Green's function estimates for a singularly perturbed convection-diffusion problem*

Sebastian Franz[†]   Natalia Kopteva[‡]

Abstract

We consider a singularly perturbed convection-diffusion problem posed in the unit square with a horizontal convective direction. Its solutions exhibit parabolic and exponential boundary layers. Sharp estimates of the Green's function and its first- and second-order derivatives are derived in the L_1 norm. The dependence of these estimates on the small diffusion parameter is shown explicitly. The obtained estimates will be used in a forthcoming numerical analysis of the considered problem.

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1 Introduction

In this paper we investigate the Green's function for the following problem posed in the unit-square domain $\Omega = (0,1)^2$:

$$L_{xy}u(x,y) := -\varepsilon(u_{xx} + u_{yy}) - (a(x,y)\,u)_x + b(x,y)\,u = f(x,y) \quad \text{for } (x,y) \in \Omega, \quad (1.1a)$$

$$u(x,y) = 0 \qquad \text{for } (x,y) \in \partial\Omega. \quad (1.1b)$$

Here ε is a small positive parameter, while the coefficients a and b are sufficiently smooth (e.g., $a, b \in C^\infty(\bar{\Omega})$). We also assume, for some positive constant α, that

$$a(x,y) \geq \alpha > 0, \qquad b(x,y) - a_x(x,y) \geq 0 \qquad \text{for all } (x,y) \in \bar{\Omega}. \quad (1.2)$$

Under these assumptions, (1.1a) is a singularly perturbed elliptic equation, frequently referred to as a convection-dominated convection-diffusion equation. This equation serves as a model for Navier-Stokes equations at large Reynolds numbers or (in the linearised case) of Oseen equations and provides an excellent paradigm for numerical techniques in the computational fluid dynamics [19].

[†]Institut für Numerische Mathematik, Technische Universität Dresden, 01062 Dresden, Germany e-mail: sebastian.franz@tu-dresden.de

[‡]Department of Mathematics and Statistics, University of Limerick, Limerick, Ireland e-mail: natalia.kopteva@ul.ie

The asymptotic analysis for problems of type (1.1) is very intricate and illustrates the complexity of their solutions [11, Section IV.1], [12]. We also refer the reader to [20, Chapter IV], [19, Chapter III.1] and [13, 14] for pointwise estimates of solution derivatives. In short, solutions of problem (1.1) typically exhibit parabolic boundary layers along the characteristic boundaries $y = 0$ and $y = 1$, and an exponential boundary layer along the outflow boundary $x = 0$. Furthermore, if a discontinuous Dirichlet boundary condition is imposed at the inflow boundary $x = 1$, then solutions also exhibit characteristic interior layers. Note that because of the complexity of the solutions, the analysis techniques [13, 14] work only for a constant-coefficient version of (1.1a). Note also that the complex solution structure is reflected in the corresponding Green's function, which is the subject of this paper.

Our interest in considering the Green's function of problem (1.1) and estimating its derivatives is motivated by the numerical analysis of this computationally challenging problem. More specifically, we shall use the obtained estimates in the forthcoming paper [7] to derive robust a posteriori error bounds for computed solutions of this problem using finite-difference methods. (This approach is related to recent articles [15, 4], which address the numerical solution of singularly perturbed equations of reaction-diffusion type.) In a more general numerical-analysis context, we note that sharp estimates for continuous Green's functions (or their generalised versions) frequently play a crucial role in a priori and a posteriori error analyses [6, 10, 18].

We shall estimate the derivatives of the Green's function in the L_1 norm (as they will be used to estimate the error in the computed solution in the dual L_∞ norm [7]). Our estimates will be *uniform in the small perturbation parameter ε* in the sense that any dependence on ε will be shown explicitly. Note also that our estimates will be *sharp* (in the sense of Theorem 2.6) up to an ε-independent constant multiplier.

As any Green's function estimate implies a certain a priori estimate for the original problem, we also refer the reader to Dörfler [5], who, for a similar problem, gives extensive a priori solution estimates that involve the right-hand side in various positive norms such as L_p and $W^{m,p}$ with $m \geq 0$. In comparison, a priori solution estimates that follow from our results, involve negative norms of the right-hand side (see Corollary 2.3 and also Remark 2.4), so they are different in nature.

Our analysis in this paper resembles those in [15, Section 3], [4, Section 3] in that, roughly speaking, we freeze the coefficients and estimate the corresponding explicit Green's function for a constant-coefficient equation, and then we investigate the difference between the original and the frozen-coefficient Green's functions. This procedure is often called the parametrix method. The two cited papers deal with equations of reaction-diffusion type, for which the Green's function in the unbounded domain is (almost) radially symmetric and exponentially decaying away from the singular point. By contrast, the Green's function for the convection-diffusion problem (1.1) exhibits a much more complex *anisotropic* structure (see Fig. 1). This is reflected in a much more intricate analysis compared to [15, 4], in particular, for the variable-coefficient case.

The paper is organised as follows. In Section 2, the Green's function associated with problem (1.1) is defined and upper bounds for its derivatives are stated in Theorem 2.2, which is the main result of the paper. The corresponding lower bounds are then given in
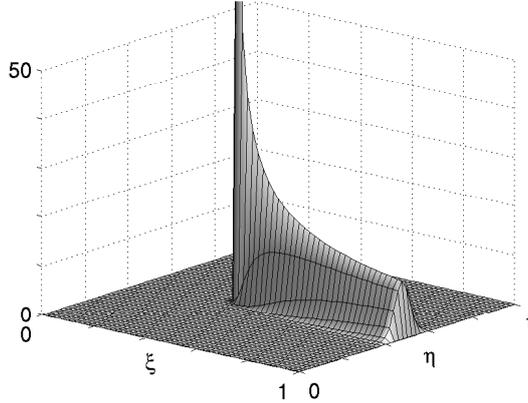
Figure 1: Typical anisotropic behaviour of the Green's function for problem (1.1): $a = 1$, $b = 0$, $(x, y) = (\frac{1}{3}, \frac{1}{2})$ and $\varepsilon = 10^{-3}$.

Theorem 2.6. In Section 3, we obtain the fundamental solution for a constant-coefficient version of (1.1a) in the domain $\Omega = \mathbb{R}^2$; this fundamental solution is bounded in Section 4. Next, in Section 5, using the method of images with an inclusion of cut-off functions, we define and estimate certain approximations of the constant-coefficient Green's functions in the domains $\Omega = (0, 1) \times \mathbb{R}$ and $\Omega = (0, 1)^2$. The difference between the frozen-coefficient approximations of Section 5 and the original variable-coefficient Green's function is estimated in Section 6; this completes the proof of Theorem 2.2. In the final Section 7 we discuss generalisation of our results to more than two dimensions.

Notation. Throughout the paper, C denotes a generic positive constant that may take different values in different formulas, but is *independent of the singular perturbation parameter ε*. A subscripted C (e.g., C_1) denotes a positive constant that takes a fixed value, and is also independent of ε. Notation such as $v = \mathcal{O}(w)$ means $|v| \leq Cw$ for some C. The standard Sobolev spaces $W^{m,p}(\Omega')$ and $L_p(\Omega')$ on any measurable subset $\Omega' \subset \mathbb{R}^2$ are used for $p \geq 1$ and $m = 1, 2$. The $L_p(\Omega')$ norm is denoted by $\|\cdot\|_{p;\Omega'}$ while the $W^{m,p}(\Omega')$ norm is denoted by $\|\cdot\|_{m,p;\Omega'}$. Sometimes the domain of interest will be an open ball $B(x', y'; \rho) := \{(x, y) \in \mathbb{R}^2 : (x - x')^2 + (y - y')^2 < \rho^2\}$ centred at (x', y') of radius ρ. For the partial derivative of a function v in a variable ξ we will use the equivalent notations v_ξ and $\partial_\xi v$. Similarly, $v_{\xi\xi}$ and $\partial_\xi^2 v$ both denote the second-order pure derivative of v in ξ, while $v_{\xi\eta}$ and $\partial_{\xi\eta}^2 v$ both denote the second-order mixed derivative of v in ξ and η.

