of the system behaviour when the BEC is non-interacting ($\Lambda = 0$), such as the bistability shown in Fig. 2 and the regular oscillations in Fig. 3 (cf. [6]), it cannot show chaotic dynamics. As shown in Sec. 3.2, chaos does occur for a non-interacting BEC, and is a dominant feature of the nonlinear dynamics in a repulsive condensate ($\Lambda > 0$) (see Sec. 4). It is then necessary to go beyond the $N_{\max} = 1$ case in order to describe the full nonlinear dynamics of the BEC-cavity interaction. We consequently show that extension of the spatial-mode model to five modes (i.e., $N_{\max} = 2$, so that $n = 0, \pm 1, \pm 2$) gives good agreement with the results of the full model of Eq. (3), even in cases where the system dynamics is chaotic.

5.2. The three-mode and the two-mode models ($n = 0, \pm 1$)

A key feature of the modal expanded equations Eqs. (8)–(10) is symmetry. In the case of three modes $n = 0, \pm 1$, for example, the six dynamical equations are invariant upon the exchange of the $n = \pm 1$ indices. When writing the modal equations in amplitude and phase by using $c_n = \mu_n \exp(i\phi_n)$ the index symmetry is reflected in the evolution of the difference of the modal amplitudes

FIG. 13: (Colour online.) Projection of the three-dimensional evolution of system (13)–(15) on the (X, D) plane showing a clear two-dimensional dynamics and no chaotic motion. The parameters are: top (red) curve ($\Lambda' = 0, \eta/\kappa = 1.2$), intermediate (black) curve ($\Lambda' = 4.275, \eta/\kappa = 1.8$) and bottom (blue) curve ($\Lambda' = 10.2, \eta/\kappa = 2.5$). **Quantities are dimensionless.**

FIG. 14: (Colour online.) Phase-space portraits as calculated from the five-mode model (Eq. (8)) when $N_{\max} = 2$ when a non-interacting condensate ($\Lambda = 0$) for the values $\eta/\kappa = 0.99$ (blue, innermost orbit), 1.08 (black, **figure-of-eight** orbit), 1.40 (red, outermost orbit). See Fig. 5 for a comparison with the results of the full model. **Note that quantities are dimensionless.**

and that of the difference of the modal phases

$$\frac{d(\mu_1 - \mu_{-1})}{dt} = \frac{\mu_0 |\alpha|^2 V_0}{2} \sin\left(\frac{\phi_1 - \phi_{-1}}{2}\right) \quad (11)$$

$$\begin{aligned} & \cos\left(\frac{\phi_1 + \phi_{-1} - 2\phi_0}{2}\right) \\ & + \Lambda' \mu_0^2 (\mu_1 - \mu_{-1}) \sin(\phi_1 + \phi_{-1} - 2\phi_0) \\ \frac{d(\phi_1 - \phi_{-1})}{dt} & = \frac{\mu_0 |\alpha|^2 V_0}{4} \\ & \left(\frac{\cos(\phi_1 - \phi_0)}{\mu_1} - \frac{\cos(\phi_{-1} - \phi_0)}{\mu_{-1}} \right) \\ & + \Lambda' (\mu_1^2 - \mu_{-1}^2) - \Lambda' \mu_0^2 \left(\frac{\mu_{-1}}{\mu_1} - \frac{\mu_1}{\mu_{-1}} \right) \\ & \cos(\phi_1 + \phi_{-1} - 2\phi_0). \end{aligned} \quad (12)$$

These equations demonstrate the existence of an invariant manifold since for initial conditions satisfying $c_{-1} = c_1$ there is no evolution of the differences of their phases and their amplitudes so that $c_{-1} = c_1$ remains valid at all times. The invariant manifold is then characterised by the dynamics of only two modes, e.g., c_0 and c_1 . The dynamics outside this invariant manifold corresponding to non-trivial initial conditions is outside the scope of this paper and will be discussed elsewhere.

By using again the decomposition in amplitudes and phases, it is also possible to demonstrate that there are only three coupled dynamical variables, the amplitudes μ_0 and μ_1 and the phase difference $\Delta = \phi_1 - \phi_0$. The evolution of the phase sum is affected by these three variables but does not, in turn, affect their dynamics. Since the three equations for μ_0 and μ_1 and Δ are complemented by the conservation condition of $|c_0|^2 + 2|c_1|^2 = 1/2$, the dynamical variables reduce from the initial six to just two variables, and chaotic dynamics becomes impossible. Note that this conclusion is slightly different from the implementation made in [4, 6] where a reduction of coupled variables is attained under the approximation $|c_0|^2 \equiv 1/2$, also used in [8]. To illustrate the two-dimensional nature of the three-mode dynamics on the invariant manifold, we have introduced the variables $R = \text{Re}(c_0 c_1^*)$, $I = \text{Im}(c_0 c_1^*)$, in agreement with [4, 6], and $D = |c_0|^2 - 2|c_1|^2$ to obtain

$$\frac{dR}{dt} = I \left[-\frac{\pi^2}{2} + \frac{\Lambda'}{4} \left(D - \frac{1}{2} \right) \right] \quad (13)$$

$$\frac{dI}{dt} = \frac{\pi^2}{2} R - \frac{|\alpha|^2 V_0 D}{4} + \frac{\Lambda' R}{4} \left(7D + \frac{1}{2} \right) \quad (14)$$

$$\frac{dD}{dt} = 2I (|\alpha|^2 V_0 - 8\Lambda' R) \quad (15)$$

where α of Eq. (4) now contains $\mathcal{O}' = 1/2 - 2R$. Note that the condition of the conservation of the total BEC mass has now been entered into the dynamical equations **and that the divergence of the flow**

$$\frac{d\dot{R}}{dR} + \frac{d\dot{I}}{dI} + \frac{d\dot{D}}{dD} = 0, \quad (16)$$

