Incorporating balance concerns in resource allocation decisions: A bi-criteria modelling approach

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We consider resource allocation problems where inputs are allocated to different entities such as activities, projects or departments. In such problems a common goal is achieving a desired *balance* in the allocation over different categories of the entities. We propose a bi-criteria framework for trading balance off against efficiency. We define and categorize indicators based on balance distribution and propose formulations and solution algorithms which provide insight into the balance-efficiency tradeoff. We illustrate our models by applying them to the data of a portfolio selection problem faced by a science funding agency and to randomly generated large-sized problem instances to demonstrate computational feasibility.

1 Introduction

Resource allocation (distribution) is a process by which resources (inputs) are allocated to different entities such as activities, projects or departments [1]. The inputs are usually allocated in a way that maximizes some output value.

A common goal in resource allocation in organizations alongside maximization of output (efficiency) is "balance" [1]. Balance can be sought in terms of various attributes such as risk (high risk vs. sure bets), internal vs. outsourced work, distribution of

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resources across industries, various markets the business is in, different project types etc. [2]. Failure to achieve a balanced portfolio is often revealed by a decision maker (DM) who claims that there is "too much" or "too little" resource going to activities of a particular type.

A related concept considered in many allocation decisions is equity (fairness). However, as we use the term, a "perfectly balanced distribution" is not necessarily a distribution where each category receives the same amount. We define balance as a more general concept, of which equity might be considered as a special case. We assume that the DM has a balance distribution based on which she evaluates the balance in a given distribution. We refer to a distribution that has the "desired proportions" shown by the balance distribution as a "perfectly balanced distribution" even if this distribution gives some categories more than the others. Equity concerns can be represented as the special case where the DM's balance distribution gives each category an equal amount.

The contributions of the current study are as follows:

- We propose a means to handle balance concerns alongside efficiency concerns in allocation problems and hence provide a bi-criteria framework to think about trading balance off against efficiency.

- We discuss ways to measure the deviation from a distribution which the DM considers as balanced and hence define and classify *imbalance indicators*.

- We propose formulations and algorithms which provide insight to the decision makers in general resource allocation settings.

Section 2 discusses an example allocation setting. Section 3 discusses alternative ways in which balance concerns have been handled in mathematical programming models and provides a brief review of related works from the literature. Section 4 introduces some imbalance indicators, which can be used to assess the degree of balance in a distribution. We introduce these indicators as another criterion to be optimized in the classical *maximize output* setting and provide bi-criteria models in section 5. In section 6 we discuss a way to solve the bi-criteria models and obtain nondominated solutions. We provide the results of our computational experiments on the performance of the suggested approach in section 7. We also provide results for 3-criteria extensions of the approach as well as a tabu search algorithm that can be used to solve large-sized problems. We conclude our discussion in Section 8.

2 The Balance Concept

Consider a setting where a DM is faced with m R&D projects and s/he will decide which ones to initiate given an available budget, B, which typically is not sufficient to initiate all projects. Each project i incurs a cost (input) c_i and returns an output value b_i. Suppose that it is possible to categorize the projects into n categories (e.g. based on the technological area or based on the department they are proposed by) and each project belongs to one (and only one) of these categories.

Each feasible portfolio corresponds to two portfolio-related distributions: a distribution of the budget B to different project categories and a distribution that shows the contribution of each category in terms of the output. Suppose that the DM wants to ensure balance in one or both of these distributions as well as having a high total output from the selected portfolio.

This is an example of an allocation problem in which the DM has concerns about ensuring balance. In this study we provide a general framework that can be used for many allocation problems. To have a structured discussion, we will illustrate the general idea using this R&D project selection example and discuss possible generalizations in the conclusion.

We distinguish the cases based on the space balance is sought, i.e. based on whether balance is sought in the input distribution or the output distribution. Which balance concept is more appropriate depends on the nature of the problem. For example in healthcare, the policy maker may want a balanced input allocation on the grounds that people should be responsible for their own health and the policy makers can only be responsible for providing them with a balanced allocation of inputs. On the other hand, the policy maker may prefer a balanced distribution of health (the output) on the grounds that health policy should aim at equal health for all.

3 Literature Review on Incorporating Balance into Models

In this section we mention some noteworthy studies that consider the balance concept in portfolio selection and allocation decisions in an explicit way. We refer the interested reader to [3] for a more detailed discussion on balance in project portfolio selection problems. There is also a broad range of applications in which equity concerns are incorporated into mathematical models, including but not limited to drug allocation [4], HIV prevention funds allocation [5], water allocation [6], bandwidth allocation [7], workload allocation [8], and location-allocation problems in homeland defense [9].

Efficiency concerns are reflected to the model by maximizing the total output. From a modelling point of view, balance concerns may be handled in two ways:

- Modifying the feasible region by introducing constraints: In this approach the analyst changes the feasible region of the problem so that the feasible allocations will ensure a certain degree of balance.

[10] considers selecting a portfolio of solar energy projects using multiattribute preference theory. As a way of ensuring a balanced portfolio, they use lower bounds on the number (or monetary value) of the projects of a certain type that are included in a portfolio. Similarly, [11] uses linear programming to maximize the total technical score of funded projects on a smoking intervention study. Balance related constraints are used to ensure geographic equity in project proposal fundings and to ensure "a spread" of changes across different quartiles of the population with respect to smoking preference and decline in smoking rate. [12] proposes an integer programming model for selecting and scheduling an optimal project portfolio. The balance related constraints enforce an upper limit on the percentage of total investment made on different project categories,

such as high risk and long term projects. The authors illustrate their approach by solving a small-size problem with 12 projects. [13] considers a multi-dimensional integer knapsack problem and introduces constraints to incorporate balance concerns into the model. The constraints are used to apply upper and lower bounds on the fraction of the resources allocated to different project categories. The authors, however, mostly focus on the linear programming relaxation of the integer programming formulation and hence assume that partial resource allocation to projects is possible. [14] develops a nonlinear integer programming model to optimize a portfolio of (possibly interdependent) product development improvement projects over multiple periods. The projects are categorized based on the strategic objectives that they support and balance over different objectives is ensured by incorporating a constraint that shows the minimum number of projects from each category. [15] discusses a multi-criteria decision analysis (MCDA) framework to allocate fishing rights to candidates in South Africa. As part of their decision support system they provide an integer formulation for the candidate selection problem in which balance concerns are reflected using constraints. These constraints ensure that the proportion of the number of candidates selected from a designated group exceeds a minimum desired level for this group. [2] propose a DEA (Data Envelopment Analysis) based methodology to construct and evaluate balanced portfolios of R&D projects with binary interactions. As part of their proposed methodology, they compute indices of risk, efficiency and balance for each project. They use a maximum threshold for risk index and minimal thresholds for efficiency and balance indices and screen the initial list of candidate projects. Only the ones that satisfy the requirements set by the indices are considered further. A similar approach is used in [16] to evaluate R&D projects in different stages of their life cycle.

[17] develop a fuzzy R&D project selection model in which balance in spending between different strategic goals is enforced in constraints. These constraints specify upper and lower bounds on the spending for each strategic goal (see also [18]).

- Modifying the objective: In this approach the analyst increases the number of criteria of the corresponding model; turning it into a bi/multi criteria problem. The approach we take falls into this category. We use this approach as it is possible to observe the trade-off between different criteria by finding different solutions to the problem.

Modifying the objective typically relies on the use of a balance indicator, $z_I(x)$, which assigns a value that shows the level of balance in a distribution x. Using the indicator one can define a balance criterion along with the efficiency criterion ($z_T(x)$).

Note that if balance is considered over different aspects such as technology areas, markets etc., it is possible to use a balance indicator for each aspect and hence generate a multi criteria model. [19], [20], [21] and [22] use multiattribute models which tackle balance concerns over multiple attributes and then use Multiattribute Value/Multiattribute Utility models to aggregate the set of attributes into a single index. One of the restrictions on the generality of the proposed models is the assumption that the number of items in the subset is constant. Moreover, an additive value function may not always be appropriate, and even when it is appropriate, determining weights may not be easy.

[23] models the concern for balance as a separate set of criteria which minimize the deviation from the ideal allocation of manpower to different project categories and also to different client categories. With additional criteria which are not balance related, he formulates a multi-criteria decision making (MCDM) model for the project portfolio selection problem. The reference point approach (see [24]) is used, which involves solving non-linear integer programming problems subject to linear resource constraints. The approach is used in an interactive setting. For various reasons including the technical difficulty due to nonlinearity, a heuristic method is used to solve the resulting optimization problems. The same approach is also used in [25].

As it expresses balance criteria as measures of deviation from a desired allocation, this approach is similar to the approach used to incorporate balance in this paper. We, however, mostly focus on a bi-criteria setting and use linear (integer) models whenever possible. The underlying reason for this choice is the ease of presentation. The ideas proposed here are easily generalizable to multidimensional settings where balance is desired over multiple attributes. Balance concern for each such attribute can be reflected as a criterion to the model and appropriate multicriteria optimization or heuristic methods can be implemented to obtain solutions. The emphasis of this paper is to introduce the idea of balance distribution based balance indicator as a way of handling balance concerns in an explicit and tractable way.

Unlike [23], we do not assume an interactive setting; we rather present the DM with a dispersed subset of nondominated portfolios. This is an alternative approach to the one used in [23]; empirical research should be performed to see which one is more appropriate in different problem environments. We provide graphical displays of the set of nondominated solutions which visualize the tradeoff between efficiency and balance. These graphs can be used as a starting point for further discussion with decision makers.

We also provide an explicit link between inequality measurement literature by making an analogy between the perfect balance line and the perfect equality line. This will be explained in detail in the next section. We discuss different indicators that can be used to assess imbalance. Our solution approach allows one to incorporate different imbalance indicators into the same model and hence observe the tradeoff between them.

4 Imbalance Indicators

In this section we propose imbalance indicators that measure how different a distribution is from an ideally balanced distribution. The indicators rely on a *balance distribution* which is provided by the DM. This balance distribution shows how the DM would allocate a certain amount of the input/output across the categories involved. This might be for example, the *status quo* or previous year's allocation.

We will use the following terminology and notation to frame our discussion:

We refer to the entities over which the balance is sought as categories. $J = \{1, 2, ..., n\}$ is the set of the categories. The vector $x \in \mathbb{R}^m$ is used to show the decision vector related to input allocation. Note that m is not necessarily equal to n unless we make explicit allocation decisions to categories themselves. For example, in the project selection problem m is the number of projects and x is the corresponding binary decision vector and we expect $n < m$ unless each project is considered as a different category.

x can be continuous or discrete and includes the decision variables which dictate the input allocation to categories (this dictation can be indirect as in the project selection problem: in that case, x shows the portfolio of projects, from which we can infer the allocation to categories). Any function defined over the input allocation is a function of the decision vector x. Let $a(x) \in \mathbb{R}^n$ be the distribution over which the balance is sought, hence it can either show the input or the output distribution to categories.