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2 Definition of the Green's function. Main result

Let $G = G(x, y; \xi, \eta)$ be the Green's function associated with problem (1.1). For each fixed $(x, y) \in \Omega$, it satisfies

$$L^*_{\xi\eta} G(x, y; \xi, \eta) = -\varepsilon(G_{\xi\xi} + G_{\eta\eta}) + a(\xi, \eta)G_\xi + b(\xi, \eta)G = \delta(x - \xi)\,\delta(y - \eta),\ (\xi, \eta) \in \Omega,$$
$$G(x, y; \xi, \eta) = 0, \qquad\qquad\qquad (\xi, \eta) \in \partial\Omega. \tag{2.1}$$

Here $L^*_{\xi\eta}$ is the adjoint differential operator to L_{xy}, while $\delta(\cdot)$ is the one-dimensional Dirac δ-distribution, so the product $\delta(x - \xi)\,\delta(y - \eta)$ is equivalent to the two-dimensional δ-distribution centred at $(\xi, \eta) = (x, y)$; see [9, Example 3.29], [21, Section 5.5]. The unique solution u of (1.1) has the representation

$$u(x, y) = \iint_\Omega G(x, y; \xi, \eta)\, f(\xi, \eta)\, d\xi\, d\eta, \tag{2.2}$$

(provided that f is sufficiently regular so that (2.2) is well-defined). Note that, for each fixed $(\xi, \eta) \in \Omega$, the Green's function G also satisfies

$$L_{xy} G(x, y; \xi, \eta) = -\varepsilon(G_{xx} + G_{yy}) - (a(x, y)G)_x + b(x, y)G = \delta(x - \xi)\delta(y - \eta),\, (x, y) \in \Omega,$$
$$G(x, y; \xi, \eta) = 0, \qquad\qquad\qquad (x, y) \in \partial\Omega. \tag{2.3}$$

Therefore, the unique solution v of the adjoint problem

$$L^*_{xy} v(x, y) = -\varepsilon(v_{xx} + v_{yy}) + a(x, y)\, v_x + b(x, y)\, v = f(x, y) \quad \text{for } (x, y) \in \Omega,$$
$$v(x, y) = 0 \qquad \text{for } (x, y) \in \partial\Omega.$$

is given by

$$v(\xi, \eta) = \iint_\Omega G(x, y; \xi, \eta)\, f(x, y)\, dx\, dy. \tag{2.4}$$

We first give a preliminary result for G.

Lemma 2.1. *Under assumptions* (1.2), *the Green's function G associated with problem* (1.1) *satisfies*

$$\int_0^1 |G(x, y; \xi, \eta)|\, d\eta \le C, \qquad \|G(x, y; \cdot)\|_{1;\Omega} \le C \qquad \text{for } (x, y) \in \Omega, \tag{2.5}$$

where C is some positive ε-independent constant.

Proof. The first estimate of (2.5) is given in the proof of [5, Theorem 2.10] (see also [19, Theorem III.1.22] and [3] for similar results). The second desired estimate follows. ☐

We now state the main result of this paper.

Theorem 2.2. *Let $\varepsilon \in (0, 1]$. The Green's function G associated with (1.1), (1.2) on the unit square $\Omega = (0, 1)^2$ satisfies, for all $(x, y) \in \Omega$, the following bounds*

$$\|\partial_\xi G(x, y; \cdot)\|_{1;\Omega} \leq C(1 + |\ln \varepsilon|), \tag{2.6a}$$

$$\|\partial_\eta G(x, y; \cdot)\|_{1;\Omega} + \|\partial_y G(x, y; \cdot)\|_{1;\Omega} \leq C\varepsilon^{-1/2}. \tag{2.6b}$$

Furthermore, for any ball $B(x', y'; \rho)$ of radius ρ centred at any $(x', y') \in \bar{\Omega}$, we have

$$\|G(x, y; \cdot)\|_{1,1;B(x',y';\rho)} \leq C\varepsilon^{-1}\rho, \tag{2.6c}$$

while for the ball $B(x, y; \rho)$ of radius ρ centred at (x, y) we have

$$\|\partial_\xi^2 G(x, y; \cdot)\|_{1;\Omega \backslash B(x,y;\rho)} \leq C\varepsilon^{-1} \ln(2 + \varepsilon/\rho), \tag{2.6d}$$

$$\|\partial_\eta^2 G(x, y; \cdot)\|_{1;\Omega \backslash B(x,y;\rho)} \leq C\varepsilon^{-1}(\ln(2 + \varepsilon/\rho) + |\ln \varepsilon|). \tag{2.6e}$$

Here C is some positive ε-independent constant.

The rest of the paper is devoted to the proof of this theorem, which is completed in Section 6.

In view of the solution representation (2.2), the bounds (2.6a), (2.6b) immediately imply the following a priori solution estimates for our original problem.

Corollary 2.3. *Let $f(x, y) = \partial_x F_1(x, y) + \partial_y F_2(x, y)$ with $F_1, F_2 \in L_\infty(\Omega)$. Then there exists a unique solution $u \in L_\infty(\Omega)$ of problem (1.1), (1.2), for which we have the bound*

$$\|u\|_{\infty;\Omega} \leq C\big[(1 + |\ln \varepsilon|) \|F_1\|_{\infty;\Omega} + \varepsilon^{-1/2} \|F_2\|_{\infty;\Omega}\big]. \tag{2.7}$$

Proof. Represent u using (2.2). Then integrate by parts and use (2.6a) and (2.6b). $\square$

Remark 2.4. *Let us associate the components $\partial_x F_1$ and $\partial_y F_2$ of f with the one-dimensional parts $-\varepsilon\partial_x^2 - \partial_x a(x, y)$ and $-\varepsilon\partial_y^2 + b(x, y)$, respectively, of the operator L_{xy}. Then, bar the weak logarithmic factor $|\ln \varepsilon|$, the bound (2.7) clearly resembles the corresponding one-dimensional a priori solution estimates. Indeed, for the one-dimensional equations $-\varepsilon u_1''(x) - (a_1(x)u_1(x))' = f_1(x)$ and $-\varepsilon u_2''(x) + b_2(x)u_2(x) = f_2(x)$ (where $a_1, b_2 \geq C > 0$) subject to $u_{1,2}(0) = u_{1,2}(1) = 0$, one has $\|u_1\|_{\infty;(0,1)} \leq C\|f_1\|_{-1,\infty;(0,1)}$, and $\|u_2\|_{\infty;(0,1)} \leq C\varepsilon^{-1/2}\|f_2\|_{-1,\infty;(0,1)}$, where $\|\cdot\|_{-1,\infty;(0,1)}$ is the norm in the negative Sobolev space $W^{-1,\infty}(0, 1)$ (see, e.g., [17, Theorem 3.25]).*

Remark 2.5. *In the proof of Corollary 2.3, the existence of a solution $u \in L_\infty(\Omega)$ of problem (1.1), (1.2) follows from the observation that the solution representation formula (2.2) yields a bounded function. Note that the existence of a bounded solution of this problem, under the additional mild assumption $b(x, y) - \frac{1}{2}\partial_x a(x, y) \geq 0$, can be shown by an application of [16, Chapter 3, Theorems 5.2 and 13.1]. In particular, the second cited theorem states that if there exists a solution $u \in W^{2,1}(\Omega)$, then it is bounded in $\bar{\Omega}$ by some ε-dependent constant. The novelty of Corollary 2.3 lies in that it explicitly shows the dependence of this constant on ε.*

Note that the upper estimates of Theorem 2.2 are *sharp* in the following sense.

Theorem 2.6 ([8])**.** *Let $\varepsilon \in (0, c_0]$ for some sufficiently small positive c_0. Set $a(x, y) := \alpha$ and $b(x, y) := 0$ in (1.1). Then the Green's function G associated with this problem on the unit square $\Omega = (0, 1)^2$ satisfies, for all $(x, y) \in [\frac{1}{4}, \frac{3}{4}]^2$, the following lower bounds:*

$$\|\partial_\xi G(x, y; \cdot)\|_{1;\Omega} \geq c \,|\ln \varepsilon|, \tag{2.8a}$$

$$\|\partial_\eta G(x, y; \cdot)\|_{1;\Omega} \geq c \,\varepsilon^{-1/2}. \tag{2.8b}$$

Furthermore, for any ball $B(x, y; \rho)$ of radius $\rho \leq \frac{1}{8}$, we have

$$\|G(x, y; \cdot)\|_{1,1;\Omega \cap B(x,y;\rho)} \geq \begin{cases} c\,\rho/\varepsilon, & \text{if } \rho \leq 2\varepsilon, \\ c\,(\rho/\varepsilon)^{1/2}, & \text{otherwise,} \end{cases} \tag{2.8c}$$

$$\|\partial_\xi^2 G(x, y; \cdot)\|_{1;\Omega \setminus B(x,y;\rho)} \geq c\,\varepsilon^{-1} \ln(2 + \varepsilon/\rho), \qquad \text{if } \rho \leq c_1 \varepsilon, \tag{2.8d}$$

$$\|\partial_\eta^2 G(x, y; \cdot)\|_{1;\Omega \setminus B(x,y;\rho)} \geq c\,\varepsilon^{-1}(\ln(2 + \varepsilon/\rho) + |\ln \varepsilon|), \qquad \text{if } \rho \leq \frac{1}{8}. \tag{2.8e}$$

Here c and c_1 are ε-independent positive constants.

This result can be anticipated from an inspection of the bounds for an explicit fundamental solution in a constant-coefficient case; see Section 4.