FIG. 15: (Colour online.) Poincaré sections as calculated from the mode model (Eq. (8)) when $N_{\max} = 2$ for the case of a repulsive condensate where $\Lambda' = 4.275$, for $\eta/\kappa = 1.43$ (blue crosses), 1.77 (black circles), and 2.22 (red squares). See Fig. 8 for a comparison with the results of the full model. Quantities are dimensionless.

as required by conservative dynamical systems. In Fig. 13 from top to bottom, we show the results of the numerical integration of Eqs. (13)–(15) with $X = -2\sqrt{2N}R$ for $(\Lambda' = 0, \eta/\kappa = 1.2)$, $(\Lambda' = 0.4275, \eta/\kappa = 1.8)$ and $(\Lambda' = 10.2, \eta/\kappa = 2.5)$. The two-dimensional structure of the invariant manifold is apparent. Note that for all these parameter values the simulation of the full equations displays chaotic motion (see Fig. 10). We can then conclude that the three-mode and two-mode models with initial conditions on the invariant manifold cannot properly describe the nonlinear interaction of the BEC with light in an optical cavity.

5.3. The five-mode model ($n = 0, \pm 1, \pm 2$)

We now consider the dynamics generated by the coupling of five modes by using Eq. (8) with $N_{\max} = 2$. The symmetry of the system reduces again the total number of coupled variables in the case of initial conditions satisfying $c_{-1} = c_1$ and $c_{-2} = c_2$ since the equations are invariant upon exchange of the ± 1 and ± 2 indices. In the five-mode case, however, it is possible to demonstrate that all phase differences are coupled with each other thus leaving a final five-dimensional system when including the mass-conservation rule. A five-dimensional nonlinear system can display quasi-periodicity and chaos.

Fig. 14 shows phase-space plots for the five-mode model with initial conditions satisfying $c_{-1} = c_1$ and $c_{-2} = c_2$. There is excellent agreement between this figure and the corresponding results from the full model (Fig. 5). In particular, we note a remarkable agreement for $\eta/\kappa = 1.08$ where the system is close to a separatrix

FIG. 16: (Colour online.) Poincaré sections as calculated from the mode model (Eq. (8)) when $N_{\max} = 2$ for the case of a repulsive condensate where $\Lambda' = 10.2$, for $\eta/\kappa = 1.80$ (blue crosses), 2.27 (black circles), and 2.84 (red squares). See Fig. 9 for a comparison with the results of the full model. Quantities are dimensionless.

and weakly chaotic motion occurs around the unstable fixed point around $(X = -4.5, P = 0)$.

Further evidence of the accuracy of the nonlinear dynamics of the five-mode model is provided in Figs. 15 and 16, where the Poincaré sections for values of the parameter Λ different from zero are presented. These figures should be compared respectively with Figs. 8 and 9 obtained by the numerical integration of the full model.

To complete the comparison between the full and the five-mode models we present in Fig. 17 the measurement of the width of the Poincaré sections performed on the five-mode model. When compared with similar measurements performed on the full model and displayed in Fig. 10, we can conclude that the five-mode model captures well all nonlinear features of relevance, from the bistable and separatrix behaviour to the quasi-periodic and chaotic dynamics.

6. CONCLUSION

We have demonstrated that a BEC enclosed in an optical cavity displays a variety of nonlinear behaviours, including bistability, regular and quasi-periodic oscillations, and chaotic dynamics. In particular we have shown that chaotic oscillations are ubiquitous when atom-atom interactions are considered beyond a critical value of the cavity pumping rate. Irregular oscillations observed in the light output of the system can be purely deterministic in nature, arising solely from the nonlinear nature of the BEC-cavity interaction. In more general terms, the results presented here represent a novel example of quantum matter exhibiting chaotic behaviour and invites fur-

FIG. 17: (Colour online.) Measurement of the orbit widths taken from the integration of the five-mode model. In each of the three cases, the results depict behaviour in agreement with that shown for the three values of Λ (i.e., 0, 85.5, and 204) given in Fig. 10. **Quantities are dimensionless.**

ther research into the nonlinear dynamics of macroscopic quantum systems.

Although the results from our analysis suggest that the behaviour of the BEC-cavity system is rather complex, it is remarkable that its dynamics are low-dimensional and describable by a simplified model involving as few as five BEC spatial modes/motional states. Our results suggest that when starting from a homogeneous state, the light-matter interaction remains confined to a few spatial modes of progressively smaller and smaller amplitude. The situation may be different if one considers initial conditions away from the homogeneous state, possibly leading to localisation and solitonic states. This will be the subject of future investigations.

Further investigations stemming from the work pre-

sented here concern the introduction of a modulated pump laser field. Our aim is to study further the behaviour of condensates in disordered and chaotic regimes and investigate novel nonlinear and quantum phenomena involving both long-range (cavity-mediated) and short-range (collision-mediated) interactions.

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