Extinction and Recurrence of Multi-group SEIR Epidemic Models with Stochastic Perturbations *

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Abstract: In this paper, we consider a class of multi-group SEIR epidemic models with stochastic perturbations. By the method of stochastic Lyapunov functions, we study their asymptotic behavior in terms of the intensity of the stochastic perturbations and the reproductive number R_0. When the perturbations are sufficiently large, the exposed and infective components decay exponentially to zero whilst the susceptible components converge weakly to a class of explicit stationary distributions regardless of the magnitude of R_0. An interesting result is that, if the perturbations are sufficiently small and $R_0 \leq 1$, then the exposed, infective and susceptible components have similar behaviors, respectively, as in the case of large perturbations. When the perturbations are small and $R_0 > 1$, we construct a new class of stochastic Lyapunov functions to show the ergodic property and the positive recurrence, and our results reveal some cycling phenomena of recurrent diseases. Computer simulations are carried out to illustrate our analytical results.

Key words: Ergodic property; extinction; positive recurrence; stochastic Lyapunov functions.

1 Introduction

Mathematical modeling in epidemiology has become a more and more useful tool in the analysis of spread and control of infectious diseases in host populations. There is an intensive literature on the mathematical epidemiology, for examples, [5, 6, 7, 9, 10, 13, 14, 23, 24, 25, 31, 37, 38, 39, 40, 45, 46, 47, 48, 49, 50, 51, 53] and the references therein. In particular, [1, 2, 43] are excellent books in this area.

One of classic epidemic models is the SIR model, which subdivides a homogeneous host population into three epidemiologically distinct types of individuals, the susceptible, the infective, and the removed, with their population sizes denoted by S, I and R, respectively. It is a reasonable approximation only for a disease which has a short incubation period compared with the time scale of disease transmission in the host population. But, for some diseases, such as HIV/AIDS, there is a long fixed period of time between the exposure and becoming infectious. In this case, rather than becoming infectious instantaneously, the susceptible enters into an exposed class, labeled by E, and remains there for a latent period of time. This leads to the SEIR model. Moreover, taking different internal structures of the host population and the transmission properties of infectious diseases into

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account, a heterogeneous host population can be partitioned into several homogeneous subgroups, according to various characteristics of individuals, such as age, contact patterns, social and economic status, profession and demographical distribution. This is known as a multi-group model. Obviously, in the multi-group model, we have to consider the interactions within a subgroup as well as among different subgroups in the course of the spread of infectious diseases. Thus, the multi-group model can produce more interesting and complicated scenarios of disease transmission. A tremendous variety of multi-group models have been formulated, analyzed, and applied to many infectious diseases, see [4, 16, 17, 18, 20, 27, 28, 32, 33, 35, 44] for examples. The classic multi-group SEIR models are governed by the following system of nonlinear ordinary differential equations

$$\left\{\begin{array}{rcl}\dfrac{dS_k}{dt} &=& \Lambda_k - d_k^S S_k - \displaystyle\sum_{j=1}^n \beta_{kj} S_k I_j,\\ \dfrac{dE_k}{dt} &=& \displaystyle\sum_{j=1}^n \beta_{kj} S_k I_j - (d_k^E + \varepsilon_k) E_k,\ 1 \le k \le n,\\ \dfrac{dI_k}{dt} &=& \varepsilon_k E_k - (d_k^I + \gamma_k) I_k,\\ \dfrac{dR_k}{dt} &=& \gamma_k I_k - d_k^R R_k,\end{array}\right. \tag{1.1}$$

where $S_k(t), E_k(t), I_k(t)$ and $R_k(t)$ are the population sizes of four distinct compartments in the j-th subgroup at time t. Here Λ_k represents the influx of individuals into the k-th group, β_{kj} represents the transmission coefficient between compartments S_k and I_j, d_k^S, d_k^E, d_k^I and d_k^R represent death rates of S, E, I and R populations in the k-th group, respectively, ε_k represents the rate of becoming infectious after a latent period in the k-th group, and γ_k represents the recovery rate of infectious individuals in the k-th group. All these parameters are assumed to be nonnegative except $\Lambda_k, d_k^S, d_k^E, d_k^I, d_k^R > 0$ for all k.

Since R_k's have no effects on the dynamics of S_k, E_k and I_k $(1 \le k \le n)$ in Eq. (1.1) and they can be solved explicitly once I_k's are known, they can be omitted in analysis and Eq. (1.1) is therefore reduced to

$$\left\{\begin{array}{rcl}\dfrac{dS_k}{dt} &=& \Lambda_k - d_k^S S_k - \displaystyle\sum_{j=1}^n \beta_{kj} S_k I_j,\\ \dfrac{dE_k}{dt} &=& \displaystyle\sum_{j=1}^n \beta_{kj} S_k I_j - (d_k^E + \varepsilon_k) E_k,\ 1 \le k \le n,\\ \dfrac{dI_k}{dt} &=& \varepsilon_k E_k - (d_k^I + \gamma_k) I_k.\end{array}\right. \tag{1.2}$$

That is why only three components S_k, E_k, I_k appear in the SRIR models in this paper and some other papers. Especially, Guo et al. [21] studied the globally asymptotic stability of Eq. (1.2) by a graph-theoretical approach. Let R_0 be the spectral radius of matrix $M_0 = \left(\dfrac{\beta_{kj}\varepsilon_k\Lambda_k}{d_k^S(d_k^E+\varepsilon_k)(d_k^I+\gamma_k)}\right)_{1\le k,j\le n}$. Under some mild conditions, Guo et al. [21] showed that R_0 is the basic reproduction number which is defined as the expected number of the secondary cases produced in an entirely susceptible population by a typical infected individual during its entire infectious period ([15]). Its biological significance is that if $R_0 < 1$, the diseases die out whilst if $R_0 > 1$, the diseases become endemic ([15]). Therefore, R_0 works as the threshold parameter which determines the extinction or the persistence of the diseases. In other words, if $R_0 \le 1$, the disease free equilibrium is globally asymptotically stable, whilst if $R_0 > 1$, there exists a unique endemic equilibrium which is globally asymptotically stable.

In practice, population systems are always subject to environmental noise. It is therefore necessary to develop stochastic population models. It has also been proved that the stochastic models can reveal some interesting properties which we observe in the real life. Moreover, instead of giving a single predicted value in the deterministic model, we can build up a distribution of the predicted outcomes by the trajectory of a stochastic model (ergodic property). Furthermore, we can obtain an approximation to the mean and the variance of the population sizes of the infective classes, and the probability of entering into a fixed subset at time t.

In this paper, we introduce random effects into Eq. (1.2) by replacing parameters d_k^S, d_k^E and d_k^I with $d_k^S + \sigma_k dB_k(t)$, $d_k^E + \theta_k d\xi_k(t)$ and $d_k^I + \rho_k d\eta_k(t)$, respectively. This is a standard technique in stochastic modeling (see e.g. [3, 11, 12, 30]). This is of course an initial stage. Ideally, we should have introduced random effects into the other parameters, but the analysis would become too complicated. In other words, we here only consider a reasonable stochastic analogue of Eq. (1.2) in the following form

$$\begin{cases} dS_k &= \left(\Lambda_k - d_k^S S_k - \sum_{j=1}^n \beta_{kj} S_k I_j\right) dt + \sigma_k S_k dB_k(t), \\ dE_k &= \left[\sum_{j=1}^n \beta_{kj} S_k I_j - (d_k^E + \varepsilon_k) E_k\right] dt + \theta_k E_k d\xi_k(t), \ 1 \le k \le n, \\ dI_k &= \left[\varepsilon_k E_k - (d_k^I + \gamma_k) I_k\right] dt + \rho_k I_k d\eta_k(t), \end{cases} \tag{1.3}$$

where $B_k(t)$, $\xi_k(t)$, $\eta_k(t)$, $1 \le k \le n$, are independent Brownian motions, and σ_k, θ_k, ρ_k, $1 \le k \le n$ are nonnegative and referred as their intensities of stochastic noises respectively which are used to describe the volatility of perturbations.

It is worth mentioning that because of mutual interactions among different groups, the multi-group epidemic models are much more complicated than a single-group model. The classical methods for a single-group epidemic model are not applicable. In this paper, we will use a graph-theoretical approach, the stochastic Lyapunov functions and the techniques in probability theory to investigate its asymptotic behavior. Especially, we construct a new class of stochastic Lyapunov functions combing with a graph-theoretical approach to obtain its ergodic property and positive recurrence. Our results provide an interesting insight into the spread of recurrent diseases.

Based on the above stochastic multi-group SEIR model (1.3), we study the transmission dynamics of infectious diseases according to the threshold value R_0 and the stochastic perturbations. We show that large perturbations can accelerate the extinction of epidemics. It makes sense in the point that the extinction of epidemics can be caused by occurrence of a catastrophe, such as earthquake, volcanic eruption or tsunami, which is considered as a large perturbation. When the perturbations are small and $R_0 \le 1$, this model has a similar dynamics as the case of large perturbations, and we can obtain an explicit limiting distribution of the susceptible in each subgroup. In addition, the ergodicity and the positive recurrence of multi-group SEIR model hold for small perturbations and $R_0 > 1$. In such a case, the invariant distributions of the sizes of infective components are obtained, and their positive densities lies in the first quadrant. Therefore, epidemics can be considered to persist in the heterogeneous host populations. We can also use the positive recurrence to illustrate characteristics of recurrent diseases in probabilistic sense, such as the cycling phenomena of the high and the lower infective levels. Furthermore, in practice, we usually make lots of records to investigate the dynamic behavior of recurrent diseases. If the numbers of records are great, we usually found that the average of records approaches a fixed positive point, but the records may fluctuate around this fixed point even if the numbers are large. In our stochastic model, under some mild conditions, we conclude that Eq. (1.3) is ergodic, that is to say, the average of records approaches the means of their invariant distributions as the numbers are large. Meanwhile, the records are recurrent, i.e., they can enter the high and the lower levels for infinite times (see Remark 6.1), which is a reason why the the records may fluctuate around their limits. From this point of review, Eq. (1.3) provide a good description of some biological phenomena of recurrent diseases.

The paper is organized as follows. In Section 2, we introduce some preliminaries used in the later parts. In Section 3, we show that there is a unique positive solution to Eq. (1.3) for any positive initial values. In Section 4, when the stochastic perturbations are large, we show that the exposed and the infective populations decay exponentially to zero whilst the suspectable populations converge weakly to a class of stationary distributions regardless of the magnitude of R_0. It is also noted that the increasing perturbations of the exposed and the infective compartments in a subgroup will accelerate the extinction of other subgroups. In Section 5, when $R_0 \le 1$ and the perturbations are small, we obtain similar results to those in Section 4. Especially, we can get the explicit exponential rates of the expectation of the sample means of the susceptible components. In Section 6, when $R_0 > 1$ and the perturbations are small, we construct a new class of stochastic Lyapunov functions to obtain the ergodic property and the positive recurrence of the epidemic models, which account for some recurring events of recurrent diseases. In Section 7, we make some computer simulations to illustrate our analytical results. In Section 8, we make the some further discussion and conclude our paper by emphasizing the difference between the large and small stochastic perturbations. In Section 8, we give the proofs of several results in the previous sections.)

2 Preliminaries

Firstly, we introduce some notations and results of graph theory ([36, 54]). It is known that a directed graph $\mathcal{G} = (V, E)$ contains a set $V = \{1, 2, \cdots, n\}$ of vertices and a set E of arcs (k, j) leading from initial vertex k to terminal vertex j. A subgraph $\mathcal{H}$ of $\mathcal{G}$ is said to be spanning if $\mathcal{H}$ and $\mathcal{G}$ have the same vertex set. A directed digraph $\mathcal{G}$ is weighted if each arc (k, j) is assigned a positive weight a_{kj}. Given a weighted digraph $\mathcal{G}$ with n vertices, define the weight matrix $A = (a_{kj})_{n\times n}$ whose entry a_{kj} equals the weight of arc (k, j) if it exists, and 0 otherwise. A weighted digraph is denoted by $(\mathcal{G}, A)$. A digraph $\mathcal{G}$ is strongly connected if for any pair of distinct vertices, there exists a directed path from one to the other and it is well known that a weighted digraph $(\mathcal{G}, A)$ is strongly connected if and only if the weight matrix A is irreducible ([8]). The Laplacian matrix of $(\mathcal{G}, A)$ is defined as

$$L_A = \begin{bmatrix} \sum_{k\neq 1} a_{1k} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \sum_{k\neq 2} a_{2k} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \sum_{k\neq n} a_{nk} \end{bmatrix}$$

Let $c_k, 1 \le k \le n$, denote the cofactor of the k-th diagonal element of L_A, by Kirchho's Matrix Tree Theorem ([42]) which can be expressed as follows.