3 Fundamental solution in a constant-coefficient case

In this section we shall explicitly solve simplifications of the two problems (2.1) and (2.3) that we have for G. To get these simplifications, we employ the parametrix method and so freeze the coefficients in these problems by replacing $a(\xi, \eta)$ by $a(x, y)$ in (2.1), and replacing $a(x, y)$ by $a(\xi, \eta)$ in (2.3), and also setting $b := 0$; the frozen-coefficient versions of the operators $L_{\xi\eta}^*$ and L_{xy} will be denoted by $\bar{L}_{\xi\eta}^*$ and $\tilde{L}_{xy}$, respectively. Furthermore, we extend the resulting equations to $\mathbb{R}^2$ and denote their solutions by $\bar{g}$ and $\tilde{g}$. Thus we get

$$\bar{L}_{\xi\eta}^* \bar{g}(x, y; \xi, \eta) = -\varepsilon(\bar{g}_{\xi\xi} + \bar{g}_{\eta\eta}) + a(x, y)\,\bar{g}_\xi = \delta(x - \xi)\,\delta(y - \eta) \quad \text{for } (\xi, \eta) \in \mathbb{R}^2, \tag{3.1}$$

$$\tilde{L}_{xy} \tilde{g}(x, y; \xi, \eta) = -\varepsilon(\tilde{g}_{xx} + \tilde{g}_{yy}) - a(\xi, \eta)\,\tilde{g}_x = \delta(x - \xi)\,\delta(y - \eta) \quad \text{for } (x, y) \in \mathbb{R}^2. \tag{3.2}$$

As the variables (x, y) appear as parameters in equation (3.1) and (ξ, η) appear as parameters in equation (3.2), we effectively have two equations with constant coefficients. A calculation (see Remark 3.1 below for details) yields explicit representations of their solutions by

$$\bar{g}(x, y; \xi, \eta) = g(x, y; \xi, \eta; q)\Big|_{q=\frac{1}{2}a(x,y)}, \qquad \tilde{g}(x, y; \xi, \eta) = g(x, y; \xi, \eta; q)\Big|_{q=\frac{1}{2}a(\xi,\eta)}. \tag{3.3}$$

Here the function g is defined, using the modified Bessel function of the second kind of order zero $K_0(\cdot)$, by

$$g = g(x, y; \xi, \eta; q) := \frac{1}{2\pi\varepsilon} e^{q\hat{\xi}_{[x]}} K_0(q\hat{r}_{[x]}), \tag{3.4a}$$

$$\hat{\xi}_{[x]} := (\xi - x)/\varepsilon, \quad \hat{\eta} := (\eta - y)/\varepsilon, \quad \hat{r}_{[x]} := \sqrt{\hat{\xi}_{[x]}^2 + \hat{\eta}^2}. \tag{3.4b}$$

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We use a subindex in $\hat{\xi}_{[x]}$ and $\hat{r}_{[x]}$ to highlight their dependence on x as in many places x will take different values; but when there is no ambiguity, we shall sometimes simply write $\hat{\xi}$ and $\hat{r}$.

Remark 3.1. *The representation* (3.4) *is given in [19, (III.1.16)]. For completeness, we sketch a proof of* (3.3), (3.4) *for* $\bar{g}$. *Set* $q = \frac{1}{2}a(x,y)$ *and* $\bar{g} = V(\xi,\eta)\,e^{q\xi/\varepsilon}$ *in* (3.1). *Now a calculation shows that*

$$-\varepsilon^2(V_{\xi\xi} + V_{\eta\eta}) + q^2 V = \varepsilon\,e^{-q\xi/\varepsilon}\,\delta(x-\xi)\,\delta(y-\eta).$$

Here in the right-hand side, one has $e^{-q\xi/\varepsilon}\,\delta(x-\xi) = e^{-qx/\varepsilon}\,\delta(x-\xi)$. *As the fundamental solution for the operator* $-\varepsilon^2(\partial_\xi^2 + \partial_\eta^2) + q^2$ *is* $\frac{1}{2\pi\varepsilon^2}K_0(qr/\varepsilon)$ *[22, Chapter VII], so* $V = \varepsilon\,e^{-qx/\varepsilon}\,\frac{1}{2\pi\varepsilon^2}K_0(qr/\varepsilon)$, *and the desired representation* (3.3), (3.4) *for* $\bar{g}$ *immediately follows.*

Remark 3.2. *Note that the solution* $\bar{g}$ *of* (3.1) *is not the fundamental solution for the operator* $\bar{L}^*_{\xi\eta}$. *Indeed, denoting the latter by* $\Gamma = \Gamma(x,y;\xi,\eta;s,t)$, *one has the equation* $\bar{L}^*_{\xi\eta}\Gamma = \delta(s-\xi)\,\delta(t-\eta)$, *in which* (x,y) *appear as parameters. So imitating the calculation in Remark 3.1, one gets* $\Gamma(x,y;\xi,\eta;s,t) = g(s,t;\xi,\eta;q)\big|_{q=\frac{1}{2}a(x,y)}$ *(compare with* (3.3)*). Similarly, the solution* $\tilde{g}$ *of* (3.2) *is not the fundamental solution for the operator* $\tilde{L}_{xy}$.

The function g and its derivatives involve the modified Bessel functions of the second kind of order zero $K_0(\cdot)$ and of order one $K_1(\cdot)$. With the notation $K_{0,1} := \max\{K_0, K_1\}$, we quote some useful properties of the modified Bessel functions [1]:

$$K_{0,1}(s) \leq Cs^{-1}e^{-s/2} \quad \forall s > 0, \qquad K_{0,1}(s) \leq Cs^{-1/2}e^{-s} \quad \forall s \geq C > 0, \tag{3.5a}$$

$$K_0(z) = K_1(z)\big[1 - \frac{1}{2z} + \mathcal{O}(z^{-2})\big]. \tag{3.5b}$$

4 Bounds for the fundamental solution $g(x,y;\xi,\eta;q)$

Throughout this section we assume that $\Omega = (0,1) \times \mathbb{R}$, but all results remain valid for $\Omega = (0,1)^2$. Here we derive a number of useful bounds for the fundamental solution g of (3.4) and its derivatives that will be used in Section 5. As sometimes $q = \frac{1}{2}a(x,y)$ or $q = \frac{1}{2}a(\xi,\eta)$ (as in (3.3)), we shall also use the full-derivative notation

$$D_\eta := \partial_\eta + \tfrac{1}{2}\partial_\eta a(\xi,\eta)\cdot\partial_q, \qquad D_y := \partial_y + \tfrac{1}{2}\partial_y a(x,y)\cdot\partial_q. \tag{4.1}$$

Lemma 4.1. *Let* $(x,y) \in [-1,1] \times \mathbb{R}$ *and* $0 < \frac{1}{2}\alpha \leq q \leq C$. *Then for the function* $g = g(x,y;\xi,\eta;q)$ *of* (3.4) *we have the following bounds*

$$\|g(x,y;\cdot;q)\|_{1;\Omega} \leq C, \tag{4.2a}$$

$$\|\partial_\xi g(x,y;\cdot;q)\|_{1;\Omega} \leq C(1 + |\ln\varepsilon|), \tag{4.2b}$$

$$\varepsilon^{1/2}\,\|\partial_\eta g(x,y;\cdot;q)\|_{1;\Omega} + \|\partial_q g(x,y;\cdot;q)\|_{1;\Omega} \leq C, \tag{4.2c}$$

$$\|(\varepsilon\hat{r}_{[x]}\,\partial_\xi g)(x,y;\cdot;q)\|_{1;\Omega} \leq C, \tag{4.2d}$$

$$\varepsilon^{1/2}\,\|(\varepsilon\hat{r}_{[x]}\,\partial^2_{\xi\eta}g)(x,y;\cdot;q)\|_{1;\Omega} + \|(\varepsilon\hat{r}_{[x]}\,\partial^2_{\xi q}g)(x,y;\cdot;q)\|_{1;\Omega} \leq C, \tag{4.2e}$$

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and for any ball $B(x', y'; \rho)$ of radius ρ centred at any $(x', y') \in [0, 1] \times \mathbb{R}$, we have

$$\|g(x, y; \cdot; q)\|_{1,1;B(x',y';\rho)} \leq C\varepsilon^{-1}\rho, \tag{4.2f}$$

while for the ball $B(x, y; \rho)$ of radius ρ centred at (x, y), we have

$$\|\partial_\xi^2 g(x, y; \cdot; q)\|_{1;\Omega \setminus B(x,y;\rho)} \leq C\varepsilon^{-1}\ln(2 + \varepsilon/\rho), \tag{4.2g}$$

$$\|\partial_\eta^2 g(x, y; \cdot; q)\|_{1;\Omega \setminus B(x,y;\rho)} \leq C\varepsilon^{-1}(\ln(2 + \varepsilon/\rho) + |\ln \varepsilon|). \tag{4.2h}$$

Furthermore, one has the bound

$$\|\partial_x g(x, y; \cdot; q)\|_{1;\Omega} \leq C(1 + |\ln \varepsilon|), \tag{4.3a}$$

and, with the full-derivative notation (4.1), the bounds

$$\|D_\eta g(x, y; \cdot; q)\|_{1;\Omega} + \|D_y g(x, y; \cdot; q)\|_{1;\Omega} \leq C\varepsilon^{-1/2}, \tag{4.3b}$$

$$\|(\varepsilon \hat{r}_{[x]} D_\eta \partial_x g)(x, y; \cdot; q)\|_{1;\Omega} + \|(\varepsilon \hat{r}_{[x]} D_y \partial_\xi g)(x, y; \cdot; q)\|_{1;\Omega} \leq C\varepsilon^{-1/2}. \tag{4.3c}$$

Proof. First, note that $\partial_x g = -\partial_\xi g$ and $\partial_y g = -\partial_\eta g$, so (4.3a) follows from (4.2b), (4.3b) follows from (4.1), (4.2c), while (4.3c) follows from (4.1), (4.2e). Thus it suffices to establish the bounds (4.2).

