Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 854743, 12 pages
http://dx.doi.org/10.1155/2013/854743

Research Article

Razumikhin-Type Theorems on Exponential Stability of SDDEs Containing Singularly Perturbed Random Processes

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Received 13 December 2012; Revised 25 April 2013; Accepted 25 April 2013

Academic Editor: Chuangxia Huang

This paper concerns Razumikhin-type theorems on exponential stability of stochastic differential delay equations with Markovian switching, where the modulating Markov chain involves small parameters. The smaller the parameter is, the rapider switching the system will experience. In order to reduce the complexity, we will "replace" the original systems by limit systems with a simple structure. Under Razumikhin-type conditions, we establish theorems that if the limit systems are pth-moment exponentially stable; then, the original systems are pth-moment exponentially stable in an appropriate sense.

1. Introduction

The stability of time delay systems is a field of intense research [1, 2]. In [2], the global uniform exponential stability independent of time delay linear and time invariant systems subjected to point and distributed delays was studied. Moreover, noise and time delay are often the sources of instability, and they may destabilize the systems if they exceed their limits [3].

Hybrid delay systems driven by continuous-time Markov chains have been used to model many practical systems in which abrupt changes may be experienced in the structure and parameters caused by phenomena such as component failures or repairs. An area of particular interest has been the automatic control of the underlying systems, with consequent emphasis on the analysis of stability of the stochastic models. For systems with time delay, there are two approaches to proving stability that correspond to the conventional Lyapunov stability theory. The first is based on Lyapunov-Krasovski functionals, the second on Lyapunov-Razumikhin functions. The latter one originated with Razumikhin [4] for the ordinary differential delay equation which is called Razumikhin-type theorem and was developed by several people [5]. In his paper, Mao [6] was the first who established a Razumikhin-type theorem for stochastic functional differential equations (SFDEs). Roughly speaking, a Razumikhin-type theorem states that if the derivative of a Lyapunov function along trajectories is negative whenever the current value of the function dominates other values over the interval of time delay; then, the Lyapunov function along trajectories will converge to zero. The Razumikhin methods have been widely used in the study of stability for functional and differential-delay systems. In this work, we shall investigate stochastic differential delay equations with Markovian switching (SDDEwMSs). The switching we shall use will be a finite-state Markov chain, which incorporates various considerations into the models and often results in the underlying Markov chain having a large state space. To overcome the difficulties and to reduce the computational complexity, much effort has been devoted to the modeling and analysis of such systems, in which one of the main ideas is to split a large-scale system into several classes and lumping the states in each class into one state; see [7–9]. Starting from the work [10], by introducing a small parameter $\varepsilon > 0$, a number of asymptotic properties of the Markov chain $r^{\varepsilon}(\cdot)$ have been established. One of the main results in [9] is that a complicated system can be replaced by the corresponding limit system having a much simpler structure. In [11, 12], long-term behavior of SDEwMSs and SDDEwMSs was investigated, respectively, while in [13, 14] the stability of random delay system with two-time-scale Markovian switching was studied. Using the stability of the limit system as a bridge, the desired asymptotic properties of the original system is obtained using perturbed Lyapunov function methods. In this work, we shall establish a Razumikhin-type theorem for SDDEwMSs.

The remainder of this work is organised as follows: in the next section, we shall begin with the formulation of the problem. Section 3 investigates the Razumikhin-type theorem for SDDEs driven by Brownian motion. The exponential stability for SDDEs driven by pure jumps is discussed in Section 4.

2. Preliminaries

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous, and $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets). Throughout the paper, we let $B(t) = (B_1(t), \dots, B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$. If A is a vector or matrix, its transpose is denoted by A^T. Let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^n$ as well as the trace norm of a matrix. For $\tau > 0$, $C([-\tau, 0]; \mathbb{R}^n)$ denotes the family of continuous functions from $[-\tau, 0]$ to $\mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{-\tau\leq\theta\leq 0}|\varphi(\theta)|$. Denote by $C^b_{\mathscr{F}}([-\tau, 0]; \mathbb{R}^n)$ the family of all $\mathscr{F}$ measurable and bounded $C([-\tau, 0]; \mathbb{R}^n)$-valued random variable. We will denote the indicator function of a set G by I_G.

Let $r(t)$ $(t \geq 0)$ be a right-continuous Markov chain on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ taking values in a finite state space $\mathbb{S} = \{1, 2, \dots, N\}$ with the generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t+\delta) = j \mid r(t) = i\} = \begin{cases} \gamma_{ij}\delta + \circ(\delta), & \text{if } i \neq j, \\ 1 + \gamma_{ii}\delta + \circ(\delta), & \text{if } i = j, \end{cases} \tag{1}$$

where $\delta > 0$ and γ_{ij} is the transition rate from i to j satisfying $\gamma_{ij} > 0$ if $i \neq j$ and $\gamma_{ii} = -\sum_{i\neq j}\gamma_{ij}$. We assume the Markov $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$. It is well known that almost every sample path $r(\cdot)$ is a right-continuous step function with finite number of simple jumps in any finite subinterval of $\mathbb{R}_+ := [0, \infty)$. As a standing hypothesis, we assume that the Markov chain is irreducible. This is equivalent to the condition that for any $i, j \in \mathbb{S}$, we can find $i_1, i_2, \dots, i_k \in \mathbb{S}$ such that

$$\gamma_{i,i_1}\gamma_{i_1,i_2}\cdots\gamma_{i_k,j} > 0. \tag{2}$$

Thus, Γ always has an eigenvalue 0. The algebraic interpretation of irreducibility is $\operatorname{rank}(\Gamma) = N - 1$. Under this condition, the Markov chain has a unique stationary (probability) distribution $\pi\Gamma = 0$, subject to $\sum_{j=1}^N \pi_j = 1$ and $\pi_j > 0$ for all $j \in \mathbb{S}$. For a real valued function $\sigma(\cdot)$ defined on $\mathbb{S}$, we define

$$\begin{aligned}\Gamma\sigma(\cdot)(\kappa) &:= \sum_{\ell\in\mathbb{S}}\gamma_{\kappa\ell}\sigma(\ell) \\ &= \sum_{\ell\neq\kappa}\gamma_{\kappa\ell}(\sigma(\ell) - \sigma(\kappa)),\end{aligned} \tag{3}$$

for each $\kappa \in \mathbb{S}$.

Consider the following stochastic delay system with Markovian swtching:

$$\begin{aligned} dx(t) &= f(x(t), x(t-\tau), r(t))\,dt \\ &\quad + g(x(t), x(t-\tau), r(t))\,dB(t), \\ x_0 &= \xi \in C([-\tau, 0]; \mathbb{R}^n), \quad r(0) \in \mathbb{S}, \end{aligned} \tag{4}$$

where $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^{n\times m}$.

To highlight the fast and slow motions, we introduce a parameter $\varepsilon > 0$ and rewrite the Markov chain $r(t)$ as $r^\varepsilon(t)$ and the generator Γ as Γ^ε. Γ^ε is given by

$$\Gamma^\varepsilon = \frac{1}{\varepsilon}\widetilde{\Gamma} + \widehat{\Gamma}, \tag{5}$$

where $\widetilde{\Gamma}/\varepsilon$ represents the fast varying motions, and $\widehat{\Gamma}$ represents the slowly changing dynamics. We denote $\Gamma^\varepsilon = (\gamma^\varepsilon_{ij})_{N\times N}$, $\widetilde{\Gamma} = (\widetilde{\gamma}_{ij})_{N\times N}$, and $\widehat{\Gamma} = (\widehat{\gamma}_{ij})_{N\times N}$. To the reduction of complexity, $\widetilde{\Gamma}$ needs to have a certain structure. Suppose that

$$\mathbb{S} = \mathbb{S}^1 \cup \mathbb{S}^2 \cup \cdots \cup \mathbb{S}^l \tag{6}$$

with $\mathbb{S}^i = \{s_{i1}, \dots, s_{iN_i}\}$ and $N = N_1 + N_2 + \cdots + N_l$, and that

$$\widetilde{\Gamma} = \operatorname{diag}\left(\widetilde{\Gamma}^1, \dots, \widetilde{\Gamma}^l\right), \tag{7}$$

where for each $k \in \{1, \dots, l\}$ and $\widetilde{\Gamma}^k$ is a generator of a Markov chain taking values in $\mathbb{S}^k$. We impose the following hypothesis:

(H1) For each $k \in \{1, \dots, l\}$, $\widetilde{\Gamma}^k$ is irreducible.

To highlight the effect of the fast switching, we rewrite the system (4) as

$$\begin{aligned} dx^\varepsilon(t) &= f\left(x^\varepsilon(t), x^\varepsilon(t-\tau), r^\varepsilon(t)\right)dt \\ &\quad + g\left(x^\varepsilon(t), x^\varepsilon(t-\tau), r^\varepsilon(t)\right)dw(t), \\ x^\varepsilon_0 &= \xi \in C([-\tau, 0]; \mathbb{R}^n), \quad r^\varepsilon = r_0. \end{aligned} \tag{8}$$

To assure the existence and uniqueness of the solution, we give the following standard assumptions.

(H2) For any integer R, there is a constant $h_R > 0$, such that

$$\begin{aligned} &\left|f(x, y, \kappa) - f(x_1, y_1, \kappa)\right| \vee \left|g(x, y, \kappa) - g(x_1, y_1, \kappa)\right| \\ &\quad \leq h_R\left(\left|x - x_1\right| + \left|y - y_1\right|\right) \end{aligned} \tag{9}$$

for all $\kappa \in \mathbb{S}$ and those $x, x_1, y, y_1 \in \mathbb{R}^n$ with $|x| \vee |x_1| \vee |y| \vee |y_1| \leq R$.

