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DISCRETE RAZUMIKHIN-TYPE TECHNIQUE AND STABILITY OF THE EULER–MARUYAMA METHOD TO STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. A discrete stochastic Razumikhin-type theorem is established to investigate whether the Euler–Maruyama (EM) scheme can reproduce the moment exponential stability of exact solutions of stochastic functional differential equations (SFDEs). In addition, the Chebyshev inequality and the Borel-Cantelli lemma are applied to show the almost sure stability of the EM approximate solutions of SFDEs. To show our idea clearly, these results are used to discuss stability of numerical solutions of two classes of special SFDEs, including stochastic delay differential equations (SDDEs) with variable delay and stochastically perturbed equations.

1. **Introduction.**

The stability analysis of numerical methods for stochastic differential equations (SDEs) has received increasing attention in recent years. Due to the presence of stochastic factors, stability here means mainly moment stability (M-stability), in particular mean-square stability, called as MS-stability, and trajectory stability (T-stability), that is almost sure stability, which is a direct extension of the deterministic stability concept. T-stability was defined in [13] for weak solutions and extended to strong solutions in [3], for which it is equivalent to the asymptotic stability property. MS-stability was defined and investigated for various kinds of numerical schemes of SDEs in [14]. In addition, [2, 6, 7] examined the almost sure and the moment stability of numerical solutions of SDEs, while [12] extended these techniques to examine almost sure and moment exponential stability for stochastic differential equations with Markovian switching (SDEwMS). The pth moment exponential stability for SDDEs with fixed delay was discussed in [1] in terms of the

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discrete Halanay inequality, while [16] investigated the almost sure exponential stability of the numerical approximations of SDDEs by using continuous and discrete semimartingale convergence theorems and in [17], the authors extended this technique to examine the almost sure exponential stability of the numerical solutions of SFDEs.

This paper will use the Razumikhin-type technique to examine the stability of numerical solutions of SFDEs. Consider the n-dimensional SFDE

$$dx(t) = f(t, x_t)dt + g(t, x_t)dw(t), \quad t \geq 0 \tag{1.1}$$

with initial data $x_0 = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$, namely, ξ is a bounded, $\mathcal{F}_0$-measurable $C([-\tau, 0]; \mathbb{R}^n)$-valued random process defined on $[-\tau, 0]$, where $x_t =: x_t(\theta) = \{x(t+\theta) : -\tau \leq \theta \leq 0\} \in C([-\tau, 0]; \mathbb{R}^n)$, $f : \mathbb{R}_+ \times C([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^n$, $g : \mathbb{R}_+ \times C([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^{n\times m}$ are Borel measurable, $w(t)$ is an m-dimensional Brownian motion. The initial data ξ satisfies $\sup_{-\tau\leq\theta\leq 0} \mathbb{E}|\xi(\theta)|^p < \infty$ for $p > 0$.

Let the step size $\triangle$ be a fraction of the delay τ, namely, $\triangle = \tau/M$ for some integer M. Then the EM method (see [11, 15, 17]) applied to (1.1) produces

$$\begin{cases} x_k = \xi(k\triangle), & -M \leq k \leq 0, \\ x_{k+1} = x_k + f(k\triangle, y_{k\triangle})\triangle + g(k\triangle, y_{k\triangle})\triangle w_k, & k \geq 0, \end{cases} \tag{1.2}$$

where $\triangle w_k = w((k+1)\triangle) - w(k\triangle)$ is the Brownian motion increment and $y_{k\triangle}$ is a $C([-\tau, 0]; \mathbb{R}^n)$-valued random variable defined by piecewise linear interpolation:

$$\begin{aligned} y_{k\triangle} \quad =: \quad & y_{k\triangle}(\theta) = x_{k+i} + \frac{\theta - i\triangle}{\triangle}(x_{k+1+i} - x_{k+i}), \\ & \text{for } i\triangle \leq \theta \leq (i+1)\triangle, \ i = -M, -M+1, \cdots, -1. \end{aligned} \tag{1.3}$$

Razumikhin-type theorems are well-known (for example, see [4, 5, 9, 10]) in the stability theory of both ordinary and stochastic differential equations. To consider the stability of deterministic difference equations, some papers (for example, [8, 18, 19]) examined the Razumikhin-type theorems for deterministic delay difference equations. Motivated by these papers, this paper establishes a discrete Razumikhin-type theorem for the pth moment exponential stability of the EM scheme (1.2). By the Chebyshev inequality and the Borel-Cantelli lemma, this paper also investigates the almost sure stability of the EM scheme (1.2). The analysis thus also includes almost sure exponential stability. To illustrate our ideas clearly, these results are applied to SDDEs with variable delay and stochastically perturbed equations and thus generalize some existing results (for example, [1] and [7]).

In the next section, we introduce some necessary notations and definitions. Section 3 reviews the Razumikhin-type theorem on the exponential stability for SFDE (1.1), and then we establish the discrete Razumikhin-type theorem. To show our idea more clearly, In Section 4, we examine stability of exact and numerical solutions of SDDEs with variable delay as an important class of SFDEs. We also examine stochastically perturbed equations in section 5 and deal with a class of linear stochastic Volterra delay-integro-differential equations (SVDIDEs) as a special case of stochastically perturbed equations.

2. Notations and definitions.

Throughout this paper, unless otherwise specified, we use the following notations. Let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^n$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$. Let $\tau > 0$ and $C([-\tau, 0], \mathbb{R}^n)$ denote the family of continuous functions from $[-\tau, 0]$ to $\mathbb{R}^n$. The inner product of x, y in $\mathbb{R}^n$ is denoted by $\langle x, y\rangle$ or $x^T y$. If $a,\ b \in \mathbb{R}$, let $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. $\mathbb{N}$ represents the set of the integer numbers, namely, $\mathbb{N} = \{0, 1, \cdots, \}$ and let $\mathbb{N}_{-M} = \{0, -1, -2, \cdots, -M\}$ for some positive integers M.

Let $(\mathbf{\Omega}, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, namely, it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $w(t) = (w^1(t), \cdots, w^m(t))^T$ be an m-dimensional Brownian motion defined on this probability space. Denote by $C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ the family of all bounded, $\mathcal{F}_0$-measurable $C([-\tau, 0]; \mathbb{R}^n)$-valued stochastic processes on $[-\tau, 0]$. Let $p > 0$ and $L^p_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^n)$ be the family of $\mathcal{F}_t$-measurable stochastic processes $\varphi = \{\varphi(\theta) : -\tau \leq \theta \leq 0\}$ such that $\|\varphi\|^p_{\mathbb{E}} := \sup_{-\tau\leq\theta\leq 0} \mathbb{E}|\varphi(\theta)|^p < \infty$. If $x(t)$ is an $\mathbb{R}^n$-valued stochastic process, define $x_t = \{x(t + \theta) : -\tau \leq \theta \leq 0\}$ for $t \geq 0$. Let $C^2(\mathbb{R}^n; \mathbb{R}_+)$ denote the family of all nonnegative functions $V(x)$ on $\mathbb{R}^n$ which are continuously twice differentiable. For a function $V \in C^2(\mathbb{R}^n; \mathbb{R}_+)$, define an operator $\mathcal{L}V$ from $\mathbb{R}_+ \times C([-\tau, 0]; \mathbb{R}^n)$ to $\mathbb{R}$ by

$$\mathcal{L}V(t, \varphi) = V_x(\varphi(0))f(t, \varphi) + \frac{1}{2}\text{trace}[g^T(t, \varphi)V_{xx}(\varphi(0))g(t, \varphi)]. \tag{2.1}$$

Then the expectation of $\mathcal{L}V$ along the solution $x(t)$ of Eq. (1.1) is given by $\mathbb{E}\mathcal{L}V(t, x_t)$. To indicate dependence on the initial data ξ, the solution will often be written $x(t) = x(t, \xi)$.

In this paper, it is assumed that there exists a unique solution $x(t, \xi)$ to Eq. (1.1) for any initial data $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ and for any $p > 0$, the pth moment of this solution is finite, namely $\mathbb{E}(\sup_{-\tau\leq t<\infty} |x(t, \xi)|^p) < \infty$. For example, the linear growth condition may guarantee the boundedness of the pth moment (see [10, Theorems 5.2.5 and 5.4.1, Pages 153 and 158]). For the purpose of stability, assume that $f(t, 0) = g(t, 0) = 0$. This implies that Eq. (1.1) admits a trivial solution $x(t, 0) \equiv 0$. The following definitions of stability for SFDEs and its EM scheme are required.

Definition 2.1. The trivial solution of Eq. (1.1) or, simply, Eq. (1.1) is said to be exponentially stable in the pth moment if for any initial data $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$,

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}|x(t, \xi)|^p < 0 \tag{2.2}$$

and almost surely exponentially stable if

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t, \xi)| < 0 \quad a.s. \tag{2.3}$$

Definition 2.2. The EM scheme (1.2) is said to be exponentially stable in the pth moment if for given stepsize $\triangle > 0$ and any bounded initial sequence $\{\xi(k\triangle)\}_{k\in\mathbb{N}_{-M}}$ if

$$\limsup_{k\to\infty} \frac{1}{k\triangle} \log \mathbb{E}|x_k|^p < 0 \tag{2.4}$$

and almost surely exponentially stable if

$$\limsup_{k\to\infty} \frac{1}{k\triangle} \log |x_k| < 0 \quad a.s. \tag{2.5}$$

In this paper, our aim is to examine whether stability of the numerical solution for (1.2) can reproduce stability of the exact solution of equation (1.1) in the senses of the moment and almost sure.