Lemma 2.1. *Assume $n \ge 2$, then*

$$c_k = \sum_{\mathcal{T}\in\mathbb{T}_k} w(\mathcal{T}), \quad 1 \le k \le n,$$

where $\mathbb{T}_k$ is the set of all spanning trees $\mathcal{T}$ of $(\mathcal{G}, A)$ that are rooted at vertex k, and $w(\mathcal{T})$ is the weight of $\mathcal{T}$. In particular, if $(\mathcal{G}, A)$ is strongly connected, then $c_k > 0$, for $1 \le k \le n$.

The following lemmas are classical results of graph theory ([36, 54]), which can be used later.

Lemma 2.2. *Assume $n \geq 2$, and the matrix A is irreducible, then the solution space of the linear system $L_A v = 0$ has dimension 1 and $(c_1, \cdots, c_n)$ is a basis of the solution space, where $c_k, 1 \leq k \leq n$, are given in Lemma 2.1.*

Lemma 2.3. *Assume $n \geq 2$, and $c_k, 1 \leq k \leq n$, are given in Lemma 2.1, then the following identity holds*

$$\sum_{k=1}^{n}\sum_{j=1}^{n} c_k a_{ij} G_k(x_k) = \sum_{k=1}^{n}\sum_{j=1}^{n} c_k a_{ij} G_j(x_j),$$

where $G_k(x_k)$, $1 \leq k \leq n$, are arbitrary functions.

Lemma 2.4. *If $n \times n$ matrix A is nonnegative and irreducible, then the spectral radius $\rho(A)$ of A is a simple eigenvalue, and A has a positive left eigenvector $w = (w_1, \cdots, w_n)$ corresponding to $\rho(A)$.*

Next, we give some criteria on the ergodic property of stochastic differential equations. Throughout this paper, unless otherwise specified, $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, P)$ denotes a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Denote

$$R^l_+ = \{x \in R^l : x_i > 0 \text{ for all } 1 \leq i \leq l\}.$$

In general, let X be a regular temporally homogeneous Markov process in $E_l \subset R^l$ described by the stochastic differential equation

$$dX(t) = b\left(X(t)\right)dt + \sum_{r=1}^{d} \sigma_r\left(X(t)\right)dB_r(t), \tag{2.1}$$

with initial value $X(t_0) = x_0 \in E_l$ and $B_r(t), 1 \leq r \leq d$, are standard Brownian motions defined on the above probability space. The diffusion matrix is defined as follows

$$A(x) = \left(a_{ij}(x)\right)_{1\leq i,j\leq l}, \quad a_{ij}(x) = \sum_{r=1}^{d} \sigma_r^i(x)\sigma_r^j(x).$$

Define the differential operator L associated with equation (2.1) by

$$L = \sum_{i=1}^{l} b_i(x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{l} A_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}.$$

If L acts on a function $V \in C^{2,1}(E_l \times R_+; R)$, then

$$LV(x) = \sum_{i=1}^{l} b_i(x)\frac{\partial V}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{l} A_{ij}(x)\frac{\partial^2 V}{\partial x_i \partial x_j},$$

where $V_x = (\frac{\partial V}{\partial x_1}, \cdots, \frac{\partial V}{\partial x_l})$ and $V_{xx} = \left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)_{l\times l}$. By Itô's formula, we have

$$dV(X(t)) = LV(X(t))dt + \sum_{r=1}^{d} V_x(X(t))\sigma_r\left(X(t)\right)dB_r(t).$$

Lemma 2.5. *([22]) We assume that there exists a bounded domain $U \subset E_l$ with regular boundary, having the following properties:*

(B.1) In the domain U and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix $A(x)$ is bounded away from zero.

(B.2) If $x \in E_l \backslash U$, the mean time τ at which a path issuing from x reaches the set U is finite, and $\sup_{x\in K} E_x \tau < \infty$ for every compact subset $K \subset E_l$.

Then, the Markov process $X(t)$ has a stationary distribution $\mu(\cdot)$ with density in E_l such that for any Borel set $B \subset E_l$

$$\lim_{t\to\infty} P(t, x, B) = \mu(B),$$

and

$$P_x \left\{ \lim_{T\to\infty} \frac{1}{T} \int_0^T f\big(x(t)\big)dt = \int_{E_l} f(x)\mu(dx) \right\} = 1,$$

for all $x \in E_l$ and $f(x)$ being a function integrable with respect to the probability measure μ.

Remark 2.1. (i) The existence of the stationary distribution with density is referred to Theorem 4.1 on page 119 and Lemma 9.4 on page 138 in [22] while the ergodicity and the weak convergence are referred to Theorem 5.1 on page 121 and Theorem 7.1 on page 130 in [22].

(ii) To verify Assumptions (B.1) and (B.2), it suffices to show that there exists a bounded domain U with regular boundary and a non-negative C^2-function V such that $A(x)$ is uniformly elliptical in U and for any $x \in E_l \backslash U$, $LV(x) \leq -C$ for some $C > 0$ (See e.g. [55], page 1163).

3 Existence and Uniqueness of the Positive Solution

By Lyapunov analysis method ([41]), we show that Eq. (1.3) has a unique global and positive solution.

Theorem 3.1. *If all the system parameters in Eq. (1.3) are nonnegative except $\Lambda_k, d_k^S, d_k^E, d_k^I > 0$ for all $1 \leq k \leq n$, then there is a unique positive solution to Eq. (1.3) on $t \geq 0$ for any initial value in R_+^{3n}, and the solution will remain in R_+^{3n} with probability 1, namely $S_k(t), E_k(t)$ and $I_k(t) \in R_+$, $1 \leq k \leq n$, for all $t \geq 0$ almost surely.*

Proof. Note that the coefficients of Eq. (1.3) are locally Lipschitz continuous, thus there exists a unique local solution on $t \in [0, \tau_e)$, where τ_e is the explosion time, thus Eq. (1.3) has a unique local solution. Assume that $m_0 \geq 0$ is sufficiently large such that $S_k(0), E_k(0), I_k(0), 1 \leq k \leq n$, all lie in the interval $[1/m_0, m_0]$. For each integer $m \geq m_0$, define the stopping time

$$\begin{aligned}\tau_m = \inf\{t \in [0, \tau_e) : &\min_{1\leq k\leq n} \{S_k(t), E_k(t), I_k(t)\} \leq 1/m \text{ or} \\ &\max_{1\leq k\leq n} \{S_k(t), E_k(t), I_k(t)\} \geq m\}.\end{aligned}$$

As usual, we set $\inf \emptyset = \infty$. Clearly, τ_m is increasing. Set $\tau_\infty = \lim\limits_{m\to\infty} \tau_m$, where $0 \leq \tau_\infty \leq \tau_e$ a.s. If we show that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ and the solution remains in R_+^{3n} for all $t \geq 0$, a.s. If this statement is false, then there is a pair of constants $T > 0$ and $\epsilon \in (0, 1)$ such that

$$P\{\tau_\infty \leq T\} > \epsilon.$$

Hence there is an integer $m_1 \geq m_0$ such that

$$P\{\tau_m \leq T\} \geq \epsilon \text{ for all } m \geq m_1. \tag{3.1}$$

Define the C^2-function $V : R_+^{3n} \to R_+$ as

$$V(S_k, E_k, I_k, 1 \leq k \leq n) = \sum_{k=1}^{n}(S_k - a_k - a_k \log \frac{S_k}{a_k}) + \sum_{k=1}^{n}(E_k - 1 - \log E_k) + \sum_{k=1}^{n}(I_k - 1 - \log I_k),$$

where a_k, $1 \leq k \leq n$, are positive constants to be determined later.
By Itô's formula, we see

$$\begin{aligned} dV &= \sum_{k=1}^{n}\left[(1 - \frac{a_k}{S_k})dS_k + \frac{a_k(dS)^2}{2S_k^2}\right] + \sum_{k=1}^{n}\left[(1 - \frac{1}{E_k})dE_k + \frac{(dE_k)^2}{2E_k^2}\right] + \sum_{k=1}^{n}\left[(1 - \frac{1}{I_k})dI_k + \frac{(dI_k)^2}{2I_k^2}\right] \\ &:= LVdt + \sum_{k=1}^{n}(1 - \frac{a_k}{S_k})\sigma_k S_k dB_k(t) + \sum_{k=1}^{n}(1 - \frac{1}{E_k})\theta_k E_k d\xi_k(t) + \sum_{k=1}^{n}(1 - \frac{1}{I_k})\rho_k I_k d\eta_k(t), \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} LV &= \sum_{k=1}^{n}\left[(1 - \frac{a_k}{S_k})(\Lambda_k - d_k^S S_k - \sum_{j=1}^{n}\beta_{kj}S_k I_j) + \frac{c_k\sigma_k^2}{2}\right] + \sum_{k=1}^{n}\left\{(1 - \frac{1}{I_k})\left[\varepsilon_k E_k - (d_k^I + \gamma_k)I_k\right] + \frac{\rho_k^2}{2}\right\} \\ &\quad + \sum_{k=1}^{n}\left\{(1 - \frac{1}{E_k})\left[\sum_{j=1}^{n}\beta_{k,j}S_k I_j - (d_k^E + \varepsilon_k)E_k\right] + \frac{\theta_k^2}{2}\right\} \\ &\leq \sum_{k=1}^{n}\left[\Lambda_k - (d_k^I + \gamma_k)I_k + a_k\sum_{j=1}^{n}\beta_{kj}I_j + a_k d_k^S + d_k^E + d_k^I + \varepsilon_k + \gamma_k + \frac{a_k\sigma_k^2}{2} + \frac{\rho_k^2}{2} + \frac{\theta_k^2}{2}\right]. \end{aligned}$$

Note that

$$\begin{aligned} &\sum_{k=1}^{n}\sum_{j=1}^{n} a_k\beta_{kj}I_j - \sum_{k=1}^{n}(d_k^I + \gamma_k)I_k \\ &= \sum_{j=1}^{n}\left[\sum_{k=1}^{n} a_k\beta_{kj}\right] I_j - \sum_{j=1}^{n}(d_j^I + \gamma_j)I_j \\ &= \sum_{j=1}^{n}\left[\sum_{k=1}^{n} a_k\beta_{kj} - (d_j^I + \gamma_j)\right] I_j. \end{aligned}$$

Choose $a_k, 1 \leq k \leq n$, such that for $1 \leq j \leq n$, $\sum_{k=1}^{n} a_k\beta_{kj} \leq d_j^I + \gamma_j$, then

$$LV \leq C, \tag{3.3}$$

where C is a generic positive constant. (3.2) and (3.3) yields

$$\begin{aligned} &\int_0^{\tau_m\wedge T} dV(S_k(r), E_k(r), I_k(r), 1\le k\le n) \\ &\le \int_0^{\tau_m\wedge T} Cdr + M_{\tau_m\wedge T}, \end{aligned} \tag{3.4}$$

where $\{M_t, t\ge 0\}$ is a continuous local martingale with initial value 0.
Taking expectation on both sides of (3.4), yields

$$\begin{aligned} &E\left[V(S_k(\tau_m\wedge T), E_k(\tau_m\wedge T), I_k(\tau_m\wedge T), 1\le k\le n)\right] \\ &\le V(S_k(0), E_k(0), I_k(0), 1\le k\le n) + E\int_0^{\tau_m\wedge T} Cdr \\ &\le V(S_k(0), E_k(0), I_k(0), 1\le k\le n) + CT. \end{aligned} \tag{3.5}$$

Set $\Omega_m = \{\tau_m \le T\}$ for $m\ge m_1$ and by (3.1), we have $P(\Omega_m)\ge \epsilon$. Note that for every $\omega\in\Omega_m$, there is at least one of $S_k(\tau_m,\omega)$, $E_k(\tau_m,\omega), I_k(\tau_m,\omega), 1\le k\le n$ equals either m or $1/m$. Consequently,

$$\begin{aligned} &V(S_k(\tau_m\wedge T), E_k(\tau_m\wedge T), I_k(\tau_m\wedge T), 1\le k\le n) \\ &\ge \min_{1\le k\le n}\left\{m - a_k - a_k\log\frac{m}{a_k}, \frac{1}{m} - a_k - a_k\log\frac{1}{a_k m}\right\}\wedge(m-1-\log m)\wedge(\frac{1}{m}-1-\log\frac{1}{m}). \end{aligned}$$

It then follows from (3.1) and (3.5) that

$$\begin{aligned} &V(S_k(0), E_k(0), I_k(0), 1\le k\le n) + CT \\ &\ge E\left[1_{\Omega_m(\omega)}V(S_k(\tau_m\wedge T), E_k(\tau_m\wedge T), I_k(\tau_m\wedge T), 1\le k\le n)\right] \\ &\ge \varepsilon\min_{1\le k\le n}\left\{m - a_k - a_k\log\frac{m}{a_k}, \frac{1}{m} - a_k - a_k\log\frac{1}{a_k m}\right\}\wedge(m-1-\log m)\wedge(\frac{1}{m}-1-\log\frac{1}{m}), \end{aligned}$$

where $1_{\Omega_m(\omega)}$ is the indicator function of Ω_m. Letting $m\to\infty$ leads to the contradiction that $V(S_k(0), E_k(0), I_k(0), 1\le k\le n) + CT = \infty$. So $\tau_\infty = \infty$ a.s. □

4 Exponential stability under large perturbations

In this section, we investigate the exponential decay of the exposed and the infective components, and the weak convergence of the susceptible components under large perturbations. It is shown that even if $R_0 > 1$ of Eq. (1.2), the random effects may make the exposed and the infective components washout more likely whilst the susceptible components converge weakly to stationary distributions with the explicit densities.