We denote the balance distribution of either input or output by $r \in \mathbb{R}^n$ where r_j is the amount allocated to category j in the balance distribution. Which of these (input/output) is intended will be clear from context. For notational simplicity we will normalize the balance distribution so as to obtain *balance shares (proportions)* for each category. Let us denote the balance share of category j as α_j. By definition $\alpha_j = r_j / \sum_{j\in J} r_j$. Hence, $\alpha \in \mathbb{R}^n$ is the balance distribution in terms of shares.

Suppose that given α, we want to assess how balanced a distribution $a(x)$ is. Using α, we can obtain a target point $\overline{r}(x)$ as follows: $\overline{r}(x)_j = \alpha_j * \sum_{j\in J} a(x)_j$.One can think of the elements of $\overline{r}(x) \in \mathbb{R}^n$ as target (desired) amounts for the different categories involved.

We denote the componentwise deviations of the distribution $a(x)$ from the corresponding $\overline{r}(x)$ as $d(x)_j$ $\forall j \in J$. That is, $d(x)_j = |a(x)_j - \overline{r}(x)_j| = \left| a(x)_j - \alpha_j * \sum_{j\in J} a(x)_j \right|$ $\forall j \in J$.

Figure 4.1 visualizes componentwise deviations in a 2 dimensional environment. Note that except r, all the terms are functions of the input allocation x. d_1 and d_2 are the componentwise distances of point a to the inflated balance point.

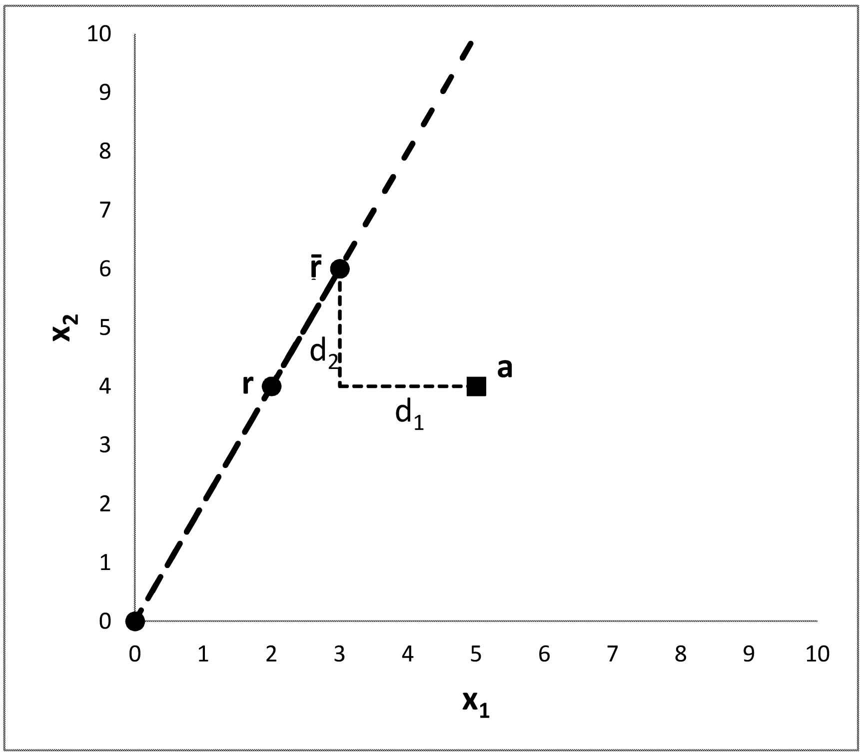

Figure 4.1: Distances from the balance point in 2D

It does not seem appropriate to capture balance using a distance measure from the balance distribution itself (r). The rescaling of r, i.e. generating $\overline{r}(x)$, is necessary to obtain an appropriate evaluation. Consider, for example, the case where $r = (1, 2)$, that is the DM considers $(1, 2)$ as a balanced distribution, and we want to assess how balanced distribution $a(x) = (2, 4)$ is. Using just a distance measure would mislead us by concluding that $(2, 4)$ is not balanced (as the componentwise absolute deviations between $(1, 2)$ and $(2, 4)$ are not equal to zero). However if $(1, 2)$ is balanced, it would seem natural to suppose that $(2, 4)$ is also balanced. This is clearly seen when we rescale $r = (1, 2)$ with respect to $(2, 4)$. We find $\overline{r}(x) = (1/3, 2/3)a(x) = (2, 4)$. Since $a(x) = \overline{r}(x)$, the componentwise deviations are all zero hence we capture that $a(x)$ has perfect balance. That is why, we avoid using just a distance measure from the given balance distribution to account for balance and instead generate a *balance line* based on the balance distribution.

The intuition behind our approach generalizes the *perfect equality line* concept used in inequality measurement theory ([26]). The *perfect equality line* consists of points whose components are equal in all dimensions, i.e, it consists of the distributions where everyone gets the same income. Despite being different in the total income, all points on this line are considered to have perfect equality, i.e. zero inequality. Similarly, we derive a line of *perfect balance* passing through the origin and the balance point (see Figure 4.1) and derive our balance indicators accordingly.

We now define the four imbalance indicators as follows.

Indicator 1: The total proportional deviation from the target.

$$I_1(x) = \frac{\sum_{j\in J} d(x)_j}{\sum_{j\in J} a(x)_j} = \frac{\sum_{j\in J} \left| a(x)_j - \alpha_j * \sum_{j\in J} a(x)_j \right|}{\sum_{j\in J} a(x)_j} = \sum_{j\in J} \left| \frac{a(x)_j}{\sum_{j\in J} a(x)_j} - \alpha_j \right|$$

This is the sum of the absolute differences between the actual share and the desired share for each category. Taken in its input oriented sense, this indicator is the fraction of input which is misallocated. Taking the proportional deviation also implies the following: of two alternative distributions with the same total absolute distance from the balance line, the one that has a larger sum will have a smaller imbalance value, hence will be favoured. Special cases of $I_1(x)$ where the balance distribution is the one with perfect equality, i.e., each category receives an equal share, have been used in the literature (e.g., in [27]).

Indicator 2: The maximum proportional deviation from the target. Unlike $I_1(x)$ this indicator focuses only on the worst-off deviation.

$$I_2(x) = \frac{Max_{j\in J}\{d(x)_j\}}{\sum_{j\in J} a(x)_j} = Max_{j\in J} \left| \frac{a(x)_j}{\sum_{j\in J} a(x)_j} - \alpha_j \right|$$

Indicator 3: The total componentwise proportional deviation. Compared to the first two indicators this is a more individual oriented measure as it is the sum of fractional misallocations to each party.

$$I_3(x) = \sum_{j\in J} \frac{d(x)_j}{\overline{r}(x)_j} = \sum_{j\in J} \frac{\left| a(x)_j - \alpha_j * \sum_{j\in J} a(x)_j \right|}{\alpha_j * \sum_{j\in J} a(x)_j} = \sum_{j\in J} \frac{1}{\alpha_j} \left| \frac{a(x)_j}{\sum_{j\in J} a(x)_j} - \alpha_j \right|$$

This measure is a weighted sum of the absolute differences between the actual share and the desired share for each category where weight for category j is $\frac{1}{\alpha_j}$. This allows one to penalize the deviations from the categories that are already assigned a low target share value. We note that for this measure to be meaningful, one should have $\alpha_j > 0$ for all j.

Indicator 4: The maximum proportional deviation from the corresponding target value over all elements of the distribution. Unlike $I_3(x)$ this indicator focuses only on the worst-off deviation.

$$I_4(x) = \underset{j \in J}{Max}\{\frac{d(x)_j}{\overline{r}(x)_j}\} = \underset{j \in J}{Max}\{\frac{1}{\alpha_j}|\frac{a(x)_j}{\sum_{j \in J} a(x)_j} - \alpha_j|\}$$

Which indicator one chooses to use, might have material significance for the solutions which are bi-criteria efficient in the biobjective formulations. However, when n is low as in Proposition 1, whichever indicator one chooses, one will get the same ordering and thus the same efficient frontier. Hence in this case, which indicator one chooses does not matter: one can choose any indicator and be confident of getting the same result.

Proposition 1 *For $n \leq 3$ we have $I_1(x) = 2 * I_2(x)$. Moreover when $n = 2$ the four indices provide us with the same order. That is, for any two distributions x^1 and x^2 where $n = 2$ (that is $a(x^1), a(x^2) \in \mathbb{R}^2$), the following holds: $I_1(x^1) \geq I_1(x^2) \iff I_2(x^1) \geq I_2(x^2) \iff I_3(x^1) \geq I_3(x^2) \iff I_4(x^1) \geq I_4(x^2)$.*

Proof is provided in Appendix A.

Remark 2 *In general, Proposition 1 no longer holds for $I_1(x)$; $I_3(x)$; $I_4(x)$ and $I_2(x)$; $I_3(x)$; $I_4(x)$ in problems where $n > 2$. As for $I_1(x)$ and $I_2(x)$ it no longer holds when $n > 3$.*

Proof is provided in Appendix A.

Table 4.1 summarizes the classification of the imbalance indicators.

Table 4.1: Imbalance Indicators

Imbalance Indicators		
Objective\Focus	Collective	Individual Oriented
Sum	$I_1(x) = \frac{\sum_{j\in J} d(x)_j}{\sum_{j\in J} a(x)_j}$	$I_3(x) = \sum\limits_{j\in J} \frac{d(x)_j}{\overline{r}(x)_j}$
Bottleneck	$I_2(x) = \frac{Max_{j\in J}\{d(x)_j\}}{\sum_{j\in J} a(x)_j}$	$I_4(x) = \underset{j\in J}{Max}\{\frac{d(x)_j}{\overline{r}(x)_j}\}$

5 Bi-criteria Models

In this section we develop bi-criteria models for allocation problems with objectives of maximizing total output and minimizing an imbalance indicator.

Although all the models we discuss are based on the same general idea, they differ in technical aspects depending on the problem type, i.e., based on whether the allocation is discrete or continuous and whether the balance is sought in the output or input distribution.

For our project selection problem we provide mixed integer formulations for the bi-criteria models, which exploit the fact that the decision variables are 0-1 variables to tackle nonlinearity due to the imbalance indicators.

We first provide a complete analysis for the case the DM desires a balanced input distribution as this problem naturally arises in many situations. It is straightforward to develop models for the case where a balanced output distribution is desired when we have a discrete setting.

5.1 Discrete Allocation

The general method proposed in this paper is applicable to different combinatorial problems that can be formulated as a binary integer problem (BIP), like some location problems. We use project selection problems as an example.

Consider the project selection problem discussed in Section 2. Suppose that the DM wants to have a portfolio where input is allocated to different project types in a balanced way and gives an example input allocation $r \in \mathbb{R}^n$, which he considers balanced. The corresponding proportional allocation is denoted as $\alpha \in \mathbb{R}^n$ as before.