3. **Stability of SFDEs and the EM scheme.** Razumikhin-type theorems are very important in stability analysis. The classical Razumikhin theorem for deterministic functional differential equations can be found in Hale *et al* [5], while Mao (see [9, 10]) extended it to SFDEs. The method was also used for deterministic delay differential equations with random delays in [4]. In this section, a discrete Razumikhin-type theorem on exponential stability of (1.2) is established and used to examine the stability of numerical solutions.

3.1. **Stability of the exact solutions of SFDEs.** For the convenience of the reader, the well-known Razumikhin-type theorem on exponential stability for SFDEs is stated (see [9, 10]). Note that $\mathbb{E}(\sup_{-\tau\le t<\infty}|x(t,\xi)|^p)<\infty$ may guarantee that $\mathbb{E}\mathcal{L}V(\varphi)$ in condition (ii) of the following theorem is well defined.

Theorem 3.1. *Let ζ, p, c_1, c_2 all be positive constants and $q>1$. Assume that there exists a function $V\in C^2(\mathbb{R}^n;\mathbb{R}_+)$ such that the following conditions hold:*

(i) *$c_1|x|^p \le V(x) \le c_2|x|^p$ for all $x\in\mathbb{R}^n$;*
(ii) *$\mathbb{E}\mathcal{L}V(t,\varphi)\le -\zeta\mathbb{E}V(\varphi(0))$ for all $t\ge 0$ and those $\varphi\in L^p_{\mathcal{F}_t}([-\tau,0];\mathbb{R}^n)$ which obeys $\mathbb{E}V(\varphi(\theta))<q\mathbb{E}V(\varphi(0))$ on $-\tau\le\theta\le 0$.*

Then for all $\xi\in C^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$, the solution of (1.1) has the property that

$$\mathbb{E}|x(t,\xi)|^p \le \frac{c_2}{c_1}\|\xi\|^p_{\mathbb{E}}e^{-\lambda t} \quad on\ t\ge 0, \tag{3.1}$$

where $\lambda=[\log(q)/\tau]\wedge\zeta$.

To explain this idea, applying the Itô formula to $e^{\lambda t}V(x(t))$, one sees that to have the pth moment exponential stability, it would require that $\mathbb{E}\mathcal{L}V(t,x_t)\le -\lambda\mathbb{E}V(x(t))$ for all $t\ge 0$. As a result, one would be forced to impose very severe restrictions on the functionals f and g. However, by this theorem, one needs to require $\mathbb{E}\mathcal{L}V(t,x_t)\le -\zeta\mathbb{E}V(x(t))$ if $\mathbb{E}V(x_t)\le q\mathbb{E}V(x(t))$. f and g can be much weakened. This is the basic idea of this theorem. Mao [9, 10] gave a classic proof for this theorem. Another proof, which is different, is given in Appendix A.

Although the pth moment exponential stability and almost sure exponential stability of the exact solution do not imply each other in general, under an irrestrictive condition the pth moment exponential stability implies almost sure exponential stability (cf. [9, 10]).

Theorem 3.2. *Let $p\ge 1$. Assume that there is a constant $K>0$ such that for every solution $x(t)$ of (1.1),*

$$\mathbb{E}(|f(t,x_t)|^p+|g(t,x_t)|^p)\le K\sup_{-\tau\le\theta\le 0}\mathbb{E}|x(t+\theta)|^p \quad on\ t\ge 0. \tag{3.2}$$

Then (3.1) implies

$$\limsup_{t\to\infty}\frac{1}{t}\log|x(t,\xi)|\le -\frac{\lambda}{p} \quad a.s. \tag{3.3}$$

In other words, the pth moment exponential stability implies almost sure exponential stability.

3.2. **Stability of numerical solutions of SFDEs.** To consider the stability of difference equations, some papers (for example, [8, 18]) examine the Razumikhin-type theorems for deterministic delay difference equations. In this subsection, we establish the Razumikhin-type theorem on exponential stability of the EM scheme (1.2).

Theorem 3.3. *Fix $\triangle > 0$. Let $\zeta_\triangle, p_\triangle, c_{1\triangle}, c_{2\triangle}$ all be positive constants, $q_\triangle > 1$ and $\triangle\zeta_\triangle < 1$. Assume that there exists a function $V_\triangle : \mathbb{R}^n \to \mathbb{R}_+$ such that the following conditions hold: for the sequence $\{x_k\}$ defined by (1.2),*

(i) $c_{1\triangle}|x_k|^{p_\triangle} \leq V_\triangle(x_k) \leq c_{2\triangle}|x_k|^{p_\triangle}$;

(ii) *for all $i \in \mathbb{N}_{-M}$, $\mathbb{E}V_\triangle(x_{k+i}) \leq q_\triangle \mathbb{E}V_\triangle(x_k)$ implies that*

$$\mathbb{E}V_\triangle(x_{k+1}) \leq (1 - \zeta_\triangle\triangle)\mathbb{E}V_\triangle(x_k);$$

(iii) *for some $\check{i} \in \mathbb{N}_{-M} - \{0\}$, $\mathbb{E}V_\triangle(x_{k+\check{i}}) > e^{(M+1)\lambda_\triangle\triangle}\mathbb{E}V_\triangle(x_k)$ implies that*

$$\mathbb{E}V_\triangle(x_{k+1}) \leq 1/q_\triangle \max_{i\in\mathbb{N}_{-M}} \mathbb{E}V_\triangle(x_{k+i}),$$

where $\lambda_\triangle = [\log q_\triangle/((M+1)\triangle)] \wedge [\log(1-\zeta_\triangle\triangle)^{-1}/\triangle]$.

Then for any bounded initial sequence $\{\xi(k\triangle)\}_{k\in\mathbb{N}_{-M}}$,

$$\mathbb{E}|x_k|^{p_\triangle} \leq \frac{c_{2\triangle}}{c_{1\triangle}}\|\xi\|_{\mathbb{E}}^{p_\triangle} e^{-\lambda_\triangle k\triangle}, \quad \text{for all } k \geq 0. \tag{3.4}$$

namely, the sequence $\{x_k\}_{k\geq 1}$ is pth moment exponentially stable.

Remark Compared with Theorem 3.1, Theorem 3.3 needs an additional condition (iii). To explain necessity of this condition, let us recall the proof of Theorem 3.1 in Appendix A. We may observe that continuity plays an important role in Theorem 3.1 and its proof. Condition (ii) of Theorem 3.1 shows that $\mathbb{E}V(x(t+\theta)) < q\mathbb{E}V(x(t))$ implies $\mathbb{E}\mathcal{L}V(t, x(t+\theta)) \leq -\zeta\mathbb{E}V(x(t))$, by which continuity gives that $\mathbb{E}V(x(t))$ is decreasing. However, the EM scheme does not hold continuity. Condition (ii) of Theorem 3.3 shows that for any $i \in \mathbb{N}_{-M}$, $\mathbb{E}V_\triangle(x_{k+i}) \leq q_\triangle\mathbb{E}V_\triangle(x_k)$ implies that $\mathbb{E}V_\triangle(x_{k+1}) \leq (1-\zeta_\triangle\triangle)\mathbb{E}V_\triangle(x_k)$, but this condition cannot guarantee the decreasing trend of $\mathbb{E}V_\triangle(x_{k+1})$ when $\mathbb{E}V_\triangle(x_{k+i}) > q_\triangle\mathbb{E}V_\triangle(x_k)$. To guarantee the decreasing trend of $\mathbb{E}V_\triangle(x_{k+1})$ when condition (ii) does not hold, it is necessary to add the additional condition (iii).

Proof. Define the sequence

$$u_k = \max_{i\in\mathbb{N}_{-M}}\{e^{\lambda_\triangle(k+i)\triangle}\mathbb{E}V_\triangle(x_{k+i})\}, \quad \text{for all } k \in \mathbb{N}.$$

It will be shown that $u_{k+1} \leq u_k$.