(H3) There is an $h > 0$, such that for any $x, y \in \mathbb{R}^n$, $\kappa \in \mathbb{S}$,

$$\begin{gathered} \left|f(x, y, \kappa)\right| \vee \left|g(x, y, \kappa)\right| \leq h\left(1 + |x| + |y|\right), \\ f(0, 0, \kappa) \equiv 0, \qquad g(0, 0, \kappa) \equiv 0. \end{gathered} \tag{10}$$

Under the assumptions (H2) and (H3), system (8) has a unique solution denoted by $x^{\varepsilon,\xi,\ell}(t)$ on $t \geq -\tau$, where the notation $x^{\varepsilon,\xi,\ell}$ emphasizes the dependence on the initial data

(ξ,ℓ). Moreover, for every $p>0$ and any compact subset K of $C([-\tau,0];\mathbb{R}^n)$, there exists a positive constant H which is independent of ε such that

$$\sup_{(\xi,\ell)\in K\times\mathbb{S}} E\left[\sup_{-\tau\le s\le t}\left|x^{\varepsilon,\xi,\ell}(s)\right|^p\right]\le H,\quad \text{on } t\ge 0. \tag{11}$$

We will consider the stability of system (8), but the state space of the Markov chain is large, and it is difficult to handle (8) directly. So we will consider the average system of (8). To proceed, lump the states in each $\mathbb{S}^k$ into a single state and define an aggregated process $\overline{r}^\varepsilon(\cdot)$ as

$$\overline{r}^\varepsilon(t)=k,\quad \text{if } r^\varepsilon(t)\in\mathbb{S}^k. \tag{12}$$

Denote the state space of $\overline{r}^\varepsilon(t)$ by $\overline{\mathbb{S}}=\{1,\dots,l\}$, the stationary distribution $\widetilde{\Gamma}^k$ by $\mu^k=(\mu_1^k,\dots,\mu_{N_k}^k)\in\mathbb{R}^{1\times N_k}$ and $\widetilde{\mu}=\operatorname{diag}(\mu^1,\dots,\mu^l)\in\mathbb{R}^{l\times N}$. Define

$$\overline{\Gamma}=\left(\overline{\gamma}_{ij}\right)_{l\times l}=\widetilde{\mu}\widehat{\Gamma}\mathbf{1} \tag{13}$$

with $\mathbf{1}=\operatorname{diag}(\mathbf{1}_{N_1},\dots,\mathbf{1}_{N_l})$ and $\mathbf{1}_{N_k}=(1,\dots,1)^T\in\mathbb{R}^{N_k\times 1}$, $k=1,\dots,l$. It has been known that $\overline{r}^\varepsilon(\cdot)$ converges weakly to $\overline{r}(\cdot)$ as $\varepsilon\to 0$, where $\overline{r}(\cdot)$ is a continuous-time Markov chain with generator $\overline{\Gamma}$ and state space $\overline{\mathbb{S}}$ (cf. [9]).

Define

$$\overline{f}(x,y,i)=\sum_{j=1}^{N_i}\mu_j^i f\left(x,y,s_{ij}\right), \tag{14}$$

$$\overline{g}(x,y,i)\overline{g}^T(x,y,i)=\sum_{j=1}^{N_i}\mu_j^i g\left(x,y,s_{ij}\right)g^T\left(x,y,s_{ij}\right) \tag{15}$$

for each $s_{ij}\in\mathbb{S}^i$ with $i\in\{1,\dots,l\}$ and $j\in\{1,\dots,N_i\}$. It is easily seen that $\overline{f}(x,y,i)$ and $\overline{g}(x,y,i)$ are the averages with respect to the stationary distribution of the Markov chain. Note that for any $(x,y)\neq(0,0)$, $g(x,y,s_{ij})g^T(x,y,s_{ij})$ are nonnegative definite matrices, so we find its "square root" of (15), which is denoted by $\overline{g}(x,y,i)$. For degenerate diffusions, we can see the argument in [15].

The averaged system of (8) is defined as follows:

$$\begin{aligned} d\overline{x}(t)&=\overline{f}(\overline{x}(t),\overline{x}(t-\tau),\overline{r}(t))\,dt\\ &\quad+\overline{g}(\overline{x}(t),\overline{x}(t-\tau),\overline{r}(t))\,dw(t),\\ \overline{x}_0&=\xi,\qquad \overline{r}=\overline{r}_0. \end{aligned} \tag{16}$$

3. Moment Exponential Stability

In this section, we shall establish the Razumikhin-type theorem on the exponential stability for (8).

Let $C^p(\mathbb{R}^n\times\overline{\mathbb{S}};\mathbb{R}_+)$ be the class of nonnegative real-valued functions defined on $\mathbb{R}^n\times\overline{\mathbb{S}}$ that are p-times continuously differentiable with respect to x. We give the following assumption about $V(x,i)\in C^p(\mathbb{R}^n\times\overline{\mathbb{S}};\mathbb{R}_+)$ for some $p\ge 4$.

(H4) For each $i\in\overline{\mathbb{S}}$, $V(x,i)\to\infty$ as $|x|\to\infty$. Moreover, $\partial^p V(x,i)=O(1)$, $\partial^\ell V(x,i)(|x|^\ell+|y|^\ell)\le K(|x|^p+|y|^p+1)$ for $1\le\ell\le p-1$, where $\partial^\ell V(x,i)$ denotes the ℓth derivative of $V(x,i)$ with respect to x and $O(y)$ denotes the function of y satisfying $\sup_y|O(y)|/y<\infty$.

Theorem 1. *Let (H1)–(H3) hold; there is a function $V(x,i)\in C^p(\mathbb{R}^n\times\overline{\mathbb{S}};\mathbb{R}_+)$ satisfying (H4), and there are positive constants λ, c_1, c_2, and $q>1$ such that*

(i) $c_1|x|^p\le V(x,i)\le c_2|x|^p$,

(ii) $\mathbb{E}[\max_{i\in\overline{\mathbb{S}}}\mathscr{L}V(x(t),x(t-\tau),i)]\le-\lambda\mathbb{E}[\max_{i\in\overline{\mathbb{S}}}V(x(t),i)]$ *provided* $\mathbb{E}[\min_{i\in\overline{\mathbb{S}}}V(x(t+\theta),i)]<q\mathbb{E}[\max_{i\in\overline{\mathbb{S}}}V(x(t),i)]$, $-\tau\le\theta\le 0$,

where

$$\begin{aligned}\mathscr{L}V(x,y,i)&=V_x(x,i)\overline{f}(x,y,i)\\ &\quad+\frac{1}{2}\operatorname{trace}\left[V_{xx}(x,i)\overline{g}(x,y,i)\overline{g}^T(x,y,i)\right]\\ &\quad+\sum_{j=1}^{l}\overline{\gamma}_{ij}V(x,j).\end{aligned} \tag{17}$$

Then, for all $\xi\in C([-\tau,0];\mathbb{R}^n)$,

$$\limsup_{\varepsilon\to 0}\mathbb{E}\left|x^\varepsilon(t)\right|^p\le \nu_2 e^{-\nu_1 t}, \tag{18}$$

where

$$\begin{aligned}\nu_1&=\min\left\{\lambda,\frac{\log q}{\tau}\right\};\\ \nu_2 &\ \textit{is a fixed constant such that}\\ \nu_2&=\frac{c_2}{c_1}\sup_{-\tau\le\theta\le 0}\mathbb{E}|\xi|^p.\end{aligned} \tag{19}$$

Remark 2. Note that the conditions of Theorem 1 are sufficient conditions for the average system (16) $\overline{x}(t)$ (or the limit process $\overline{x}(t)$). However the conclusion of Theorem 1 is about the process $x^\varepsilon(t)$. Since the structure of the the average system (16) is much simpler than that of $x^\varepsilon(t)$, this theorem has reduced the computational complexity for the system (8).

Remark 3. $\limsup_{\varepsilon\to 0}\mathbb{E}|x^\varepsilon(t)|^p$ does exist by (11).

Proof of Theorem 1. Define

$$\overline{V}(x,\zeta)=\sum_{i=1}^{l}V(x,i)I_{\{\zeta\in\mathbb{S}^i\}}=V(x,i),\quad \text{if } \zeta\in\mathbb{S}^i. \tag{20}$$

Note that

$$\begin{aligned}\overline{V}\left(x^\varepsilon(t),r^\varepsilon(t)\right)&=V\left(x^\varepsilon(t),\overline{r}^\varepsilon(t)\right),\\ \sum_{\kappa=1}^{N}\widetilde{\gamma}_{l\kappa}\overline{V}(x,\kappa)&=\sum_{\kappa=1}^{N}\widetilde{\gamma}_{l\kappa}\sum_{i=1}^{l}V(x,i)I_{\{\kappa\in\mathbb{S}^i\}}=0.\end{aligned} \tag{21}$$

We extend $r(t)$ to $[-\tau, 0]$ by setting $r(t) = r(0)$; then, $\mathbb{E}\overline{V}(x^{\varepsilon}(t), r^{\varepsilon}(t))$ is right continuous on $t \geq -\tau$.