We use an $m \times n$ incidence matrix G with elements g_{ij} for $i \in I$ and $j \in J$ as follows:

$$g_{ij} = \begin{cases} 1 \text{ if project } i \text{ belongs to category } j \\ 0 \text{ otherwise} \end{cases}$$

The binary variable associated with each project is as follows:

$$x_i = \begin{cases} 1 \text{ if project } i \text{ is initiated} \\ 0 \text{ otherwise} \end{cases} \quad \text{for } i \in I$$

Note that we seek balance in the input space and $a(x)_j = \sum\limits_{i \in I} c_i g_{ij} x_i$ for all $j \in J$, that is, the input allocated to a certain category is the sum of the costs of the initiated projects in that category. In what follows, we assume that at least one of the projects will be initiated in a feasible solution. We also have $\sum_{j \in J} a(x)_j = \sum_{i \in I} c_i x_i$.

We now provide an example model that uses the indicator $I_1(x)$ as the second objective. For the project selection problem $I_1(x)$ is as follows:

$$I_1(x) = \frac{\sum_{j \in J} \left| \alpha_j * \sum_{j \in J} a(x)_j - a(x)_j \right|}{\sum_{j \in J} a(x)_j} = \frac{\sum_{j \in J} \left| \sum_{i \in I} \alpha_j c_i x_i - \sum_{i \in I} c_i g_{ij} x_i \right|}{\sum_{i \in I} c_i x_i}.$$

We have the following model where we use variables Z_T and Z_I to denote $z_T(x)$ and $z_I(x)$, respectively.

$$Max \ \{Z_T, -Z_I\} \tag{5.1a}$$

$$s.t. \ \sum_{i \in I} c_i x_i \leq B \tag{5.1b}$$

$$Z_T = \sum_{i \in I} b_i x_i \tag{5.1c}$$

$$Z_I = \frac{\sum_{j \in J} \left| \sum_{i \in I} \alpha_j c_i x_i - \sum_{i \in I} c_i g_{ij} x_i \right|}{\sum_{i \in I} c_i x_i} \tag{5.1d}$$

$$x_i \in \{0, 1\} \ \forall i \in I \tag{5.1e}$$

The above model is nonlinear due to constraint set 5.1d. We linearize it by introducing auxiliary variables d_j, y_j and t_j and obtain the following MIP model (See [28] and [29] for more information on such linearizations):

Model 1

$$Max\ \{Z_T, -Z_I\} \tag{5.2a}$$

$$s.t.\ \sum_{i \in I} c_i x_i \leq B \tag{5.2b}$$

$$Z_T = \sum_{i \in I} b_i x_i \tag{5.2c}$$

$$\sum_{i \in I} c_i(\alpha_j - g_{ij})x_i \leq d_j\ \forall j \in J \tag{5.2d}$$

$$\sum_{i \in I} c_i(g_{ij} - \alpha_j)x_i \leq d_j\ \forall j \in J \tag{5.2e}$$

$$d_j - \sum_{i \in I} c_i(\alpha_j - g_{ij})x_i \leq 2 * d^{UB} * y_j\ \forall j \in J \tag{5.2f}$$

$$d_j - \sum_{i \in I} c_i(g_{ij} - \alpha_j)x_i \leq 2 * d^{UB} * (1 - y_j)\ \forall j \in J \tag{5.2g}$$

$$Z_I^{LB} x_i \leq t_i \leq Z_I^{UB} x_i\ \ \forall i \in I \tag{5.2h}$$

$$Z_I^{LB}(1 - x_i) \leq Z_I - t_i \leq Z_I^{UB}(1 - x_i)\ \ \forall i \in I \tag{5.2i}$$

$$\sum_{j \in J} d_j = \sum_{i \in I} c_i t_i \tag{5.2j}$$

$$x_i \in \{0, 1\}\ \forall i \in I \tag{5.2k}$$

$$y_j \in \{0, 1\}\ \forall j \in J \tag{5.2l}$$

$$t_i \geq 0\ \forall i \in I \tag{5.2m}$$

Constraint set 5.2b ensures that the total budget is not exceeded and the constraint set 5.2c defines Z_T, total output of the portfolio. We define new variables d_js that show the absolute distances, i.e. $d_j = \left|\sum_{i \in I} c_i(\alpha_j - g_{ij})x_i\right|\ \forall i$. Constraint sets 5.2d, 5.2e, 5.2f, 5.2g and auxiliary binary variables (y_js) are used to define the absolute distances (d_js) and tackle the nonlinearity due to the absolute function. d^{UB} is an upper bound for the continuous d_j variables. We use the same upper bound for all the d_j variables and calculate the bound as follows: $d^{UB} = \sum_{i \in I} c_i$. Constraint sets 5.2h, 5.2i and 5.2j are used to tackle the nonlinearity due to the ratio terms in the definition of Z_I (see constraint set

5.1d) as follows: In constraint set 5.1d we have $\sum_{j\in J} d_j = Z_I * \sum_{i\in I} c_i x_i = \sum_{i\in I} c_i Z_I x_i$. We define auxiliary continuous variables t_i such that $t_i = Z_I * x_i$ $\forall i \in I$ hence obtain constraint set 5.2j. Constraint sets 5.2h and 5.2i ensure that $t_i = Z_I * x_i$ $\forall i$ hold. Z_I^{UB} and Z_I^{LB} are upper and lower bound parameters for Z_I, respectively. From the definition of Z_I, $Z_I^{LB} = 0$. We define Z_I^{UB} as follows: $Z_I^{UB} = n * d^{UB}/\underset{i}{Min}\{c_i\}$.

Model 1 has $2m+2n+2$ variables and $2m+4n+3$ constraints excluding set constraints.

Remark 1 *d^{UB} is an upper bound for all d_j.*

Proof. $d_j = \left|\sum_{i\in I} c_i(\alpha_j - g_{ij})x_i\right| = \sum_{i\in I} c_i \left|\alpha_j - g_{ij}\right| x_i$. Since both $0 \leq \alpha_j \leq 1$ $\forall j$ and $0 \leq g_{ij} \leq 1$ $\forall i, j$ we have $|\alpha_j - g_{ij}| \leq 1$. Hence $\left|\sum_{i\in I} c_i(\alpha_j - g_{ij})x_i\right| \leq \sum_{i\in I} c_i x_i \leq \sum_{i\in I} c_i$. ■

It is possible to include additional constraints in cases where certain projects are mutually exclusive for some underlying technical reasons. We note that these are easily handled computationally, hence for ease of presentation we do not include such constraints into the formulation explicitly. The models involving $I_2(x)$, $I_3(x)$ and $I_4(x)$ are very similar hence are provided in Appendix B.

It is straightforward to develop models for the case where a balanced output distribution is desired when we have a discrete setting. The model will be the same except the following: We use b_i instead of c_i in constraint sets 5.2d, 5.2e, 5.2f, 5.2g and 5.2j and change d^{UB} and Z_I^{UB} accordingly.

5.2 Continuous Allocation

Suppose that a DM should decide how to allocate a given input B among m projects but this time the allocation can be performed in a continuous manner. We use the same notation as in the discrete case with a difference in the decision variable and output definition.

Let x_i be the allocated input to project i and let $f_i(x_i)$ be the resulting output. The input allocated to category j denoted as $a(x)_j$ is a linear function of x such that $a(x)_j = \sum\limits_{i\in I} g_{ij} x_i$ for all $j \in J$. In such cases the total input allocation is always B, i.e.,

$\sum_{j \in J} a(x)_j = B$. Note that the properties of the production functions $f_i(x_i)$ will affect the complexity of the problem and the resulting models may be difficult to solve when e.g. these functions are nonlinear. However, if production functions are concave it is possible to use piecewise linearization and obtain a linear problem as we show in the example in the next section.

Recall that the indicators in the discrete setting have decision variables in the denominator and hence require linearization. As Remark **??** shows the balance criterion no longer requires such linearization in the input oriented continuous setting.

Remark 2 *For the continuous allocation the indicators ($I_1(x)$,$I_2(x)$,$I_3(x)$ and $I_4(x)$) in the input oriented setting reduce to linear functions of deviations.*

Proof. Given a balance resource distribution α, $I_1(x)$ is as follows: $I_1(x) = \frac{\sum_{j \in J} |\alpha_j B - a(x)_j|}{B} = \frac{\sum_{j \in J} d(x)_j}{B}$. Hence minimizing $I_1(x)$ is equivalent to minimizing $\sum_{j \in J} d(x)_j$. Similarly, it is possible to show that minimizing $I_2(x)$, $I_3(x)$ and $I_4(x)$ are equivalent to minimizing $\underset{j \in J}{Max}\{d(x)_j\}$, $\sum\limits_{j \in J} \prod\limits_{i \neq j} \alpha_i d(x)_j$, and $\underset{j}{Max}\{\prod\limits_{i \neq j} \alpha_i d(x)_j\}$, respectively. Also note that one does not need the auxiliary binary variables (e.g. $y_j s$ in model 1) to linearize the nonlinearity due to the absolute function as we directly minimize linear functions of the absolute distances. ■

6 Solution Approach

Our models are bi-criteria versions of the knapsack problem. In the discrete case knapsack problem is considered to be a nondeterministic polynomial-time hard (NP-hard) problem ([30]).

We define set Z as follows: $Z = \{(Z_T, Z_I) : Z_T = z_T(x) \text{ and } Z_I = z_I(x),\ x \in X\}$.

Definition 1 *For two points (Z_T, Z_I) and (Z'_T, Z'_I), (Z_T, Z_I) dominates (Z'_T, Z'_I) if $Z_T \geq Z'_T$ and $Z_I \leq Z'_I$ with strict inequality holding at least once.*

Definition 2 *A point (Z_T, Z_I) is nondominated and the corresponding solution (x) is efficient if there is no other point in Z that dominates it.*

We call all the nondominated solutions for a problem the *nondominated set.*

We use the epsilon constraint method to obtain nondominated (/efficient) solutions for the bi-criteria problems considered here. This method is based on sequentially solving single objective problems in which the value of the second objective is controlled using a constraint (see [31] and [32] for a discussion of the epsilon constraint method).

The general algorithm is as follows (note that lex max refers to lexicographic maximization).

Step 0. Solve lex max $(z_T(x), -z_I(x))$

s.t. $x \in X$

Let the optimal value for $z_I(x)$ be Z_I^*

Step 1. If $Z_I^* \leq Z_I^{LB}$ Stop.

Otherwise, set $k = Z_I^* - Stepsize$.

Step 2. Solve lex max $(z_T(x), -z_I(x))$

s.t. $x \in X$

$z_I(x) \leq k$

Let the optimal value for $z_I(x)$ be Z_I^*

Go to Step 1.

When the objective function values are integer, it is possible to generate all nondominated points with this method. In this paper we use the method to generate a subset of the nondominated set as our objective function values are not necessarily integer. We first generate the solution that has the maximum output (Z_T) value and obtain a nondominated solution at each iteration until we generate the one that has the minimum imbalance (Z_I) value. We use a parameter *Stepsize* to control the maximum difference between two consecutively generated nondominated points in terms of their imbalance values. The smaller the *Stepsize*, the higher the number of nondominated solutions found. On the other hand, the higher the computational time is. Note that it is also possible to modify the algorithm such that it starts with the solution that has the minimum imbalance and moves toward the ones with higher total output values by controlling $z_T(x)$ by a constraint.