Also define

$$\bar{i} = \bar{i}(k) = \max\{i \in \mathbb{N}_{-M}, u_k = e^{\lambda_\triangle(k+i)\triangle}\mathbb{E}V_\triangle(x_{k+i})\}.$$

Then

$$u_k = e^{\lambda_\triangle(k+\bar{i})\triangle}\mathbb{E}V_\triangle(x_{k+\bar{i}}).$$

When $\bar{i} \leq -1$, then for any $i \in \mathbb{N}_{-M} - \{0\}$,

$$e^{\lambda_\triangle(k+i+1)\triangle}\mathbb{E}V_\triangle(x_{k+i+1}) \leq e^{\lambda_\triangle(k+\bar{i})\triangle}\mathbb{E}V_\triangle(x_{k+\bar{i}}), \tag{3.5}$$

which implies that

$$\max_{i\in\mathbb{N}_{-M}-\{0\}}\{e^{\lambda_\triangle(k+i+1)\triangle}\mathbb{E}V_\triangle(x_{k+i+1})\} \leq e^{\lambda_\triangle(k+\bar{i})\triangle}\mathbb{E}V_\triangle(x_{k+\bar{i}}) = u_k. \tag{3.6}$$

In fact,

$$e^{\lambda_\triangle(k+1)\triangle}\mathbb{E}V_\triangle(x_{k+1}) \leq e^{\lambda_\triangle(k+\bar{i})\triangle}\mathbb{E}V_\triangle(x_{k+\bar{i}}) = u_k \tag{3.7}$$

also holds, which will now be shown. By the definition of $\bar{i}$,

$$u_k = e^{\lambda_\triangle(k+\bar{i})\triangle}\mathbb{E}V_\triangle(x_{k+\bar{i}}) > e^{\lambda_\triangle k\triangle}\mathbb{E}V_\triangle(x_k),$$

which implies that

$$\mathbb{E}V_\vartriangle(x_{k+\bar{i}}) > e^{-\lambda_\vartriangle \bar{i}\triangle}\mathbb{E}V_\vartriangle(x_k) \geq e^{\lambda_\vartriangle \triangle}\mathbb{E}V_\vartriangle(x_k)$$

since $\bar{i} \leq -1$. By the definition of $\lambda_\vartriangle$, it follows from condition (iii) that

$$\begin{aligned}\max_{i\in\mathbb{N}_{-M}} \mathbb{E}V_\vartriangle(x_{k+i}) &\geq q_\vartriangle \mathbb{E}V_\vartriangle(x_{k+1}) \\ &\geq e^{\lambda_\vartriangle(M+1)\triangle}\mathbb{E}V_\vartriangle(x_{k+1}),\end{aligned}$$

which implies that

$$\begin{aligned}e^{\lambda_\vartriangle(k+1)\triangle}\mathbb{E}V_\vartriangle(x_{k+1}) &\leq e^{(k+1)\lambda_\vartriangle\triangle}e^{-(M+1)\lambda_\vartriangle\triangle}\max_{i\in\mathbb{N}_{-M}}\{\mathbb{E}V_\vartriangle(x_{k+i})\} \\ &\leq \max_{i\in\mathbb{N}_{-M}}\{e^{\lambda_\vartriangle(k-M)\triangle}\mathbb{E}V_\vartriangle(x_{k+i})\} \\ &\leq \max_{i\in\mathbb{N}_{-M}}\{e^{\lambda_\vartriangle(k+i)\triangle}\mathbb{E}V_\vartriangle(x_{k+i})\} = u_k.\end{aligned}$$

Hence, (3.7) holds. This, together with (3.6) yields that

$$u_{k+1} \leq u_k \quad \text{when } \bar{i} \leq -1. \tag{3.8}$$

If $\bar{i} = 0$, then for any $i \in \mathbb{N}_{-M}$, by the definition of $\bar{i}$,

$$e^{\lambda_\vartriangle(k+i)\triangle}\mathbb{E}V_\vartriangle(x_{k+i}) \leq e^{\lambda_\vartriangle k\triangle}\mathbb{E}V_\vartriangle(x_k).$$

Hence, by the definition of $\lambda_\vartriangle$,

$$\begin{aligned}\mathbb{E}V_\vartriangle(x_{k+i}) &\leq e^{-\lambda_\vartriangle i\triangle}\mathbb{E}V_\vartriangle(x_k) \\ &\leq e^{\lambda_\vartriangle M\triangle}\mathbb{E}V_\vartriangle(x_k) \\ &< e^{\lambda_\vartriangle(M+1)\triangle}\mathbb{E}V_\vartriangle(x_k) \\ &\leq q_\vartriangle\mathbb{E}V_\vartriangle(x_k).\end{aligned}\tag{3.9}$$

Condition (ii) gives

$$\mathbb{E}V_\vartriangle(x_{k+1}) \leq (1-\zeta_\vartriangle\triangle)\mathbb{E}V_\vartriangle(x_k). \tag{3.10}$$

Therefore

$$\begin{aligned}u_{k+1}-u_k &= e^{\lambda_\vartriangle(k+1)\triangle}\mathbb{E}V_\vartriangle(x_{k+1}) - e^{\lambda_\vartriangle k\triangle}\mathbb{E}V_\vartriangle(x_k) \\ &\leq e^{\lambda_\vartriangle(k+1)\triangle}[\mathbb{E}V_\vartriangle(x_{k+1}) - \mathbb{E}V_\vartriangle(x_k)] + e^{\lambda_\vartriangle k\triangle}(e^{\lambda_\vartriangle\triangle}-1)\mathbb{E}V_\vartriangle(x_k) \\ &\leq -\zeta_\vartriangle\triangle e^{\lambda_\vartriangle(k+1)\triangle}\mathbb{E}V_\vartriangle(x_k) + e^{\lambda_\vartriangle k\triangle}(e^{\lambda_\vartriangle\triangle}-1)\mathbb{E}V_\vartriangle(x_k) \\ &= e^{\lambda_\vartriangle k\triangle}(e^{\lambda_\vartriangle\triangle} - 1 - \zeta_\vartriangle e^{\lambda_\vartriangle\triangle}\triangle)\mathbb{E}V_\vartriangle(x_k).\end{aligned}$$

The definition of $\lambda_\vartriangle$ gives $e^{\lambda_\vartriangle\triangle}(1-\zeta_\vartriangle\triangle) \leq 1$. Thus $u_{k+1} \leq u_k$ also holds for $\bar{i} = 0$. This, together with (3.8), yields $u_{k+1} \leq u_k$ for all $k \in \mathbb{N}$. By the definition of u_k and condition (i), for all $i \in \mathbb{N}_{-M}$,

$$\begin{aligned}e^{\lambda_\vartriangle(k+i)\triangle}\mathbb{E}V_\vartriangle(x_{k+i}) &\leq u_0 \\ &= \max_{i\in\mathbb{N}_{-M}}\{e^{\lambda_\vartriangle i\triangle}\mathbb{E}V_\vartriangle(x_i)\} \\ &\leq \max_{i\in\mathbb{N}_{-M}}\{\mathbb{E}V_\vartriangle(x_i)\} \\ &\leq c_{2\vartriangle}\max_{i\in\mathbb{N}_{-M}}\{\mathbb{E}|x_i|^{p_\vartriangle}\} \\ &= c_{2\vartriangle}\|\xi\|_{\mathbb{E}}^{p_\vartriangle}.\end{aligned}$$

Hence

$$e^{\lambda_\vartriangle k\triangle}\mathbb{E}V_\vartriangle(x_k) \leq c_{2\vartriangle}\|\xi\|_{\mathbb{E}}^{p_\vartriangle}.$$

Condition (i) gives

$$\mathbb{E}|x_k|^{p_\triangle} \le \frac{c_{2\triangle}}{c_{1\triangle}} \|\xi\|_{\mathbb{E}}^{p_\triangle} e^{-\lambda_\triangle k\triangle}, \tag{3.11}$$

as required. □

In the special case when both $\zeta_\triangle$ and $q_\triangle$ are independent of $\triangle$, namely $\zeta_\triangle = \zeta$ and $q_\triangle = q$, since $\tau = M\triangle$ and $\lim_{\triangle\to 0}(1-\zeta\triangle)^{-1/\triangle} = e^\zeta$, it is easy to see that

$$\lim_{\triangle\to 0} \lambda_\triangle = \frac{\log q}{\tau} \wedge \zeta,$$

which is λ specified in Theorem 3.1.

The following simple theorem gives a link between the pth moment exponential stability and almost sure exponential stability of the EM scheme.

Theorem 3.4. *For any $p_\triangle > 0$, (3.4) implies*

$$\limsup_{k\to\infty} \frac{1}{k\triangle} \log|x_k| \le -\frac{\lambda_\triangle}{p_\triangle}, \quad a.s. \tag{3.12}$$

In other words, the pth moment exponential stability of the EM scheme implies the almost sure exponential stability.