Theorem 4.1. *If $B = (\beta_{kj})_{1\le kj\le n}$ is irreducible, then*

$$\begin{aligned} &\max_{1\le k\le n}\left\{\limsup_{T\to\infty}\frac{1}{T}\log E_k(T), \limsup_{T\to\infty}\frac{1}{T}\log I_k(T)\right\} \\ &\le (R_0-1)^+\max_{1\le k\le n}\{d_k^I+\gamma_k\} + R_0\sum_{j=1}^n(d_j^I+\gamma_j) - \frac{1}{2\sum_{k=1}^n\left(\frac{1}{\theta_k^2}+\frac{1}{\rho_k^2}\right)}. \end{aligned} \tag{4.1}$$

Especially, if I_k converge exponentially to 0 and $2d_k^S > \sigma_k^2$, $1 \le k \le n$, then

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T S_k(u)du = \frac{\Lambda_k}{d_k^S},\ a.s, \quad \lim_{T\to\infty} \frac{1}{T}\int_0^T E\left(S_k(t) - \frac{\Lambda_k}{d_k^S}\right)^2 dt = \frac{\sigma_k^2\Lambda_k^2}{(2d_k^S - \sigma_k^2)(d_k^S)^2},$$

and $S_k(T) \to^w \nu_k$, as $T \to \infty$, where $\to^w$ means the convergence in distribution and ν_k is a probability measure in R_+ such that $\int_0^\infty x\nu_k(dx) = \frac{\Lambda_k}{d_k^S}$. In particularly, ν_k has density $(A_k x^2 p_k(x))^{-1}$, where A_k is a normal constant and

$$p_k(x) = x^{\frac{2d_k^S}{\sigma_k^2}} \exp\left(\frac{2\Lambda_k}{\sigma_k^2 x}\right), \quad x > 0. \tag{4.2}$$

The proof is given in the Appendix.

Remark 4.1. In the deterministic model, when $R_0 > 1$, the positive solution converges to the endemic equilibrium P^*. However, if the intensities of white noise θ_k^2 and ρ_k^2 are large enough, the exposed and the infective populations of Eq. (1.3) will always die out exponentially regardless of the magnitude of R_0. Meanwhile, the increasing perturbation size of a group of populations speeds up the extinction of the other groups of populations. Thus the asymptotic behavior of perturbed epidemic models can be very different from that of the deterministic counterpart.

To explain such a phenomena, we may regard the large perturbation as the occurrence of a catastrophe, such as earthquake, volcanic eruption or tsunami, which brings depopulation of the exposed and the infective individuals. Therefore, Theorem 4.1 fits the scenario of such extinction well.

5 Exponential stability under small perturbations and $R_0 \le 1$

In this section, we investigate the asymptotic behavior under small perturbations and $R_0 \le 1$. We can obtain the similar results as that in Theorem 4.1, which are given in the following theorem.

Theorem 5.1. *If $B = (\beta_{kj})_{1\le k,j\le n}$ is irreducible, $R_0 \le 1$ and $2d_k^S > \sigma_k$, $1 \le k \le n$, then*

$$\begin{aligned} &\max_{1\le k\le n}\left\{\limsup_{T\to\infty}\frac{1}{T}\log E_k(T), \limsup_{T\to\infty}\frac{1}{T}\log I_k(T)\right\} \\ &\le \max_{1\le k\le n}\left\{\frac{\sigma_k}{\sqrt{2d_k^S - \sigma_k^2}}\right\}\sum_{j=1}^n R_0(d_j^I + \gamma_j) - \frac{1}{2\sum_{k=1}^n\left(\frac{1}{\theta_k^2} + \frac{1}{\rho_k^2}\right)}. \end{aligned} \tag{5.1}$$

Especially, if $I_k, 1 \le k \le n$, converge exponentially to 0, then $S_k(T) \to^w \nu_k$, as $T \to \infty$, where ν_k is the probability measure in R_+ defined in Theorem 4.1 and for any $1 \le k \le n$,

$$\lim_{T\to\infty}\frac{1}{T}\log\left(\left|ES_k(T) - \frac{\Lambda_k}{d_k^S}\right|\right) = -d_k^S. \tag{5.2}$$

The proof can be seen in the Appendix.

Remark 5.1. If $R_0 \le 1$ and σ_k^S is small enough, then the exposed and the infective components always have exponential stability whilst the susceptible components converge weakly to a class of stationary distributions instead of P_0 on the effect of white noises. Furthermore, the expectation of sample means of the susceptible components have also the exponential convergence with the explicit rates.

6 Ergodic Property under small perturbations and $R_0 > 1$

By a new class of stochastic Lyapunov functions, we obtain the ergodic property and the positive recurrence of Eq. (1.3) to illustrate the cycling phenomena of recurrent diseases.

Theorem 6.1. *If $B = (\beta_{kj})_{1\le k,j\le n}$ is irreducible, $R_0 > 1$, $\sigma_k, \theta_k, \rho_k, 1 \le k \le n$, are positive and*

$$\min_{1\le k\le n}\left\{\frac{c_k(d_k^S-\sigma_k^2)S_k^*}{2}, a_k(d_k^E-2\theta_k^2)(E_k^*)^2, a_k(d_k^I+\gamma_k-2\rho_k^2)(I_k^*)^2\right\} > \sum_{k=1}^n (A_k\sigma_k^2+B_k\theta_k^2+C_k\rho_k^2), \tag{6.1}$$

then Eq. (1.3) has the ergodic property and converges to the unique stationary distribution μ. Here $P^* = (S_1^*, \cdots, S_n^*, E_1^*, \cdots, E_n^*, I_1^*, \cdots, I_n^*)$ is the unique endemic equilibrium of (1.2), c_k, $1 \le k \le n$, denotes the cofactor of the k-th diagonal element of $L_{\bar B}$, respectively, $\bar B = (\bar B_{k,j})_{n\times n} = (\beta_{kj}S_k^*I_j^*)_{n\times n}$, and

$$\begin{aligned}
a_k &= \left(d_k^E + d_k^I + \gamma_k + 2d_k^S + (d_k^S)^2\frac{d_k^E+d_k^I+\gamma_k}{(d_k^I+\gamma_k)d_k^E} + 2\sigma_k^2\right)^{-1}\frac{c_k(d_k^S-\sigma_k^2)}{2S_k^*},\\
\kappa &= \max_{1\le k\le n}\left\{\frac{\beta_{k,j}I_k^*}{d_k^S}\right\},\\
A_k &= \left(\frac{\kappa}{2}+1\right)c_kS_k^* + 2a_k(S_k^*)^2,\\
B_k &= \frac{(\kappa+1)c_kE_k^*}{2} + 2a_k(E_k^*)^2,\\
C_k &= \frac{(\kappa+1)(d_k^E+\varepsilon_k)c_kI_k^*}{2\varepsilon_k} + 2a_k(I_k^*)^2\left(\frac{d_k^I+\gamma_k+d_k^E}{\varepsilon_k}+1\right).
\end{aligned}$$

Especially, we have

$$\begin{aligned}
\lim_{T\to\infty}\frac{1}{T}E\int_0^T\sum_{k=1}^n\Big[&\frac{c_k(d_k^S-\sigma_k^2)}{2S_k^*}(S_k(t)-S_k^*)^2 + a_k(d_k^E-2\theta_k^2)(E_k(t)-E_k^*)^2\\
&+a_k(d_k^I+\gamma_k-2\rho_k^2)(I_k(t)-I_k^*)^2\Big]\,dt \le \sum_{k=1}^n(A_k\sigma_k^2+B_k\theta_k^2+C_k\rho_k^2).
\end{aligned}$$

Proof. If $B = (\beta_{kj})_{1\le k,j\le n}$ is irreducible and $R_0 > 1$, Guo et al. ([21]) pointed out there is a unique endemic equilibrium $P^* = (S_1^*, \cdots, S_n^*, E_1^*, \cdots, E_n^*, I_1^*, \cdots, I_n^*)$ in Eq. (1.2) such that for $1 \le k \le n$,

$$\begin{aligned}
\Lambda_k &= d_k^S S_k^* + \sum_{j=1}^n \beta_{kj}S_k^*I_j^*,\\
\sum_{j=1}^n \beta_{kj}S_k^*I_j^* &= (d_k^E+\varepsilon_k)E_k^*,\\
\varepsilon_kE_k^* &= (d_k^I+\gamma_k)I_k^*.
\end{aligned} \tag{6.2}$$

Firstly, define the C^2 function V_1: $R_+^{3n} \to R_+$ by

$$V_1(S_k, E_k, I_k, 1 \le k \le n) = \sum_{k=1}^{n} c_k \left[S_k - S_k^* - S_k^* \log \frac{S_k}{S_k^*} + E_k - E_k^* - E_k^* \log \frac{E_k}{E_k^*} \right.$$
$$\left. + \frac{d_k^E + \varepsilon_k}{\varepsilon_k} \left(I_k - I_k^* - I_k^* \log \frac{I_k}{I_k^*} \right) \right].$$

We compute

$$\begin{aligned} LV_1 = & \sum_{k=1}^{n} c_k \left[\Lambda_k - d_k^S S_k - \frac{(d_k^E + \varepsilon_k)(d_k^I + \gamma_k) I_k}{\varepsilon_k} - \frac{\Lambda_k S_k^*}{S_k} + \sum_{j=1}^{n} \beta_{kj} S_k^* I_j + d_k^S S_k^* \right. \\ & \left. - \sum_{j=1}^{n} \frac{\beta_{kj} S_k I_j E_k^*}{E_k} + (d_k^E + \varepsilon_k) E_k^* - \frac{(d_k^E + \varepsilon_k) E_k I_k^*}{I_k} + \frac{(d_k^E + \varepsilon_k)(d_k^I + \gamma_k) I_k^*}{\varepsilon_k} \right] \\ & + \sum_{k=1}^{n} c_k \left[\frac{\sigma_k^2 S_k^*}{2} + \frac{\theta_k^2 E_k^*}{2} + \frac{\rho_k^2 I_k^* (d_k^E + \varepsilon_k)}{2\varepsilon_k} \right]. \end{aligned} \tag{6.3}$$

Substituting (6.2) into (6.3), we have

$$\begin{aligned} LV_1 = & \sum_{k=1}^{n} c_k \left[2 d_k^S S_k^* - d_k^S S_k - \frac{d_k^S (S_k^*)^2}{S_k} + 3 \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* - \sum_{j=1}^{n} \frac{\beta_{kj} (S_k^*)^2 I_j^*}{S_k} + \sum_{j=1}^{n} \beta_{kj} S_k^* I_j \right. \\ & \left. - \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* \frac{S_k I_j E_k^*}{S_k^* I_j^* E_k} - \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* \frac{E_k I_k^*}{E_k^* I_k} - \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* \frac{I_k}{I_k^*} \right] \\ & + \sum_{k=1}^{n} c_k \left[\frac{\sigma_k^2 S_k^*}{2} + \frac{\theta_k^2 E_k^*}{2} + \frac{\rho_k^2 I_k^* (d_k^E + \varepsilon_k)}{2\varepsilon_k} \right] \\ = & \sum_{k=1}^{n} c_k d_k^S S_k^* \left(2 - \frac{S_k}{S_k^*} - \frac{S_k^*}{S_k} \right) + \sum_{k=1}^{n} c_k \left[3 \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* - \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* \frac{S_k^*}{S_k} \right. \\ & \left. - \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* \frac{S_k I_j E_k^*}{S_k^* I_j^* E_k} - \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* \frac{E_j I_j^*}{E_j^* I_j} \right] \\ & + \sum_{k=1}^{n} c_k \left[\sum_{j=1}^{n} \beta_{kj} S_k^* I_j - \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* \frac{I_k}{I_k^*} \right] + \sum_{k=1}^{n} c_k \left[\frac{\sigma_k^2 S_k^*}{2} + \frac{\theta_k^2 E_k^*}{2} + \frac{\rho_k^2 I_k^* (d_k^E + \varepsilon_k)}{2\varepsilon_k} \right]. \end{aligned}$$

By Lemma 2.3,

$$\sum_{k=1}^{n} c_k \left[\sum_{j=1}^{n} \beta_{kj} S_k^* I_j - \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* \frac{I_k}{I_k^*} \right] = \sum_{k=1}^{n} \sum_{j=1}^{n} c_k \beta_{kj} S_k^* I_j^* \frac{I_j}{I_j^*} - \sum_{k=1}^{n} \sum_{j=1}^{n} c_k \beta_{kj} S_k^* I_j^* \frac{I_k}{I_k^*} = 0. \tag{6.4}$$