Throughout the proof, x and y are fixed so we employ the notation $\hat{\xi} := \hat{\xi}_{[x]}$ and $\hat{r} := \hat{r}_{[x]}$. A calculation shows that the first-order derivatives of $g(x, y; \xi, \eta; q)$ are given by

$$\partial_\xi g = \tfrac{q}{2\pi\varepsilon^2} e^{q\hat{\xi}} \left[K_0(q\hat{r}) - \frac{\hat{\xi}}{\hat{r}} K_1(q\hat{r}) \right], \tag{4.4a}$$

$$\partial_\eta g = -\tfrac{q}{2\pi\varepsilon^2} e^{q\hat{\xi}} \left[\frac{\hat{\eta}}{\hat{r}} K_1(q\hat{r}) \right], \tag{4.4b}$$

$$\partial_q g = \tfrac{1}{2\pi\varepsilon} \hat{r} e^{q\hat{\xi}} \left[\frac{\hat{\xi}}{\hat{r}} K_0(q\hat{r}) - K_1(q\hat{r}) \right]. \tag{4.4c}$$

Here we used $K_0' = -K_1$, [1], and then $\partial_\xi \hat{r} = \varepsilon^{-1}\hat{\xi}/\hat{r}$ and $\partial_\eta \hat{r} = \varepsilon^{-1}\hat{\eta}/\hat{r}$. In a similar manner, but additionally using $K_1'(s) = -K_0(s) - K_1(s)/s$ [1], and also $\partial_\xi(\hat{\eta}/\hat{r}) = -\varepsilon^{-1}\hat{\xi}\hat{\eta}/\hat{r}^3$ and $\partial_\eta(\hat{\eta}/\hat{r}) = \varepsilon^{-1}\hat{\xi}^2/\hat{r}^3$, one gets the second-order derivatives

$$\partial_{\xi\eta}^2 g = \tfrac{q}{2\pi\varepsilon^3} e^{q\hat{\xi}} \frac{\hat{\eta}}{\hat{r}^2} \left[q\hat{r}\left(\frac{\hat{\xi}}{\hat{r}} K_0(q\hat{r}) - K_1(q\hat{r})\right) + 2\frac{\hat{\xi}}{\hat{r}} K_1(q\hat{r}) \right], \tag{4.5a}$$

$$\partial_{\xi q}^2 g = \tfrac{q}{2\pi\varepsilon^2} e^{q\hat{\xi}} \hat{r}^{-1} \left[\hat{\xi}\hat{r}\{2K_0(q\hat{r}) + \tfrac{1}{q\hat{r}} K_1(q\hat{r})\} - (\hat{\xi}^2 + \hat{r}^2)K_1(q\hat{r}) \right] + q^{-1}\partial_\xi g, \tag{4.5b}$$

$$\partial_\eta^2 g = \tfrac{q}{2\pi\varepsilon^3} e^{q\hat{\xi}} \left[q\frac{\hat{\eta}^2}{\hat{r}^2} K_0(q\hat{r}) + \frac{\hat{\eta}^2 - \hat{\xi}^2}{\hat{r}^3} K_1(q\hat{r}) \right]. \tag{4.5c}$$

Finally, combining $\partial_\xi^2 g = -\partial_\xi^2 g + \frac{2q}{\varepsilon} \partial_\xi g$ with (4.4a) and (4.5c) yields

$$\partial_\xi^2 g = \tfrac{q}{2\pi\varepsilon^3} e^{q\hat{\xi}} \left[q\left(K_0(q\hat{r}) + \frac{\hat{\xi}^2}{\hat{r}^2} K_0(q\hat{r}) - 2\frac{\hat{\xi}}{\hat{r}} K_1(q\hat{r}) \right) + \frac{\hat{\xi}^2 - \hat{\eta}^2}{\hat{r}^3} K_1(q\hat{r}) \right]. \tag{4.5d}$$

Now we proceed to estimating the above derivatives of g. Note that $d\xi \, d\eta = \varepsilon^2 d\hat{\xi} \, d\hat{\eta}$, where $(\hat{\xi}, \hat{\eta}) \in \hat{\Omega} := \varepsilon^{-1}(-x, 1-x) \times \mathbb{R} \subset (-\infty, 2/\varepsilon) \times \mathbb{R}$. Consider the domains

$$\hat{\Omega}_1 := \{\hat{\xi} < 1 + \tfrac{1}{4}|\hat{\eta}|\}, \qquad \hat{\Omega}_2 := \{\max\{1, \tfrac{1}{4}|\hat{\eta}|\} < \hat{\xi} < 2/\varepsilon\}.$$

As $\hat{\Omega} \subset \hat{\Omega}_1 \cup \hat{\Omega}_2$ for any $x \in [-1, 1]$, it is convenient to consider integrals over these two subdomains separately.

(i) Consider $(\hat{\xi}, \hat{\eta}) \in \hat{\Omega}_1$. Then $\hat{\xi} \leq 1 + \tfrac{1}{4}\hat{r}$ so, with the notation $K_{0,1} := \max\{K_0, K_1\}$, one gets

$$\varepsilon^2 \big[(1+\hat{r})(\varepsilon^{-1}|g| + |\partial_\xi g| + |\partial_\eta g| + |\partial_q g| + |\partial_{\xi q}^2 g|) + \varepsilon \hat{r}|\partial_{\xi\eta}^2 g|\big] \leq C e^{q\hat{\xi}}(1 + \hat{r} + \hat{r}^2) K_{0,1}(q\hat{r})$$
$$\leq C\hat{r}^{-1} e^{-q\hat{r}/8}, \qquad (4.6)$$

where we combined $e^{q\hat{\xi}} \leq e^{q(1+\hat{r}/4)}$ with $1 + \hat{r} + \hat{r}^2 \leq C e^{q\hat{r}/8}$ (which follows from $q \geq \tfrac{1}{2}\alpha$) and $K_{0,1}(q\hat{r}) \leq C(q\hat{r})^{-1} e^{-q\hat{r}/2}$ (see (3.5a)). This immediately yields

$$\iint_{\hat{\Omega}_1} \big[(1+\hat{r})(\varepsilon^{-1}|g| + |\partial_\xi g| + |\partial_\eta g| + |\partial_q g| + |\partial_{\xi q}^2 g|) + \varepsilon \hat{r}|\partial_{\xi\eta}^2 g|\big] \left(\varepsilon^2 d\hat{\xi} \, d\hat{\eta}\right) \leq C \int_0^\infty e^{-q\hat{r}/8} \, d\hat{r}$$
$$\leq C. \qquad (4.7)$$

Similarly,

$$\varepsilon^2 \big[|\partial_\xi^2 g| + |\partial_\eta^2 g|\big] \leq C\varepsilon^{-1} e^{q\hat{\xi}} \left(1 + \hat{r}^{-1}\right) K_{0,1}(q\hat{r}) \leq C\varepsilon^{-1} \hat{r}^{-2} e^{-q\hat{r}/8},$$

so

$$\iint_{\hat{\Omega}_1 \setminus B(0,0;\hat{\rho})} \big[|\partial_\xi^2 g| + |\partial_\eta^2 g|\big] \left(\varepsilon^2 d\hat{\xi} \, d\hat{\eta}\right) \leq C\varepsilon^{-1} \int_{\hat{\rho}}^\infty \hat{r}^{-1} e^{-q\hat{r}/8} \, d\hat{r} \leq C\varepsilon^{-1} \ln(2 + \hat{\rho}^{-1}). \quad (4.8)$$

Furthermore, for an arbitrary ball $\hat{B}_{\hat{\rho}}$ of radius $\hat{\rho}$ in the coordinates $(\hat{\xi}, \hat{\eta})$, we get

$$\iint_{\hat{\Omega}_1 \cap \hat{B}_{\hat{\rho}}} \big[|\partial_\xi g| + |\partial_\eta g| + |g|\big] \left(\varepsilon^2 d\hat{\xi} \, d\hat{\eta}\right) \leq C \int_0^{\hat{\rho}} e^{-q\hat{r}/8} \, d\hat{r} \leq C \min\{\hat{\rho}, 1\}. \qquad (4.9)$$