Let $\overline{\nu} \in (0, \nu_1)$ be arbitrary, and define

$$\begin{aligned} U(t) &:= e^{\overline{\nu}t} \limsup_{\varepsilon \to 0} \mathbb{E}V(x^{\varepsilon}(t), \overline{r}^{\varepsilon}(t)) \\ &= e^{\overline{\nu}t} \limsup_{\varepsilon \to 0} \mathbb{E}\overline{V}(x^{\varepsilon}(t), r^{\varepsilon}(t)). \end{aligned} \tag{22}$$

If we can show that $U(t) \leq c_1\nu_2$, then the proof is completed.

If $t \in [-\tau, 0]$, by condition (i),

$$\begin{aligned} U(t) &\leq \lim_{\varepsilon \to 0} \mathbb{E}\overline{V}(x^{\varepsilon}(t), r^{\varepsilon}(t)) = \mathbb{E}V(\xi, 0) \leq c_2 \mathbb{E}|\xi(0)|^p \\ &\leq c_2 \sup_{-\tau \leq \theta \leq 0} \mathbb{E}|\xi(\theta)|^p = c_1\nu_2. \end{aligned} \tag{23}$$

If $t \geq 0$, we will prove that $U(t) \leq c_1\nu_2$. Otherwise, there exists the smallest $\rho \in (0, \infty)$ such that all $t \in [-\tau, \rho)$, $U(t) \leq c_1\nu_2$ and $U(\rho) \geq c_1\nu_2$ as well as $U(\rho + \overline{\delta}) > U(\rho)$ for all suffieciently small $\overline{\delta}$.

For $t \in [\rho - \tau, \rho)$,

$$\begin{aligned} &\limsup_{\varepsilon \to 0} \mathbb{E}V(x^{\varepsilon}(t), \overline{r}^{\varepsilon}(t)) \\ &\quad = e^{-\overline{\nu}t} U(t) \\ &\quad \leq e^{-\overline{\nu}t} U(\rho) = e^{-\overline{\nu}t} e^{\overline{\nu}\rho} \limsup_{\varepsilon \to 0} \mathbb{E}V(x^{\varepsilon}(\rho), \overline{r}^{\varepsilon}(\rho)) \\ &\quad \leq e^{\overline{\nu}\tau} \limsup_{\varepsilon \to 0} \mathbb{E}V(x^{\varepsilon}(\rho), \overline{r}^{\varepsilon}(\rho)). \end{aligned} \tag{24}$$

If $\limsup_{\varepsilon \to 0} \mathbb{E}V(x^{\varepsilon}(\rho), \overline{r}^{\varepsilon}(\rho)) = 0$, then $\limsup_{\varepsilon \to 0} \mathbb{E}V(x^{\varepsilon}(t), \overline{r}^{\varepsilon}(t)) = 0$, $t \in [\rho - \tau, \rho)$.

Since $(x^{\varepsilon}(t), \overline{r}^{\varepsilon}(t))$ converges to $(\overline{x}(t), \overline{r}(t))$ with probability one (see Lemma 2.3 in [12]), by condition (i), we can derive

$$\overline{x}(t) = 0, \quad t \in [\rho - \tau, \rho). \tag{25}$$

Recalling the fact $\overline{f}(0, 0, i) \equiv 0$, $\overline{g}(0, 0, i) \equiv 0$ and using the uniqueness of the equation, we then have $\overline{x}(t) = 0$, a.e. $t > 0$. Therefore we have

$$\limsup_{\varepsilon \to 0} \mathbb{E}V(x^{\varepsilon}(t), \overline{r}^{\varepsilon}(t)) = 0, \quad t > 0. \tag{26}$$

Then $U(\rho) = 0$, which is a contradiction. Hence we see that $\lim_{\varepsilon \to 0} \mathbb{E}V(x^{\varepsilon}(\rho), \overline{r}^{\varepsilon}(\rho)) \neq 0$. For $t \in [\rho - \tau, \rho)$, there exists a $q > 1$ such that

$$\begin{aligned} &\limsup_{\varepsilon \to 0} \mathbb{E}V(x^{\varepsilon}(t), \overline{r}^{\varepsilon}(t)) \\ &\quad \leq q \limsup_{\varepsilon \to 0} \mathbb{E}V(x^{\varepsilon}(\rho), \overline{r}^{\varepsilon}(\rho)), \ \overline{\nu} < \frac{\log q}{\tau}. \end{aligned} \tag{27}$$

Consequently, there exists a sufficiently small $\varepsilon_0 > 0$, such that, for any $\varepsilon \in (0, \varepsilon_0)$,

$$\begin{aligned} &\mathbb{E}\left[\min_{i \in \overline{\mathbb{S}}} V(x^{\varepsilon}(\rho + \theta), i)\right] \\ &\quad \leq q\mathbb{E}\left[\max_{i \in \overline{\mathbb{S}}} V(x^{\varepsilon}(\rho), i)\right], \quad \theta \in [-\tau, 0]. \end{aligned} \tag{28}$$

By condition (ii),

$$\mathbb{E}\left[\max_{i \in \overline{\mathbb{S}}} \mathscr{L}V(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), i)\right] \leq -\lambda \mathbb{E}\left[\max_{i \in \overline{\mathbb{S}}} V(x^{\varepsilon}(t), i)\right]; \tag{29}$$

then,

$$\mathbb{E}\left[\mathscr{L}V(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \overline{r}^{\varepsilon}(t))\right] \leq -\lambda \mathbb{E}\left[V(x^{\varepsilon}(t), \overline{r}(t))\right]. \tag{30}$$

Noting that $\overline{\nu} < \nu \leq \lambda$, we have

$$\mathbb{E}\left[\mathscr{L}V(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \overline{r}^{\varepsilon}(t))\right] \leq -\overline{\nu} \mathbb{E}\left[V(x^{\varepsilon}(t), \overline{r}(t))\right]. \tag{31}$$

We now consider

$$\begin{aligned} &U(\rho + \overline{\delta}) - U(\rho) \\ &\quad = \limsup_{\varepsilon \to 0} \Big[e^{\overline{\nu}(\rho + \overline{\delta})} \mathbb{E}\left[V\left(x^{\varepsilon}(\rho + \overline{\delta}), \overline{r}^{\varepsilon}(\rho + \overline{\delta})\right)\right] \\ &\qquad - e^{\overline{\nu}\rho} \mathbb{E}\left[V(x^{\varepsilon}(\rho), \overline{r}^{\varepsilon}(\rho))\right]\Big] \\ &\quad = \limsup_{\varepsilon \to 0} \mathbb{E} \int_{\rho}^{\rho + \overline{\delta}} e^{\overline{\nu}t} \left[\mathscr{L}V(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \overline{r}^{\varepsilon}(t)) \right. \\ &\qquad \left. + \overline{\nu}V(x^{\varepsilon}(t), \overline{r}^{\varepsilon}(t))\right] dt \\ &\quad = \limsup_{\varepsilon \to 0} \mathbb{E} \int_{\rho}^{\rho + \overline{\delta}} e^{\overline{\nu}t} \left[\mathscr{L}\overline{V}(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)) \right. \\ &\qquad \left. + \overline{\nu}V(x^{\varepsilon}(t), \overline{r}^{\varepsilon}(t))\right] dt. \end{aligned} \tag{32}$$

By the definition of operator $\mathscr{L}$, we have

$$\begin{aligned} &\mathscr{L}\overline{V}(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)) \\ &\quad = \overline{V}_x(x^{\varepsilon}(t), r^{\varepsilon}(t)) f(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)) \\ &\qquad + \frac{1}{2}\operatorname{trace}\left[\overline{V}_{xx}(x^{\varepsilon}(t), r^{\varepsilon}(t))\right. \end{aligned}$$