Proof. By the Chebyshev inequality and (3.4), for any integer $k > 0$,

$$\mathbb{P}\{|x_k|^{p_\triangle} > k^2 e^{-\lambda_\triangle k\triangle}\} \le \frac{c_{2\triangle}\|\xi\|_{\mathbb{E}}^{p_\triangle}}{c_{1\triangle}k^2}.$$

Then, by the Borel-Cantelli lemma, it follows that for almost all $\omega \in \mathbf{\Omega}$ there exists a random variable $k_0(\omega)$ such that for any $k > k_0(\omega)$,

$$|x_k|^{p_\triangle} \le k^2 e^{-\lambda_\triangle k\triangle},$$

which implies that for almost all $\omega \in \mathbf{\Omega}$,

$$\frac{1}{k\triangle}\log|x_k| \le -\frac{\lambda_\triangle}{p_\triangle} + \frac{2\log k}{\lambda_\triangle p_\triangle k\triangle}$$

whenever $k \ge k_0(\omega)$. Letting $k\to\infty$ yields (3.12), as required. □

4. Stability of exact and numerical solutions of SDDEs with variable delay.

Although Theorems 3.1 and 3.3 have similar expressions, it is not easy to observe whether stability of the numerical solution (1.2) may reproduce stability of the exact solution to Eq. (1.1) since these two theorems are not related to coefficients f and g explicitly. To show this property clearly, this section considers stability of exact and numerical solutions of SDDEs with variable delay, which are a very important class of SFDEs. The pth moment stability of the numerical solution of SDDEs with fixed delay was considered in [1] using the discrete Halanay inequality and the almost sure exponential stability of the numerical approximations of SDDEs with fixed delay was considered in [16] by using continuous and discrete semimartingale convergence theorems. Here the time delay is allowed to be a function of time, namely variable delay. In particular, the SDDE

$$dx(t) = F(x(t), x(t-\delta(t)))dt + G(x(t), x(t-\delta(t)))dw(t), \tag{4.1}$$

will be considered, where $\delta : [0,\infty) \to [0,\tau]$ represents the variable delay and $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $G = [G_{ij}]_{n\times m} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are Borel measurable functions with $F(0,0) = 0$, $G(0,0) = 0$ for the purpose of stability. Note that F and G could be rewritten as $F(\varphi(0), \varphi(-\delta(t)))$ and $G(\varphi(0), \varphi(-\delta(t)))$ in functional form. By letting $f(t,\varphi) = F(\varphi(0), \varphi(-\delta(t)))$ and $g(t,\varphi) = G(\varphi(0), \varphi(-\delta(t)))$, the operator (2.1) becomes

$$\mathcal{L}V(x,y) = V_x(x)F(x,y) + \frac{1}{2}\mathrm{trace}[G^T(x,y)V_{xx}(x)G(x,y)] \tag{4.2}$$

for any $x, y \in \mathbb{R}^n$. Applying Theorem 3.1 gives the following result (see [10, Theorem 6.4, p177]).

Theorem 4.1. *Let λ_1, λ_2, p, c_1, c_2 all be positive numbers. Assume that there exists a function $V \in C^2(\mathbb{R}^n; \mathbb{R}_+)$ such that condition (i) of Theorem 3.1 holds and*

$$\mathcal{L}V(x,y) \le -\lambda_1 V(x) + \lambda_2 V(y) \tag{4.3}$$

for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, where $\mathcal{L}V(x,y)$ is defined by (4.2). If $\lambda_1 > \lambda_2$, then Eq. (4.1) is pth moment exponentially stable, namely

$$\limsup_{t\to\infty} \frac{\log \mathbb{E}|x(t,\xi)|^p}{t} \le -(\lambda_1 - q\lambda_2),$$

where $q \in (1, \lambda_1/\lambda_2)$ is the unique root of $\lambda_1 - q\lambda_2 = \log(q)/\tau$. If, in addition, there is a $K > 0$ such that for any $x, y \in \mathbb{R}^n$,

$$|F(x,y)| \vee |G(x,y)| \le K(|x| + |y|), \tag{4.4}$$

then Eq. (4.1) is also almost surely exponentially stable, specifically with

$$\limsup_{t\to\infty} \frac{\log |x(t,\xi)|}{t} \le -\frac{\lambda_1 - q\lambda_2}{p}.$$

Now consider the stability of the numerical solution of Eq. (4.1). The EM scheme of (4.1) is

$$\begin{cases} x_k = \xi(k\triangle), & k \in \mathbb{N}_{-M}, \\ x_{k+1} = x_k + F(x_k, x_{k-\delta_k})\triangle + G(x_k, x_{k-\delta_k})\triangle w_k, & k \ge 0, \end{cases} \tag{4.5}$$

where $\delta_k = \lfloor \delta(t) \rfloor \in \{0, 1, 2, \cdots, M\}$ in which $\lfloor x \rfloor$ represents the integer part of x, that is, the nearest grid-point on the left of the delayed argument is used to replace the exact point. Applying Theorem 3.3 to the EM approximation (4.5) yields the following result:

Theorem 4.2. *Fix $\triangle > 0$. Let $\lambda_{1\triangle}, \lambda_{2\triangle}, p_\triangle, c_{1\triangle}, c_{2\triangle}$ all be positive constants. Assume that there exists a function $V_\triangle : \mathbb{R}^n \to \mathbb{R}_+$ such that condition (i) of Theorem 3.3 holds and for the sequence $\{x_k\}_{k\ge 1}$ determined by (4.5),*

$$\mathbb{E}V_\triangle(x_{k+1}) \le (1 - \lambda_{1\triangle}\triangle)\mathbb{E}V_\triangle(x_k) + \lambda_{2\triangle}\triangle\mathbb{E}V_\triangle(x_{k-\delta_k}) \tag{4.6}$$

for all $\delta_k \in \{0, 1, \cdots, M\}$. If $\lambda_{1\triangle} > \lambda_{2\triangle}$ and $(\lambda_{1\triangle} - \lambda_{2\triangle})\triangle < 1$, then for any bounded initial sequence $\{\xi(k\triangle)\}_{k\in\mathbb{N}_{-M}}$,

$$\limsup_{k\to\infty} \frac{\log \mathbb{E}|x_k|^{p_\triangle}}{k\triangle} \le -(\lambda_{1\triangle} - q_\triangle\lambda_{2\triangle}) < 0, \tag{4.7}$$

and

$$\limsup_{k\to\infty} \frac{\log |x_k|}{k\triangle} \le -\frac{\lambda_{1\triangle} - q_\triangle\lambda_{2\triangle}}{p_\triangle} < 0, \quad a.s. \tag{4.8}$$

where $q_\triangle \in (1, \lambda_{1\triangle}/\lambda_{2\triangle})$ is the unique root of the equation

$$\lambda_{1\triangle} - q_\triangle\lambda_{2\triangle} = \frac{\log q_\triangle}{(M+1)\triangle}.$$

Proof. The key steps in applying Theorem 3.3 are to verify conditions (ii) and (iii). Since $\lambda_{1\triangle} > \lambda_{2\triangle}$, for any $\delta_k \in \{0, 1, \cdots, M\}$, if $V_\triangle(x_{k-\delta_k}) \le q_\triangle \mathbb{E}V_\triangle(x_k)$, by condition (4.6),

$$\begin{aligned}\mathbb{E}V_\triangle(x_{k+1}) &\le (1-\lambda_{1\triangle}\triangle)\mathbb{E}V_\triangle(x_k) + \lambda_{2\triangle}\triangle\mathbb{E}V_\triangle(x_{k-\delta_k})\\ &= [1-(\lambda_{1\triangle} - q_\triangle\lambda_{2\triangle})\triangle]\mathbb{E}V_\triangle(x_k).\end{aligned}$$

Choosing $\zeta_\triangle = \lambda_{1\triangle} - q_\triangle\lambda_{2\triangle}$ and noting that $\zeta_\triangle\triangle < 1$, it is obvious that condition (ii) of Theorem 3.3 holds. Noting that $\lambda_{1\triangle} > \lambda_{2\triangle}$, by the definition of $q_\triangle$,

$$1 - \lambda_{1\triangle}\triangle + q_\triangle\lambda_{2\triangle}\triangle \in (0, 1). \tag{4.9}$$

Let

$$\lambda_\triangle = \lambda_{1\triangle} - q_\triangle\lambda_{2\triangle} = \frac{\log q_\triangle}{(M+1)\triangle}.$$

This, together with (4.9), yields

$$(1-\lambda_{1\triangle}\triangle)e^{-(M+1)\lambda_\triangle\triangle} + \lambda_{2\triangle}\triangle \le \frac{1}{q_\triangle}. \tag{4.10}$$

Hence, for some $\check{i} \in \mathbb{N}_{-M} - \{0\}$, if $\mathbb{E}V_\triangle(x_{k-\check{i}}) > e^{(M+1)\lambda_\triangle\triangle}\mathbb{E}V_\triangle(x_k)$, by condition (4.6) and inequality (4.10),

$$\begin{aligned}V_\triangle(x_{k+1}) &\le (1-\lambda_{1\triangle}\triangle)e^{-(M+1)\lambda_\triangle\triangle}\mathbb{E}V_\triangle(x_{k-\check{i}}) + \lambda_{2\triangle}\triangle\mathbb{E}V_\triangle(x_{k-\delta_k})\\ &\le [(1-\lambda_{1\triangle}\triangle)e^{-(M+1)\lambda_\triangle\triangle} + \lambda_{2\triangle}\triangle]\max_{i\in\mathbb{N}_{-M}}\mathbb{E}V_\triangle(x_{k+i})\\ &\le \frac{1}{q_\triangle}\max_{i\in\mathbb{N}_{-M}}\mathbb{E}V_\triangle(x_{k+i}),\end{aligned}$$

which implies that condition (iii) of Theorem 3.3 holds. Hence Theorem 3.3 implies the desired assertion (4.7). By condition (4.4), applying Theorem 3.4 gives (4.8), which completes the proof. □