(ii) Next consider $(\hat{\xi}, \hat{\eta}) \in \hat{\Omega}_2$. In this subdomain, it is convenient to rewrite the integrals in terms of $(\hat{\xi}, t)$, where

$$t := \hat{\xi}^{-1/2} \hat{\eta} \qquad \text{so} \qquad \hat{\xi}^{-1/2} d\hat{\eta} = dt, \qquad \hat{r} - \hat{\xi} = \frac{\hat{\eta}^2}{\hat{r} + \hat{\xi}} \leq t^2. \qquad (4.10)$$

Note that $q\hat{r} \geq q \geq \tfrac{1}{2}\alpha$ in $\hat{\Omega}_2$, so $K_{0,1}(q\hat{r}) \leq C(q\hat{r})^{-1/2} e^{-q\hat{r}}$ by the second bound in (3.5a). We also note that $\hat{\xi} \leq \hat{r} \leq \sqrt{17}\hat{\xi}$ in $\hat{\Omega}_2$ so $\hat{r} - \hat{\xi} = \hat{\eta}^2/(\hat{r} + \hat{\xi}) \geq c_0 \hat{\eta}^2/\hat{\xi} = c_0 t^2$, where $c_0 := (1 + \sqrt{17})^{-1}$. Consequently $e^{-q(\hat{r}-\hat{\xi})} \leq e^{-qc_0 t^2}$, so

$$e^{q\hat{\xi}} K_{0,1}(q\hat{r}) \leq CQ \quad \text{for } (\hat{\xi}, \hat{\eta}) \in \hat{\Omega}_2, \qquad \text{where} \quad Q := \hat{\xi}^{-1/2} e^{-qc_0 t^2} \qquad (4.11)$$

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and

$$\int_{\mathbb{R}} \left(1 + |t| + t^2 + |t|^3 + t^4\right) Q \, d\hat{\eta} \le C \int_{\mathbb{R}} \left(1 + |t| + t^2 + |t|^3 + t^4\right) e^{-qc_0 t^2} \, dt \le C. \qquad (4.12)$$

We now claim that for g and its derivatives in $\hat{\Omega}_2$ one has

$$\varepsilon^2 |g| \le C \, \varepsilon \, Q, \qquad (4.13a)$$

$$\varepsilon^2 |\partial_\eta g| \le C \xi^{-1/2} \, |t| \, Q, \qquad (4.13b)$$

$$\varepsilon^2 |\partial_\eta^2 g| \le C \, \varepsilon^{-1} \hat{\xi}^{-1} \, [t^2 + 1] \, Q. \qquad (4.13c)$$

Here (4.13a) is straightforward, and (4.13b) immediately follows from (4.4b) as $|\hat{\eta}|/\hat{r} \le |\hat{\eta}|/\hat{\xi} = \xi^{-1/2}|t|$. The next bound (4.13c) is obtained from (4.5c) using $\hat{\eta}^2/\hat{r}^2 \le \xi^{-1}t^2$ and $|\hat{\eta}^2 - \hat{\xi}^2|/\hat{r}^3 \le \hat{r}^{-1} \le \hat{\xi}^{-1}$.

Furthermore, we claim that in $\hat{\Omega}_2$ one also has

$$\varepsilon^2 (\varepsilon\hat{r}|\partial_\xi g| + |\partial_q g|) \;\le\; C \, \varepsilon[t^2 + 1] \, Q, \qquad (4.13d)$$

$$\varepsilon^2 |\partial_\xi g| \;\le\; C \, \hat{\xi}^{-1} \, [t^2 + 1] \, Q, \qquad (4.13e)$$

$$\varepsilon^2 (\varepsilon\hat{r}|\partial_{\xi\eta}^2 g|) \;\le\; C \, \hat{\xi}^{-1/2} \, |t| \, [t^2 + 1] Q, \qquad (4.13f)$$

$$\varepsilon^2 (\varepsilon\hat{r}|\partial_{\xi q}^2 g|) \;\le\; C \, \varepsilon[t^4 + 1] \, Q + q^{-1}\varepsilon^2(\varepsilon\hat{r}|\partial_\xi g|), \qquad (4.13g)$$

$$\varepsilon^2 |\partial_\xi^2 g| \;\le\; C \, \varepsilon^{-1}\hat{\xi}^{-2} \, [t^4 + 1] \, Q. \qquad (4.13h)$$

To get (4.13d), we combine (4.4a) and (4.4c) with the observation that

$$\left|K_\nu(q\hat{r}) - \frac{\hat{\xi}}{\hat{r}} K_\mu(q\hat{r})\right| = \left|1 - \frac{\hat{\xi}}{\hat{r}} + \mathcal{O}(\hat{r}^{-1})\right| K_1(q\hat{r}) \qquad \text{for } \hat{r} \ge 1, \qquad (4.14a)$$

$$\le C\hat{r}^{-1}[t^2 + 1]K_1(q\hat{r}) \qquad \text{for } \hat{\xi} \ge 1, \qquad (4.14b)$$

where $\nu, \mu = 0, 1$. Note that (4.14a) and (4.14b) are easily verified using (3.5b) and $\hat{r} - \hat{\xi} \le t^2$ from (4.10), respectively. The bound (4.13e) follows from the bound for $\partial_\xi g$ in (4.13d) as $\hat{r}^{-1} \le \hat{\xi}^{-1}$. We now proceed to (4.13f), which is obtained from (4.5a) again using $|\hat{\eta}|/\hat{r} \le \xi^{-1/2}|t|$ and then (4.14b) and $\hat{\xi}/\hat{r} \le 1$. Next, one gets (4.13g) from (4.5b) using $\{2K_0(q\hat{r}) + \frac{1}{q\hat{r}}K_1(q\hat{r})\} = 2K_1(q\hat{r})[1 + \mathcal{O}(\hat{r}^{-2})]$ (which follows from (3.5b)) and then $(\hat{r} - \hat{\xi})^2 \le t^4$. The final bound (4.13h) is derived in a similar manner by employing (3.5b) to rewrite the term in square-brackets of (4.5d) as $\left[q\left(1 - \frac{\hat{\xi}}{\hat{r}}\right)^2 - \frac{3\hat{\eta}^2}{2\hat{r}^3} + \mathcal{O}(\hat{r}^{-2})\right]K_1(q\hat{r})$. Thus all the bounds (4.13) are now established.

Combining the obtained estimates (4.13) with (4.12) yields

$$\iint_{\hat{\Omega}_2} \left[|g| + \varepsilon^{1/2}|\partial_\eta g| + \varepsilon\hat{r}|\partial_\xi g| + |\partial_q g| + \varepsilon^{1/2}\varepsilon\hat{r}|\partial_{\xi\eta}^2 g| + \varepsilon\hat{r}|\partial_{\xi q}^2 g| + \varepsilon|\partial_\xi^2 g|\right] \left(\varepsilon^2 d\hat{\xi} \, d\hat{\eta}\right)$$

$$\le C \int_1^{2/\varepsilon} [\varepsilon + \varepsilon^{1/2}\hat{\xi}^{-1/2} + \hat{\xi}^{-2}] \, d\hat{\xi} \le C. \quad (4.15)$$

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Similarly, combining $(4.13c), (4.13e)$ with (4.12) yields

$$\iint_{\hat{\Omega}_2} \left[|\partial_\xi g| + \varepsilon|\partial_\eta^2 g|\right] \left(\varepsilon^2 d\hat{\xi}\, d\hat{\eta}\right) \leq C \int_1^{2/\varepsilon} \hat{\xi}^{-1}\, d\hat{\xi} \leq C(1 + |\ln \varepsilon|). \qquad (4.16)$$

Furthermore, by $(4.13b), (4.13e)$, for an arbitrary ball $\hat{B}_{\hat\rho}$ of radius $\hat{\rho}$ in the coordinates $(\hat{\xi}, \hat{\eta})$, we get

$$\iint_{\hat{\Omega}_2 \cap \hat{B}_{\hat\rho}} \left[|\partial_\xi g| + |\partial_\eta g| + |g|\right] \left(\varepsilon^2 d\hat{\xi}\, d\hat{\eta}\right) \leq C \int_1^{1+\hat\rho} [\hat{\xi}^{-1} + \hat{\xi}^{-1/2} + \varepsilon]\, d\hat{\xi} \leq C\hat{\rho}. \qquad (4.17)$$

To complete the proof, we now recall that $\hat{\Omega} \subset \hat{\Omega}_1 \cup \hat{\Omega}_2$ and combine estimates (4.7) and (4.8) (that involve integration over $\hat{\Omega}_1$) with (4.15) and (4.16), which yields the desired bounds $(4.2a)$–$(4.2e)$ and $(4.2g), (4.2h)$. To get the latter two bounds we also used the observation that the ball $B(x, y; \rho)$ in the coordinates (ξ, η) becomes the ball $B(0, 0; \hat{\rho})$ of radius $\hat{\rho} = \varepsilon^{-1}\rho$ in the coordinates $(\hat{\xi}, \hat{\eta})$. The remaining assertion $(4.2f)$ is obtained by combining (4.9) with (4.17) and noting that an arbitrary ball $B(x', y'; \rho)$ of radius ρ in the coordinates (ξ, η) becomes a ball $\hat{B}_{\hat\rho}$ of radius $\hat{\rho} = \varepsilon^{-1}\rho$ in the coordinates $(\hat{\xi}, \hat{\eta})$. $\qquad \square$

Remark 4.2. *The first bound* $(4.2a)$ *of Lemma 4.1 can be also obtained by noting that* $I_g(\xi) := \int_{\mathbb{R}} g\, d\eta$ *satisfies the differential equation* $[-\varepsilon\partial_\xi^2 + 2q\partial_\xi]I_g = \delta(x - \xi)$ *(this follows from an equation of type* (3.1) *for* g*) and the conditions* $I_g(-\infty) = 0$ *and* $I_g(x) = (2q)^{-1}$. *From this, one can easily deduce that* $\int_0^1 I_g(\xi) \leq C$, *which yields* $(4.2a)$ *in view of* $g > 0$.