$$\times g\left(x^{\varepsilon}(t),\ x^{\varepsilon}(t-\tau),r^{\varepsilon}(t)\right)$$
$$\times g^{T}\left(x^{\varepsilon}(t),x^{\varepsilon}(t-\tau),r^{\varepsilon}(t)\right)\Big]$$
$$+\sum_{\kappa=1}^{N}\gamma^{\varepsilon}_{r^{\varepsilon}(t)\kappa}\overline{V}\left(x^{\varepsilon}(t),\kappa\right)$$
$$=\overline{V}_{x}\left(x^{\varepsilon}(t),r^{\varepsilon}(t)\right)f\left(x^{\varepsilon}(t),x^{\varepsilon}(t-\tau),r^{\varepsilon}(t)\right)$$
$$+\frac{1}{2}\operatorname{trace}\Big[\overline{V}_{xx}\left(x^{\varepsilon}(t),r^{\varepsilon}(t)\right)$$
$$\times g\left(x^{\varepsilon}(t),\ x^{\varepsilon}(t-\tau),r^{\varepsilon}(t)\right)$$
$$\times g^{T}\left(x^{\varepsilon}(t),x^{\varepsilon}(t-\tau),r^{\varepsilon}(t)\right)\Big]$$
$$+\sum_{\kappa=1}^{N}\widehat{\gamma}_{r^{\varepsilon}(t)\kappa}\overline{V}\left(x^{\varepsilon}(t),\kappa\right)$$
$$=V_{x}\left(x^{\varepsilon}(t),\overline{r}^{\varepsilon}(t)\right)\overline{f}\left(x^{\varepsilon}(t),x^{\varepsilon}(t-\tau),\overline{r}^{\varepsilon}(t)\right)$$
$$+\frac{1}{2}\operatorname{trace}\Big[V_{xx}\left(x^{\varepsilon}(t),\overline{r}^{\varepsilon}(t)\right)$$
$$\times \overline{g}\left(x^{\varepsilon}(t),\ x^{\varepsilon}(t-\tau),\overline{r}^{\varepsilon}(t)\right)$$
$$\times \overline{g}^{T}\left(x^{\varepsilon}(t),x^{\varepsilon}(t-\tau),\overline{r}^{\varepsilon}(t)\right)\Big]$$
$$+\sum_{j=1}^{l}\overline{\gamma}^{\varepsilon}_{\overline{r}^{\varepsilon}(t)j}V\left(x^{\varepsilon}(t),j\right)$$
$$+V_{x}\left(x^{\varepsilon}(t),\overline{r}^{\varepsilon}(t)\right)\Big[f\left(x^{\varepsilon}(t),x^{\varepsilon}(t-\tau),r^{\varepsilon}(t)\right)$$
$$-\overline{f}\left(x^{\varepsilon}(t),x^{\varepsilon}(t-\tau),\overline{r}^{\varepsilon}(t)\right)\Big]$$
$$+\frac{1}{2}\operatorname{trace}\Big[V_{xx}\left(x^{\varepsilon}(t),\overline{r}^{\varepsilon}(t)\right)$$
$$\times\Big(g\left(x^{\varepsilon}(t),\ x^{\varepsilon}(t-\tau),r^{\varepsilon}(t)\right)$$
$$\times g^{T}\left(x^{\varepsilon}(t),x^{\varepsilon}(t-\tau),r^{\varepsilon}(t)\right)$$
$$-\overline{g}\left(x^{\varepsilon}(t),x^{\varepsilon}(t-\tau),\overline{r}^{\varepsilon}(t)\right)$$
$$\times \overline{g}^{T}\left(x^{\varepsilon}(t),x^{\varepsilon}(t-\tau),\overline{r}^{\varepsilon}(t)\right)\Big)\Big]$$
$$+\sum_{\kappa=1}^{N}\widehat{\gamma}_{r^{\varepsilon}(t)\kappa}\overline{V}\left(x^{\varepsilon}(t),\kappa\right)$$
$$-\sum_{j=1}^{l}\overline{\gamma}_{\overline{r}^{\varepsilon}(t)j}V\left(x^{\varepsilon}(t),j\right)$$
$$=\mathscr{L}V\left(x^{\varepsilon}(t),x^{\varepsilon}(t-\tau),\overline{r}^{\varepsilon}(t)\right)$$
$$+V_{x}\left(x^{\varepsilon}(t),\overline{r}^{\varepsilon}(t)\right)$$
$$\times\Big[f\left(x^{\varepsilon}(t),x^{\varepsilon}(t-\tau(t)),\right.$$
$$\left.r^{\varepsilon}(t)\right)-\overline{f}\left(x^{\varepsilon}(t),x^{\varepsilon}(t-\tau(t)),\overline{r}^{\varepsilon}(t)\right)\Big]$$
$$+\frac{1}{2}\operatorname{trace}\Big[V_{xx}\left(x^{\varepsilon}(t),\overline{r}^{\varepsilon}(t)\right)$$
$$\times\Big(g\left(x^{\varepsilon}(t),\ x^{\varepsilon}(t-\tau),r^{\varepsilon}(t)\right)$$
$$\times g^{T}\left(x^{\varepsilon}(t),x^{\varepsilon}(t-\tau),r^{\varepsilon}(t)\right)$$
$$-\overline{g}\left(x^{\varepsilon}(t),x^{\varepsilon}(t-\tau),\overline{r}^{\varepsilon}(t)\right)$$
$$\times \overline{g}^{T}\left(x^{\varepsilon}(t),x^{\varepsilon}(t-\tau),\overline{r}^{\varepsilon}(t)\right)\Big)\Big]$$
$$+\sum_{\kappa=1}^{N}\widehat{\gamma}_{r^{\varepsilon}(t)\kappa}\overline{V}\left(x^{\varepsilon}(t),\kappa\right)$$
$$-\sum_{j=1}^{l}\overline{\gamma}_{\overline{r}^{\varepsilon}(t)j}V\left(x^{\varepsilon}(t),j\right). \tag{33}$$

So

$$U\left(\rho+\overline{\delta}\right)-U\left(\rho\right)$$
$$=\limsup_{\varepsilon\to 0}\mathbb{E}\int_{\rho}^{\rho+\overline{\delta}}e^{\overline{\nu}t}\Big[\mathscr{L}V\left(x^{\varepsilon}(t),x^{\varepsilon}(t-\tau),\right.$$
$$\left.\overline{r}^{\varepsilon}(t)\right)+\overline{\nu}V\left(x^{\varepsilon}(t),\overline{r}^{\varepsilon}(t)\right)\Big]dt$$
$$+\limsup_{\varepsilon\to 0}\mathbb{E}\int_{\rho}^{\rho+\overline{\delta}}e^{\overline{\nu}t}V_{x}\left(x^{\varepsilon}(t),\overline{r}^{\varepsilon}(t)\right)$$
$$\times\Big[f\left(x^{\varepsilon}(t),x^{\varepsilon}(t-\tau),r^{\varepsilon}(t)\right)$$
$$-\overline{f}\left(x^{\varepsilon}(t),x^{\varepsilon}(t-\tau),\overline{r}^{\varepsilon}(t)\right)\Big]dt$$
$$+\frac{1}{2}\limsup_{\varepsilon\to 0}\mathbb{E}\int_{\rho}^{\rho+\overline{\delta}}e^{\overline{\nu}t}\ \operatorname{trace}$$
$$\times\Big[V_{xx}\left(x^{\varepsilon}(t),\ \overline{r}^{\varepsilon}(t)\right)$$
$$\times\Big(g\left(x^{\varepsilon}(t),x^{\varepsilon}(t-\tau),r^{\varepsilon}(t)\right)$$
$$\times g^{T}\left(x^{\varepsilon}(t),x^{\varepsilon}(t-\tau),r^{\varepsilon}(t)\right)$$
$$-\overline{g}\left(x^{\varepsilon}(t),x^{\varepsilon}(t-\tau(t)),\overline{r}^{\varepsilon}(t)\right)$$
$$\times \overline{g}^{T}\left(x^{\varepsilon}(t),x^{\varepsilon}(t-\tau),\overline{r}^{\varepsilon}(t)\right)\Big]dt$$
$$+\limsup_{\varepsilon\to 0}\mathbb{E}\int_{\rho}^{\rho+\overline{\delta}}e^{\overline{\nu}t}\left(\sum_{\kappa=1}^{N}\widehat{\gamma}^{\varepsilon}_{r^{\varepsilon}(t)\kappa}\overline{V}\left(x^{\varepsilon}(t),\kappa\right)\right.$$
$$\left.-\sum_{j=1}^{l}\overline{\gamma}^{\varepsilon}_{\overline{r}^{\varepsilon}(t)j}V\left(x^{\varepsilon}(t),j\right)\right)dt$$
$$=: I_1+I_2+I_3+I_4. \tag{34}$$

By the definition of $\overline{f}$,

$$f\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right)-\overline{f}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \overline{r}^{\varepsilon}(t)\right) = \sum_{i=1}^{l}\sum_{j=1}^{N_i} f\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), s_{ij}\right) \times\left[I_{\{r^{\varepsilon}(t)=s_{ij}\}}-\mu_j^i I_{\{\overline{r}^{\varepsilon}(t)=i\}}\right]. \tag{35}$$

This, together with assumption (H2), implies

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\mathbb{E}\int_{\rho}^{\rho+\overline{\delta}} e^{\overline{\nu}t}V_x\left(x^{\varepsilon}(t),\overline{r}^{\varepsilon}(t)\right) \times\left[f\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right) -\overline{f}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \overline{r}^{\varepsilon}(t)\right)\right]dt\\
&\leq \lim_{\varepsilon\to 0}\left[\mathbb{E}\left|\int_{\rho}^{\rho+\overline{\delta}} e^{\overline{\nu}t}V_x\left(x^{\varepsilon}(t),\overline{r}^{\varepsilon}(t)\right) \times\left[f\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right) -\overline{f}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \overline{r}^{\varepsilon}(t)\right)\right]dt\right|^2\right]^{1/2}\\
&= \lim_{\varepsilon\to 0}\left[\mathbb{E}\left|\int_{\rho}^{\rho+\overline{\delta}} e^{\overline{\nu}t}V_x\left(x^{\varepsilon}(t),\overline{r}^{\varepsilon}(t)\right) \times\sum_{i=1}^{l}\sum_{j=1}^{N_i} f\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), s_{ij}\right) \times\left[I_{\{r^{\varepsilon}(t)=s_{ij}\}}-\mu_j^i I_{\{\overline{r}^{\varepsilon}(t)=i\}}\right]dt\right|^2\right]^{1/2}\\
&\leq \lim_{\varepsilon\to 0}\left[\mathbb{E}\left|\int_{\rho}^{\rho+\overline{\delta}} \sum_{i=1}^{l}\sum_{j=1}^{N_i} e^{\overline{\nu}t}h\left(1+\left|x^{\varepsilon}(t)\right|^p+\left|x^{\varepsilon}(t-\tau)\right|^p\right) \times\left[I_{\{r^{\varepsilon}(t)=s_{ij}\}}-\mu_j^i I_{\{\overline{r}^{\varepsilon}(t)=i\}}\right]dt\right|^2\right]^{1/2}.
\end{aligned} \tag{36}$$