Meanwhile, the inequality $x \ge 1 + \ln x$, $x > 0$ implies

$$
\begin{aligned}
&\sum_{k=1}^{n} c_k \left[3\sum_{j=1}^{n}\beta_{kj}S_k^*I_j^* - \sum_{j=1}^{n}\beta_{kj}S_k^*I_j^*\frac{S_k^*}{S_k} - \sum_{j=1}^{n}\beta_{kj}S_k^*I_j^*\frac{S_kI_jE_k^*}{S_k^*I_j^*E_k} - \sum_{j=1}^{n}\beta_{kj}S_k^*I_j^*\frac{E_jI_j^*}{E_j^*I_j}\right]\\
&= \sum_{k=1}^{n} c_k \sum_{j=1}^{n}\beta_{k,j}S_k^*I_j^*\left[3 - \frac{S_k^*}{S_k} - \frac{S_kI_jE_k^*}{S_k^*I_j^*E_k} - \frac{E_jI_j^*}{E_j^*I_j}\right]\\
&\le \sum_{k=1}^{n} c_k \sum_{j=1}^{n}\beta_{kj}S_k^*I_j^*\left[\ln\frac{E_k}{E_k^*} - \ln\frac{E_j}{E_j^*}\right]\\
&= \sum_{k=1}^{n}\sum_{j=1}^{n} c_k\beta_{kj}S_k^*I_j^*\ln\frac{E_k}{E_k^*} - \sum_{k=1}^{n}\sum_{j=1}^{n} c_k\beta_{kj}S_k^*I_j^*\ln\frac{E_j}{E_j^*} = 0,
\end{aligned}
\tag{6.5}
$$

where the last equality is derived from Lemma 2.3.
Therefore, (6.4) and (6.5) yield

$$
LV_1 \le \sum_{k=1}^{n} c_k d_k^S S_k^*\left(2 - \frac{S_k}{S_k^*} - \frac{S_k^*}{S_k}\right) + \sum_{k=1}^{n} c_k\left[\frac{\sigma_k^2 S_k^*}{2} + \frac{\theta_k^2E_k^*}{2} + \frac{\rho_k^2I_k^*(d_k^E+\varepsilon_k)}{2\varepsilon_k}\right]. \tag{6.6}
$$

Secondly, define the C^2 function V_2: $R_+^{2n} \to R_+$ by

$$
V_2(S_k, E_k, I_k, 1\le k\le n) = \sum_{k=1}^{n} c_k\left[E_k - E_k^* - E_k^*\log\frac{E_k}{E_k^*} + \frac{d_k^E+\varepsilon_k}{\varepsilon_k}\left(I_k - I_k^* - I_k^*\log\frac{I_k}{I_k^*}\right)\right].
$$

By computation,

$$
\begin{aligned}
LV_2 &= \sum_{k=1}^{n} c_k\left[\sum_{j=1}^{n}\beta_{kj}S_kI_j - \frac{(d_k^E+\varepsilon_k)(d_k^I+\gamma_k)}{\varepsilon_k}I_k - \sum_{j=1}^{n}\frac{\beta_{kj}S_kI_jE_k^*}{E_k} + (d_k^E+\varepsilon_k)E_k^*\right.\\
&\quad\left. - \frac{(d_k^E+\varepsilon_k)E_k}{I_k} + \frac{(d_k^E+\varepsilon_k)(d_k^I+\gamma_k)}{\varepsilon_k}I_k^*\right] + \sum_{k=1}^{n} c_k\left(\frac{\theta_k^2E_k^*}{2} + \frac{(d_k^E+\varepsilon_k)\rho_k^2I_k^*}{2\varepsilon_k}\right)\\
&= \sum_{k=1}^{n} c_k\left[\sum_{j=1}^{n}\beta_{kj}(S_k-S_k^*)(I_j-I_j^*) + \sum_{j=1}^{n}\beta_{kj}S_kI_k^* + \sum_{j=1}^{n}\beta_{kj}S_k^*I_j - \sum_{j=1}^{n}\beta_{kj}S_k^*I_k^*\right.\\
&\quad - \frac{(d_k^E+\varepsilon_k)(d_k^I+\gamma_k)}{\varepsilon_k}I_k - \sum_{j=1}^{n}\frac{\beta_{kj}S_kI_jE_k^*}{E_k} + (d_k^E+\varepsilon_k)E_k^* - \frac{(d_k^E+\varepsilon_k)E_k}{I_k}\\
&\quad\left. + \frac{(d_k^E+\varepsilon_k)(d_k^I+\gamma_k)}{\varepsilon_k}I_k^*\right] + \sum_{k=1}^{n} c_k\left(\frac{\theta_k^2E_k^*}{2} + \frac{(d_k^E+\varepsilon_k)\rho_k^2I_k^*}{2\varepsilon_k}\right)\\
&= \sum_{k=1}^{n}\sum_{j=1}^{n} c_k\beta_{kj}(S_k-S_k^*)(I_j-I_j^*) + \sum_{k=1}^{n}\sum_{j=1}^{n} c_k\beta_{kj}S_k^*I_k^*\left[1 + \frac{S_k}{S_k^*} - \frac{S_kI_jE_k^*}{S_k^*I_j^*E_k} - \frac{E_kI_k^*}{E_k^*I_k}\right]\\
&\quad + \sum_{k=1}^{n}\sum_{j=1}^{n} c_k\beta_{kj}S_k^*I_j^*\frac{I_j}{I_j^*} - \sum_{k=1}^{n} c_k\beta_{kj}S_k^*I_j^*\frac{I_k}{I_k^*} + \sum_{k=1}^{n} c_k\left(\frac{\theta_k^2E_k^*}{2} + \frac{(d_k^E+\varepsilon_k)\rho_k^2I_k^*}{2\varepsilon_k}\right),
\end{aligned}
$$

where the last equality is derived from (6.2).
Applying inequality $x \geq 1 + \log x$, $x > 0$ again, yields

$$
\begin{aligned}
&\sum_{k=1}^{n}\sum_{j=1}^{n} c_k\beta_{kj}S_k^*I_k^*\left[1 + \frac{S_k}{S_k^*} - \frac{S_kI_jE_k^*}{S_k^*I_j^*E_k} - \frac{E_kI_k^*}{E_k^*I_k}\right] \\
&\leq \sum_{k=1}^{n}\sum_{j=1}^{n} c_k\beta_{kj}S_k^*I_k^*\left[\frac{S_k}{S_k^*} - 1 - \log\frac{S_kI_jE_k^*}{S_k^*I_j^*E_k} - \log\frac{E_kI_k^*}{E_k^*I_k}\right] \\
&\leq \sum_{k=1}^{n}\sum_{j=1}^{n} c_k\beta_{kj}S_k^*I_k^*\left[\frac{S_k}{S_k^*} - 1 + \log\frac{S_k^*}{S_k} - \log\frac{I_j}{I_j^*} - \log\frac{I_k^*}{I_k}\right] \\
&\leq \sum_{k=1}^{n}\sum_{j=1}^{n} c_k\beta_{kj}S_k^*I_k^*\left[\frac{S_k}{S_k^*} + \frac{S_k^*}{S_k} - 2\right] - \sum_{k=1}^{n}\sum_{j=1}^{n} c_k\beta_{kj}S_k^*I_k^*\left[\log\frac{I_j}{I_j^*} - \log\frac{I_k^*}{I_k}\right] \\
&\leq \sum_{k=1}^{n}\sum_{j=1}^{n} c_k\beta_{kj}S_k^*I_k^*\left[\frac{S_k}{S_k^*} + \frac{S_k^*}{S_k} - 2\right],
\end{aligned}
\tag{6.7}
$$

where the last inequality is derived from Lemma 2.3 such that

$$
\sum_{k=1}^{n}\sum_{j=1}^{n} c_k\beta_{kj}S_k^*I_k^*\log\frac{I_j}{I_j^*} - \sum_{k=1}^{n}\sum_{j=1}^{n} c_k\beta_{kj}S_k^*I_k^*\log\frac{I_k^*}{I_k} = 0.
$$

Similarly, we get

$$
\sum_{k=1}^{n}\sum_{j=1}^{n} \beta_{kj}S_k^*I_j^*\frac{I_j}{I_j^*} - \sum_{k=1}^{n}\sum_{j=1}^{n} \beta_{kj}S_k^*I_j^*\frac{I_k}{I_k^*} = 0. \tag{6.8}
$$

Hence, (6.7) and (6.8) imply

$$
\begin{aligned}
LV_2 \leq &\sum_{k=1}^{n}\sum_{j=1}^{n} c_k\beta_{kj}(S_k - S_k^*)(I_j - I_j^*) + \sum_{k=1}^{n}\sum_{j=1}^{n} c_k\beta_{kj}S_k^*I_k^*\left[\frac{S_k}{S_k^*} + \frac{S_k^*}{S_k} - 2\right] \\
&+ \sum_{k=1}^{n} c_k\left(\frac{\theta_k^2E_k^*}{2} + \frac{(d_k^E + \varepsilon_k)\rho_k^2I_k^*}{2\varepsilon_k}\right).
\end{aligned}
\tag{6.9}
$$

Thirdly, define the C^2 function V_3: $R_+^n \to R_+$ by

$$
V_3(S_k, 1 \leq k \leq n) = \sum_{k=1}^{n} \frac{c_k(S_k - S_k^*)^2}{2S_k^*}.
$$

By computation and (6.2),

$$
\begin{aligned}
LV_3 &= \sum_{k=1}^{n} \frac{c_k(S_k - S_k^*)}{S_k^*}(\Lambda_k - d_k^S S_k - \sum_{j=1}^{n} \beta_{kj} S_k I_j) + \sum_{k=1}^{n} \frac{c_k \sigma_k^2 S_k^2}{2S_k^*} \\
&= -\sum_{k=1}^{n} \frac{c_k d_k^S (S_k - S_k^*)^2}{S_k^*} - \sum_{k=1}^{n}\sum_{j=1}^{n} \frac{c_k \beta_{kj}(S_k - S_k^*)^2 I_j}{S_k^*} \\
&\quad - \sum_{k=1}^{n}\sum_{j=1}^{n} c_k \beta_{kj}(S_k - S_k^*)(I_j - I_j^*) + \sum_{k=1}^{n} \frac{c_k \sigma_k^2 S_k^2}{2S_k^*} \\
&\le -\sum_{k=1}^{n} \frac{c_k(d_k^S - \sigma_k^2)(S_k - S_k^*)^2}{S_k^*} - \sum_{k=1}^{n}\sum_{j=1}^{n} c_k \beta_{kj}(S_k - S_k^*)(I_j - I_j^*) + \sum_{k=1}^{n} c_k S_k^* \sigma_k^2.
\end{aligned} \tag{6.10}
$$

Fourthly, define the C^2 function V_4: $R_+^{3n} \to R_+$ by

$$
V_4(S_k, E_k, I_k, 1 \le k \le n) = \sum_{k=1}^{n} a_k(S_k - S_k^* + E_k - E_k^* + I_k - I_k^*)^2.
$$

By computation and (6.2),

$$
\begin{aligned}
LV_4 &= 2\sum_{k=1}^{n} a_k(S_k - S_k^* + E_k - E_k^* + I_k - I_k^*)\left(\Lambda_k - d_k^S S_k - d_k^E E_k - (d_k^I + \gamma_k) I_k\right) \\
&\qquad + \sum_{k=1}^{n} a_k(\sigma_k^2 S_k^2 + \theta_k^2 E_k^2 + \rho_k^2 I_k^2) \\
&= -2\sum_{k=1}^{n} a_k \left[d_k^S (S_k - S_k^*)^2 + d_k^E (E_k - E_k^*)^2 + (d_k^I + \gamma_k)(I_k - I_k^*)^2 \right.\\
&\quad + (d_k^S + d_k^E)(S_k - S_k^*)(E_k - E_k^*) + (d_k^I + \gamma_k + d_k^S)(S_k - S_k^*)(I_k - I_k^*) \\
&\quad \left. + (d_k^I + \gamma_k + d_k^E)(E_k - E_k^*)(I_k - I_k^*)\right] + \sum_{k=1}^{n} a_k(\sigma_k^2 S_k^2 + \theta_k^2 E_k^2 + \rho_k^2 I_k^2).
\end{aligned}
$$

Since $2ab \le a^2 + b^2$, we obtain

$$
\begin{aligned}
LV_4 &\le \sum_{k=1}^{n} a_k m_k (S_k - S_k^*)^2 - \sum_{k=1}^{n} a_k(d_k^E - 2\theta_k^2)(E_k - E_k^*)^2 - \sum_{k=1}^{n} a_k(d_k^I + \gamma_k - 2\rho_k^2)(I_k - I_k^*)^2 \\
&\quad - 2\sum_{k=1}^{n} a_k(d_k^I + \gamma_k + d_k^E)(E_k - E_k^*)(I_k - I_k^*) + 2\sum_{k=1}^{n} a_k(\sigma_k^2 (S_k^*)^2 + \theta_k^2 (E_k^*)^2 + \rho_k^2 (I_k^*)^2),
\end{aligned} \tag{6.11}
$$

where $m_k = d_k^E + d_k^I + \gamma_k + 2d_k^S + (d_k^S)^2 \dfrac{d_k^E + d_k^I + \gamma_k}{(d_k^I + \gamma_k) d_k^E} + 2\sigma_k^2$.