Our next result shows that for $x \geq 1$, one gets stronger bounds for g and its derivatives. These bounds involve the weight function

$$\lambda := e^{2q(x-1)/\varepsilon} \qquad (4.18)$$

and show that, although λ is exponentially large in ε, this is compensated by the smallness of g and its derivatives.

Lemma 4.3. *Let* $(x, y) \in [1, 3] \times \mathbb{R}$ *and* $0 < \frac{1}{2}\alpha \leq q \leq C$. *Then for the function* $g = g(x, y; \xi, \eta; q)$ *of* (3.4) *and the weight* λ *of* (4.18), *one has the following bounds*

$$\|([1 + \varepsilon\hat{r}_{[x]}]\, \lambda g)(x, y; \cdot; q)\|_{1;\Omega} \leq C\varepsilon, \qquad (4.19a)$$

$$\|(\lambda\, \partial_\xi g)(x, y; \cdot; q)\|_{1;\Omega} + \|(\lambda\, \partial_q g)(x, y; \cdot; q)\|_{1;\Omega} \leq C, \qquad (4.19b)$$

$$\|([1 + \varepsilon^{1/2}\hat{r}_{[x]}]\, \lambda\, \partial_\eta g)(x, y; \cdot; q)\|_{1;\Omega} + \varepsilon^{1/2}\|(\varepsilon\hat{r}_{[x]}\, \lambda\, \partial_{\xi\eta}^2 g)(x, y; \cdot; q)\|_{1;\Omega} \leq C, \qquad (4.19c)$$

$$\|\hat{r}_{[x]}\, \partial_q(\lambda\, g)(x, y; \cdot; q)\|_{1;\Omega} + \|\varepsilon\hat{r}_{[x]}\, \partial_q(\lambda\, \partial_\xi g)(x, y; \cdot; q)\|_{1;\Omega} \leq C, \qquad (4.19d)$$

and for any ball $B(x', y'; \rho)$ *of radius* ρ *centred at any* $(x', y') \in [0, 1] \times \mathbb{R}$, *one has*

$$\|(\lambda\, g)(x, y; \cdot; q)\|_{1,1;B(x',y';\rho)} \leq C\varepsilon^{-1}\rho, \qquad (4.19e)$$

while for the ball $B(x, y; \rho)$ *of radius* ρ *centred at* (x, y), *one has*

$$\|(\lambda\, \partial_\xi^2 g)(x, y; \cdot; q)\|_{1;\Omega \setminus B(x,y,\rho)} + \|(\lambda\, \partial_\eta^2 g)(x, y; \cdot; q)\|_{1;\Omega \setminus B(x,y,\rho)} \leq C\varepsilon^{-1}\ln(2 + \varepsilon/\rho). \qquad (4.19f)$$

Furthermore, with the differential operators (4.1), *we have*

$$\|\partial_x(\lambda g)(x,y;\cdot;q)\|_{1;\Omega} + \|D_y(\lambda g)(x,y;\cdot;q)\|_{1;\Omega} + \|D_\eta(\lambda g)(x,y;\cdot;q)\|_{1;\Omega} \le C, \qquad (4.20a)$$

$$\|\varepsilon \hat{r}_{[x]} D_y(\lambda \, \partial_\xi g)(x,y;\cdot;q)\|_{1;\Omega} + \|\varepsilon \hat{r}_{[x]} D_\eta \partial_x(\lambda g)(x,y;\cdot;q)\|_{1;\Omega} \le C\varepsilon^{-1/2}. \quad (4.20b)$$

Proof. Throughout the proof we use the notation $A = A(x) := (x-1)/\varepsilon \ge 0$. Then (4.18) becomes $\lambda = e^{2qA}$. We partially imitate the proof of Lemma 4.1. Again $d\xi \, d\eta = \varepsilon^2 \, d\hat{\xi} \, d\hat{\eta}$, but now $(\hat{\xi}, \hat{\eta}) \in \hat{\Omega} = \varepsilon^{-1}(-x, 1-x) \times \mathbb{R} \subset (-3/\varepsilon, -A) \times \mathbb{R}$. So $\hat{\xi} < -A \le 0$ immediately yields

$$\lambda \, e^{q\hat{\xi}} = e^{2q(A-|\hat{\xi}|)} \, e^{q|\hat{\xi}|} \le e^{q|\hat{\xi}|}. \qquad (4.21)$$

Consider the domains

$$\hat{\Omega}_1' := \left\{ |\hat{\xi}| < 1 + \tfrac{1}{4}|\hat{\eta}|, \ \hat{\xi} < -A \right\}, \qquad \hat{\Omega}_2' := \left\{ |\hat{\xi}| > \max\{1, \tfrac{1}{4}|\hat{\eta}|\}, \ -3/\varepsilon < \hat{\xi} < -A \right\}.$$

As $\hat{\Omega} \subset \hat{\Omega}_1' \cup \hat{\Omega}_2'$ for any $x \in [1,3]$, we estimate integrals over these two domains separately.

(i) Let $(\hat{\xi}, \hat{\eta}) \in \hat{\Omega}_1'$. Then $|\hat{\xi}| \le 1 + \tfrac{1}{4}\hat{r}$ so, by (4.21), one has $\lambda \, e^{q\hat{\xi}} \le e^{q(1+\hat{r}/4)}$. The first line in (4.6) remains valid, but now we combine it with

$$\lambda \, e^{q\hat{\xi}} \, (1 + \hat{r} + \hat{r}^2) \, K_{0,1}(q\hat{r}) \le C\hat{r}^{-1} \, e^{-q\hat{r}/8} \qquad (4.22)$$

(which is obtained similarly to the second line in (4.6)). This leads to a version of (4.7) that involves the weight λ:

$$\iint_{\hat{\Omega}_1'} \lambda \left[(1+\hat{r})(\varepsilon^{-1}|g| + |\partial_\xi g| + |\partial_\eta g| + |\partial_q g| + |\partial_{\xi q}^2 g|) + \varepsilon\hat{r}|\partial_{\xi\eta}^2 g| \right] \left(\varepsilon^2 d\hat{\xi} \, d\hat{\eta} \right) \le C. \quad (4.23)$$

In a similar manner, we obtain versions of estimates (4.8) and (4.9), that also involve the weight λ:

$$\iint_{\hat{\Omega}_1' \setminus B(0,0;\hat{\rho})} \lambda \, |\partial_\eta^2 g| \left(\varepsilon^2 d\hat{\xi} \, d\hat{\eta} \right) \le C\varepsilon^{-1} \int_{\hat{\rho}}^\infty \hat{r}^{-1} e^{-q\hat{r}/8} \, d\hat{r} \le C\varepsilon^{-1} \ln(2 + \hat{\rho}^{-1}), \qquad (4.24)$$

$$\iint_{\hat{\Omega}_1' \cap \hat{B}_{\hat{\rho}}} \lambda \left[|\partial_\xi g| + |\partial_\eta g| + |g| \right] \left(\varepsilon^2 d\hat{\xi} \, d\hat{\eta} \right) \le C \int_0^{\hat{\rho}} e^{-q\hat{r}/8} \, d\hat{r} \le C \min\{\hat{\rho}, 1\}, \qquad (4.25)$$

where $\hat{B}_{\hat{\rho}}$ is an arbitrary ball of radius $\hat{\rho}$ in the coordinates $(\hat{\xi}, \hat{\eta})$. Furthermore, (4.23) combined with $|\partial_q(\lambda \, g)| \le \lambda(2A|g| + |\partial_q g|)$ and $|\partial_q(\lambda \, \partial_\xi g)| \le \lambda(2A|\partial_\xi g| + |\partial_{\xi q}^2 g|)$ and then with $A \le 2/\varepsilon$ yields

$$\iint_{\hat{\Omega}_1'} \hat{r} \left[|\partial_q(\lambda \, g)| + \varepsilon|\partial_q(\lambda \, \partial_\xi g)| \right] \left(\varepsilon^2 d\hat{\xi} \, d\hat{\eta} \right) \le C. \qquad (4.26)$$

(ii) Now consider $(\hat{\xi}, \hat{\eta}) \in \hat{\Omega}_2'$. In this subdomain (similarly to $\hat{\Omega}_2$ in the proof of Lemma 4.1) one has $|\hat{\xi}| \le \hat{r} \le \sqrt{17}|\hat{\xi}|$ and $c_0 t^2 \le \hat{r} - |\hat{\xi}| \le t^2$, where $t := |\hat{\xi}|^{-1/2}\hat{\eta}$ (compare with (4.10)). We also introduce a new barrier Q such that

$$e^{q\hat{\xi}} K_{0,1}(q\hat{r}) \le CQ \quad \text{for } (\hat{\xi}, \hat{\eta}) \in \hat{\Omega}_2', \quad \text{where } Q := \lambda^{-1} e^{2q(A-|\hat{\xi}|)} \left\{ |\hat{\xi}|^{-1/2} e^{-qc_0 t^2} \right\}, \quad (4.27)$$

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(compare with (4.11)). Note that the inequality in (4.27) is obtained similarly to the one in (4.11), as (4.21) implies $e^{q\hat{\xi}} K_{0,1}(q\hat{r}) = \lambda^{-1} e^{2q(A-|\hat{\xi}|)} \{e^{q|\hat{\xi}|} K_{0,1}(q\hat{r})\}$.