By the argument of Lemma 7.14 in [9], the right side of above inequality is equivalent to to 0; that is, $I_2=0$. Similarly, we can show

$$\begin{aligned}
I_3 &= \frac{1}{2}\limsup_{\varepsilon\to 0}\mathbb{E}\int_{\rho}^{\rho+\overline{\delta}} e^{\overline{\nu}t} \operatorname{trace}\times\left[V_{xx}\left(x^{\varepsilon}(t),\overline{r}^{\varepsilon}(t)\right) \times\left(g\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right)\right.\right.\\
&\quad\times g^T\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right) -\overline{g}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \overline{r}^{\varepsilon}(t)\right)\\
&\quad\left.\left.\times\overline{g}^T\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \overline{r}^{\varepsilon}(t)\right)\right)\right]dt=0.
\end{aligned} \tag{37}$$

By the definition of $\widehat{\Gamma}$ and $\overline{\Gamma}$, we have

$$\begin{aligned}
&\sum_{\kappa=1}^{N}\widehat{\gamma}_{r^{\varepsilon}(t)\kappa}\overline{V}\left(x^{\varepsilon}(t),\kappa\right)=\widehat{\Gamma}\overline{V}\left(x^{\varepsilon}(t),\cdot\right)\left(r^{\varepsilon}(t)\right),\\
&\sum_{j=1}^{l}\overline{\gamma}_{\overline{r}^{\varepsilon}(t)j}V\left(x^{\varepsilon}(t),j\right)=\overline{\Gamma}V\left(x^{\varepsilon}(t),\cdot\right)\left(\overline{r}^{\varepsilon}(t)\right),
\end{aligned} \tag{38}$$

hence

$$\begin{aligned}
I_4 &= \limsup_{\varepsilon\to 0}\mathbb{E}\int_{\rho}^{\rho+\overline{\delta}} e^{\overline{\nu}t}\left(\sum_{\kappa=1}^{N}\widehat{\gamma}^{\varepsilon}_{r^{\varepsilon}(t)\kappa}\overline{V}\left(x^{\varepsilon}(t),\kappa\right) -\sum_{j=1}^{l}\overline{\gamma}^{\varepsilon}_{\overline{r}^{\varepsilon}(t)j}V\left(x^{\varepsilon}(t),j\right)\right)dt\\
&= \limsup_{\varepsilon\to 0}\mathbb{E}\int_{\rho}^{\rho+\overline{\delta}} e^{\overline{\nu}t}\left(\widehat{\Gamma}\overline{V}\left(x^{\varepsilon}(t),\cdot\right)\left(r^{\varepsilon}(t)\right) -\overline{\Gamma}V\left(x^{\varepsilon}(t),\cdot\right)\left(\overline{r}^{\varepsilon}(t)\right)\right)dt\\
&= \limsup_{\varepsilon\to 0}\mathbb{E}\int_{\rho}^{\rho+\overline{\delta}} e^{\overline{\nu}t}\sum_{i=1}^{l}\sum_{j=1}^{N_i}\widehat{\Gamma}\overline{V}\left(x^{\varepsilon}(t),\cdot\right)\left(s_{ij}\right) \times\left[I_{\{r^{\varepsilon}(t)=s_{ij}\}}-\mu_j^i I_{\{\overline{r}^{\varepsilon}(t)=i\}}\right]dt\\
&\leq \limsup_{\varepsilon\to 0}\left[\mathbb{E}\left|\int_{\rho}^{\rho+\overline{\delta}} e^{\overline{\nu}t}\sum_{i=1}^{l}\sum_{j=1}^{N_i}\widehat{\Gamma}\overline{V}\left(x^{\varepsilon}(t),\cdot\right)\left(s_{ij}\right) \times\left[I_{\{r^{\varepsilon}(t)=s_{ij}\}}-\mu_j^i I_{\{\overline{r}^{\varepsilon}(t)=i\}}\right]\right|^2\right]^{1/2}.
\end{aligned} \tag{39}$$

By assumption (H4) and the argument of Lemma 7.14 in [9], we have the right side of above inequality is equivalent to 0, that is, $I_4=0$.

Therefore by the condition (ii)

$$U\left(\rho+\overline{\delta}\right)-U(\rho) = \lim_{\varepsilon\to 0}\mathbb{E}\int_{\rho}^{\rho+\overline{\delta}} e^{\overline{\nu}t}\left[\mathscr{L}V\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right) +\overline{\nu}V\left(x^{\varepsilon}(t),\overline{r}^{\varepsilon}(t)\right)\right]dt\leq 0; \tag{40}$$

this is

$$U\left(\rho+\overline{\delta}\right)\leq U(\rho). \tag{41}$$

This contradicts the definition of ρ. The proof is now completed. □

Example 4. Let $r^\varepsilon(\cdot)$ be a Markov chain generated by Γ^ε given in (5) with

$$\widetilde{\Gamma} = \begin{pmatrix} -2 & 2 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 3 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}, \quad \widehat{\Gamma} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{42}$$

The generator $\widetilde{\Gamma}$ consists of two irreducible blocks. The stationary distributions are $\mu^1 = (0.5, 0.5)$, $\mu^2 = (1/7, 2/7, 4/7)$, and

$$\overline{\Gamma} = \begin{pmatrix} -1 & 1 \\ \dfrac{6}{7} & -\dfrac{6}{7} \end{pmatrix}. \tag{43}$$

Consider a one-dimensional equation

$$dx^\varepsilon(t) = f\left(x^\varepsilon(t), r^\varepsilon(t)\right) dt + g\left(x^\varepsilon(t-\tau), r^\varepsilon(t)\right) dw(t) \tag{44}$$

with

$$\begin{gathered} f(x, s_{11}) = \frac{x}{8}, \qquad f(x, s_{12}) = \frac{x}{8}, \\ g(x, s_{11}) = \frac{x\cos x}{8\sqrt{2}}, \qquad g(x, s_{12}) = \frac{x \sin x}{8\sqrt{2}}, \\ f(x, s_{21}) = -28(x + \sin x), \\ f(x, s_{22}) = 7x + 14\sin x, \qquad f(x, s_{23}) = -\frac{7}{4}x, \\ g(x, s_{21}) = \frac{\sqrt{7}}{4} x \sin x, \\ g(x, s_{22}) = -\frac{\sqrt{7}}{4} x\cos x, \qquad g(x, s_{23}) = \frac{\sqrt{7}}{8}x. \end{gathered} \tag{45}$$

Then the limit equation is

$$d\overline{x}(t) = f\left(\overline{x}(t), \overline{r}(t)\right) dt + g\left(\overline{x}(t-\tau), \overline{r}(t)\right) dw(t), \tag{46}$$

where $\overline{r}$ is the Markov chain generated by $\overline{\Gamma}$ and

$$\begin{gathered} \overline{f}(x, 1) = \frac{x}{8}, \qquad \overline{f}(x, 2) = -3x, \\ \overline{g}(x, 1) = \frac{x}{16}, \qquad \overline{g}(x, 2) = \frac{x}{4}. \end{gathered} \tag{47}$$

Let $V(x, 1) = 2x^2$, $V(x, 2) = x^2$; then,

$$\begin{gathered} \mathscr{L}V(x, y, 1) \le -\frac{1}{2}|x|^2 + \frac{|y|^2}{128}, \\ \mathscr{L}V(x, y, 2) \le -\frac{36}{7}|x|^2 + \frac{|y|^2}{16}, \end{gathered} \tag{48}$$

Consequently

$$\max_{i=1,2} \mathscr{L}V(x, y, i) \le -\frac{1}{2}|x|^2 + \frac{1}{16}|y|^2 \tag{49}$$

$$= -\frac{1}{4}\left[\max_{i=1,2} V(x, i)\right] + \frac{1}{16}\left[\min_{i=1,2} V(y, i)\right]. \tag{50}$$

It is easy to see that we can find a $q > 1$ such that $(1/4) - (q/16) > 0$. Therefore, for any $\phi \in L^2_{\mathscr{F}_t}([-\tau, 0]; \mathbb{R}^n)$ satisfying $\mathbb{E}[\min_{i\in\mathbb{S}}\phi(\theta)] \le q\mathbb{E}[\max_{i\in\mathbb{S}}\phi(0)]$ on $-\tau \le \theta \le 0$, (49) yields

$$\mathbb{E}\left[\max_{i\in\mathbb{S}} \mathscr{L}V(x, y, i)\right] \le -\left(\frac{1}{4} - \frac{q}{16}\right)\mathbb{E}\left[\max_{i=1,2} V(x, i)\right]. \tag{51}$$

Hence, by Theorem 1, the solution $x^\varepsilon(t)$ is mean square stable when ε is sufficient small.