In Theorem 4.1, the key condition is (4.3), which may deduce condition (ii) of Theorem 3.1. In Theorem 3.3, the key condition is (4.6), which may deduce conditions (ii) and (iii). This implies that the additional condition (iii) of Theorem 3.3 disappears. Note that conditions (4.3) and (4.6) are similar. This admits us to further examine stability of the exact and numerical solution of Eq. (4.1). The rest of this section examines the condition under which the mean square and almost sure exponential stability of the stochastic sequence $\{x_k\}_{k\ge0}$ defined by (4.5) may reproduce the corresponding stability of the solution $x(t)$ of Eq. (4.1). The following assumptions on coefficients F and G will be imposed:

Assumption 1. *Assume that there is a $\hat{\zeta} > 0$ such that*

$$x^T F(x, 0) \le -\hat{\zeta}|x|^2 \quad \textit{for all } x \in \mathbb{R}^n. \tag{4.11}$$

Assume also that there are nonnegative numbers α_0, α_1, β_0 and β_1 such that for any $x, \bar{x}, y \in \mathbb{R}$,

$$|F(x,0) - F(\bar{x}, y)| \le \alpha_0|x-\bar{x}| + \alpha_1|y|$$

and

$$\mathrm{trace}[G^T(x,y)G(x,y)] \le \beta_0|x|^2 + \beta_1|y|^2.$$

Assumption 2. *For any $x, y \in \mathbb{R}^n$, there exists a constant $K > 0$ such that*

$$|F(x,y)|^2 \le K(|x|^2 + |y|^2).$$

Note that Assumptions 1 and 2 implies that both F and G satisfy the linear growth condition. These two assumptions, together with the local Lipschitz condition for F and G may guarantee that there is a unique solution to Eq. (4.1) and this solution is pth bounded. Assumptions 1 and 2 may also guarantee the following stability of the trivial solution.

Theorem 4.3. *Let Assumption 1 hold and define $\lambda_1 = 2\hat{\zeta} - \beta_0 - \alpha_1$ and $\lambda_2 = \alpha_1 + \beta_1$. If*

$$\lambda_1 > \lambda_2, \tag{4.12}$$

then for any initial data $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$,

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}|x(t,\xi)|^2 \le -(\lambda_1 - q\lambda_2), \tag{4.13}$$

where $q \in (1, \lambda_1/\lambda_2)$ is the unique root of the equation

$$\lambda_1 - q\lambda_2 = \frac{\log q}{\tau}, \tag{4.14}$$

namely, the solution $x(t,\xi)$ of Eq. (4.1) is mean square exponentially stable. In addition, if Assumption 2 also holds,

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t,\xi)| \le -\frac{\lambda_1 - q\lambda_2}{2} \quad a.s. \tag{4.15}$$

Proof. Choose $V(x) = |x|^2$. By Assumption 1, applying the operator (4.2) yields that for any $x, y \in \mathbb{R}^n$,

$$\begin{aligned}
\mathcal{L}V(x,y) &= 2x^T F(x,y) + \mathrm{trace}[G^T(x,y)G(x,y)] \\
&= 2x^T F(x,0) + 2x^T[F(x,y) - F(x,0)] + \mathrm{trace}[G^T(x,y)G(x,y)] \\
&\le -2\hat{\zeta}|x|^2 + 2\alpha_1|x||y| + \beta_0|x|^2 + \beta_1|y|^2 \\
&\le -2\hat{\zeta}|x|^2 + \alpha_1|x|^2 + \alpha_1|y|^2 + \beta_0|x|^2 + \beta_1|y|^2 \\
&\le -\lambda_1|x|^2 + \lambda_2|y|^2.
\end{aligned}$$

Noting that $\lambda_1 > \lambda_2$, applying Theorem 4.1 yields the desired results (4.13) and (4.15), as required. □

Consider now whether the EM scheme (4.5) reproduces stability of the exact solution of Eq. (4.1).

Theorem 4.4. *Under Assumptions 1 and 2, if (4.12) holds, then for any bounded initial sequence $\{\xi(k\triangle)\}_{k\in\mathbb{N}_{-M}}$, for any $\varepsilon \in (0, \lambda_1 - q\lambda_2)$, where λ_1, λ_2 and q are defined by Theorem 4.3, there exists a $\triangle^* > 0$, for any $\triangle < \triangle^*$, the EM approximation (4.5) has properties*

$$\limsup_{k\to\infty} \frac{1}{k\triangle} \log \mathbb{E}|x_k|^2 \le -(\lambda_1 - q\lambda_2 - \varepsilon) \tag{4.16}$$

and

$$\limsup_{k\to\infty} \frac{1}{k\triangle} \log |x_k| \le -\frac{\lambda_1 - q\lambda_2 - \varepsilon}{2}, \quad a.s. \tag{4.17}$$

Proof. By the EM approximation (4.5),

$$\begin{aligned}|x_{k+1}|^2 &= |x_k|^2 + 2\langle x_k, F(x_k, x_{k-\delta_k})\triangle\rangle + |F(x_k, x_{k-\delta_k})|^2\triangle^2 \\ &\quad + \sum_{i=1}^{n}\Big(\sum_{j=1}^{m} G_{ij}(x_k, x_{k-\delta_k})\triangle w_k^j\Big)^2 + M_k \\ &= |x_k|^2 + 2\langle x_k, F(x_k, 0)\triangle\rangle + 2\langle x_k, F(x_k, x_{k-\delta_k})\triangle - F(x_k, 0)\triangle\rangle \\ &\quad + |F(x_k, x_{k-\delta_k})|^2\triangle^2 + \sum_{i=1}^{n}\Big(\sum_{j=1}^{m} G_{ij}(x_k, x_{k-\delta_k})\triangle w_k^j\Big)^2 + M_k,\end{aligned}$$

where $M_k = 2\langle x_k + F(x_k, x_{k-\delta_k})\triangle, G(x_k, x_{k-\delta_k})\triangle w_k\rangle$. It is obvious that $\mathbb{E}(M_k|\mathcal{F}_{k\triangle}) = 0$. Assumption 2 gives

$$2\langle x_k, F(x_k, 0)\triangle\rangle \le -2\hat{\zeta}|x_k|^2\triangle$$

and Assumption 1 gives

$$\begin{aligned}2\langle x_k, F(x_k, x_{k-\delta_k})\triangle - F(x_k, 0)\triangle\rangle &\le 2|x_k||F(x_k, x_{k-\delta_k}) - F(x_k, 0)|\triangle \\ &\le 2\alpha_1|x_k||x_{k-\delta_k}|\triangle \\ &\le \alpha_1|x_k|^2\triangle + \alpha_1|x_{k-\delta_k}|^2\triangle.\end{aligned}$$

Moreover,

$$\begin{aligned}\mathbb{E}\Big[\sum_{i=1}^{n}\Big(\sum_{j=1}^{m} G_{ij}(x_k, x_{k-\delta_k})\triangle w_k^j\Big)^2\Big|\mathcal{F}_{k\triangle}\Big] &\le \operatorname{trace}[G^T(x_k, x_{k-\delta_k})G(x_k, x_{k-\delta_k})\triangle] \\ &\le \beta_0|x_k|^2\triangle + \beta_1|x_{k-\delta_k}|^2\triangle,\end{aligned}$$

so

$$\mathbb{E}(|x_{k+1}|^2|\mathcal{F}_{k\triangle}) \le (1-\lambda_1\triangle+K\triangle^2)|x_k|^2+\alpha_1\triangle|x_{k-\delta_k}|^2+K\triangle^2|x_{k-\delta_k}|^2+\beta_1\triangle|x_{k-\delta_k}|^2.$$

Taking expectation on both sides then gives

$$\mathbb{E}|x_{k+1}|^2 \le [1 - (\lambda_1 - K\triangle)\triangle]\mathbb{E}|x_k|^2 + (\lambda_2 + K\triangle)\triangle\mathbb{E}|x_{k-\delta_k}|^2. \tag{4.18}$$

Define $\lambda_{1\triangle} = \lambda_1 - K\triangle$ and $\lambda_{2\triangle} = \lambda_2 + K\triangle$. Let $\triangle_1^* = (\lambda_1 - \lambda_2)/(2K)$. Then for any $\triangle \le \triangle_1^*$, $\lambda_{1\triangle} > \lambda_{2\triangle}$ and condition (4.6) of Theorem 4.1 holds. Applying Theorem 4.1 yields

$$\limsup_{k\to\infty}\frac{\log\mathbb{E}|x_k|^2}{k\triangle} \le -\hat{\lambda}_\triangle < 0, \tag{4.19}$$

and

$$\limsup_{k\to\infty}\frac{\log|x_k|}{k\triangle} \le -\frac{\hat{\lambda}_\triangle}{2} < 0, \quad a.s. \tag{4.20}$$

where $\hat{\lambda}_\triangle = \lambda_{1\triangle} - q_\triangle\lambda_{2\triangle}$ and $q_\triangle \in (1, \lambda_{1\triangle}/\lambda_{2\triangle})$ is the unique root of the equation $\lambda_{1\triangle} - q_\triangle\lambda_{2\triangle} = \log(q_\triangle)/((M+1)\triangle)$. Define the function

$$h_\triangle(x) = \lambda_{1\triangle} - \lambda_{2\triangle}x - \frac{\log x}{\tau+\triangle}.$$