Finally, define the C^2 function V_5: $R_+^n \to R_+$ by

$$
V_5(I_k, 1 \le k \le n) = \sum_{k=1}^{n} \frac{a_k(d_k^I + \gamma_k + d_k^E)}{\varepsilon_k}(I_k - I_k^*)^2.
$$

We compute

$$
\begin{aligned}
LV_5 &= 2\sum_{k=1}^{n} a_k(d_k^I+\gamma_k+d_k^E)(E_k-E_k^*)(I_k-I_k^*) + \sum_{k=1}^{n}\frac{a_k(d_k^I+\gamma_k+d_k^E)\rho_k^2 I_k^2}{\varepsilon_k} \\
&\qquad - 2\sum_{k=1}^{n}\frac{a_k(d_k^I+\gamma_k+d_k^E)(d_k^I+\gamma_k)}{\varepsilon_k}(I_k-I_k^*)^2 \\
&\le 2\sum_{k=1}^{n} a_k(d_k^I+\gamma_k+d_k^E)(E_k-E_k^*)(I_k-I_k^*) + 2\sum_{k=1}^{n}\frac{a_k(d_k^I+\gamma_k+d_k^E)\rho_k^2 (I_k^*)^2}{\varepsilon_k} \\
&\qquad - 2\sum_{k=1}^{n}\frac{a_k(d_k^I+\gamma_k+d_k^E)(d_k^I+\gamma_k-\rho_k^2)}{\varepsilon_k}(I_k-I_k^*)^2 \\
&\le 2\sum_{k=1}^{n} a_k(d_k^I+\gamma_k+d_k^E)(E_k-E_k^*)(I_k-I_k^*) + 2\sum_{k=1}^{n}\frac{a_k(d_k^I+\gamma_k+d_k^E)\rho_k^2 (I_k^*)^2}{\varepsilon_k}.
\end{aligned}
\tag{6.12}
$$

Hence, (6.11) and (6.12) imply

$$
\begin{aligned}
L(V_4+V_5) &\le \sum_{k=1}^{n} a_k m_k(S_k-S_k^*)^2 - \sum_{k=1}^{n} a_k(d_k^E-2\theta_k^2)(E_k-E_k^*)^2 - \sum_{k=1}^{n} a_k(d_k^I+\gamma_k-2\rho_k^2)(I_k-I_k^*)^2 \\
&\qquad + 2\sum_{k=1}^{n} a_k(\sigma_k^2(S_k^*)^2+\theta_k^2(E_k^*)^2+\rho_k^2(I_k^*)^2) + 2\sum_{k=1}^{n}\frac{a_k(d_k^I+\gamma_k+d_k^E)\rho_k^2 (I_k^*)^2}{\varepsilon_k}.
\end{aligned}
\tag{6.13}
$$

(6.6) together with (6.9), (6.10) and (6.13), yield

$$
\begin{aligned}
L\left(\kappa V_1+V_2+V_3+V_4+V_5\right) &\le -\sum_{k=1}^{n}\frac{c_k(d_k^S-\sigma_k^2)}{2S_k^*}(S_k-S_k^*)^2 - \sum_{k=1}^{n} a_k(d_k^E-2\theta_k^2)(E_k-E_k^*)^2 \\
&\quad - \sum_{k=1}^{n} a_k(d_k^I+\gamma_k-2\rho_k^2)(I_k-I_k^*)^2 + \kappa\sum_{k=1}^{n} c_k\left[\frac{\sigma_k^2 S_k^*}{2}+\frac{\theta_k^2 E_k^*}{2}+\frac{\rho_k^2 I_k^*(d_k^E+\varepsilon_k)}{2\varepsilon_k}\right] \\
&\quad + \sum_{k=1}^{n} c_k\left(\frac{\theta_k^2 E_k^*}{2}+\frac{(d_k^E+\varepsilon_k)\rho_k^2 I_k^*}{2\varepsilon_k}\right) + 2\sum_{k=1}^{n}\frac{a_k(d_k^I+\gamma_k+d_k^E)\rho_k^2 (I_k^*)^2}{\varepsilon_k} \\
&\quad + 2\sum_{k=1}^{n} a_k(\sigma_k^2(S_k^*)^2+\theta_k^2(E_k^*)^2+\rho_k^2(I_k^*)^2) + \sum_{k=1}^{n} c_k S_k^*\sigma_k^2 \\
&= -\sum_{k=1}^{n}\left[\frac{c_k(d_k^S-\sigma_k^2)}{2S_k^*}(S_k-S_k^*)^2 + a_k(d_k^E-2\theta_k^2)(E_k-E_k^*)^2 + a_k(d_k^I+\gamma_k-2\rho_k^2)(I_k-I_k^*)^2\right] \\
&\quad + \sum_{k=1}^{n}(A_k\sigma_k^2+B_k\theta_k^2+C_k\rho_k^2),
\end{aligned}
$$

Note that if (6.1) holds, then the ellipsoid

$$\sum_{k=1}^{n}\left[\frac{c_k(d_k^S-\sigma_k^2)}{2S_k^*}(S_k-S_k^*)^2+a_k(d_k^E-2\theta_k^2)(E_k-E_k^*)^2+a_k(d_k^I+\gamma_k-2\rho_k^2)(I_k-I_k^*)^2\right]$$
$$=\sum_{k=1}^{n}(A_k\sigma_k^2+B_k\theta_k^2+C_k\rho_k^2)$$

lies entirely in R_+^{3n}. We can take U to be any neighborhood of the ellipsoid with $\bar{U}\subseteq E_l=R_+^{3n}$, so for $x\in U\setminus E_l$, $LV\le -C$, which implies condition (B.2) in Lemma (2.5) is satisfied. Besides, there is a constant $C>0$ such that for $x\in\bar{U},\xi\in R^{3n}$,

$$\sum_{i,j=1}^{3n}\left(\sum_{k=1}^{3n}a_{ik}(x)a_{jk}(x)\right)\xi_i\xi_j=\sum_{k=1}^{n}\sigma_k^2x_k^2\xi_k^2+\sum_{k=1}^{n}\theta_k^2x_{n+k}^2\xi_{n+k}^2+\sum_{k=1}^{n}\rho_k^2x_{2n+k}^2\xi_{2n+k}^2\ge C.$$

Applying Rayleigh's principle ([52], P349), condition (B.1) is satisfied. Therefore, Eq. (1.3) has a unique stationary distribution μ in R_+^{3n} and it is ergodic. □

Corollary 6.1. *Under the above assumptions, Eq.* (1.3) *is positive recurrent.*

Proof. In the proof of Theorem 6.1, condition (B.2) of Lemma 2.5 is verified. Thus, Eq. (1.3) is positive recurrent by the definition of positive recurrence and lemma 3.1 in [22], on page 116-117. □

Remark 6.1. For fixed α_1 and α_2 such that $\alpha_1>\alpha_2>0$, let $U_1=\{x\in R_+;x\ge\alpha_1\}$ and $U_2=\{x\in R_+;x\le\alpha_2\}$ denote the high and the lower infective levels, respectively. Set $\tau_0^i=\inf\{t\ge 0;I_i(t)\in U_1\}$ and $\tau_1^i=\inf\{t\ge\tau_0^i;I_i(t)\in U_2\}$, we define the following sequence of stopping times recursively for every $1\le i\le n$:

$$\begin{aligned}\tau_{2k}^i&=\inf\{t\ge\tau_{2k-1}^i;I_i(t)\in U_1\},\quad k\ge1;\\ \tau_{2k+1}^i&=\inf\{t\ge\tau_{2k}^i;I_i(t)\in U_2\},\quad k\ge1.\end{aligned}$$

By Corollary 1 and strong Markov property, $\tau_k^i<\infty,k\ge0$, a.s. This means the recurring phenomena of high and lower infective levels in j-th group, which can be used to illustrate some cycling events of recurrent diseases and provide a biological insight of recurrent diseases.

7 Simulations

In this section, we make simulations to confirm our analytical results. Using the Milstein's higher order method ([26]), we simulate the positive solution to Eq. (1.3) with the given positive initial value and parameters. The corresponding discretization equations are

$$\begin{cases}S_{k,i+1}=S_{k,i}+(\Lambda_k-d_k^SS_{k,i}-\sum_{j=1}^{n}\beta_{kj}S_{k,i}I_{j,i})\Delta t+\sigma_kS_{k,i}B_{k,i}\sqrt{\Delta t}+\dfrac{\sigma_k^2S_{k,i}}{2}(B_{k,i}^2\Delta t-\Delta t),\\ E_{k,i+1}=E_{k,i}+\left(\sum_{j=1}^{n}\beta_{kj}S_{k,i}I_{j,i}-(d_k^E+\varepsilon_k)E_{k,i}\right)\Delta t+\theta_kE_{k,i}\xi_{k,i}\sqrt{\Delta t}+\dfrac{\theta_k^2E_{k,i}}{2}(\xi_{k,i}^2\Delta t-\Delta t),\\ I_{k,i+1}=I_{k,i}+\left(\varepsilon_kE_{k,i}-(\gamma_k+d_k^I)I_{k,i}\right)\Delta t+\rho_kI_{k,i}\eta_{k,i}\sqrt{\Delta t}+\dfrac{\rho_k^2I_{k,i}}{2}(\eta_{k,i}^2\Delta t-\Delta t),\end{cases}$$

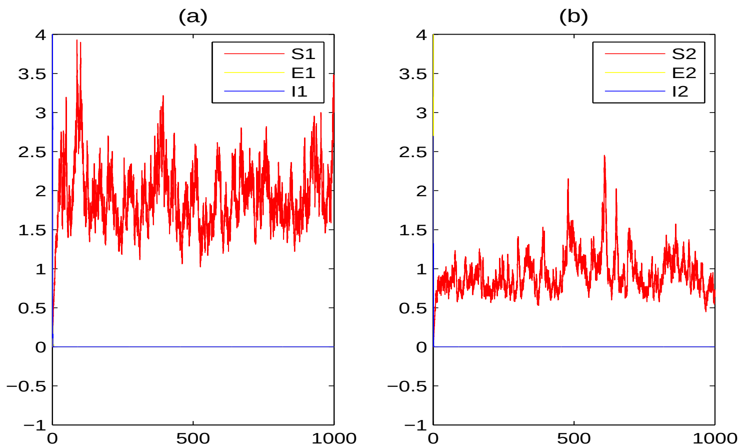

Figure 1

where $B_{k,i}, \xi_{k,i}, \eta_{k,i}$, $1 \le k \le n$, $1 \le j \le m$, are independent Gaussian random variables with distribution $N(0,1)$. Let $n = 2$, we consider two groups of infective populations. In Eq. (1.3), we choose the positive initial value $(S_1(0), S_2(0), E_1(0), E_2(0), I_1(0), I_2(0)) = (3.6, 2.4, 2.7, 3.2, 2.1, 2.7)$ and the parameters $\Lambda_1 = 0.2$, $\Lambda_2 = 0.1$, $\varepsilon_1 = 0.3$, $\varepsilon_2 = 0.2$, $\gamma_1 = 0.3$, $\gamma_2 = 0.2$, $d_k^S = d_k^E = d_k^I = 0.1$, $1 \le k \le 2$. Using Matlab software, we simulate the solution to Eq. (1.3) with different values of β_{kj}, σ_k^S, θ_k^E and ρ_k^I, $1 \le k, j \le 2$.

In Fig.1-Fig.6, we choose parameters such that $R_0 > 1$ and the conditions in Theorem 4.1 are satisfied, i.e., $\beta_{11} = \beta_{22} = \beta_{12} = \beta_{21} = 2$, $\sigma_1 = \sigma_2 = 0.1$, $\theta_1 = \theta_2 = 2.4$, $\rho_1 = \rho_2 = 3.0$. In (a) and (b), note that the exposed and the infective populations tend exponentially to 0. In (c) and (d), the average of susceptible populations $\frac{1}{T}\int_0^T S_k(t)dt$ converges to $\frac{\Lambda_k}{d_k^S}$, $1 \le k \le 2$. In Fig.3-Fig.6, we represent the histograms of S_k and ν_k, $1 \le k \le 2$. We use statistical software R to record the values of $S_k, 1 \le k \le 2$, at large time $t = 50000$, and $\Delta t = 0.01$. Comparing these figures we know that when the time is large, the kernel density of S_k looks very like the one of ν_k, $1 \le k \le 2$. Thus S_k is a good approximation to ν_k, $1 \le k \le 2$.