4. Stochastic Delay System with Pure Jumps

In this section we discuss the stability of the following stochastic delay system with pure jumps:

$$\begin{aligned} dx^\varepsilon(t) &= f\left(x^\varepsilon(t), x^\varepsilon(t-\tau), r^\varepsilon(t)\right) dt \\ &\quad + \int_{\mathbb{R}^m} b\left(x^\varepsilon(t-), x^\varepsilon((t-\tau)-), r^\varepsilon(t), z\right) \widetilde{N}(dt, dz), \\ x_0 &= \xi \in C\left([-\tau, 0]; \mathbb{R}^n\right), \quad r(0) \in \mathbb{S}, \end{aligned} \tag{52}$$

where $x^\varepsilon(t-) = \lim_{s\uparrow t} x^\varepsilon(s)$, $b : \mathbb{R}^n\times\mathbb{R}^n\times\mathbb{S}\times\mathbb{R}^m \to \mathbb{R}^{n\times m}$. We assume that the each column $b^{(l)}$ of the $n\times m$ matrix $b = [b_{ij}]$ depends on z only through the lth coordinate z_l; that is,

$$\begin{gathered} b^{(k)}(x, y, \kappa, z) = b^{(k)}(x, y, \kappa, z_k); \\ z = (z_1, \ldots, z_m) \in \mathbb{R}^m, \quad \kappa \in \mathbb{S}. \end{gathered} \tag{53}$$

$N(t, z)$ is a m-dimensional Poisson process, and the compensated Poisson, process is defined by

$$\begin{aligned} \widetilde{N}(dt, dz) &= \left(\widetilde{N}_1(dt, dz_1), \ldots, \widetilde{N}_d(dt, dz_m)\right) \\ &= \left(N_1(dt, dz_1) - \lambda_1(dz_1)dt, \ldots,\right. \\ &\qquad \left. N_m(dt, dz_m) - \lambda_m(dz_m)dt\right), \end{aligned} \tag{54}$$

where $\{N_j, j = 1, \ldots, m\}$ are independent one-dimensional Poisson random measures with characteristic measure $\{\lambda_j, j = 1, \ldots, m\}$ coming from m independent one-dimensional Poisson point processes.

The averaged system of (18) is defined as follows:

$$\begin{aligned} d\overline{x}(t) &= \overline{f}\left(\overline{x}(t), \overline{x}(t-\tau), \overline{r}(t)\right) dt \\ &\quad + \int_{\mathbb{R}^m} \overline{b}\left(\overline{x}(t-), \overline{x}((t-\tau)-), \overline{r}(t), z\right) \\ &\qquad \times \widetilde{N}(dt, dz), \\ \overline{x}_0 &= \xi \in C\left([-\tau, 0]; \mathbb{R}^n\right), \quad \overline{r}(0) \in \mathbb{S}, \end{aligned} \tag{55}$$

where $\overline{x}(t-) = \lim_{s\uparrow t}\overline{x}(s)$, $\overline{b} : \mathbb{R}^n \times \mathbb{R}^n \times \overline{\mathbb{S}} \times \mathbb{R}^m \rightarrow \mathbb{R}^{n\times m}$. Similar to the definition of $\overline{f}$, we define

$$\overline{b}(x, y, i, z) = \sum_{j=1}^{N_i} \mu_j^i b(x, y, s_{ij}, z). \tag{56}$$

For each $s_{ij} \in \mathbb{S}^i$ with $i \in \{1, \ldots, l\}$ and $j \in \{1, \ldots, N_i\}$.

To assure the existence and uniqueness of the solution of (52), we also give the following standard assumptions.

(H2′) For any integer R, there is a constant $h_R > 0$, such that

$$\begin{aligned} &|f(x, y, i) - f(x_1, y_1, i)| \\ &\quad \vee \sum_{k=1}^{m} \int_{\mathbb{R}} \left|b^{(k)}(x_2, y_2, \kappa, z_k) - b^{(k)}(x_1, y_1, \kappa, z_k)\right| \lambda_k(dz_k) \\ &\quad \le h_R(|x_2 - x_1| + |y_2 - y_1|) \end{aligned} \tag{57}$$

for all $i \in \mathbb{S}$ and those $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$ with $|x_1| \vee |x_2| \vee |y_1| \vee |y_2| \le R$.

(H3′) There is an $h > 0$, such that for any $x, y \in \mathbb{R}^n$, $i \in \mathbb{S}$,

$$\begin{aligned} &|f(x, y, i)| \vee \sum_{k=1}^{m} \int_{\mathbb{R}} \left|b^{(k)}(x, y, \kappa, z_k)\right| \lambda_k(dz_k) \\ &\quad \le h(1 + |x| + |y|), \quad f(0, 0, \kappa) \equiv 0, \quad b(0, 0, \kappa, z) \equiv 0. \end{aligned} \tag{58}$$

Given $V \in C^p(\mathbb{R}^n \times \mathbb{S}; \mathbb{R}_+)$, we define the operator $\mathbb{L}V$ by

$$\begin{aligned} &\mathbb{L}V(x, y, i) \\ &\quad = V_x(x, i) f(x, y, i) + \sum_{j=1}^{N} \gamma_{ij} V(x, j) \\ &\quad + \int_{\mathbb{R}} \sum_{k=1}^{m} \left\{V\left(x + b^{(k)}(x, y, \kappa, z_k), \kappa\right) - V(x, i) \right. \\ &\quad \left. - V_x(x, i) b^{(k)}(x, y, \kappa, z_k)\right\} \lambda_k(dz_k), \end{aligned} \tag{59}$$

where

$$V_x(x, i) = \left(\frac{\partial V(x, i)}{\partial x_1}, \ldots, \frac{\partial V(x, i)}{\partial x_m}\right). \tag{60}$$

We need the following lemma, for details see [16].

Lemma 5. *Let (H1) and (H2′), (H3′) hold, as $\varepsilon \rightarrow 0$; then, $(x^{\varepsilon}(\cdot), \overline{r}^{\varepsilon}(\cdot))$ converges weakly to $(\overline{x}(\cdot), \overline{r}(\cdot))$ in $D([0, \infty), \mathbb{R}^n \times \overline{\mathbb{S}})$, where $D([0, \infty), \mathbb{R}^n \times \overline{\mathbb{S}})$ is the space of functions defined on $[0, \infty)$ that are right continuous and have left limits taking values in $\mathbb{R}^n \times \overline{\mathbb{S}}$ and endowed with the Skorohod topology.*

We now state our main result in this section.

Theorem 6. *Let (H1) and (H2′), (H3′) hold; there is a function $V(x, i) \in C^p(\mathbb{R}^n \times \overline{\mathbb{S}}; \mathbb{R}_+)$ satisfying (H4), and there are positive constants λ, c_1, c_2, and $q > 1$ such that*

(i) $c_1|x|^p \le V(x, i) \le c_2|x|^p$,

(ii) $\mathbb{E}[\max_{i\in\overline{\mathbb{S}}}\mathbb{L}V(x(t), x(t-\tau), i)] \le -\lambda\mathbb{E}[\max_{i\in\overline{\mathbb{S}}}V(x(t), i)]$ *provided* $\mathbb{E}[\min_{i\in\overline{\mathbb{S}}}V(x(t+\theta), i)] < q\mathbb{E}[\max_{i\in\overline{\mathbb{S}}}V(x(t), i)]$, $-\tau \le \theta \le 0$,

Then, for all $\xi \in C([-\tau, 0]; \mathbb{R}^n)$,

$$\limsup_{\varepsilon\rightarrow 0}\mathbb{E}|x^{\varepsilon}(t)|^p \le \nu_4 e^{-\nu_3 t}, \tag{61}$$

where

$$\begin{aligned} &\nu_3 = \min\left\{\lambda, \frac{\log q}{\tau}\right\}, \text{ and} \\ &\nu_4 \text{ is a fixed constant such that} \\ &\nu_4 = \frac{c_2}{c_1}\sup_{-\tau\le\theta\le 0} E|\xi|^p. \end{aligned} \tag{62}$$

Proof. As the proof of Theorem 1, define

$$\overline{V}(x, \zeta) = \sum_{i=1}^{l} V(x, i) I_{\{\zeta\in\mathbb{S}^i\}} = V(x, i) \quad \text{if } \zeta \in \mathbb{S}^i. \tag{63}$$

We extend $r(t)$ to $[-\tau, 0]$ by setting $r(t) = r(0)$. Then, $\mathbb{E}\overline{V}(x^{\varepsilon}(t), r^{\varepsilon}(t))$ is right continuous on $t \ge -\tau$.

Let $\overline{\nu} \in (0, \nu_3)$ be arbitrary, and define

$$\begin{aligned} U(t) &:= e^{\overline{\nu}t}\limsup_{\varepsilon\rightarrow 0}\mathbb{E}V(x^{\varepsilon}(t), \overline{r}^{\varepsilon}(t)) \\ &= e^{\overline{\nu}t}\limsup_{\varepsilon\rightarrow 0}\mathbb{E}\overline{V}(x^{\varepsilon}(t), r^{\varepsilon}(t)). \end{aligned} \tag{64}$$

If we can show that $U(t) \le c_1\nu_4$, then the proof is completed.

If $t \in [-\tau, 0]$, by condition (i), is the same as the proof of Theorem 1, we have $U(t) \le c_1\nu_4$.

In the following we shall prove that $U(t) \le c_1\nu_4$ if $t \ge 0$. Otherwise, there exists the smallest $\rho \in (0, \infty)$ such that all $t \in [-\tau, \rho)$, $U(t) \le c_1\nu_4$, and $U(\rho) \ge c_1\nu_4$ as well as $U(\rho+\overline{\delta}) > U(\rho)$ for all suffieciently small $\overline{\delta}$.