It is clear that $h(\cdot)$ is a monotone decreasing function and for any $x \in (1, \lambda_1/\lambda_2)$,

$$\lim_{\triangle\to 0} h_\triangle(x) = h_0(x) = \lambda_1 - \lambda_2 x - \frac{\log x}{\tau}.$$

By definitions of $q_\triangle$ and q, $h_\triangle(q_\triangle) = 0$ and $h_0(q) = 0$. Since $h(\cdot)$ is a monotone function,

$$\lim_{\triangle \to 0} q_\triangle = q$$

and

$$\lim_{\triangle \to 0} \hat{\lambda}_\triangle = \hat{\lambda},$$

where $\hat{\lambda} = \lambda_1 - q\lambda_2$. Hence, for any $\varepsilon \in (0, \hat{\lambda})$, there exists a $\triangle_2^* > 0$ such that for any $\triangle < \triangle_2^*$,

$$\hat{\lambda}_\triangle > \hat{\lambda} - \varepsilon.$$

Choosing $\triangle^* = \triangle_1^* \wedge \triangle_2^*$, for any $\triangle < \triangle^*$, (4.19) and (4.20) therefore yield the desired assertions (4.16) and (4.17). $\square$

Theorems 4.3 and 4.4 can be applied to the linear SDDE system

$$dx(t) = [Ax(t) + Bx(t - \delta(t))]dt + [Cx(t) + Dx(t - \delta(t))]dw(t) \tag{4.21}$$

with the initial data $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$, where $A, B, C, D \in \mathbb{R}^{n\times n}$ and $\delta(t) \in [0, \tau]$. The corresponding EM approximation of Eq. (4.21) is

$$\begin{cases} x_k = \xi(k\triangle), & k \in \mathbb{N}_{-M}, \\ x_{k+1} = x_k + [Ax_k + Bx_{k-\delta_k}]\triangle + [Cx_k + Dx_{k-\delta_k}]\triangle w_k, & k \geq 0, \end{cases} \tag{4.22}$$

Let $\lambda_{\max}(A + A^T)$ represent the maximum eigenvalue of the symmetric matrix $A + A^T$ and note that

$$x^T Ax \leq \frac{1}{2}\lambda_{\max}(A + A^T)|x|^2.$$

Also note that

$$|Ax - A\bar{x} - By| \leq |A||x - \bar{x}| + |B||y|$$

and

$$|Cx + Dy|^2 \leq 2|C|^2|x|^2 + 2|D|^2|y|^2.$$

It is obvious that $F(x, y) = Ax + By$ and $G(x, y) = Cx + Dy$ satisfy Assumptions 1 and 2 if $\lambda_{\max}(A + A^T) < 0$. Applying Theorems 4.3 and 4.4 gives the following result directly.

Theorem 4.5. *If $-\lambda_{\max}(A^T + A) > 2(|B| + |C|^2 + |D|^2)$, then for any initial data $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$, the solution $x(t, \xi)$ of (4.21) satisfies*

$$\limsup_{t\to\infty} \frac{1}{t}\mathbb{E}|x(t, \xi)|^2 \leq -(\lambda_1 - q\lambda_2) \tag{4.23}$$

and

$$\limsup_{t\to\infty} \frac{1}{t}\log|x(t, \xi)| \leq -\frac{\lambda_1 - q\lambda_2}{2}, \quad a.s. \tag{4.24}$$

where $\lambda_1 = -\lambda_{\max}(A^T + A) - 2|C|^2 - |B|$, $\lambda_2 = |B| + 2|D|^2$ and $q \in (1, \lambda_1/\lambda_2)$ is the unique root of the equation

$$\lambda_1 - q\lambda_2 = \frac{\log q}{\tau}.$$

For any $\varepsilon \in (0, \lambda_1 - q\lambda_2)$, there exists a $\triangle^* > 0$ such that for any $\triangle < \triangle^*$, the EM approximation x_k of (4.22) has the properties

$$\limsup_{k\to\infty} \frac{1}{k\triangle}\mathbb{E}|x_k|^2 \leq -(\lambda_1 - q\lambda_2) + \varepsilon \tag{4.25}$$

and

$$\limsup_{k\to\infty} \frac{1}{k\triangle} \log |x_k| \le -\frac{\lambda_1 - q\lambda_2 - \varepsilon}{2}, \quad a.s. \tag{4.26}$$

5. **Exponential stability of exact and numerical solutions to stochastically perturbed equations.** In general, any systems are often subject to environmental noise and history. If a system is asymptotically stable, it is therefore interesting to determine how much stochastic perturbation and history effect this system can tolerate without losing the property of asymptotic stability. Such a point of view is described as the problem of robust stability, which is an important issue for stochastically perturbed equations. To examine the problem whether the EM scheme can reproduce this property of stochastically perturbed equations, let us consider the equation

$$dx(t) = (\psi(x(t)) + F(x_t))dt + g(x_t)dw(t) \tag{5.1}$$

with the initial data $x_0 = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$, where

$$\psi : \mathbb{R}^n \to \mathbb{R}^n,\ F : C([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^n,\ g = [g_{ij}]_{n\times m} : C([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^{n\times m}$$

and $\psi(0) \equiv 0$, $F(0) \equiv 0$ and $g(0) \equiv 0$ for the purpose of stability. Eq. (5.1) can be regarded as the perturbed equation by stochastic noise and time delay of the ordinary differential equation

$$\dot{x}(t) = \psi(x(t)).$$

Define $f(\varphi) = \psi(\varphi(0)) + F(\varphi)$. (5.1) may return to (1.1). Moreover, for the function $V \in C^2(\mathbb{R}^n; \mathbb{R}_+)$, $\mathcal{L}V$ becomes

$$\mathcal{L}V(\varphi) = V_x(\varphi(0))[\psi(\varphi(0)) + F(\varphi)] + \frac{1}{2}\text{trace}[g^T(\varphi)V_{xx}(\varphi(0))g(\varphi)]. \tag{5.2}$$

The EM method (1.2) applied to (5.1) has the form

$$\begin{cases} x_k = \xi(k\triangle), & k \in \mathbb{N}_{-M}, \\ x_{k+1} = x_k + [\psi(x_k) + F(y_{k\triangle})]\triangle + g(y_{k\triangle})\triangle w_k, & k \ge 0. \end{cases} \tag{5.3}$$

Let η and μ be two probability measures on $[-\tau, 0]$, namely, $\int_{-\tau}^0 d\eta = \int_{-\tau}^0 d\mu = 1$ (in this paper, probability measures may be extended to any right-continuous nondecreasing functions). We impose the following assumptions for coefficients ψ, F and G.

Assumption 3. *There is a constant $\bar{K}$ such that $|\psi(x)|^2 \le \bar{K}|x|^2$.*

Assumption 4. *There is a constant $\bar{\zeta} > 0$ such that*

$$\langle x, \psi(x)\rangle \le -\bar{\zeta}|x|^2 \text{ for all } x \in \mathbb{R}^n, \tag{5.4}$$

and there are constants K_1 and K_2 and two probability measures η and μ on $[-\tau, 0]$ such that

$$|F(\varphi)| \le K_1 \int_{-\tau}^0 |\varphi(\theta)|d\eta \quad \text{and} \quad \text{trace}[g^T(\varphi)g(\varphi)] \le K_2 \int_{-\tau}^0 |\varphi(\theta)|^2 d\mu \tag{5.5}$$

for all $\varphi \in C([-\tau, 0]; \mathbb{R}^n)$.

(5.4) is from the one-sided Lipschitz condition since $\psi(0) = 0$. (5.5) is the linear growth condition on F and g (cf. [10, 11]). Assumptions 3 and 4, together with the local Lipschitz conditions on ψ, F and g, may guarantee that Eq. (5.1) exists a unique solution and this solution is pth bounded. Under these conditions, let us examine the mean square and almost sure exponential stability of the exact solution to (5.1) and the numerical solution to (5.3).

Theorem 5.1. *Let $\bar{\alpha} = 2\bar{\zeta} - K_1$, $\bar{\beta} = K_1 + K_2$. Under Assumption 4, if*

$$\bar{\alpha} > \bar{\beta}, \tag{5.6}$$

then for any initial data $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$,

$$\mathbb{E}|x(t,\xi)|^2 \le \|\xi\|^2_{\mathbb{E}} e^{-\bar{\lambda} t}, \tag{5.7}$$

where $\bar{\lambda} = \bar{\alpha} - q\bar{\beta}$ and $q \in (1, \bar{\alpha}/\bar{\beta})$ is the unique solution of the equation

$$\bar{\alpha} - q\bar{\beta} = \frac{\log q}{\tau}. \tag{5.8}$$

In addition, if Assumption 3 holds, then

$$\limsup_{t\to\infty} \frac{1}{t} \log|x(t,\xi)| \le -\frac{\bar{\lambda}}{2} \quad a.s. \tag{5.9}$$

that is, the trivial solution of Eq. (5.1) is also almost surely exponentially stable.