In Fig. 7 and Fig. 8, we only change the values of the intensities: $\sigma_k = \theta_k = \rho_k = 0.1$, $1 \le k \le 2$. In (e) and (f), the simulating solutions fluctuate around the endemic equilibrium. In (g) and (h), we give the simulations of $\frac{1}{t}\int_0^t S_k(s)ds$, $\frac{1}{t}\int_0^t E_k(s)ds$ and $\frac{1}{t}\int_0^t I_k(s)ds$, $1 \le k \le 2$, which conform the ergodicity of Eq. (1.3).

8 Discussion and concluding remarks

It is seen that when the perturbations are very large, the exposed and the infective components will be forced to expire. It makes sense in the point that the extinction of epidemics can be caused by the occurrence of a catastrophe, such as earthquake, volcanic eruption, or tsunami, which can be considered as a large perturbation. Hence, R_0 will not act as the threshold to determine the extinction or the persistence of epidemics as that of the deterministic model. In such a case, the exposed and the infective components decay exponentially to zero in every group regardless of the magnitude of R_0. We also obtain the weak convergence of the susceptible components. In particular, the expectation of limiting sample means, limiting sample variances and the densities of invariant distributions of the

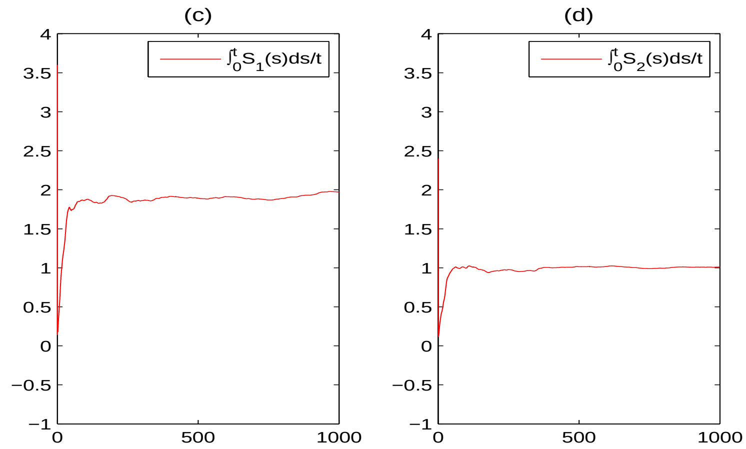

Figure 2

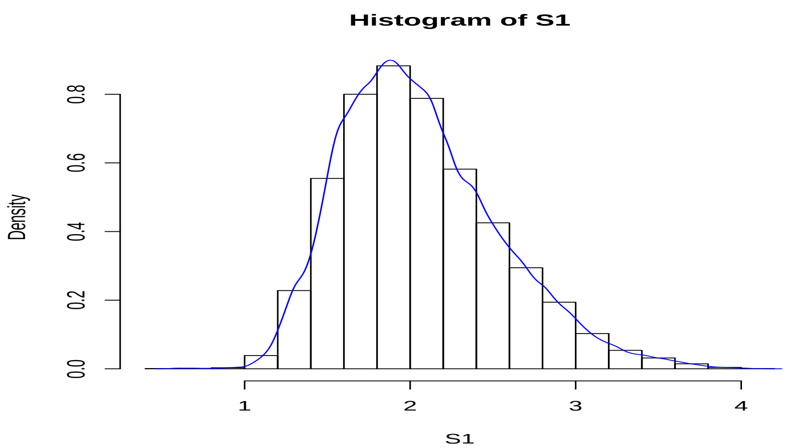

Figure 3

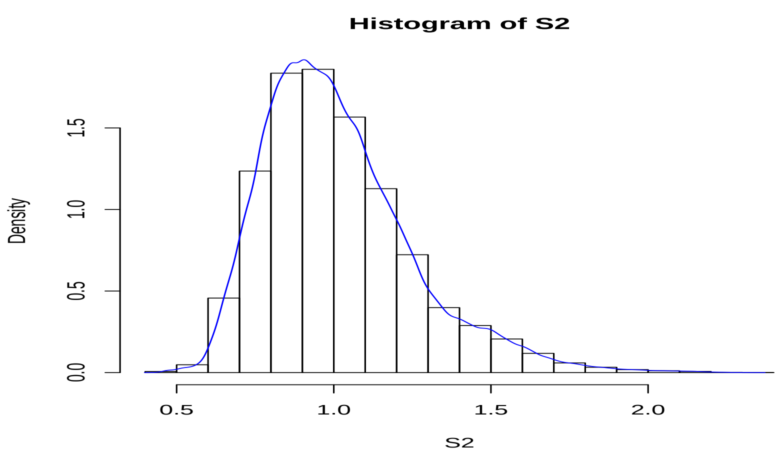

Figure 4

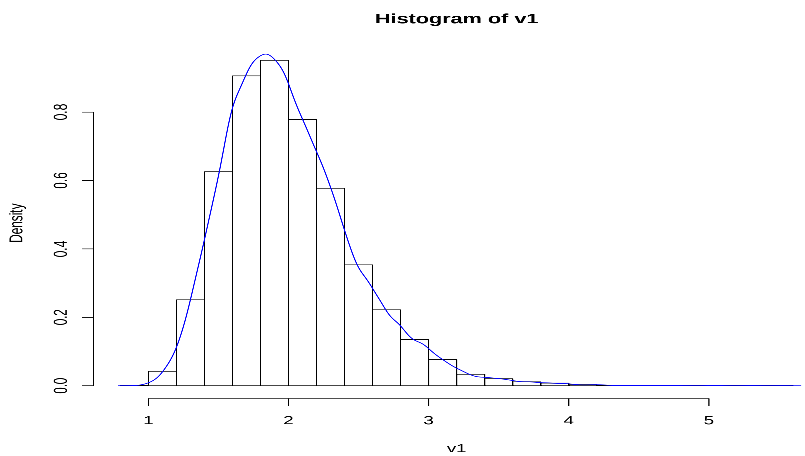

Figure 5

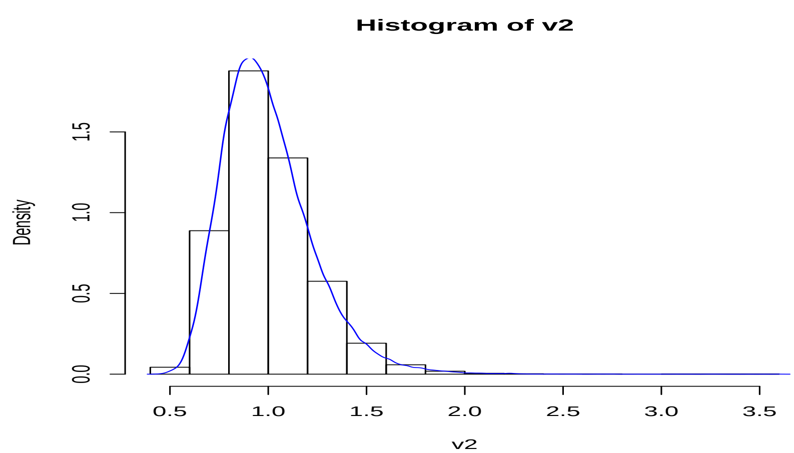

Figure 6

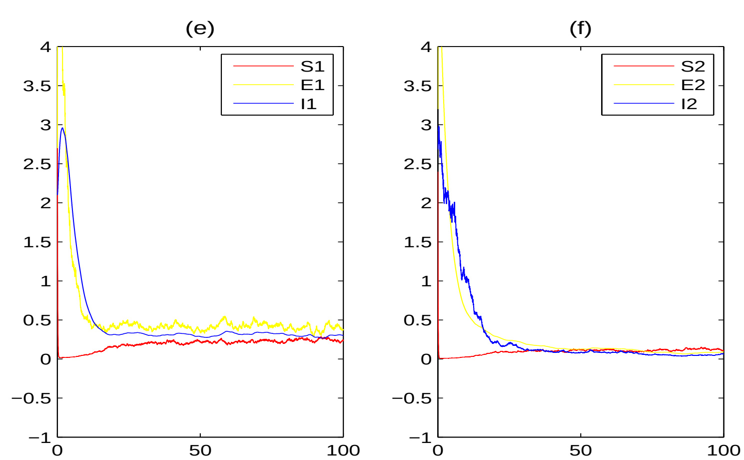

Figure 7

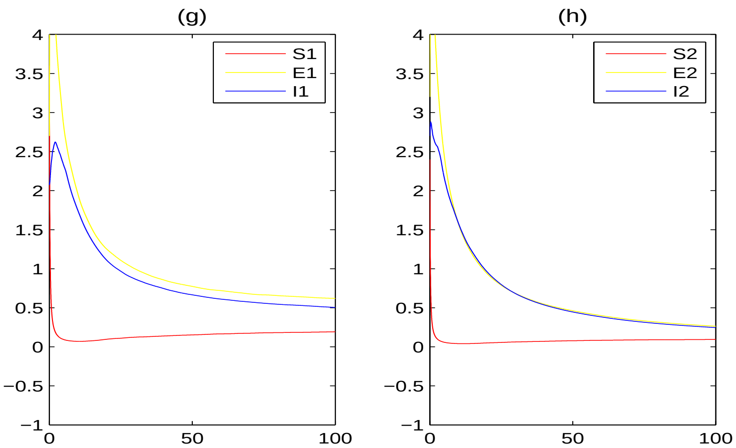

Figure 8

susceptible components are denoted explicitly by parameters in this model.

Under small perturbations, if $R_0 \leq 1$ and some mild conditions hold, the results in the case of large perturbations also hold. In addition, we obtain the explicit exponential rates of the expectation of sample means of the susceptible components. If $R_0 > 1$, we study the ergodicity and the positive recurrence of Eq. (1.3) by a new class of stochastic Lyapunov functions. Furthermore, we use the positive recurrence and the strong Markov property to illustrate the recurrent phenomena of high and lower infective levels of recurrent diseases, which provides a biological insight of recurrent diseases (see Remark 6.1). In such a case, R_0 plays a role similar to the threshold of the deterministic model. Therefore, large perturbation surpasses the effect of R_0 as a threshold value, and small perturbation retains some role of R_0 in stochastic sense.

9 Appendix

In this part, we will give the proofs of several results in the previous sections.

Proof of Theorem 4.1 By comparison theorem (Theorem 1.1 of [29], on page 352), $S_k \leq X_k$, where X_k is the positive solution with initial value $X_k(0) = S_k(0)$ such that

$$dX_k = (\Lambda_k - d_k^S X_k)dt + \sigma_k X_k dB_k(t). \tag{9.1}$$

Since $0 < S_k \leq X_k$, X_k is positive.

First, we show (9.1) is stable in distribution and ergodic. Let $Y_k = X_k - \dfrac{\Lambda_k}{d_k^S}$, then Y_k satisfies

$$dY_k = -d_k^S Y_k dt + \sigma_k (Y_k + \frac{\Lambda_k}{d_k^S}) dB_k(t).$$

Theorem 2.1 (a) in [19] with $C = 1$ implies that the diffusion process Y_k is stable in distribution as $t \to \infty$, so does X_k.

By Theorem 1.16 in [34], we see X_k is ergodic, and with respect to the Lebesgue measure its unique invariant measure ν_k has density $(A_k x^2 p_k(x))^{-1}$, where $A_k = M_k \sigma_k^2 \exp\left(-\frac{2\Lambda_k}{\sigma_k^2}\right)$. Since X_k

is stable in distribution, and it is clear that the stability in distribution implies that the limiting distribution is just the invariant distribution. Therefore, $X_k(t)$ converges weakly to ν_k as $t \to \infty$.