As the same in the proof of Theorem 1 we can have that $\lim_{\varepsilon\rightarrow 0}\mathbb{E}V(x^{\varepsilon}(\rho), \overline{r}^{\varepsilon}(\rho)) \ne 0$. Hence for $t \in [\rho - \tau, \rho)$, there exists a q such that

$$\begin{aligned} &\limsup_{\varepsilon\rightarrow 0}\mathbb{E}V(x^{\varepsilon}(t), \overline{r}^{\varepsilon}(t)) \\ &\quad < q\limsup_{\varepsilon\rightarrow 0}\mathbb{E}V(x^{\varepsilon}(\rho), \overline{r}^{\varepsilon}(\rho)), \overline{\nu} < \frac{\log q}{\tau}. \end{aligned} \tag{65}$$

Consequently, there exists a sufficiently small $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$,

$$\mathbb{E}\left[\min_{i\in\overline{\mathbb{S}}} V(x^{\varepsilon}(\rho+\theta), i)\right] \le q\mathbb{E}\left[\max_{i\in\overline{\mathbb{S}}} V(x^{\varepsilon}(\rho), i)\right], \quad \theta \in [-\tau, 0]. \tag{66}$$

By condition (ii),

$$\mathbb{E}\left[\max_{i\in\overline{\mathbb{S}}}\mathbb{L}V(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), i)\right] \le -\lambda\mathbb{E}\left[\max_{i\in\overline{\mathbb{S}}} V(x^{\varepsilon}(t), i)\right], \tag{67}$$

we then have for $\overline{\nu} < \nu \le \lambda$,

$$\mathbb{E}\left[\mathbb{L}V\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \overline{r}^{\varepsilon}(t)\right)\right] \le -\overline{\nu}\mathbb{E}\left[V\left(x^{\varepsilon}(t), \overline{r}(t)\right)\right]. \tag{68}$$

We now consider

$$\begin{aligned} U\left(\rho+\overline{\delta}\right) - U(\rho) &= \limsup_{\varepsilon\to 0}\left[e^{\overline{\nu}(\rho+\overline{\delta})}\mathbb{E}\left[V\left(x^{\varepsilon}\left(\rho+\overline{\delta}\right), \overline{r}^{\varepsilon}\left(\rho+\overline{\delta}\right)\right)\right]\right. \\ &\qquad \left. - e^{\overline{\nu}\rho}\mathbb{E}V\left[\left(x^{\varepsilon}(\rho), \overline{r}^{\varepsilon}(\rho)\right)\right]\right] \\ &= \limsup_{\varepsilon\to 0}\mathbb{E}\int_{\rho}^{\rho+\overline{\delta}} e^{\overline{\nu}t}\left[\mathbb{L}V\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \overline{r}^{\varepsilon}(t)\right)\right. \\ &\qquad \left. + \overline{\nu}V\left(x^{\varepsilon}(t), \overline{r}^{\varepsilon}(t)\right)\right] dt \\ &= \limsup_{\varepsilon\to 0}\mathbb{E}\int_{\rho}^{\rho+\overline{\delta}} e^{\overline{\nu}t}\left[\mathbb{L}\overline{V}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right)\right. \\ &\qquad \left. + \overline{\nu}V\left(x^{\varepsilon}(t), \overline{r}^{\varepsilon}(t)\right)\right] dt. \end{aligned} \tag{69}$$

By the definition of the operator $\mathbb{L}$ similar to that of the proof of Theorem 1, we have

$$\begin{aligned} &\mathbb{L}\overline{V}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right) \\ &\quad = \overline{V}_x\left(x^{\varepsilon}(t), r^{\varepsilon}(t)\right) f\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right) \\ &\qquad + \sum_{k=1}^{m}\int_{\mathbb{R}}\left\{\overline{V}\left(x^{\varepsilon}(t) + b^{(k)}\left(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-),\right.\right.\right. \\ &\qquad\qquad \left.\left. r^{\varepsilon}(t), z_k\right), r^{\varepsilon}(t)\right) \\ &\qquad\qquad - \overline{V}\left(x^{\varepsilon}(t), r^{\varepsilon}(t)\right) - \overline{V}_x\left(x^{\varepsilon}(t), r^{\varepsilon}(t)\right) b^{(k)} \\ &\qquad\qquad \times \left.\left(x^{\varepsilon}(t-), \ x^{\varepsilon}((t-\tau)-), r^{\varepsilon}(t), z_k\right)\right\} \\ &\qquad\qquad \times \lambda_k(dz_k) \\ &\qquad + \sum_{j=1}^{N}\gamma^{\varepsilon}_{r^{\varepsilon}(t)j}\overline{V}\left(x^{\varepsilon}(t), j\right) \\ &\quad = \mathbb{L}V\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \overline{r}^{\varepsilon}(t)\right) \\ &\qquad + V_x\left(x^{\varepsilon}(t), \overline{r}^{\varepsilon}(t)\right) \\ &\qquad\quad \times \left[f\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right)\right. \\ &\qquad\qquad \left. - \overline{f}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \overline{r}^{\varepsilon}(t)\right)\right] \\ &\qquad + \sum_{k=1}^{m}\int_{\mathbb{R}}\left\{V\left(x^{\varepsilon}(t) + b^{(k)}\right.\right. \\ &\qquad\qquad \times \left(x^{\varepsilon}(t-), \ x^{\varepsilon}((t-\tau)-), r^{\varepsilon}(t), z_k\right), \overline{r}^{\varepsilon}(t)\right) \\ &\qquad\qquad - V\left(x^{\varepsilon}(t) + \overline{b}^{(k)}\right. \\ &\qquad\qquad\quad \times \left(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-), \overline{r}^{\varepsilon}(t), z_k\right), \\ &\qquad\qquad\quad \left.\left.\overline{r}^{\varepsilon}(t)\right)\right\}\lambda_k(dz_k) \\ &\qquad - \sum_{k=1}^{m}\int_{\mathbb{R}}\left\{V_x\left(x^{\varepsilon}(t), \overline{r}^{\varepsilon}(t)\right)\right. \\ &\qquad\qquad \times \left(b^{(k)}\left(x^{\varepsilon}(t-), \ x^{\varepsilon}((t-\tau)-), r^{\varepsilon}(t), z_k\right)\right) \\ &\qquad\qquad - \overline{b}^{(k)}\left(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-),\right. \\ &\qquad\qquad\quad \left.\left.\overline{r}^{\varepsilon}(t), z_k\right)\right\}\lambda_k(dz_k) \\ &\qquad + \sum_{j=1}^{N}\widehat{\gamma}_{r^{\varepsilon}(t)j}\overline{V}\left(x^{\varepsilon}(t), j\right) - \sum_{j=1}^{l}\overline{\gamma}_{\overline{r}^{\varepsilon}(t)j}V\left(x^{\varepsilon}(t), j\right). \end{aligned} \tag{70}$$

This implies

$$\begin{aligned} &U\left(\rho+\overline{\delta}\right) - U(\rho) \\ &\quad = \limsup_{\varepsilon\to 0}\mathbb{E}\int_{\rho}^{\rho+\overline{\delta}} e^{\overline{\nu}t}\left[\mathscr{L}V\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \overline{r}^{\varepsilon}(t)\right)\right. \\ &\qquad\qquad \left. + \overline{\nu}V\left(x^{\varepsilon}(t), \overline{r}^{\varepsilon}(t)\right)\right] dt \\ &\qquad + \limsup_{\varepsilon\to 0}\mathbb{E}\int_{\rho}^{\rho+\overline{\delta}} e^{\overline{\nu}t}V_x\left(x^{\varepsilon}(t), \overline{r}^{\varepsilon}(t)\right) \\ &\qquad\qquad \times \left[f\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), r^{\varepsilon}(t)\right)\right. \\ &\qquad\qquad\quad \left. - \overline{f}\left(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \overline{r}^{\varepsilon}(t)\right)\right] dt \\ &\qquad + \limsup_{\varepsilon\to 0}\mathbb{E}\int_{\rho}^{\rho+\overline{\delta}} e^{\overline{\nu}t} \\ &\qquad\qquad \times \left\{\sum_{k=1}^{m}\int_{\mathbb{R}}\left\{V\left(x^{\varepsilon}(t)\right.\right.\right. \\ &\qquad\qquad\qquad + b^{(k)}\left(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-),\right. \\ &\qquad\qquad\qquad\quad \left.\left. r^{\varepsilon}(t), z_k\right), \overline{r}^{\varepsilon}(t)\right) \\ &\qquad\qquad\qquad - V\left(x^{\varepsilon}(t) + \overline{b}^{(k)}\left(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-),\right.\right. \\ &\qquad\qquad\qquad\quad \left.\left.\left.\left.\overline{r}^{\varepsilon}(t), z_l\right), \overline{r}^{\varepsilon}(t)\right)\right\}\lambda_k(dz_k)\right\} dt \\ &\qquad - \limsup_{\varepsilon\to 0}\mathbb{E}\int_{\rho}^{\rho+\overline{\delta}} e^{\overline{\nu}t} \\ &\qquad\qquad \times \left\{\sum_{k=1}^{m}\int_{\mathbb{R}}\left\{V_x\left(x^{\varepsilon}(t), \overline{r}^{\varepsilon}(t)\right)\right.\right. \end{aligned}$$

$$\times \Big(b^{(k)}\left(x^{\varepsilon}(t-),\; x^{\varepsilon}((t-\tau)-),\; r^{\varepsilon}(t), z_k\right)) - \bar{b}^{(k)}\left(x^{\varepsilon}(t-),\; x^{\varepsilon}((t-\tau)-),\; \bar{r}^{\varepsilon}(t), z_k\right))\} \times \lambda_k(dz_k)\Big\}\, dt$$

$$+ \limsup_{\varepsilon\to 0}\mathbb{E}\int_{\rho}^{\rho+\bar{\delta}} e^{\bar{\gamma}t}\left(\sum_{j=1}^{N}\hat{\gamma}_{r^{\varepsilon}(t)j}\bar{V}\left(x^{\varepsilon}(t),j\right) - \sum_{j=1}^{l}\bar{\gamma}_{\bar{r}^{\varepsilon}(t)j}V\left(x^{\varepsilon}(t),j\right)\right)dt$$

$$=: J_1 + J_2 + J_3 + J_4 + J_5. \quad (71)$$

By the definition of $\bar{b}$,

$$\begin{aligned} &b^{(k)}\left(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-), r^{\varepsilon}(t), z_k\right) - \bar{b}^{(k)}\left(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-), \bar{r}^{\varepsilon}(t), z_k\right) \\ &= \sum_{i=1}^{l}\sum_{j=1}^{N_i} b^{(k)}\left(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-), s_{ij}, z_k\right) \times \left[I_{\{r^{\varepsilon}(t)=s_{ij}\}} - \mu_j^i I_{\{\bar{r}^{\varepsilon}(t)=i\}}\right]. \end{aligned} \quad (72)$$