With the new definition (4.27) of Q, the bounds (4.13a)–(4.13c) remain valid in $\hat{\Omega}_2'$ only with $\hat{\xi}$ replaced by $|\hat{\xi}|$. Note that the bounds (4.13d)–(4.13g) are not valid in $\hat{\Omega}_2'$, (as they were obtained using $\hat{r} - \hat{\xi} \leq t^2$, which is not the case for $\hat{\xi} < 0$). Instead, we claim that in $\hat{\Omega}_2'$ one has

$$\varepsilon^2|\partial_\xi g| \leq C Q, \tag{4.28a}$$

$$\varepsilon^2|\partial_q g| \leq C \varepsilon|\hat{\xi}| Q, \tag{4.28b}$$

$$\varepsilon^2(\varepsilon\hat{r}|\partial_{\xi\eta}g|) \leq C |\hat{\xi}|^{1/2} |t| Q, \tag{4.28c}$$

$$\varepsilon^2(|\partial_q(\lambda g)| + \varepsilon|\partial_q(\lambda \partial_\xi g)|) \leq C \varepsilon \lambda [(|\hat{\xi}| - A) + t^2 + 1] Q. \tag{4.28d}$$

Here (4.28a) immediately follows from (4.4a) as $|\hat{\xi}|/\hat{r} \leq 1$. The bound (4.28b) is obtained from (4.4c) in a similar way, also using $\hat{r} \leq \sqrt{17}|\hat{\xi}|$. The next bound (4.28c), is deduced from (4.5a) using $\eta = |\hat{\xi}|^{1/2}t$ and again $|\hat{\xi}|/\hat{r} \leq 1$, and also $\hat{r} + 1 \leq 2\hat{r}$.

To establish (4.28d), note that $\partial_q(\lambda g) = \lambda[2A g + \partial_q g]$ and $\partial_q(\lambda \partial_\xi g) = \lambda[2A \partial_\xi g + \partial_{\xi q}^2 g]$. Using (3.5b), (4.14a) and $\{2K_0(q\hat{r}) + \frac{1}{q\hat{r}}K_1(q\hat{r})\} = 2K_1(q\hat{r})[1 + \mathcal{O}(\hat{r}^{-2})]$ (which follows from (3.5b)), one can rewrite the definition of g and relations (4.4c), (4.4a), (4.5b) as

$$g = \tfrac{1}{2\pi\varepsilon} e^{q\hat{\xi}} \left[1 + \mathcal{O}(\hat{r}^{-1})\right] K_1(q\hat{r}),$$

$$\partial_q g = \tfrac{1}{2\pi\varepsilon} e^{q\hat{\xi}} \left[-(\hat{r} + |\hat{\xi}|) + \mathcal{O}(1)\right] K_1(q\hat{r}),$$

$$\partial_\xi g = \tfrac{q}{2\pi\varepsilon^2} e^{q\hat{\xi}} \hat{r}^{-1}\left[(\hat{r} + |\hat{\xi}|) + \mathcal{O}(1)\right] K_1(q\hat{r}),$$

$$\partial_{\xi q}^2 g = \tfrac{q}{2\pi\varepsilon^2} e^{q\hat{\xi}} \hat{r}^{-1}\left[-(\hat{r} + |\hat{\xi}|)^2 + \mathcal{O}(1)\right] K_1(q\hat{r}) + q^{-1}\partial_\xi g.$$

Next note that

$$S := (\hat{r} + |\hat{\xi}|) - 2A = 2(|\hat{\xi}| - A) + (\hat{r} - |\hat{\xi}|) \leq 2(|\hat{\xi}| - A) + t^2.$$

Consequently, a calculation shows that

$$\lambda^{-1} \partial_q(\lambda g) = \tfrac{1}{2\pi\varepsilon} e^{q\hat{\xi}} \left[-S + \hat{r}^{-1}\mathcal{O}(A + \hat{r})\right] K_1(q\hat{r}),$$

$$\lambda^{-1} \partial_q(\lambda \partial_\xi g) = \tfrac{q}{2\pi\varepsilon^2} e^{q\hat{\xi}} \left[-S \hat{r}^{-1}(\hat{r} + |\hat{\xi}|) + \hat{r}^{-1}\mathcal{O}(A + 1)\right] K_1(q\hat{r}) + q^{-1}\partial_\xi g.$$

In view of $\hat{r}^{-1}(A + \hat{r} + 1) \leq C$ and $\hat{r}^{-1}(\hat{r} + |\hat{\xi}|) \leq 2$, and also (4.28a), the final bound (4.28d) in (4.28) follows.

Next, note that (4.12) is valid with Q replaced by the multiplier $\{e^{q|\hat{\xi}|} K_{0,1}(q\hat{r})\}$ from the current definition (4.27) of Q. Combining this observation with the bounds (4.13a)–(4.13c) and (4.28a)–(4.28c), and also with $\hat{r} \leq \sqrt{17}|\hat{\xi}|$, yields

$$\iint_{\hat{\Omega}_2'} \lambda \left[(\varepsilon^{-1} + \hat{r})|g| + |\partial_\xi g| + (1 + \varepsilon^{1/2}\hat{r})|\partial_\eta g| + |\partial_q g| + \varepsilon^{1/2}(\varepsilon\hat{r}|\partial_{\xi\eta}^2 g|) + \varepsilon|\partial_\eta^2 g|\right] \left(\varepsilon^2 d\hat{\xi}\, d\hat{\eta}\right)$$

$$\leq C \int_{-3/\varepsilon}^{-\max\{A,1\}} \left[1 + \varepsilon|\hat{\xi}| + |\hat{\xi}|^{-1/2} + (\varepsilon|\hat{\xi}|)^{1/2} + |\hat{\xi}|^{-1}\right] e^{2q(A-|\hat{\xi}|)} d\hat{\xi} \leq C. \tag{4.29}$$

Similarly, from (4.28d) combined with $\hat{r}\varepsilon \leq \sqrt{17}|\hat{\xi}|\,\varepsilon \leq 3\sqrt{17}$, one gets

$$\iint_{\hat{\Omega}_2'} \hat{r}\left[|\partial_q(\lambda\,g)| + \varepsilon|\partial_q(\lambda\,\partial_\xi g)|\right]\left(\varepsilon^2 d\hat{\xi}\,d\hat{\eta}\right) \leq C\int_{-3/\varepsilon}^{-\max\{A,1\}}\left[(|\hat{\xi}| - A) + 1\right]e^{2q(A-|\hat{\xi}|)}\,d\hat{\xi} \leq C.$$
(4.30)

Furthermore, by (4.13b), (4.28a), for an arbitrary ball $\hat{B}_{\hat{\rho}}$ of radius $\hat{\rho}$ in the coordinates $(\hat{\xi}, \hat{\eta})$, we get

$$\iint_{\hat{\Omega}_2' \cap \hat{B}_{\hat{\rho}}} \lambda\left[|\partial_\xi g| + |\partial_\eta g| + |g|\right]\left(\varepsilon^2 d\hat{\xi}\,d\hat{\eta}\right) \leq C\int_{-\max\{A,1\}-\hat{\rho}}^{-\max\{A,1\}}\left[1 + |\hat{\xi}|^{-1/2}\right]e^{2q(A-|\hat{\xi}|)}\,d\hat{\xi} \leq C\hat{\rho}.$$
(4.31)

To complete the proof of (4.19), we now recall that $\hat{\Omega} \subset \hat{\Omega}_1' \cup \hat{\Omega}_2'$ and combine estimates (4.23), (4.24), (4.26) (that involve integration over $\hat{\Omega}_1'$) with (4.29), (4.30), which yields the desired bounds (4.19a)–(4.19d) and the bound for $\partial_\eta^2 g$ in (4.19f). To get the latter bound we also used the observation that the ball $B(x,y;\rho)$ in the coordinates (ξ, η) becomes the ball $B(0,0;\hat{\rho})$ of radius $\hat{\rho} = \varepsilon^{-1}\rho$ in the coordinates $(\hat{\xi}, \hat{\eta})$. The bound for $\partial_\xi^2 g$ in (4.19f) follows as $\partial_\xi^2 g = -\partial_\eta^2 g + \frac{2q}{\varepsilon}\partial_\xi g$ for $(\xi, \eta) \neq (x, y)$. The remaining assertion (4.19e) is obtained by combining (4.25) with (4.31) and noting that an arbitrary ball $B(x', y'; \rho)$ of radius ρ in the coordinates (ξ, η) becomes a ball $\hat{B}_{\hat{\rho}}$ of radius $\hat{\rho} = \varepsilon^{-1}\rho$ in the coordinates $(\hat{\xi}, \hat{\eta})$. Thus we have established all the bounds (4.19).