6.1 An Example Problem

We now provide a real life example for the input oriented discrete case based on data given to us by a public sector agency whose R&D portfolio selection problem provided the immediate motivation for the current work. The problem is a project selection problem subject to the available budget. The cost and value figures for each project are tabulated below (see Table 6.1). The values are a weighted average of performances of each project over multiple criteria. Note that the projects are of three types and the cost values are normalized to protect confidentiality. The budget and value correspond to input and output, respectively.

Table 6.1: Data for the example problem

Project Index	Project Type	Cost	Overall Value	Project Index	Project Type	Cost	Overall Value
1	Type 1	0.19	1.39	21	Type 2	0.88	1.71
2	Type 1	0.16	1.13	22	Type 2	0.86	1.34
3	Type 1	0.30	1.67	23	Type 3	0.05	2.15
4	Type 1	0.29	1.48	24	Type 3	0.18	2.47
5	Type 1	0.55	2.13	25	Type 3	0.16	1.96
6	Type 1	0.57	1.43	26	Type 3	0.31	3.42
7	Type 1	0.96	1.50	27	Type 3	0.43	3.92
8	Type 1	0.99	1.44	28	Type 3	0.42	3.42
9	Type 1	0.74	0.99	29	Type 3	0.42	2.97
10	Type 1	0.67	0.85	30	Type 3	0.33	2.29
11	Type 2	0.21	3.13	31	Type 3	0.37	1.67
12	Type 2	0.28	2.52	32	Type 3	0.59	2.60
13	Type 2	0.28	2.11	33	Type 3	0.42	1.79
14	Type 2	0.40	2.43	34	Type 3	0.96	4.08
15	Type 2	0.24	1.49	35	Type 3	0.54	2.11
16	Type 2	0.58	2.91	36	Type 3	0.54	2.08
17	Type 2	0.95	3.15	37	Type 3	0.90	3.25
18	Type 2	0.89	2.82	38	Type 3	0.75	2.20
19	Type 2	0.91	2.47	39	Type 3	0.81	2.06
20	Type 2	0.61	1.57				

Suppose that the agency has a budget (total input) of 9.31 units, which is about 45% of the total cost of all the projects available. Given this budget, the portfolio that maximizes the total output has a total output of 59.32 and requires an input of 9.2 units.

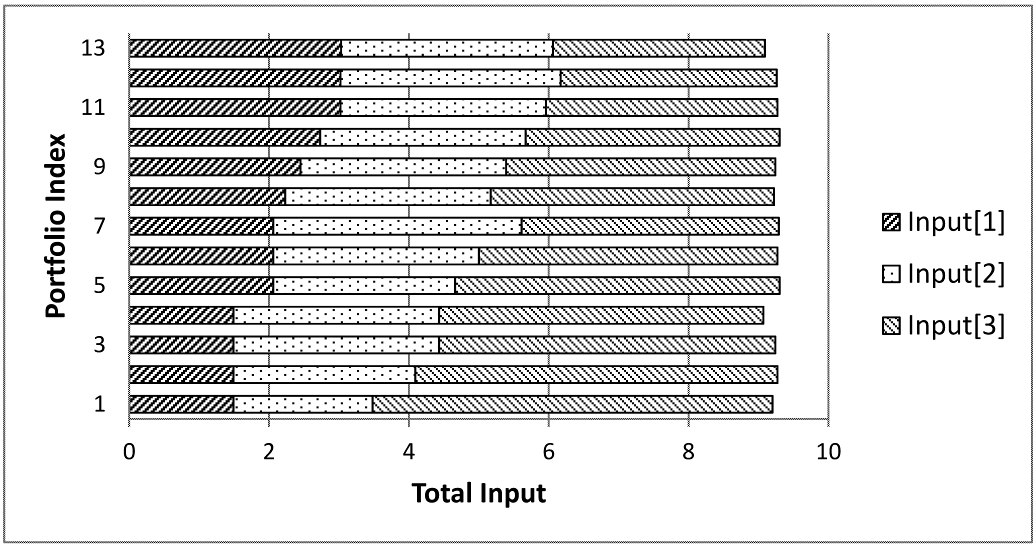

Figure 6.1: Solutions obtained using $I_3(x)$

The allocation of this total input to the three type of projects are 1.49 units, 1.99 units, 5.72 units, for types 1, 2 and 3 respectively.

Suppose that the DM considers an input allocation that has equal percentages as balanced. That is, in a perfectly balanced portfolio the total amount allocated to each project type should be 33% of the overall input.

For explanatory purposes we will use one of the indicators, $I_3(x)$, and show the portfolios obtained by solving the corresponding bi-criteria problems. A subset of the efficient portfolios obtained using $I_3(x)$ using $Stepsize$ of 0.05 are visualized in Figure 6.1. The figure shows 13 portfolios each of which is obtained through one iteration of the algorithm.

The first portfolio is the one that gives the maximum total output and type 3 projects are allocated more input than the other two types in this portfolio. It is seen that in each new solution the algorithm returns a portfolio where the three types are closer in input usage. In the first iterations, balance is increased via increasing the input allocation to type 2 projects. As we restrict the solution to become more and more balanced, the allocation to type 1 projects increases. One can also see the amount of sacrifice from

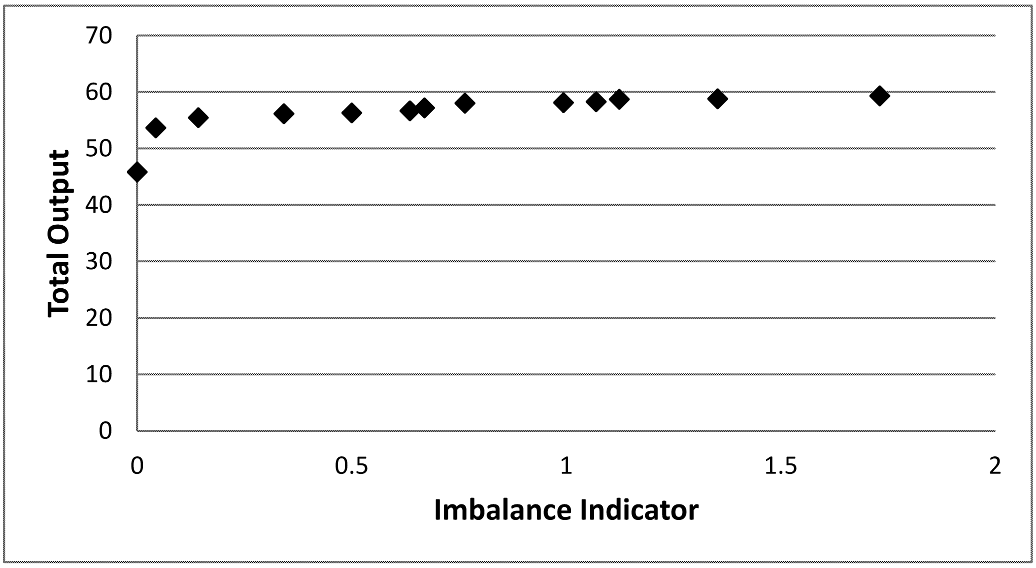

Figure 6.2: Efficiency vs. Balance

efficiency (total output) by moving towards more balanced portfolios in Figure 6.2, which is the total output vs. imbalance graph.

The epsilon constraint approach allows us to visit the whole nondominated set in a uniform way, i.e., we provide representative portfolios for different parts of the whole nondominated set. Seeing such a uniform subset of the nondominated set has advantages in terms of clarity and transparency. The results show the tradeoff between the efficiency and balance criteria. For example, moving from the first solution to the second one sacrifices from efficiency around 0.8% and this increases balance around 20%. On the other hand, it is seen that as we restrict the solution to become more and more balanced, the efficiency sacrifice that we have to make may increase significantly. In addition to seeing the tradeoff between the two criteria, one can have more information about the solution structure. A quick review would give us an idea about the more "powerful" projects, the ones that occur in most of the nondominated solutions. In the above example we observed that the projects 1,3,4,5; 11,16; and 23, 24, 25, 26, 28, 29 are included in most of the portfolios. As a way of simplifying the decision making process, these projects

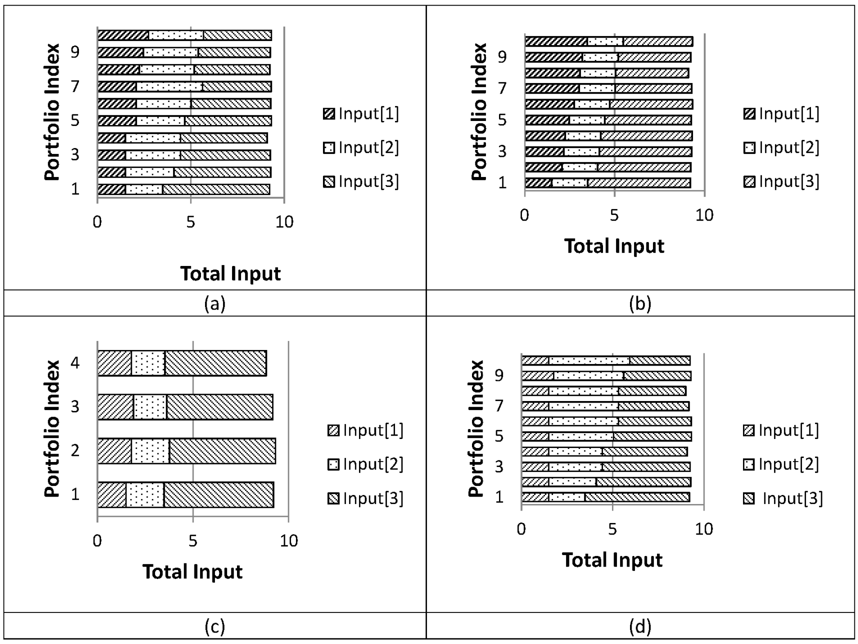

Figure 6.3: Different Balance Points

can be fixed and the others might be analyzed in more detail.

We have also generated solutions by using $I_1(x)$, $I_2(x)$ and $I_4(x)$ using a *Stepsize* of 0.05. We observed a similar trend as in Figure 6.1.

It is also possible to use other balance distributions and see the resulting solutions. Figure 6.3 (a)-(d) show the first 10 solutions obtained using the balance distributions (33, 33, 33); (60, 20, 20); (20, 20, 60); (20, 60, 20) respectively with $I_3(x)$. Note that in the third case we report only 4 solutions as these were the only solutions obtained using a *Stepsize* of 0.05 in the algorithm. This is because the output maximizing solution already has low imbalance with respect to the given balance distribution. No significant adjustment was necessary in this case.