Now, we show that $f_k(t) := EX_k^p(t)$ is uniformly bounded for some $p > 1$ to be determined later. Applying Itô's formula to X_k^p, we have

$$dX_k^p = \left(p\Lambda_k X_k^{p-1} - pd_k^S X_k^p + \frac{\sigma_k^2 p(p-1) X_k^p}{2} \right) dt + p\sigma_k X_k^p dB_k(t).$$

Taking expectation of the above equation, and using the fact $a^{\frac{1}{p}} b^{\frac{p-1}{p}} \leq \frac{a}{p} + \frac{b(p-1)}{p}$, $a, b > 0$, yields

$$\begin{aligned} f_k'(t) &\leq \Lambda_k^p + (p-1) f_k(t) - p\left[d_k^S - \frac{\sigma_k^2 (p-1)}{2} \right] f_k(t) \\ &\leq \Lambda_k^p + p\left[\frac{p-1}{p} - \left(d_k^S - \frac{\sigma_k^2(p-1)}{2} \right) \right] f_k(t). \end{aligned}$$

Choose $p > 1$ close enough to 1 such that

$$\frac{p-1}{p} - \left(d_k^S - \frac{\sigma_k^2(p-1)}{2} \right) < 0,$$

then $\sup\limits_{t \geq 0} EX_k^p(t) = \sup\limits_{t \geq 0} f_k(t) < \infty$, and $\int_0^\infty x^p \nu_k(dx) < \infty$.
By ergodic theorem, we have

$$P_x \left\{ \lim_{T \to \infty} \frac{1}{T} \int_0^T X_k(t) dt = \int_0^\infty x \nu_k(dx) \right\} = 1, \tag{9.2}$$

for all $x \in R_+$. Whilst Jensen's inequality yields

$$E\left[\frac{1}{T} \int_0^T X_k(t) dt \right]^p \leq E \frac{1}{T} \int_0^T X_k^p(t) dt \leq \sup_{t \geq 0} EX_k^p(t) < \infty.$$

Therefore, $\left\{ \frac{1}{T} \int_0^T X_k(t) dt, t \geq 0 \right\}$ is uniformly integrable and together with (9.2), we have

$$E \frac{1}{T} \int_0^T X_k(t) dt \to \int_0^\infty x \nu_k(dx). \tag{9.3}$$

Taking expectation on both sides of (9.1), yields

$$\frac{EX_k(t)}{t} = \Lambda_k - \frac{d_k^S}{t} E \int_0^t X_k(s) ds.$$

Let $t \to \infty$ and taking (9.3) into account, we have

$$\int_0^\infty x \nu_k(dx) = \frac{\Lambda_k}{d_k^S}.$$

Next, define C^2 function V: $R_+^{2n} \to R_+$ by

$$V(E_k, I_k, 1 \le k \le n) = \sum_{k=1}^{n} e_k \left(E_k + \frac{d_k^E + \varepsilon_k}{\varepsilon_k} I_k \right),$$

where $e_k = \dfrac{w_k \varepsilon_k}{(d_k^E + \varepsilon_k)(d_k^I + \gamma_k)}$, $1 \le k \le n$, and $(w_1, \cdots, w_n)$ is the positive left eigenvector of M_0 corresponding to R_0. We compute

$$dV = \left[\sum_{k=1}^{n} \sum_{j=1}^{n} e_k \beta_{kj} S_k I_j - \sum_{k=1}^{n} w_k I_k \right] dt + \sum_{k=1}^{n} e_k \left[\theta_k E_k d\xi_k(t) + \frac{(d_k^E + \varepsilon_k)\rho_k}{\varepsilon_k} I_k d\eta_k(t) \right].$$

Itô's formula implies

$$\begin{aligned}
d \log V = {} & V^{-1} \left[\sum_{k=1}^{n} \sum_{j=1}^{n} e_k \beta_{kj} S_k I_j - \sum_{k=1}^{n} w_k I_k \right] dt - (2V^2)^{-1} \sum_{k=1}^{n} e_k^2 \left(\theta_k^2 E_k^2 + \frac{(d_k^E + \varepsilon_k)^2 \rho_k^2}{\varepsilon_k^2} I_k^2 \right) dt \\
& + V^{-1} \sum_{k=1}^{n} e_k \left[\theta_k E_k d\xi_k(t) + \frac{(d_k^E + \varepsilon_k)\rho_k}{\varepsilon_k} I_k d\eta_k(t) \right] \\
= {} & V^{-1} \left[\sum_{k=1}^{n} \sum_{j=1}^{n} \frac{e_k \beta_{kj} \Lambda_k}{d_k^S} I_j - \sum_{k=1}^{n} w_k I_k \right] dt + V^{-1} \sum_{k=1}^{n} \sum_{j=1}^{n} e_k \beta_{kj} \left(S_k - \frac{\Lambda_k}{d_k^S} \right) I_j dt \\
& - (2V^2)^{-1} \sum_{k=1}^{n} e_k^2 \left(\theta_k^2 E_k^2 + \frac{(d_k^E + \varepsilon_k)^2 \rho_k^2}{\varepsilon_k^2} I_k^2 \right) dt + V^{-1} \sum_{k=1}^{n} e_k \left[\theta_k E_k d\xi_k(t) + \frac{(d_k^E + \varepsilon_k)\rho_k}{\varepsilon_k} I_k d\eta_k(t) \right].
\end{aligned}$$

Note that

$$\sum_{k=1}^{n} \sum_{j=1}^{n} \frac{e_k \beta_{kj} \Lambda_k}{d_k^S} I_j - \sum_{k=1}^{n} w_k I_k = (R_0 - 1) \sum_{k=1}^{n} w_k I_k,$$

and

$$\begin{aligned}
V^2 & \le \left[\sum_{k=1}^{n} \left(e_k \theta_k E_k \frac{1}{\theta_k} + \frac{e_k (d_k^E + \varepsilon_k) \rho_k I_k}{\varepsilon_k} \frac{1}{\rho_k} \right) \right]^2 \\
& \le \sum_{k=1}^{n} e_k^2 \left(\theta_k^2 E_k^2 + \frac{(d_k^E + \varepsilon_k)^2 \rho_k^2}{\varepsilon_k^2} I_k^2 \right) \sum_{k=1}^{n} \left(\frac{1}{\theta_k^2} + \frac{1}{\rho_k^2} \right),
\end{aligned}$$

then

$$\begin{aligned}
d \log V \le {} & \left[V^{-1}(R_0 - 1) \sum_{k=1}^{n} w_k I_k + V^{-1} \sum_{k=1}^{n} \sum_{j=1}^{n} e_k \beta_{kj} \left(S_k - \frac{\Lambda_k}{d_k^S} \right) I_j \right] dt \\
& - \frac{1}{2 \sum_{k=1}^{n} \left(\frac{1}{\theta_k^2} + \frac{1}{\rho_k^2} \right)} + V^{-1} \sum_{k=1}^{n} e_k \left[\theta_k E_k d\xi_k(t) + \frac{(d_k^E + \varepsilon_k)\rho_k}{\varepsilon_k} I_k d\eta_k(t) \right].
\end{aligned}$$

Since

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T V(t)^{-2} \sum_{k=1}^n e_k^2 \left(\theta_k^2 E_k^2(t) + \frac{(d_k^E + \varepsilon_k)^2 \rho_k^2}{\varepsilon_k^2} I_k^2(t) \right) dt < \infty,$$

the strong large number law of martingales implies

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T V(t)^{-1} \sum_{k=1}^n e_k \left[\theta_k E_k d\xi_k(t) + \frac{(d_k^E + \varepsilon_k)\rho_k}{\varepsilon_k} I_k d\eta_k(t) \right] = 0, \quad a.s.$$

Therefore,

$$\begin{aligned}\limsup_{T\to\infty} \frac{\log V(T)}{T} \leq \limsup_{T\to\infty} \frac{1}{T} \int_0^T V(t)^{-1} (R_0 - 1) \sum_{k=1}^n w_k I_k(t) dt - \frac{1}{2\sum_{k=1}^n \left(\frac{1}{\theta_k^2} + \frac{1}{\rho_k^2} \right)} \\ + \limsup_{T\to\infty} \frac{1}{T} \int_0^T V(t)^{-1} \sum_{k=1}^n \sum_{j=1}^n e_k \beta_{kj} \left(S_k(t) - \frac{\Lambda_k}{d_k^S} \right) I_j(t) dt.\end{aligned} \tag{9.4}$$

If $R_0 > 1$, then

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T V(t)^{-1} (R_0 - 1) \sum_{k=1}^n w_k I_k(t) dt \leq (R_0 - 1) \max_{1\leq k\leq n} \{d_k^I + \gamma_k\}. \tag{9.5}$$

Also, note that

$$\begin{aligned}&\limsup_{T\to\infty} \frac{1}{T} \int_0^T V(t)^{-1} \sum_{k=1}^n \sum_{j=1}^n e_k \beta_{kj} \left(X_k(t) - \frac{\Lambda_k}{d_k^S} \right) I_j(t) dt \\ &\leq \limsup_{T\to\infty} \frac{1}{T} \int_0^T V(t)^{-1} \sum_{k=1}^n \sum_{j=1}^n e_k \beta_{kj} X_k(t) I_j(t) dt \\ &\leq \limsup_{T\to\infty} \frac{1}{T} \int_0^T V(t)^{-1} \sum_{k=1}^n \sum_{j=1}^n \frac{e_k \beta_{kj} (d_j^I + \gamma_j)}{w_j} X_k(t) dt \leq M_1,\end{aligned} \tag{9.6}$$

where

$$\begin{aligned}M_1 &= \sum_{k=1}^n \sum_{j=1}^n \frac{w_k \beta_{kj} \varepsilon_k \Lambda_k (d_j^I + \gamma_j)}{d_k^S w_j (d_k^E + \varepsilon_k)(d_k^I + \gamma_k)} \\ &= \sum_{j=1}^n \sum_{k=1}^n \frac{w_k \beta_{kj} \varepsilon_k \Lambda_k}{d_k^S (d_k^E + \varepsilon_k)(d_k^I + \gamma_k)} \cdot \frac{d_j^I + \gamma_j}{w_j} \\ &= \sum_{j=1}^n R_0 w_j \cdot \frac{d_j^I + \gamma_j}{w_j} \\ &= \sum_{j=1}^n R_0 (d_j^I + \gamma_j).\end{aligned}$$

Therefore, if $R_0 > 1$,

$$\begin{aligned}&\limsup_{T\to\infty}\frac{1}{T}\log\left[\sum_{k=1}^n \frac{w_k\varepsilon_k}{(d_k^E+\varepsilon_k)(d_k^I+\gamma_k)}\left(E_k+\frac{d_k^E+\varepsilon_k}{\varepsilon_k}I_k\right)\right]\\ &\le (R_0-1)\max_{1\le k\le n}\{d_k^I+\gamma_k\}+R_0\sum_{j=1}^n(d_j^I+\gamma_j)-\frac{1}{2\sum_{k=1}^n\left(\frac{1}{\theta_k^2}+\frac{1}{\rho_k^2}\right)}.\end{aligned}$$

If $R_0 \le 1$, taking (9.4) and (9.6) into account, then

$$\begin{aligned}&\limsup_{T\to\infty}\frac{1}{T}\log\left[\sum_{k=1}^n \frac{w_k\varepsilon_k}{(d_k^E+\varepsilon_k)(d_k^I+\gamma_k)}\left(E_k+\frac{d_k^E+\varepsilon_k}{\varepsilon_k}I_k\right)\right]\\ &\le R_0\sum_{j=1}^n(d_j^I+\gamma_j)-\frac{1}{2\sum_{k=1}^n\left(\frac{1}{\theta_k^2}+\frac{1}{\rho_k^2}\right)}.\end{aligned}$$

Finally, since

$$\limsup_{T\to\infty}\frac{1}{T}\log\sum_{k=1}^n \frac{w_k\varepsilon_k}{(d_k^E+\varepsilon_k)(d_k^I+\gamma_k)}\left(E_k+\frac{d_k^E+\varepsilon_k}{\varepsilon_k}I_k\right)=\max_{1\le k\le n}\left\{\limsup_{T\to\infty}\frac{1}{T}\log E_k,\limsup_{T\to\infty}\frac{1}{T}\log I_k\right\},$$

(4.1) holds.

At last, we concentrate on S_k. We next show that S_k is stable in distribution. To do this, we introduce a new stochastic process $S_{k,\varepsilon(t)}$ which is defined by its initial condition $S_{k,\varepsilon}(0) = S_k(0)$ and the stochastic differential equation

$$dS_{k,\varepsilon} = [\Lambda_k - (d_k^S+\varepsilon)S_{k,\varepsilon}]dt + \sigma_k S_{k,\varepsilon}dB_k(t).$$

First, we prove

$$\liminf_{t\to\infty}(S_k(t)-S_{k,\varepsilon}(t))\ge 0,\ a.e. \tag{9.7}$$

Consider

$$d(S_k-S_{k,\varepsilon})=\left[(\varepsilon-\sum_{j=1}^n\beta_{k,j}I_j)S_k-(d_k^S+\varepsilon)(S_k-S_{k,\varepsilon})\right]dt+\sigma_k(S_k-S_{k,\varepsilon})dB_k(t). \tag{9.8}$$

The solution is given by

$$\begin{aligned}S_k(t)-S_{k,\varepsilon}(t)&=\exp\left\{-(d_k^S+\varepsilon+\frac{\sigma_k^2}{2})t+\sigma_kB_k(t)\right\}\\ &\cdot\int_0^t\exp\left\{(d_k^S+\varepsilon+\frac{\sigma_k^2}{2})s-\sigma_kB_k(s)\right\}\left(\varepsilon-\sum_{j=1}^n\beta_{k,j}I_j(s)\right)S_k(s)ds.\end{aligned}$$

If $I_k \to 0$, $1\le k\le n$, a.e., then for almost $\omega\in\Omega$, $\exists\ T = T(\omega)$ such that

$$\varepsilon > \sum_{j=1}^n \beta_{k,j}I_j(t),\ \forall t > T.$$

Hence, for almost $\omega \in \Omega$, for any $t > T$,

$$S_k(t) - S_{k,\varepsilon}(t) \geq \exp\left\{-(d_k^S + \varepsilon + \frac{\sigma_k^2}{2})t + \sigma_k B_k(t)\right\}$$
$$\cdot \int_0^T \exp\left\{(d_k^S + \varepsilon + \frac{\sigma_k^2}{2})s - \sigma_k B_k(s)\right\}\left(\varepsilon - \sum_{j=1}^n \beta_{k,j} I_j(s)\right) S_k(s) ds.$$