By assumption (H2′), we have

$$\begin{aligned} J_4 &= \limsup_{\varepsilon\to 0}\sum_{k=1}^{m}\mathbb{E}\int_{\rho}^{\rho+\bar{\delta}} e^{\bar{\gamma}t}V_x\left(x^{\varepsilon}(t),\bar{r}^{\varepsilon}(t)\right) \times \int_{\mathbb{R}}\left[b^{(k)}\left(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-), r^{\varepsilon}(t), z_k\right) - \bar{b}^{(k)}\left(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-), \bar{r}^{\varepsilon}(t), z_k\right)\right] \times \lambda_k(dz_k)\, dt \\ &= \limsup_{\varepsilon\to 0}\sum_{k=1}^{m}\sum_{i=1}^{l}\sum_{j=1}^{N_i}\mathbb{E}\int_{\rho}^{\rho+\bar{\delta}} e^{\bar{\gamma}t}V_x\left(x^{\varepsilon}(t),\bar{r}^{\varepsilon}(t)\right) \times \int_{\mathbb{R}} b^{(k)} \times \left(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-), s_{ij}, z_k\right) \times \left[I_{\{r^{\varepsilon}(t)=s_{ij}\}} - \mu_j^i I_{\{\bar{r}^{\varepsilon}(t)=i\}}\right] \times \lambda_k(dz_k)\, dt \\ &\le \limsup_{\varepsilon\to 0}\sum_{k=1}^{m}\sum_{i=1}^{l}\sum_{j=1}^{N_i}\left[\mathbb{E}\left|\int_{\rho}^{\rho+\bar{\delta}} e^{\bar{\gamma}t}V_x \times \left(x^{\varepsilon}(t),\bar{r}^{\varepsilon}(t)\right) \times \int_{\mathbb{R}} b^{(k)} \times \left(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-), s_{ij}, z_k\right) \times \left[I_{\{r^{\varepsilon}(t)=s_{ij}\}} - \mu_j^i I_{\{\bar{r}^{\varepsilon}(t)=i\}}\right] \times \lambda_k(dz_k)\, dt\right|^2\right]^{1/2}. \end{aligned} \quad (73)$$

By the argument of Lemma 7.14 in [9], the right side of the inequality above is equivalent to 0, that is, $J_4 = 0$. Similarly, by mean-value theorem, we can show that there exists $\eta^{(k)}(t)$ which is between $x^{\varepsilon}(t)+b^{(k)}(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-), r^{\varepsilon}(t), z_k)$ and $x^{\varepsilon}(t)+\bar{b}^{(k)}(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-), r^{\varepsilon}(t), z_k)$ such that

$$\begin{aligned} J_3 &= \lim_{\varepsilon\to 0}\sum_{k=1}^{m}\mathbb{E}\int_{\rho}^{\rho+\bar{\delta}} e^{\bar{\gamma}t} \times \Big\{\int_{\mathbb{R}}\{V_x(\eta(t)) \times \left[b^{(k)}\left(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-), r^{\varepsilon}(t), z_k\right) - \bar{b}^{(k)}\left(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-), \bar{r}^{\varepsilon}(t), z_k\right]\right\}\lambda_k(dz_k)\Big\}\, dt \\ &= \lim_{\varepsilon\to 0}\sum_{k=1}^{m}\sum_{i=1}^{l}\sum_{j=1}^{N_i}\mathbb{E}\int_{\rho}^{\rho+\bar{\delta}} e^{\bar{\gamma}t} \times V_x(\eta(t))\int_{\mathbb{R}} b^{(k)}\left(x^{\varepsilon}(t-), x^{\varepsilon}((t-\tau)-), s_{ij}, z_k\right) \times \left[I_{\{r^{\varepsilon}(t)=s_{ij}\}} - \mu_j^i I_{\{\bar{r}^{\varepsilon}(t)=i\}}\right]\lambda_k(dz_k)\, dt \\ &\le \lim_{\varepsilon\to 0}\sum_{k=1}^{m}\sum_{i=1}^{l}\sum_{j=1}^{N_i}\left[\mathbb{E}\left|\int_{\rho}^{\rho+\bar{\delta}} e^{\bar{\gamma}t}V_x(\eta(t)) \times \int_{\mathbb{R}} b^{(k)}\left(x^{\varepsilon}(t-),\right.\right.\right. \end{aligned}$$

$$x^{\varepsilon}((t-\tau)-), s_{ij}, z_k)$$
$$\times \left[I_{\{r^{\varepsilon}(t)=s_{ij}\}} - \mu_j^i I_{\{\overline{r}^{\varepsilon}(t)=i\}}\right]$$
$$\times \lambda_k(dz_k)\, dt\big|^2\Big]^{1/2}. \tag{74}$$

By the argument of Lemma 7.14 in [9], we have $J_3 = 0$. Similar to the proof of Theorem 1, we can derive $J_2 = 0, J_5 = 0$.

Therefore we arrive at

$$\begin{aligned} &U(\rho+\overline{\delta}) - U(\rho) \\ &= \lim_{\varepsilon\to 0}\mathbb{E}\int_{\rho}^{\rho+\overline{\delta}} e^{\overline{\gamma}t}\left[\mathbb{L}V(x^{\varepsilon}(t), x^{\varepsilon}(t-\tau), \overline{r}^{\varepsilon}(t))\right. \\ &\qquad \left.+\overline{\gamma}V(x^{\varepsilon}(t), \overline{r}^{\varepsilon}(t))\right] dt \le 0; \end{aligned} \tag{75}$$

then,

$$U(\rho+\overline{\delta}) \le U(\rho). \tag{76}$$

This contradicts the definition of ρ. The proof is therefore completed. □

We shall give an example to illustrate our theory:

Example 7. Let $r^{\varepsilon}(\cdot)$ be a Markov chain generated by

$$\Gamma^{\varepsilon} = \frac{1}{\varepsilon}\widetilde{\Gamma} + \widehat{\Gamma} = \frac{1}{\varepsilon}\begin{pmatrix} -1 & 0 & 1 & 0 \\ \frac{1}{2} & -1 & 0 & \frac{1}{2} \\ 0 & 2 & -2 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & -1 \end{pmatrix}; \tag{77}$$

here we set $\widehat{\Gamma} = 0$. The stationary distribution is $\mu = (4/19, 8/19, 3/19, 4/19)$. Consider a one-dimensional equation

$$\begin{aligned} dx^{\varepsilon}(t) &= f(x^{\varepsilon}(t), r^{\varepsilon}(t))\, dt \\ &\quad + \int_0^{\infty}\sigma(r^{\varepsilon}(t), z)\, x^{\varepsilon}((t-\tau)-)\, \widetilde{N}(dt, dz) \end{aligned} \tag{78}$$

with

$$\begin{aligned} f(x,1) &= 2\sin x, \qquad f(x,2) = -\frac{19}{8}x, \\ f(x,3) &= -\frac{19}{6}x, \qquad f(x,4) = -2\sin x. \end{aligned} \tag{79}$$

Let

$$\begin{gathered} \beta(z) = \frac{4}{19}\sigma(1,z) + \frac{8}{19}\sigma(2,z) + \frac{3}{19}\sigma(3,z) + \frac{4}{19}\sigma(4,z), \\ \int_0^{\infty}\beta^2(z)\,\lambda(dz) < 2. \end{gathered} \tag{80}$$

Then the limit equation is

$$d\overline{x}(t) = -\frac{3}{2}\overline{x}(t)\, dt + \int_0^{\infty}\beta(z)\,\overline{x}((t-\tau)-)\,\widetilde{N}(dt, dz). \tag{81}$$

Let $V(x) = x^2$; then,

$$\mathbb{L}V(x,y) \le -3|x|^2 + \int_0^{\infty}\beta^2(z)\,\lambda(dz)\,|y|^2. \tag{82}$$

We can find a $q > 1$ such that $3-2q > 0$. Therefore, for any $\phi \in L^2_{\mathcal{F}_t}([-\tau,0];\mathbb{R}^n)$ satisfying $\mathbb{E}[\min_{i\in\overline{\mathbb{S}}}\phi(\theta)] \le q\mathbb{E}[\max_{i\in\overline{\mathbb{S}}}\phi(0)]$ on $-\tau \le \theta \le 0$, (49) yields

$$\mathbb{E}\left[\max_{i\in\overline{\mathbb{S}}}\mathscr{L}V(x,y,i)\right] \le -(3-2q)\,\mathbb{E}\left[\max_{i=1,2}V(x,i)\right]. \tag{83}$$

Hence, by Theorem 6, the solution $x^{\varepsilon}(t)$ is mean square stable.

Acknowledgment

This paper was supported by the National Science Foundation of China with Grant no. 60904005.

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