As another example, we consider a linearized version of the above problem to analyze the case where we have continuous allocations. We assume that the production functions $f_i(x_i)$ are concave of the form $f_i(x_i) = s_i * x_i^{\theta_i}$ for all i. We generate s_i and θ_i values from uniform distributions $U(0, 5)$ and $U(0, 0.5)$, respectively. We solve the problem using piecewise linear approximation for the concave production functions. Figure 6.4 shows

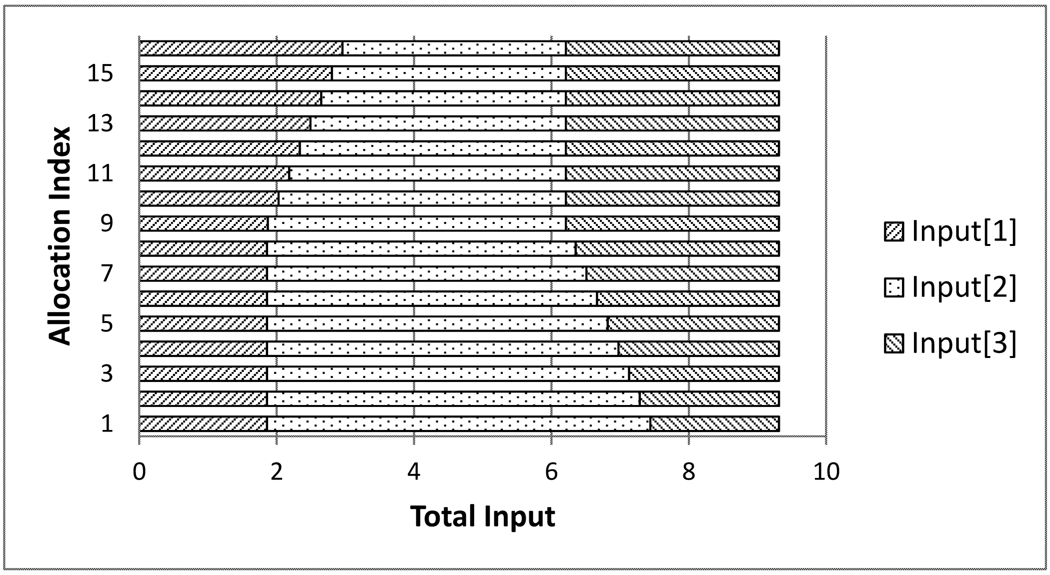

Figure 6.4: Continuous Case using $I_3(x)$ with balance distribution (0.33,0.33,0.33)

a set of efficient allocations which are obtained setting *Stepsize* to 0.1. One can clearly see that we move to more balanced allocations towards the end of the spectrum. Also note that unlike the discrete case, all of the allocations have the same total input value. The tradeoff between the two criteria is clearly seen in Figure 6.5, which shows the total output value vs. imbalance value for this continuous case.

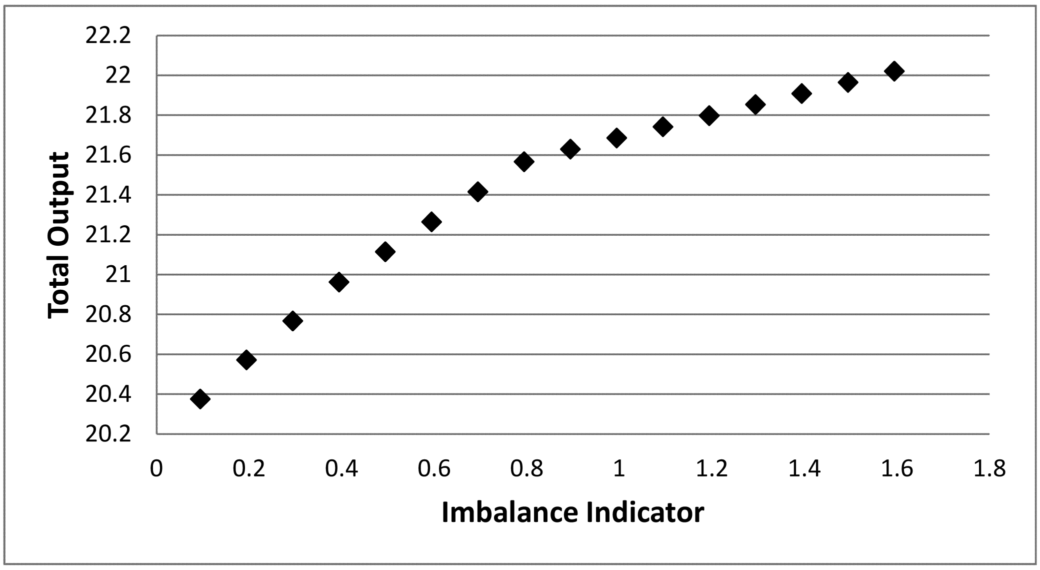

Figure 6.5: Efficiency vs. Balance in the Continuous Case

7 Computational Study

In this part we discuss the computational aspects of the recommended epsilon constraint approach by providing the results of an experimental study. For the experimental study we again use the project selection problem. The aim of this section is to see the size of problems for which we can obtain a subset of the nondominated solutions to present the DM in reasonable time, using the formulations developed in the previous sections of this paper.

We consider the (discrete) project portfolio selection problem where m and n denote the number of projects and the number of project categories, respectively. As in [32] the output (b_i) and the input (c_i) values are randomly generated integers between 10 and 100. We set $B = 0.5 \sum c_i$. We start with m =50 increasing in increments of 50. As for n, we use 3 and 5. For each m and n combination, we generate 10 problem instances.

We use the (adaptive) epsilon constraint approach discussed in [32], which is a generalization of the scheme we discussed in section 6 for arbitrary numbers of objectives. The algorithms are coded in Visual C++ and solved by a dual core (Intel Core i5 2.27

GHz) computer with 4 GB RAM. The optimal solutions are found by CPLEX 12.2. The solution times are expressed in central processing unit (CPU) seconds. We set a time limit of 1 hour for the execution of the epsilon constraint approach.

We first discuss the results for problems where we seek balance in the input space as in our experience most applications involve concerns about ensuring balance in the input distribution to categories. In most cases we report the results for the models using I_3 ($I_3(x)$) as the imbalance indicator. That is because I_3 is likely to be computationally more complex than the others. We also report results for the cases where we introduce 2 imbalance indicators, in which case we use I_3 and I_2 ($I_2(x)$) as the two indicators. The balance distributions (r) are taken as $(50, 30, 20)$ and $(50, 30, 20, 10, 80)$ for the $n = 3$ and $n = 5$ settings, respectively. Hence α is $(0.5, 0.3, 0.2)$ and $(0.26, 0.16, 0.11, 0.05, 0.42)$ for these two settings.

We ran extensive experiments and we show a sample of the more interesting results in Table 7.1. In this table we report the average and maximum values for solution times and number of calls to CPLEX. We also report the average and minimum number of nondominated solutions (|ND|) returned by the algorithm. Note that the number of instances for which the algorithm could not terminate in 1 hour are indicated in parenthesis for the settings where the maximum solution time is 3600 seconds and these settings are reported in italics. The table also reports the value of the parameter *Stepsize*, which is used to adjust the right hand side of the constraint restricting the criterion value that is treated in the constraint for the bi-criteria problems. We report results for problems with a single type of input (indicated as Inp=1) and with two types of inputs (indicated as Inp=2).

For the instances where we obtain the nondominated set we set the optimality gap (θ) to 0.01% for CPLEX. We optimize the model that minimizes the imbalance while restricting the total output value with a constraint. As we assume integer values for the total output, setting *Stepsize* to 1 ensures that none of the nondominated points is missed.

As seen from Table 7.1 the cardinality of the nondominated set and the solution time required to solve the single-objective subproblems increase as the number of categories increases. These results also indicate that for larger problems, the size of the nondominated set or the solution time of the single-objective subproblems may be prohibitively large to allow us to obtain the whole nondominated frontier in reasonable time. However, in many cases presenting a large number of solutions to the decision maker may be neither necessary nor desirable. We rather suggest obtaining a moderate number of solutions which approximate the nondominated solutions and spread over the nondominated region in a uniform way.

To determine an appropriate *Stepsize* value we find the two nondominated solutions at the two ends of the nondominated frontier: The solution that has the largest Z_T value and the solution that has the smallest Z_I value. These solutions provide us the maximum and minimum total output values in the nondominated set: $Z_{TMax} = Max\{Z_T : (Z_T, Z_I) \in Z\}$ and $Z_{TMin} = Min\{Z_T : (Z_T, Z_I) \in Z\}$, respectively. We set $Stepsize = (Z_{TMax} - Z_{TMin})/40$. We solve the single objective sub-problems with a predefined optimality gap θ; hence find approximate nondominated solutions with a worst case quality guarantee. We report the results in Table 7.1. We also report results for $\theta = 5\%$ case with a fixed *Stepsize* value of 1.

The results indicate that the solution times increase as n increases for fixed m although the number of solutions returned decreases or stays similar. It indicates that the single objective subproblems become more difficult when n increases. For fixed n, the solution times and the average number of solutions increase as we increase m. As expected, increasing the optimality gap parameter (θ) leads to a decrease in solution times. The number of calls to CPLEX also decreases as θ increases, this is because the algorithm starts with a solution that has larger imbalance values and hence returns solutions which lie at the center of the frontier.

We next perform experiments for 3-criteria problems where there is a single input and there are 2 different indicators. For these experiments we use I_2 (collective-bottleneck indicator) and I_3 (individual oriented-sum indicator) as the two additional criteria to the total output criterion. We express the two imbalance criteria in the form of constraints and set the corresponding *Stepsize* values as 0.05 and 0.5 for I_2 and I_3, respectively.

It is possible to observe the effect of the number of categories (n) to the solution times. The effect of the number of projects on solution times does not seem to be as predictable as the effect of the number of categories. In some settings we even observe smaller solution times as m gets larger for fixed n.

We also note that the correlation coefficient between the values of the indicators I_2 and I_3 is quite high: it is between 0.8 and 0.96 for all settings. This indicates that for most cases if a portfolio has a high I_2 value, it is likely to have a high I_3 value as well. As expected, highest correlation is observed for the settings with $n = 3$ categories. This is because, for such cases if the worst-off category has high deviation from the target it is likely that the sum of all deviations will be high as well. As the number of categories increases the effect of the worst-off value to the total deviation decreases, resulting in differences between sum-oriented and bottleneck-oriented indicators.

Finally, we consider the case where there are two inputs. In this setting the projects consume two inputs and return a single output. The output and input values are randomly generated integers between 10 and 100, as before. The resulting model is a 3-criteria model, where we have the total output criteria and two imbalance criteria corresponding to the distributions of the two inputs over project categories. We report the results of our experiments in Table 7.1 for $\theta = 5\%$, where we use I_3 as the imbalance indicator. The two imbalance criteria are incorporated in the form of constraints with the same *Stepsize* values of 0.5.

It is seen from Table 7.1 that the solution times and the number of solutions increase considerably when the number of categories increases. Moreover, as there are two different inputs, the correlation between the values of the imbalance indicator that correspond to distributions of these two inputs is expected to decrease compared to the previous 3-criteria instances. The correlation coefficients were between 0.46 and 0.78 for the all the settings.

Our computational results indicate that the heuristic version of the epsilon constraint approach with appropriate *Stepsize* and optimality gap parameters can be used for small to medium-size problems. We observe that the solution times tend to increase significantly as the number of categories increases. For large-size problems with more than 150 projects different heuristic algorithms can be employed to obtain solutions in reasonable time.