Therefore,

$$\liminf_{t\to\infty} (S(t) - S_{k,\varepsilon}(t)) \geq 0, \ a.e.$$

Next, note that $X_k \geq S_{k,\varepsilon}$, a.e. and

$$d(X_k - S_{k,\varepsilon}) = [\varepsilon S_{k,\varepsilon} - d_k^S(X_k - S_{k,\varepsilon})]dt + \sigma_k(X_k - S_{k,\varepsilon})dB_k(t).$$

Taking expectation of the equation above, yields

$$\begin{aligned} E\left|X_k(T) - S_{k,\varepsilon}(T)\right| &= \int_0^T \left[\varepsilon S_{k,\varepsilon}(t) - d_k^S(X_k(t) - S_{k,\varepsilon}(t))\right] dt \\ &\leq \int_0^T \left[\varepsilon X_k(t) - d_k^S\left|X_k(t) - S_{k,\varepsilon}\right|\right] dt. \end{aligned}$$

Hence

$$E\left|X_k(T) - S_{k,\varepsilon}(T)\right| \leq \frac{\varepsilon \sup_{u\geq 0} EX_k(u)}{d_k^S}\left(1 - \exp\{-d_k^S T\}\right).$$

This implies

$$\liminf_{\varepsilon\to 0} \lim_{T\to\infty} E|X_k(T) - S_{k,\varepsilon}(T)| = 0. \tag{9.9}$$

Combing (9.7), (9.9) and the fact that $S_k(t) \leq X_k(t)$, we get

$$\lim_{T\to\infty} (X_k(T) - S_k(T)) = 0, \text{ in probability.}$$

Since $X_k(T)$ converges weakly to distribution ν_k, so does $S_k(T)$ as $T \to \infty$. The proof of

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T E\left(S_k(t) - \frac{\Lambda_k}{d_k^S}\right)^2 dt = \frac{\sigma_k^2 \Lambda_k^2}{(2d_k^S - \sigma_k^2)(d_k^S)^2}$$

is left in Theorem 5.1.

Proof of Theorem 5.1 First, by ergodic property of (9.1), we get

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \left|X_k(t) - \frac{\Lambda_k}{d_k^S}\right| dt = \int_0^\infty \left|x - \frac{\Lambda_k}{d_k^S}\right| \nu_k(dx). \tag{9.10}$$

Note that $\int_0^\infty \left|x - \frac{\Lambda_k}{d_k^S}\right| \nu_k(dx) \leq \left(\int_0^\infty \left(x - \frac{\Lambda_k}{d_k^S}\right)^2 \nu_k(dx)\right)^{\frac{1}{2}}$ by Hölder inequality, and for any $m > 0$, the ergodicity of X_k implies

$$\int_0^\infty \left(x - \frac{\Lambda_k}{d_k^S}\right)^2 \wedge m\nu_k(dx) = \lim_{T\to\infty} \frac{1}{T}\int_0^T E\left(X_k(t) - \frac{\Lambda_k}{d_k^S}\right)^2 \wedge m dt \leq \limsup_{T\to\infty} \frac{1}{T}\int_0^T E\left(X_k(t) - \frac{\Lambda_k}{d_k^S}\right)^2 dt.$$

Let $m \to \infty$, we have

$$\int_0^\infty \left(x - \frac{\Lambda_k}{d_k^S}\right)^2 \nu_k(dx) \le \limsup_{T\to\infty} \frac{1}{T}\int_0^T E\left(X_k(t) - \frac{\Lambda_k}{d_k^S}\right)^2 dt. \tag{9.11}$$

Applying Itô's formula to $\left(X_k(t) - \frac{\Lambda_k}{d_k^S}\right)^2$, then

$$\begin{aligned}
\left(X_k(t) - \frac{\Lambda_k}{d_k^S}\right)^2 &= \int_0^t \left[2\left(X_k(s) - \frac{\Lambda_k}{d_k^S}\right)(\Lambda_k - d_k^S X_k(s)) + \sigma_k^2 X_k^2(s)\right] ds \\
&\quad + 2\int_0^t \sigma_k \left(X_k(s) - \frac{\Lambda_k}{d_k^S}\right) X_k(s) dB_k(s) \\
&= -2d_k^S \int_0^t \left(X_k(s) - \frac{\Lambda_k}{d_k^S}\right)^2 ds + \sigma_k^2 \int_0^t X_k^2(s) ds \\
&\quad + 2\int_0^t \sigma_k \left(X_k(s) - \frac{\Lambda_k}{d_k^S}\right) X_k(s) dB_k(s) \\
&= -(2d_k^S - \sigma_k^2)\int_0^t \left(X_k(s) - \frac{\Lambda_k}{d_k^S}\right)^2 ds + \frac{2\Lambda_k \sigma_k}{d_k^S}\int_0^t \left(X_k(s) - \frac{\Lambda_k}{d_k^S}\right) ds \\
&\quad + \sigma_k^2 \left(\frac{\Lambda_k}{d_k^S}\right)^2 t + 2\int_0^t \sigma_k \left(X_k(s) - \frac{\Lambda_k}{d_k^S}\right) X_k(s) dB_k(s).
\end{aligned}$$

Taking expectation on both sides of the above equation and using the fact that $\lim_{T\to\infty} \frac{1}{T}\int_0^T EX_k(t)dt = \frac{\Lambda_k}{d_k^S}$, if $2d_k^S > \sigma_k^2$, then $\sup_{t\ge 0} E\left(X_k(t) - \frac{\Lambda_k}{d_k^S}\right)^2 < \infty$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T E\left(X_k(t) - \frac{\Lambda_k}{d_k^S}\right)^2 dt = \frac{\sigma_k^2 \Lambda_k^2}{(2d_k^S - \sigma_k^2)(d_k^S)^2}. \tag{9.12}$$

and by $\lim_{t\to\infty}(X_k(t) - S_k(t)) = 0$ in probability, we have

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T E\left(S_k(t) - \frac{\Lambda_k}{d_k^S}\right)^2 dt = \frac{\sigma_k^2 \Lambda_k^2}{(2d_k^S - \sigma_k^2)(d_k^S)^2}.$$

Taking (9.10), (9.11) and (9.12) into account, yields

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \left|X_k(t) - \frac{\Lambda_k}{d_k^S}\right| dt = \int_0^\infty \left|x - \frac{\Lambda_k}{d_k^S}\right| \nu_k(dx) \le \frac{\sigma_k \Lambda_k}{d_k^S \sqrt{2d_k^S - \sigma_k^2}}. \tag{9.13}$$

Secondly, in (9.4), note that

$$\begin{aligned}
V(t)^{-1}\sum_{k=1}^n \sum_{j=1}^n e_k \beta_{kj}\left(S_k(t) - \frac{\Lambda_k}{d_k^S}\right) I_j(t) &\le V(t)^{-1}\sum_{k=1}^n \sum_{j=1}^n e_k \beta_{kj}\left(X_k(t) - \frac{\Lambda_k}{d_k^S}\right) I_j(t) \\
&\le \sum_{k=1}^n \sum_{j=1}^n \frac{w_k \beta_{kj}\varepsilon_k (d_j^I + \gamma_j)}{w_j (d_k^E + \varepsilon_k)(d_k^I + \gamma_k)} \left|X_k(t) - \frac{\Lambda_k}{d_k^S}\right|,
\end{aligned}$$

together with (9.13), implying

$$\limsup_{T\to\infty}\frac{1}{T}\int_0^T V(t)^{-1}\sum_{k=1}^n\sum_{j=1}^n e_k\beta_{k,j}\left(S_k(t)-\frac{\Lambda_k}{d_k^S}\right)I_j(t)dt\le M_2. \tag{9.14}$$

where

$$\begin{aligned}M_2 &= \sum_{k=1}^n\sum_{j=1}^n\frac{\sigma_k w_k\beta_{kj}\varepsilon_k\Lambda_k(d_j^I+\gamma_j)}{d_k^S w_j(d_k^E+\varepsilon_k)(d_k^I+\gamma_k)\sqrt{2d_k^S-\sigma_k^2}}\\
&\le \max_{1\le k\le n}\left\{\frac{\sigma_k}{\sqrt{2d_k^S-\sigma_k^2}}\right\}\sum_{j=1}^n\sum_{k=1}^n\frac{w_k\beta_{kj}\varepsilon_k\Lambda_k}{d_k^S(d_k^E+\varepsilon_k)(d_k^I+\gamma_k)}\cdot\frac{d_j^I+\gamma_j}{w_j}\\
&\le \max_{1\le k\le n}\left\{\frac{\sigma_k}{\sqrt{2d_k^S-\sigma_k^2}}\right\}\sum_{j=1}^n R_0 w_j\cdot\frac{d_j^I+\gamma_j}{w_j}\\
&= \max_{1\le k\le n}\left\{\frac{\sigma_k}{\sqrt{2d_k^S-\sigma_k^2}}\right\}\sum_{j=1}^n R_0(d_j^I+\gamma_j).\end{aligned}$$

Taking (9.4) and (9.14) into account, we get (5.1).

Thirdly, we study the exponential bounds of $\left|ES_k(t)-\frac{\Lambda_k}{d_k^S}\right|$. Let $h_k(t):=EX_k(t)-\dfrac{\Lambda_k}{d_k^S}$ and take the expectation on both sides of Eq. (9.1), we have

$$h_k'(t)=-d_k^S h(t).$$

Then

$$EX_k(t)-\frac{\Lambda_k}{d_k^S}=h_k(t)=h_k(0)\exp\{-d_k^S t\}, \tag{9.15}$$

together with comparison theorem, implying

$$ES_k(t)-\frac{\Lambda_k}{d_k^S}\le EX_k(t)-\frac{\Lambda_k}{d_k^S}=h_k(0)\exp\{-d_k^S t\}. \tag{9.16}$$

Meanwhile, (9.8) yields

$$g_k^\varepsilon(t):=E\left(S_k(t)-S_{k,\varepsilon}(t)\right)=\int_0^t E(\varepsilon-\sum_{j=1}^n\beta_{k,j}I_j(s))S_k(s)ds-(d_k^S+\varepsilon)\int_0^t E(S_k(s)-S_{k,\varepsilon}(s))ds.$$

Since $I_j, 1\le j\le n$ converge exponentially to 0, $ES_k(t)$ converges to $\dfrac{\Lambda_k}{d_k^S}$ and $\liminf_{t\to\infty}E\left(S_k(t)-S_{k,\varepsilon}(t)\right)\ge E\liminf_{t\to\infty}\left(S_k(t)-S_{k,\varepsilon}(t)\right)\ge 0$ (see (9.7)), for any $\delta>0$, there exists $T_0>0$ such that for any $t\ge T_0$,

$$E(\varepsilon-\sum_{j=1}^n\beta_{k,j}I_j(t))S_k(t)\ge\frac{\varepsilon\Lambda_k}{2d_k^S}$$

and

$$g_k^\varepsilon(t) = E\left(S_k(t) - S_{k,\varepsilon}(t)\right) \geq -\delta.$$

By computation, for any $t \geq T_0$,

$$E\left(S_k(t) - S_{k,\varepsilon}(t)\right) \geq g_k^\varepsilon(T_0)\exp\{-(d_k^S+\varepsilon)(t-T_0)\} + \frac{\varepsilon\Lambda_k}{2d_k^S(d_k^S+\varepsilon)}\left(1-\exp\{-(d_k^S+\varepsilon)(t-T_0)\}\right),$$

together with (9.15), implies for any $t \geq T_0$,

$$\begin{aligned} ES_k(t) - \frac{\Lambda_k}{d_k^S} &= E\left(S_k(t) - S_{k,\varepsilon}(t)\right) + ES_{k,\varepsilon}(t) - \frac{\Lambda_k}{d_k^S+\varepsilon} + \frac{\Lambda_k}{d_k^S+\varepsilon} - \frac{\Lambda_k}{d_k^S} \\ &\geq g_k^\varepsilon(T_0)\exp\{-(d_k^S+\varepsilon)(t-T_0)\} + \frac{\varepsilon\Lambda_k}{2d_k^S(d_k^S+\varepsilon)}\left(1-\exp\{-(d_k^S+\varepsilon)(t-T_0)\}\right) \\ &\quad + h_k(0)\exp\{-(d_k^S+\varepsilon)t\} + \frac{\Lambda_k}{d_k^S+\varepsilon} - \frac{\Lambda_k}{d_k^S}. \end{aligned}$$

By $g_k^\varepsilon(T_0) \geq -\delta$, let $\varepsilon \to 0$, then for any $t \geq T_0$,

$$ES_k(t) - \frac{\Lambda_k}{d_k^S} \geq (h_k(0)-\delta)\exp\{-d_k^S t\},$$

together with (9.16), yields (5.2).

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