Proof. Choose $V(x) = |x|^2$. For any $\varepsilon > 0$, recall the elementary inequality $2|xy| \le \varepsilon|x|^2 + \varepsilon^{-1}|y|^2$. By the Hölder inequality, for any $\varepsilon > 0$, applying Assumption 4 gives

$$\begin{aligned}
\mathcal{L}V(\varphi) &= 2\langle \varphi(0), \psi(\varphi(0)) + F(\varphi)\rangle + \mathrm{trace}[g^T(\varphi)g(\varphi)] \\
&\le -2\bar{\zeta}|\varphi(0)|^2 + \varepsilon^{-1}|\varphi(0)|^2 + \varepsilon|F(\varphi)|^2 + \mathrm{trace}[g^T(\varphi)g(\varphi)] \\
&\le (-2\bar{\zeta} + \varepsilon^{-1})|\varphi(0)|^2 + \varepsilon K_1^2 \int_{-\tau}^0 |\varphi(\theta)|^2 d\eta + K_2 \int_{-\tau}^0 |\varphi(\theta)|^2 d\mu
\end{aligned}$$

For all $\varphi \in L^2_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^n)$ satisfying $\mathbb{E}|\varphi(\theta)|^2 < q|\varphi(0)|^2$, choosing $\varepsilon = K_1^{-1}$, we therefore have

$$\mathbb{E}\mathcal{L}V(\varphi) \le [-2\bar{\zeta} + K_1 + (K_1 + K_2)q]\mathbb{E}|\varphi(0)|^2 = -(\bar{\alpha} - \bar{\beta}q)\mathbb{E}|\varphi(0)|^2.$$

Note that condition (5.6) implies $\bar{\alpha} > \bar{\beta}q$. This shows that condition (ii) in Theorem 3.1 holds. By definition of q, applying Theorem 3.1 gives the assertion (5.7). By Assumptions 3 and 4, applying the Hölder inequality yields

$$\begin{aligned}
\mathbb{E}|f(\varphi)|^2 &\le 2\mathbb{E}|\psi(\varphi(0))|^2 + 2\mathbb{E}|F(\varphi)|^2 \\
&\le 2\bar{K}\mathbb{E}|\varphi(0)|^2 + 2K_1^2 \int_{-\tau}^0 \mathbb{E}|\varphi(\theta)|^2 d\eta \\
&\le 2(\bar{K} + K_1^2) \sup_{-\tau\le\theta\le 0} \mathbb{E}|\varphi(\theta)|^2
\end{aligned}$$

and

$$\mathbb{E}|g(\varphi)|^2 \le K_2 \int_{-\tau}^0 \mathbb{E}|\varphi(\theta)|^2 d\mu \le K_2 \sup_{-\tau\le\theta\le 0} \mathbb{E}|\varphi(\theta)|^2,$$

which implies that condition (3.2) holds. Applying Theorem 3.2 therefore gives the almost sure stability (5.9), as required. □

Theorem 5.1 gives a criterion on how large the stochastic noise and time delay perturbation that equation $\dot{x}(t) = \psi(x(t))$ can tolerate so that the perturbed equation Eq. (5.1) remains exponential stability. We now examine the stability of the numerical solutions.

Theorem 5.2. *Under Assumptions 3 and 4, if condition (5.6) holds, then for any* $\varepsilon \in (0,\ \bar{\lambda})$*, where* $\bar{\lambda}$ *is defined by Theorem 5.1, there is a* $\triangle^* > 0$ *such that for any* $\triangle < \triangle^*$*, the EM approximation (5.3) has the properties*

$$\mathbb{E}|x_k|^2 \le \|\xi\|_{\mathbb{E}}^2 e^{-(\bar{\lambda}-\varepsilon)k\triangle}, \tag{5.10}$$

and

$$\limsup_{k\to\infty} \frac{1}{k\triangle} \log|x_k| \le -\frac{\bar{\lambda}-\varepsilon}{2},\ \ a.s. \tag{5.11}$$

Proof. By (5.3), applying Assumption 3 yields

$$\begin{aligned} |x_{k+1}|^2 &= |x_k|^2 + 2\langle x_k, \psi(x_k)\triangle\rangle + 2\langle x_k, F(y_{k\triangle})\triangle\rangle + |\psi(x_k) + F(y_{k\triangle})|^2\triangle^2 \\ &\quad + \sum_{i=1}^n \Big(\sum_{j=1}^m g_{ij}(y_{k\triangle})\triangle w_k^j\Big)^2 + M_k \\ &\le (1-2\bar{\zeta}\triangle)|x_k|^2 + 2|x_k||F(y_{k\triangle})|\triangle + 2(|\psi(x_k)|^2 + |F(y_{k\triangle})|^2)\triangle^2 \\ &\quad + \sum_{i=1}^n \Big(\sum_{j=1}^m g_{ij}(y_{k\triangle})\triangle w_k^j\Big)^2 + M_k, \end{aligned} \tag{5.12}$$

where $M_k = 2\langle x_k + (\psi(x_k) + F(y_{k\triangle}))\triangle, g(y_{k\triangle})\triangle w_k\rangle$. Obviously, $\mathbb{E}(M_k|\mathcal{F}_{k\triangle}) = 0$. For any $\varepsilon > 0$,

$$2|x_k||F(y_{k\triangle})| \le \varepsilon^{-1}|x_k|^2 + \varepsilon|F(y_{k\triangle})|^2.$$

By Assumption 4 and the Hölder inequality, we have

$$|F(y_{k\triangle})|^2 \le K_1^2 \int_{-\tau}^0 |y_{k\triangle}(\theta)|^2 d\eta.$$

By Assumption 4, we may estimate that

$$\begin{aligned} \mathbb{E}\Big[\sum_{i=1}^n \Big(\sum_{j=1}^m g_{ij}(y_{k\triangle})\triangle w_k^j\Big)^2 \Big| \mathcal{F}_{k\triangle}\Big] &= \sum_{i=1}^n \sum_{j=1}^m g_{ij}^2(y_{k\triangle})\triangle \\ &= \text{trace}[g^T(y_{k\triangle})g(y_{k\triangle})]\triangle \\ &\le K_2\triangle \int_{-\tau}^0 |y_{k\triangle}(\theta)|^2 d\mu. \end{aligned}$$

By these estimates, taking the conditional expectation in (5.12) and choosing $\varepsilon = K_1^{-1}$ yield

$$\begin{aligned} \mathbb{E}(|x_{k+1}|^2|\mathcal{F}_{k\triangle}) &\le [1 - (2\bar{\zeta} - K_1)\triangle + 2\bar{K}\triangle^2]|x_k|^2 + K_2\triangle \int_{-\tau}^0 |y_{k\triangle}(\theta)|^2 d\mu \\ &\quad + (K_1\triangle + 2K_1^2\triangle^2)\int_{-\tau}^0 |y_{k\triangle}(\theta)|^2 d\eta. \end{aligned} \tag{5.13}$$

By definition (1.3) of $y_{k\triangle}(\theta)$, for any $\theta \in [i\triangle, (i+1)\triangle],\ i \in \mathbb{N}_{-M} - \{0\}$, we have

$$\begin{aligned} \mathbb{E}|y_{k\triangle}(\theta)|^2 &\le \mathbb{E}\Big|\frac{\theta - i\triangle}{\triangle}x_{k+i+1} + \frac{(i+1)\triangle - \theta}{\triangle}x_{k+i}\Big|^2 \\ &\le \mathbb{E}\Big(\frac{\theta - i\triangle}{\triangle}|x_{k+i+1}|^2 + \frac{(i+1)\triangle - \theta}{\triangle}|x_{k+i}|^2\Big) \\ &\le \sup_{i\in\mathbb{N}_{-M}} \mathbb{E}|x_{k+i}|^2. \end{aligned}$$

So, taking expectation on the both sides in (5.13) gives

$$\mathbb{E}|x_{k+1}|^2 \le (1 - \bar{\alpha}\triangle + 2\bar{K}\triangle^2)\mathbb{E}|x_k|^2 + (\bar{\beta}\triangle + 2K_1^2\triangle^2) \sup_{i\in\mathbb{N}_{-M}} \mathbb{E}|x_{k+i}|^2. \tag{5.14}$$

Let $\bar{\alpha}_\triangle = \bar{\alpha} - 2\bar{K}\triangle$ and $\bar{\beta}_\triangle = \bar{\beta} + 2K_1^2\triangle$. Note that $\bar{\alpha} > \bar{\beta}$. Letting $\triangle_1^* = (\bar{\alpha} - \bar{\beta})/[2(\bar{K} + K_1^2)]$, then for any $\triangle < \triangle_1^*$, $\bar{\alpha}_\triangle > \bar{\beta}_\triangle$. It is obvious that there exists a $\triangle_2^* > 0$ such that for any $\triangle < \triangle_1^* \wedge \triangle_2^*$ such that there exists $q_\triangle \in (1, \bar{\alpha}_\triangle/\bar{\beta}_\triangle)$ satisfying $1 - (\bar{\alpha}_\triangle - q_\triangle\beta)\triangle \in (0, 1)$, which implies that condition (ii) of Theorem 3.3 holds. Define

$$\bar{\lambda}_\triangle = \bar{\alpha}_\triangle - q_\triangle\bar{\beta}_\triangle = \frac{\log q_\triangle}{(M+1)\triangle}.$$