We have also attempted to obtain nondominated solutions for problems for which the imbalance criterion is defined over the output distribution to categories. We observe that these problems are harder to solve. Even for the smallest problems considered (m=50, n=3) the epsilon-constraint based heuristic with 5% optimality gap fails to return solutions in 1 hour for some instances. To obtain solutions to these problems in reasonable time heuristic algorithms can be explored. One such approach is described here and some preliminary results are provided.

We designed a tabu search (TS) heuristic that starts with the solution that maximizes the total output. Using this initial feasible solution, we try to find solutions with improved balance values by searching its neighborhood. Given a solution, we search its neighborhood by switching the status of the projects in a pairwise manner. That is, for each pair of projects one of which is in the portfolio and the other is not, we exclude the former and include the latter if such an interchange is feasible. We calculate the potential improvement in balance for each such move, and perform the move that leads to the maximum improvement. We terminate when the number of non-improving moves reaches to 250 or number of the iterations reaches to 1000. We set tabu tenure to 50, i.e., we do not undo a selected move for the next 50 iterations. We use aspiration criterion as the best solution, i.e., tabu status of the moves that improve the best solution is overridden. We keep the candidate solutions in a set called *incumbent set*.

The TS algorithm executes to improve the Z_I value. Meanwhile we keep track of the corresponding Z_T values and update the incumbent set whenever we find an eligible solution, i.e., a solution which is non-dominated by the incumbent set.

Our experiments showed that the TS has satisfactory performance in terms of solution quality and computational time. We now report computational results for TS. As it was not possible to obtain the exact nondominated set for the case where we seek balance in the output space we report the performance of the TS algorithm for the input-oriented

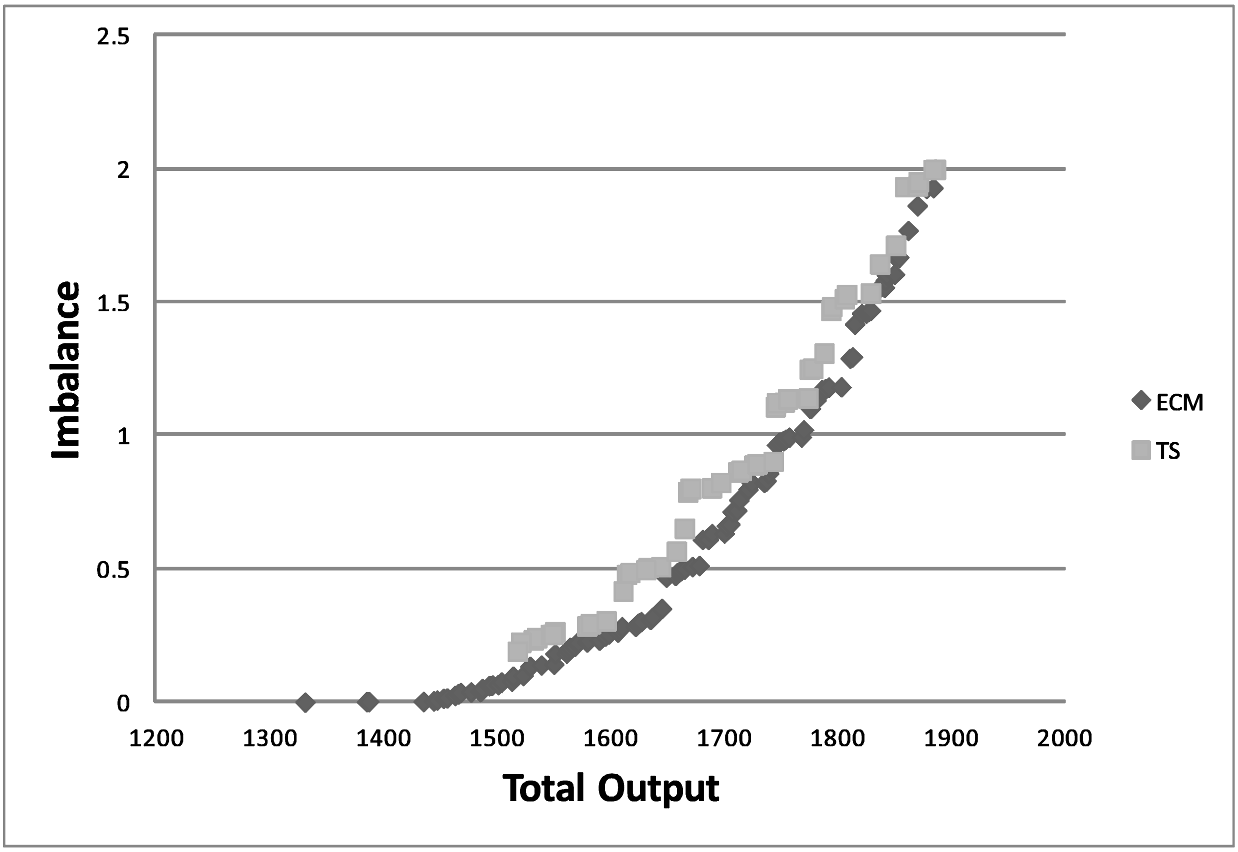

Figure 7.1: TS algorithm vs. ECM for a problem with m=50 n=3

case. Figure 7.1 shows the solutions obtained by the TS algorithm and the (exact) epsilon constraint method (ECM) in an example instance for m=50 n=3 case. As seen the TS approximates the nondominated set quite well.

Table 7.1: Performance results for the epsilon constraint approach

Inp=1

				CPU Time		|ND|		*Stepsize*		Calls to CPLEX	
Criteria	Gap	m	n	Avg	Max	Avg	Min	Avg	Max	Avg	Max
2	0	50	3	363.92	930.81	64	31	1.0	1.0	128	219
			5	*2024.80*	*3600 (1)*	*103*	*70*	*1.0*	*1.0*	*207*	*283*
	0.01	50	3	4.31	13.61	13	5	9.9	15.9	27	37
			5	39.65	249.96	10	5	50.1	56.5	21	45
		100	3	6.46	10.70	15	8	13.1	29.3	30	39
			5	*831.51*	*3600 (2)*	*16*	*2*	*53.1*	*95.0*	*40*	*51*
		150	3	8.04	11.78	16	13	16.3	24.8	33	43
	0.05	50	3	0.69	0.98	7	4	9.9	15.9	14	19
			5	3.98	16.07	6	3	50.1	56.5	14	21
		100	3	1.94	3.77	10	4	13.1	29.3	21	41
			5	10.01	24.21	12	5	44.4	57.1	26	39
		150	3	3.88	5.48	12	6	16.3	24.8	25	39
			5	25.15	74.90	15	9	55.4	81.6	31	43
		50	3	1.24	1.86	11	8	1.0	1.0	24	37
			5	20.68	46.29	35	22	1.0	1.0	71	103
		100	3	4.31	6.53	20	5	1.0	1.0	40	69
			5	94.50	271.34	70	26	1.0	1.0	142	257
		150	3	8.82	14.97	26	15	1.0	1.0	53	95
			5	181.16	664.59	77	23	1.0	1.0	154	333
3	0.01	50	3	10.11	30.51	10	2	-	-	62	136
			5	332.66	1060.22	40	8	-	-	261	779
		100	3	49.27	372.90	11	2	-	-	56	84
			5	*2507.664*	*3600 (4)*	*65*	*9*	-	-	*426*	*785*
	0.05	50	3	6.21	38.75	8	1	-	-	47	83
			5	612.30	3311.74	31	7	-	-	190	444
		100	3	7.93	11.36	9	3	-	-	60	83
			5	*461.85*	*3600 (1)*	*29*	*9*	-	-	*163*	*286*
		150	3	12.09	23.48	7	2	-	-	55	100
			5	395.25	1363.90	40	9	-	-	224	585

Inp=2

3	0.05	50	3	138.80	1230.46	21	3	-	-	97	263
			5	*2885.29*	*3600 (7)*	*84*	*11*	-	-	*992*	*2095*
		100	3	23.44	143.64	15	4	-	-	87	342
			5	*3253.64*	*3600 (9)*	*61*	*4*	-	-	*328*	*328*
		150	3	15.02	31.25	9	5	-	-	51	84

We also compared TS with our (heuristic) epsilon constraint method (ECM) with 1% optimality gap (ECM(1%)) with variable *Stepsize* values (As reported in Table 7.1). In terms of solution time TS massively outperforms ECM as it takes less than 2 seconds for TS to return a set of candidate solutions even for the largest problem instances considered as opposed to 363 seconds for ECM. However, the TS method clearly does not give guarantees of optimality and so knowing how good the generated solutions are in general is problematic. To assess the quality of solutions returned by the algorithms in this particular case, we use three performance metrics, namely P, D1 and D2. We denote the solutions returned by the TS or the heuristic ECM as the ANS (approximate non-dominated set). P is the percentage of exact non-dominated objective vectors returned by the TS (or heuristic ECM). D1 and D2 give information about the average and maximum distances between the points of the nondominated set and the points in set ANS, respectively (See [33] for the formulations of these metrics).

Table 7.2 shows the results. To give an idea about the scale of the distance metric we provide a graphical display of the solutions returned by the algorithms for an example instance which has the average distance values for both TS and ECM. For this instance TS has values of 0.07 and 0.13 and ECM(1%) has 0.01 and 0.03 for distance metrics D1 and D2, respectively.

As seen from the Table 7.2 and Figure 7.2, ECM outperforms the TS but the performance of TS algorithm is still satisfactory for these problems.

Table 7.2: Performance results for TS and ECM with 1 optimality gap

			P		D1		D2	
m	n	Algorithm	Avg	Min	Avg	Max	Avg	Max
50	3	TS	1.18	0	0.06	0.1	0.13	0.22
		ECM (1%)	1.69	0	0.01	0.02	0.04	0.05
50	5	TS	0.68	0	0.05	0.09	0.12	0.24
		ECM (1%)	13.83	0	0.01	0.02	0.05	0.13

For the output case we were unable to obtain the (approximate) nondominated set in reasonable time using the epsilon constraint approach with 5% optimality gap. Figure 7.3 shows the results of the ECM with 5% optimality gap and TS for an example in-

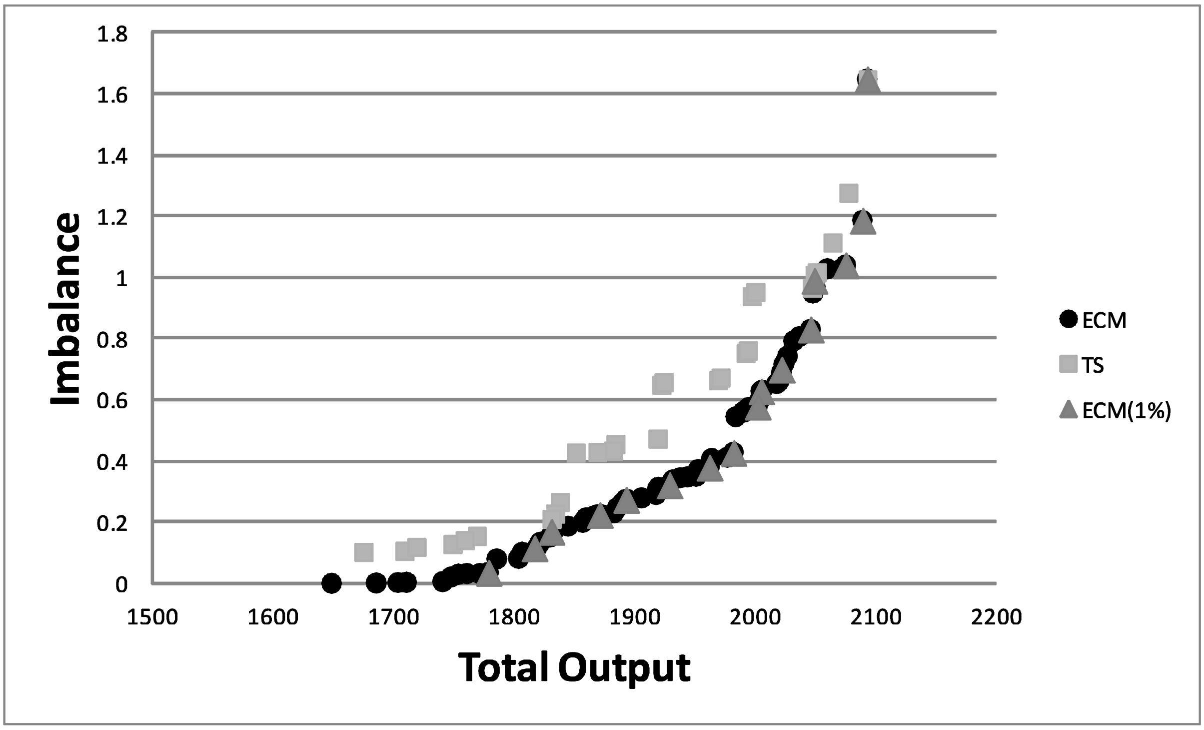

Figure 7.2: Comparison of ECM (Heuristic) and TS for an instance with average distance values

stance where m=50 and n=3. We leave a detailed comparative study of different solution approaches for the output-oriented case to future research.

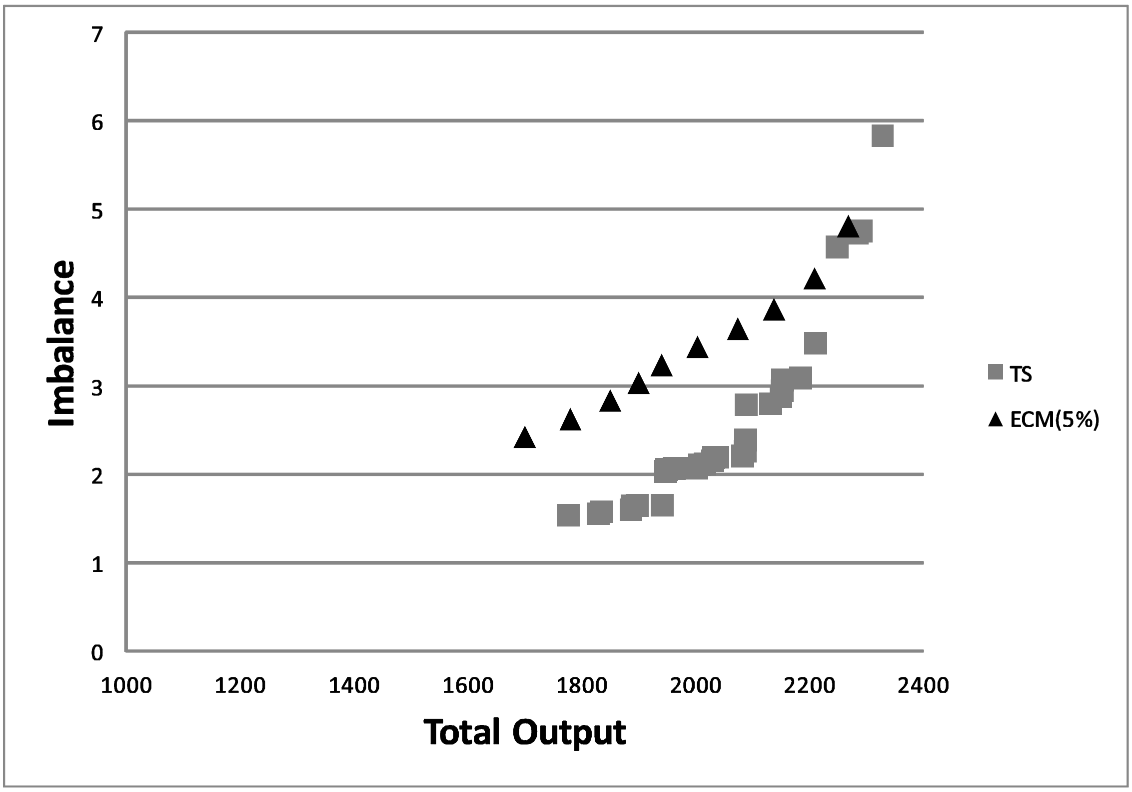

Figure 7.3: TS Algorithm vs ECM with 5% optimality gap

We observe that the TS algorithm returns a set of good solutions in negligible time for the input-oriented cases. For the output-oriented case the algorithm finds a set of solutions in negligible time. We have also done some explorations to extend the TS algorithm for multicriteria problems and observed that the solution times are negligible. However, further research should focus on generating a diverse set of solutions for multicriteria cases using algorithms that are computationally efficient. We hope this interesting and challenging question stimulates further research.

8 Conclusion

Allocation problems include a wide range of applications where inputs are allocated to entities so as to maximize the total output. Taking our motivation from various real life cases where a balanced (input/ output) distribution over categories is considered important as well as total output maximization we provide a framework to trade balance off against efficiency in such problems.

We define and categorize balance distribution based (im)balance indicators and show a way to incorporate these measures into the mathematical formulations of different allocation problems. We propose bi-criteria modelling by introducing balance as another criterion to the model alongside the total output criterion. We discuss an approach to obtain a subset of nondominated solutions. The solutions obtained are distributed over the entire nondominated set in a uniform way and range from the solution that has the maximum total output to the solution that has maximum balance.

We illustrate the approach by solving a real life project selection problem. Considering balance explicitly as another criterion and showing a subset of the efficient solutions to the DM has many advantages like bringing transparency to decisions and facilitating communication with the stakeholders. The generated graphs can help to initialize a structured discussion on balance. Observing how much one has to sacrifice to get closer to an ideally balanced distribution can provide justification for the decisions made for the final allocation.

We discuss the performance of the epsilon constraint approach by providing experimental results for larger bi-criteria and 3-criteria project selection problems. We are able to obtain a subset of (approximate) nondominated solutions that spread uniformly over the nondominated frontier, hence represent different regions of the frontier. We also suggest a TS approach for large-size problems and those with output-oriented imbalance criteria. We provide initial experimental results on the performance of the TS approach.

It is possible to use this modelling approach in other types of allocation problems where we allocate a homogeneous good to multiple entities. We note that the nature of the allocation, i.e. whether it is discrete or continuous, has an effect on the type of models developed. For example, for problems where the input allocation is continuous and balance is sought in the output space there is no obvious way to transform the decision model to a tractable mixed integer program when one is using the imbalance indicators discussed here: in these problems we cannot assume that the denominator is constant nor do we have binary variables and so cannot linearize using the idea we deployed in the discrete cases.

We have taken an initial step to bring in the perfect equality line concept to consider balance in resource (or output) distributions. There are possible further steps that can be taken. For example:

- Further research can be done on generalizing the proposed approach to a multi-criteria case where balance concerns are defined over multiple aspects and on developing ways to present the problem to the DM in a way that is easily communicated. Related algorithmic challenges can be addressed using appropriate methods such as metaheuristics.

- The balance line concept can also be extended by allowing a piecewise linear structure for the balance line. For example when the total amount available is very low the DM might tend to desire an equitable allocation, and as the total amount distributed increases, some other allocations may become more desirable than the equal allocation. The balance line approach can also be generalized to a *balance cone* approach where the extreme points and rays of the cone are generated based on the information given by the DM. Regarding any allocation within the cone as perfectly balanced one might assess the balance of alternative allocations and provide a subset of nondominated points to the DM.

- Axiomatic discussion of the difference imbalance indicators is another research area that we believe would be interesting. Presumably a key idea in axiomatizing balance would involve the observation that the points on the balance line are equally balanced and that as one moves towards the balance line one gets points which are better in terms of balance.

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A Proofs of Propositions 1 and 2

A.1 Proof of Proposition 1

We first prove the first part of the proposition and show that for $n = 2, 3$ we have $I_1(x) = 2I_2(x)$.

Let $n = 3$, and let the input/output distribution over which balance is sought be $a(x)$. $a(x) = (a(x)_1, a(x)_2, a(x)_3)$ and $a(x)_1 + a(x)_2 + a(x)_3 = a_T$. Suppose the balance distribution is $(\alpha_1, \alpha_2, \alpha_3)$. Let $d(x)_1$, $d(x)_2$ and $d(x)_3$ be the componentwise absolute deviations from the rescaled balance distribution. The following holds: $d(x)_1 + d(x)_2 + d(x)_3 = 2 * Max\{d(x)_1, d(x)_2, d(x)_3\}$. To see this, without loss of generality (w.l.o.g.) assume that $Max\{d(x)_1, d(x)_2, d(x)_3\} = d(x)_1$. Observe that the total negative componentwise deviation of $a(x)$ from $\overline{r}(x)$ should be equal to the total positive componentwise deviation. Hence we have $Max\{d(x)_1, d(x)_2, d(x)_3\} = d(x)_1 = d(x)_2 + d(x)_3$ and $d(x)_1 + d(x)_2 + d(x)_3 = 2 * Max\{d(x)_1, d(x)_2, d(x)_3\}$. Hence

$$I_1(x) = \frac{d(x)_1 + d(x)_2 + d(x)_3}{a_T} = \frac{2Max\{d(x)_1, d(x)_2, d(x)_3\}}{a_T} = 2I_2(x).$$

Note that it is easy to verify that $I_1(x) = 2I_2(x)$ for $n = 2$ in the same way.

We now prove the second part of the proposition: For $n = 2$, $I_1(x^1) \geq I_1(x^2) \iff I_2(x^1) \geq I_2(x^2) \iff I_3(x^1) \geq I_3(x^2) \iff I_4(x^1) \geq I_4(x^2)$. Note that $I_1(x^1) = 2I_2(x^1)$ (and $I_1(x^2) = 2I_2(x^2)$), hence $I_1(x^1) \geq I_1(x^2) \iff I_2(x^1) \geq I_2(x^2)$ for $n = 2$.