Using the similar technique to the proof of Theorem 4.2 shows that condition (iii) of Theorem 3.3 also holds. By definition of $\lambda_\triangle$, applying Theorem 3.3 yields

$$\limsup_{k\to\infty} \frac{\log \mathbb{E}|x_k|^2}{k\triangle} \le -\bar{\lambda}_\triangle < 0 \tag{5.15}$$

and

$$\limsup_{k\to\infty} \frac{\log |x_k|}{k\triangle} \le -\frac{\bar{\lambda}_\triangle}{2} < 0, \quad a.s. \tag{5.16}$$

Define

$$\bar{h}_\triangle(x) = \bar{\alpha}_\triangle - \bar{\beta}_\triangle x - \frac{\log x}{\tau + \triangle}.$$

Using the similar techniques to the proof of Theorem 4.4, $\lim_{\triangle\to 0} \bar{\lambda}_\triangle = \bar{\lambda}$. Hence, for any $\varepsilon \in (0, \bar{\lambda})$, there exists a $\triangle_3^* > 0$ such that for any $\triangle < \triangle_3^*$,

$$\bar{\lambda}_\triangle > \bar{\lambda} - \varepsilon.$$

Choosing $\triangle^* = \triangle_1^* \wedge \triangle_2^* \wedge \triangle_3^*$, for any $\triangle < \triangle^*$, (5.15) and (5.16) therefore yield the desired assertions (5.10) and (5.11), as required. □

As another special SFDEs, SVDIDEs arise widely in scientific fields such biology, ecology, medicine and physics (cf. [1, 18]). The rest of this section considers the following scalar linear SVDIDE

$$dx(t) = \Big[-\alpha x(t) + \beta \int_{t-\tau}^{t} x(s)ds \Big] dt + \Big[\sigma x(t) + \rho \int_{t-\tau}^{t} x(s)ds \Big] dw(t) \tag{5.17}$$

with initial data $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R})$, where α, β, σ, ρ are constants and $\alpha > 0$. This equation may also be seen as a perturbed equation by stochastic noise and time delay of the linear system $\dot{x}(t) = -\alpha x(t)$. Let $d\mu = ds/\tau$ and η be a Dirac measure in the origin and then $d\nu = (\sigma d\eta + \rho\tau d\mu)/(\sigma + \rho\tau)$. It is obvious that both μ and ν are the probability measures on $[-\tau, 0]$. Eq. (5.17) may therefore be rewritten as

$$dx(t) = \Big[-\alpha x(t) + \beta\tau \int_{-\tau}^{0} x(t+\theta)d\mu \Big] dt + \Big[(\sigma + \rho\tau) \int_{-\tau}^{0} x(t+\theta)d\nu \Big] dw(t). \tag{5.18}$$

Define $\psi(x) = -\alpha x$ and

$$F(\varphi) = \beta\tau \int_{-\tau}^{0} \varphi(\theta)d\mu, \quad g(\varphi) = (\sigma + \rho\tau) \int_{-\tau}^{0} \varphi(\theta)d\nu.$$

Eq. (5.17) may be rewritten as the stochastically perturbed equation (5.1). It is obvious that Assumptions 3 and 4 hold with $\bar{K} = \alpha^2$, $\bar{\zeta} = \alpha$, $K_1 = |\beta|\tau$ and $K_2 = |\sigma + \rho\tau|$.

The EM method (1.2) applied to Eq. (5.17) has the form

$$\begin{cases} x_k = \xi(k\triangle), & k \in \mathbb{N}_{-M}, \\ x_{k+1} = (1-\alpha\triangle)x_k + \beta\sum_{i=0}^{M} x_{k-i}\triangle^2 + \Big(\sigma x_k + \rho\sum_{i=0}^{M} x_{k-i}\triangle\Big)\triangle w_k, & k \geq 0. \end{cases} \tag{5.19}$$

Then applying Theorems 5.1 and 5.2 gives stability of the exact and numerical solutions of (5.17).

Theorem 5.3. *If $2\alpha > 2|\beta|\tau+|\sigma+\rho\tau|$, then for any initial data $\xi \in C^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$, the solution $x(t,\xi)$ of (5.17) satisfies*

$$\limsup_{t\to\infty} \frac{1}{t}\mathbb{E}|x(t,\xi)|^2 \leq -(\tilde{\alpha} - q\tilde{\beta}) \tag{5.20}$$

and

$$\limsup_{t\to\infty} \frac{1}{t}\log|x(t,\xi)| \leq -\frac{\tilde{\alpha} - q\tilde{\beta}}{2}, \quad a.s. \tag{5.21}$$

where $\tilde{\alpha} = 2\alpha - |\beta|\tau$, $\tilde{\beta} = |\beta|\tau + |\sigma+\rho\tau|$ and $q \in (1, \tilde{\alpha}/\tilde{\beta})$ is the unique root of the equation

$$\tilde{\alpha} - q\tilde{\beta} = \frac{\log q}{\tau}.$$

For any $\varepsilon \in (0, (\tilde{\alpha} - q\tilde{\beta}))$, there exists a $\triangle^* > 0$ such that for any $\triangle < \triangle^*$, the EM approximation x_k of (5.19) has the properties

$$\limsup_{k\to\infty} \frac{1}{k\triangle}\mathbb{E}|x_k|^2 \leq -(\tilde{\alpha} - q\tilde{\beta}) + \varepsilon \tag{5.22}$$

and

$$\limsup_{k\to\infty} \frac{1}{k\triangle}\log|x_k| \leq -\frac{(\tilde{\alpha} - q\tilde{\beta}) - \varepsilon}{2}, \quad a.s. \tag{5.23}$$

Appendix A: Proof of Theorem 3.1. If we can prove (3.1) for any $\lambda \in (0, \zeta \wedge \log(q)/\tau)$, we will complete the proof. This is equivalent to prove

$$e^{\lambda t}\mathbb{E}|x(t)|^p \leq \frac{c_2}{c_1}\|\xi\|^p_{\mathbb{E}} =: \kappa \;\text{ on } (t \geq 0).$$

From the condition (i) we will complete the proof if we can prove that for any $t \geq -\tau$,

$$W(t) := e^{\lambda t}\mathbb{E}V(x(t)) \leq \kappa c_1. \tag{A.1}$$

When $t \in [-\tau, 0]$,

$$W(t) \leq \mathbb{E}V(x(t)) \leq c_2\mathbb{E}|x(t)|^p \leq c_2\|\xi\|^p_{\mathbb{E}} = \kappa c_1.$$

Then we claim (A.1) for all $t \geq 0$. Otherwise, by the continuity of $W(t)$, there exists the smallest $\rho \in [0,\infty)$ such that for $t \in [-\tau, \rho]$, $W(t) \leq \kappa c_1$ and $W(\rho) = \kappa c_1$ as well as $W(\rho+\delta) > W(\rho)$ for all sufficiently small δ. Then for any $t \in [\rho-\tau, \rho]$,

$$\begin{aligned} \mathbb{E}V(x(t)) &= e^{-\lambda t}W(t) \\ &\leq e^{-\lambda t}W(\rho) \\ &= e^{\lambda(\rho-t)}\mathbb{E}V(x(\rho)) \\ &\leq e^{\lambda\tau}\mathbb{E}V(x(\rho)). \end{aligned} \tag{A.2}$$

If $\mathbb{E}V(x(\rho)) = 0$, by the condition (i), for any $t \in [\rho-\tau, \rho]$,

$$c_1\mathbb{E}|x(t)|^p \leq \mathbb{E}V(x(t)) = 0,$$

which means $x_\rho = 0, a.s.$ From the existence and uniqueness of the solution, $x(t) = 0, a.s.$ which contradicts the definition of ρ. We therefore have $\mathbb{E}V(x(\rho)) > 0$. By $\lambda < \log(q)/\tau$, which implies $q > e^{\lambda\tau}$, (A.2) gives that

$$\mathbb{E}V(x_\rho(\theta)) < q\mathbb{E}V(x_\rho(0)) \text{ on } -\tau \le \theta \le 0.$$

Choosing $\varphi = x_\rho$, applying the condition (ii) yields

$$\mathbb{E}\mathcal{L}V(\rho, x_\rho) \le -\zeta\mathbb{E}V(x(\rho)).$$

From $\lambda < \zeta$ and continuity of $V(\cdot)$, $t \in [\rho - \delta, \rho + \delta]$ for sufficient small δ,

$$\mathbb{E}\mathcal{L}V(t, x_t) \le -\lambda\mathbb{E}V(x(t)).$$

We therefore have

$$W(\rho + \delta) - W(\rho) = \int_\rho^{\rho+\delta} e^{\lambda t}[\mathbb{E}\mathcal{L}V(t, x_t) + \lambda\mathbb{E}V(x(t))]dt \le 0,$$

which implies $W(\rho + \delta) \le W(\rho)$. This contradicts the definition of ρ. Therefore, for any $t \ge -\tau$, (3.1) is satisfied, as desired.

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