Let $a(x^1) = (a(x^1)_1, a(x^1)_2)$ and $a(x^2) = (a(x^2)_1, a(x^2)_2)$. Let $a(x^1)_1 + a(x^1)_2 = a_T^1$ and $a(x^2)_1 + a(x^2)_2 = a_T^2$. Suppose the balance allocation is (α_1, α_2). Let $\overline{r}(x^1)$ and $\overline{r}(x^2)$ be the corresponding (adjusted) balance distributions, i.e., $\overline{r}(x^1) = (a_T^1 * \alpha_1, a_T^1 * \alpha_2)$ and $\overline{r}(x^2) = (a_T^2 * \alpha_1, a_T^2 * \alpha_2)$. Note that $d(x^1)_1 = d(x^1)_2$ and $d(x^2)_1 = d(x^2)_2$.

1. We will first show that $I_1(x^1) \geq I_1(x^2) \iff I_3(x^1) \geq I_3(x^2)$.

   (a) $I_1(x^1) \geq I_1(x^2) \implies I_3(x^1) \geq I_3(x^2)$

   Suppose that $I_1(x^1) \geq I_1(x^2)$ while $I_3(x^1) < I_3(x^2)$.

If $I_1(x^1) \geq I_1(x^2)$ then

$$\frac{2d(x^1)_1}{a_T^1} \geq \frac{2d(x^2)_1}{a_T^2} \tag{A.1}$$

If $I_3(x^1) < I_3(x^2)$ then

$$\begin{aligned}
&\frac{d(x^1)_1}{a_T^1 * \alpha_1} + \frac{d(x^1)_1}{a_T^1 * \alpha_2} < \frac{d(x^2)_1}{a_T^2 * \alpha_1} + \frac{d(x^2)_1}{a_T^2 * \alpha_2} \implies \\
&\frac{d(x^1)_1}{a_T^1}\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right) < \frac{d(x^2)_1}{a_T^2}\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right) \implies \\
&\frac{d(x^1)_1}{a_T^1} < \frac{d(x^2)_1}{a_T^2} \implies \frac{2d(x^1)_1}{a_T^1} < \frac{2d(x^2)_1}{a_T^2}
\end{aligned} \tag{A.2}$$

From equations A.1 and A.2 we have a contradiction hence there is no x^1 and x^2 such that $I_1(x^1) \geq I_1(x^2)$ while $I_3(x^1) < I_3(x^2)$ for $n = 2$. It is easy to verify $I_3(x^1) \geq I_3(x^2) \implies I_1(x^1) \geq I_1(x^2)$ in the same way.

2. We will now show that $I_1(x^1) \geq I_1(x^2) \iff I_4(x^1) \geq I_4(x^2)$.

   $I_1(x^1) \geq I_1(x^2) \implies I_4(x^1) \geq I_4(x^2)$

   Suppose that $I_1(x^1) \geq I_1(x^2)$ while $I_4(x^1) < I_4(x^2)$. From previous result if $I_1(x^1) \geq I_1(x^2)$ equation A.1 holds.

   If $I_4(x^1) < I_4(x^2)$ then

$$\begin{aligned}
Max\{\frac{d(x^1)_1}{a_T^1 * \alpha_1}, \frac{d(x^1)_2}{a_T^1 * \alpha_2}\} \quad &< \quad Max\{\frac{d(x^2)_1}{a_T^2 * \alpha_1}, \frac{d(x^2)_2}{a_T^2 * \alpha_2}\} \\
\frac{d(x^1)_1}{Min\{a_T^1 * \alpha_1, a_T^1 * \alpha_2\}} \quad &< \quad \frac{d(x^2)_1}{Min\{a_T^2 * \alpha_1, a_T^2 * \alpha_2\}}
\end{aligned}$$

   Without loss of generality let $\alpha_1 < \alpha_2$. Then we have $\frac{d(x^1)_1}{a_T^1 * \alpha_1} < \frac{d(x^2)_1}{a_T^2 * \alpha_1} \implies \frac{d(x^1)_1}{a_T^1} < \frac{d(x^2)_1}{a_T^2}$. This is equation A.2, hence the rest follows as in part 1 leading to a contradiction. Similarly, it is easy to show that $I_4(x^1) \geq I_4(x^2) \implies I_1(x^1) \geq I_1(x^2)$ also holds.

A.2 Proof of Remark 2

Consider the following counterexamples:

Example 1 *Consider two allocations x^1 and x^2 which have $a(x^1)$ and $a(x^2)$ as shown in the table below and suppose that the balance distribution is r. The pairwise comparisons of the two alternatives are different under $I_4(x)$. We have $I_1(x^1) < I_1(x^2)$; $I_3(x^1) < I_3(x^2)$ but $I_4(x^1) > I_4(x^2)$.*

Allocation	$I_1(x)$	$I_3(x)$	$I_4(x)$
$a(x^1) =$*(16,16,13)*	*0.21*	*0.67*	*0.42*
$a(x^2) =$*(18,20,20)*	*0.28*	*0.84*	*0.38*
$r =$*(36,20,24)*			

In the example case given below the pairwise comparisons of the two alternatives are different under $I_1(x)$. Note that for $n = 3$ we have $I_1(x^1) \geq I_1(x^2) \iff I_2(x^1) \geq I_2(x^2)$ so $I_2(x)$ is also not consistent with $I_3(x)$ and $I_4(x)$.

Allocation	$I_1(x)$	$I_3(x)$	$I_4(x)$
$a(x^1) =$ *(11,12,18)*	*0.17*	*0.49*	*0.24*
$a(x^2) =$ *(20,10,17)*	*0.16*	*0.51*	*0.28*
$r =$*(30,25,30)*			

In the example case given below the pairwise comparisons of the two alternatives are different under $I_3(x)$.

Allocation	$I_1(x)$	$I_3(x)$	$I_4(x)$
$a(x^1) =$*(18,20,10)*	*0.25*	*0.80*	*0.42*
$a(x^2) =$ *(12,12,17)*	*0.26*	*0.78*	*0.47*
$r =$*(39,27,26)*			

Example 2 *To show that $I_1(x) = 2 * I_2(x)$ no longer holds when $n > 3$; consider the below example where $I_1(x^1) < I_1(x^2)$ but $I_2(x^1) > I_2(x^2)$.*

Allocation	$I_1(x)$	$I_2(x)$
$a(x^1) =$*(18,13,10,17)*	*0.25*	*0.13*
$a(x^2) =$*(19,11,20,15)*	*0.28*	*0.11*
$r =$*(39,33,28,20)*		

B Models using other indicators

B.1 Using $I_2(x)$:

This model is very similar to Model 1, except for the constraints related to Z_I. We use decision variables I_j to denote componentwise misallocations, i.e., $I_j = d_j / \sum_{i \in I} c_i x_i$. We find upper and lower bounds for I_j. We use the same bounds for all I_j and denote them as I^{UB} and I^{LB} , respectively. The bounds are as follows (d^{UB} is as defined in Model 1):

$$\begin{aligned} I^{UB} &= \frac{d^{UB}}{Min_i\{c_i\}} \\ I^{LB} &= \frac{d^{LB}}{\sum_{i \in I} c_i x_i} = 0 \end{aligned}$$

Z_I is the maximum componentwise deviation, i.e. $I_j \leq Z_I$ for all $j \in J$ and we minimize Z_I, hence $Z_I^{UB} = I^{UB}$. We have nonlinear terms in the equation defining I_js. We use the same techniques used in model 1 and obtain the following model.

Model 2

$Max\ \{Z_T, -Z_I\}$

Constraint sets 5.2b, 5.2c, 5.2d, 5.2e,5.2f, 5.2g

$I_j \leq Z_I \quad \forall j \in J$

$d_j = \sum_{i \in I} c_i t_{ij}\ \forall j \in J$

$I^{LB} x_i \leq t_{ij} \leq I^{UB} x_i\ \forall i \in I,\ j \in J$

$I^{LB}(1 - x_i) \leq I_j - t_{ij} \leq I^{UB}(1 - x_i)\ \forall i \in I,\ j \in J$

$x_i \in \{0, 1\}\ \forall i \in I$

$y_j \in \{0, 1\}\ \forall j \in J$

$d_j \geq 0\ \forall j,\ t_{ij} \geq 0\ \forall i \in I,\ j \in J$

Model 2 has $mn + m + 2n + 2$ variables and $2mn + 6n + 2$ constraints excluding the set constraints.

B.2 Using $I_3(x)$:

This model uses $I_3(x)$ as the balance criterion. Recall that this indicator is the sum of the componentwise proportional deviations. We use decision variables I_j to denote the componentwise proportional deviations for the categories in the model. That is, $I_j = d_j/\alpha_j \sum_{i\in I} c_i x_i$. We use the following upper and lower bounds for I_j in the model, denoted as I_j^{UB} and I_j^{LB}, respectively (We set d^{UB} as before):

$$\begin{aligned} I_j^{UB} &= (\frac{\sum_{j\in J} r_j}{r_j})\frac{d_j^{UB}}{Min_i\{c_i\}} = (\frac{\sum_{j\in J} r_j}{r_j})\frac{d^{UB}}{Min_i\{c_i\}} \text{ for all } j \in J \\ I_j^{LB} &= \frac{d^{LB}}{\alpha_j \sum_{i\in I} c_i x_i} = 0 \text{ for all } j \in J. \end{aligned}$$

Using I_j^{UB} and I_j^{LB} we can set $Z_I^{UB} = \sum_{j\in J} I_j^{UB}$ and $Z_I^{LB} = 0$.

The resulting model is the following:

Model 3

$Max\ \{Z_T, -Z_I\}$

Constraint sets 5.2b, 5.2c, 5.2d, 5.2e,5.2f, 5.2g

$$\sum_{j\in J} I_j = Z_I$$

$$d_j = \sum_{i\in I} c_i \alpha_j t_{ij}\ \forall j \in J$$

$$I_j^{LB} x_i \leq t_{ij} \leq I_j^{UB} x_i\ \forall i \in I,\ j \in J$$

$$I_j^{LB}(1 - x_i) \leq I_j - t_{ij} \leq I_j^{UB}(1 - x_i)\ \forall i \in I,\ j \in J$$

$$x_i \in \{0, 1\}\ \forall i \in I$$

$$y_j \in \{0, 1\}\ \forall j \in J$$

$$d_j \geq 0\ \forall j, t_{ij} \geq 0\ \forall i \in I,\ j \in J$$

Model 3 has $mn + m + 2n + 2$ variables and $2mn + 5n + 3$ constraints excluding the set constraints.

B.3 Using $I_4(x)$:

This model uses $I_4(x)$ in the objective function. It is very similar to model 3 with a slight change in the constraint defining Z_I. We change it as follows:

$$I_j \leq Z_I \ \forall j \in J$$

Where I_j^{UB}, I_j^{LB} and d^{UB} are as in model 3 and $Z_I^{UB} = \underset{j}{Max}\{\frac{\sum_{j\in J} r_j}{r_j}\}d^{UB} = \underset{j}{Max}\{\frac{1}{\alpha_j}\}d^{UB}$. The resulting model has $mn+m+2n+2$ variables and $2mn+6n+2$ constraints excluding the set constraints.

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