We now proceed to the proof of the bounds (4.20). Note that $\partial_x g = -\partial_\xi g$ and $\partial_y g = -\partial_\eta g$. Combining these with (4.19b) and the bound for $\|\lambda\,\partial_\eta g\|_{1;\Omega}$ in (4.19c), yields $\|\lambda\,\partial_x g\|_{1;\Omega} + \|\lambda\,D_y g\|_{1;\Omega} + \|\lambda\,D_\eta g\|_{1;\Omega} \leq C$. Now, combining $\partial_x\lambda = 2q\varepsilon^{-1}\lambda$ and $\partial_q\lambda = 2A\lambda \leq 4\varepsilon^{-1}\lambda$ with (4.19a), yields $\|g\,\partial_x\lambda\|_{1;\Omega} + \|g\,D_y\lambda\|_{1;\Omega} + \|g\,D_\eta\lambda\|_{1;\Omega} \leq C$. Consequently, we get (4.20a).

To estimate $\varepsilon\hat{r}_{[x]}\,D_y(\lambda\,\partial_\xi g)$, note that it involves $\varepsilon\hat{r}_{[x]}\,\partial_y(\lambda\,\partial_\xi g) = -\varepsilon\hat{r}_{[x]}\,\lambda\,\partial_{\xi\eta}^2 g$ (as λ is independent of y and $\partial_y g = -\partial_\eta g$), for which we have a bound in (4.19c), and also $\varepsilon\hat{r}_{[x]}\,\partial_q(\lambda\,\partial_\xi g)$, for which we have a bound in (4.19d). The desired bound for $\varepsilon\hat{r}_{[x]}\,D_y(\lambda\,\partial_\xi g)$ in (4.20b) follows.

For the second bound in (4.20b), a calculation yields $\varepsilon\hat{r}_{[x]}\,D_\eta\partial_x(\lambda g) = \varepsilon\hat{r}_{[x]}\,D_\eta(\lambda\,\partial_x g) + 2\hat{r}_{[x]}\,D_\eta(q\lambda g)$. The first term is estimated similarly to $\varepsilon\hat{r}_{[x]}\,D_y(\lambda\,\partial_\xi g)$ in (4.20b). The remaining term $\hat{r}_{[x]}\,D_\eta(q\lambda g)$ involves $\hat{r}_{[x]}\,\partial_\eta(q\lambda g) = q\,\hat{r}_{[x]}\,\lambda\,\partial_\eta g$, for which we have a bound in (4.19c), and also $\hat{r}_{[x]}\,\partial_q(q\lambda g) = q\,\hat{r}_{[x]}\,\partial_q(\lambda g) + \hat{r}_{[x]}\,\lambda g$, for which we have bounds in (4.19d) and (4.19a). Consequently we get the second bound in (4.20b). $\qquad\square$

Lemma 4.4. *Under the conditions of Lemma 4.3, for some positive constant c_1 one has*

$$\|\lambda g(x,y;\cdot)\|_{2,1;[0,\frac{1}{3}]\times\mathbb{R}} + \|D_y(\lambda g)(x,y;\cdot)\|_{1,1;[0,\frac{1}{3}]\times\mathbb{R}} \leq Ce^{-c_1\alpha/\varepsilon}.$$
(4.32)

Proof. We imitate the proof of Lemma 4.3, only now $\xi < \frac{1}{3}$ or $\hat{\xi} < (\frac{1}{3} - x)/\varepsilon \leq -\frac{2}{3}/\varepsilon$. Thus instead of the subdomains $\hat{\Omega}_1'$ and $\hat{\Omega}_2'$ we now consider $\hat{\Omega}_1''$ and $\hat{\Omega}_2''$ defined by $\hat{\Omega}_k'' := \hat{\Omega}_k' \cap \{\hat{\xi} < -(x - \frac{1}{3})/\varepsilon\}$. Thus in $\hat{\Omega}_1''$ (4.22) remains valid with $q \geq \frac{1}{2}\alpha$, but now $\hat{r} > \frac{2}{3}/\varepsilon$. Therefore, when we integrate over $\hat{\Omega}_1''$ (instead of $\hat{\Omega}_1'$), the integrals of type (4.23), (4.24) become bounded by $Ce^{-c_1\alpha/\varepsilon}$ for any fixed $c_1 < \frac{1}{16}$. Next, when considering integrals over

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$\hat{\Omega}_2''$ (instead of $\hat{\Omega}_2'$), note that $A - |\hat{\xi}| \leq -\frac{2}{3}/\varepsilon$ so the quantity $e^{2q(A-|\hat{\xi}|)}$ in the definition (4.27) of Q is now bounded by $e^{-\frac{2}{3}\alpha/\varepsilon}$. Consequently, the integrals of type (4.29) over $\hat{\Omega}_2''$ also become bounded by $Ce^{-c_1\alpha/\varepsilon}$. $\qquad\square$

Remark 4.5. *All the estimates of Lemmas 4.1, 4.3 and 4.4 remain valid if one sets* $q := \frac{1}{2}a(x,y)$ *or* $q := \frac{1}{2}a(\xi,\eta)$ *in* g, λ, *and their derivatives (after the differentiation is performed).*

5 Approximations $\bar{G}$ and $\tilde{G}$ for Green's function G

We shall use two related cut-off functions ω_0 and ω_1 defined by

$$\omega_0(t) \in C^2(0,1), \quad \omega_0(t) = 1 \ \text{for} \ t \leq \tfrac{2}{3}, \quad \omega_0(t) = 0 \ \text{for} \ t \geq \tfrac{5}{6}; \quad \omega_1(t) := \omega_0(1-t), \quad (5.1)$$

so $\omega_k(k) = 1$, $\omega_k(1-k) = 0$ and $\frac{d^m}{dt^m}\omega_k(0) = \frac{d^m}{dt^m}\omega_k(1) = 0$ for $k = 0, 1$ and $m = 1, 2$.

Recall that solutions $\bar{g}$ and $\tilde{g}$ of the frozen-coefficient equations (3.1) and (3.2) in the domain $\mathbb{R}^2$ are explicitly given by (3.3), (3.4). Now consider these two equations in some domain $\Omega \subset \mathbb{R}^2$ subject to homogeneous Dirichlet boundary conditions on $\partial\Omega$. For such problems, one can employ $\bar{g}$ and $\tilde{g}$ to construct solution approximations using the method of images with an inclusion of the above cut-off functions. First we construct such solution approximations, denoted by $\bar{G}$ and $\tilde{G}$, for the domain $\Omega = (0,1) \times \mathbb{R}$ (in Section 5.1), then for our domain of interest $\Omega = (0,1)^2$ (in Section 5.2).

Note that although $\bar{G}$ and $\tilde{G}$ are constructed as solution approximations for the frozen-coefficient equations, we shall see in Section 6 that they, in fact, provide approximations to the Green's function G for our original variable-coefficient problem.

5.1 Approximations $\bar{G}$ and $\tilde{G}$ for the domain $\Omega = (0,1) \times \mathbb{R}$

As outlined earlier in this Section 5, for the domain $\Omega = (0,1) \times \mathbb{R}$, we define $\bar{G}$ and $\tilde{G}$ by

$$\bar{G}(x,y;\xi,\eta) := \bar{\mathcal{G}}\big|_{q=\frac{1}{2}a(x,y)}, \qquad \tilde{G}(x,y;\xi,\eta) := \tilde{\mathcal{G}}\big|_{q=\frac{1}{2}a(\xi,\eta)}, \qquad (5.2)$$

$$\bar{\mathcal{G}}(x,y;\xi,\eta;q) := \tfrac{1}{2\pi\varepsilon}e^{q\hat{\xi}_{[x]}}\Big\{\big[K_0(q\hat{r}_{[x]}) - K_0(q\hat{r}_{[-x]})\big] - \big[K_0(q\hat{r}_{[2-x]}) - K_0(q\hat{r}_{[2+x]})\big]\omega_1(\xi)\Big\}, \quad (5.3\text{a})$$

$$\tilde{\mathcal{G}}(x,y;\xi,\eta;q) := \tfrac{1}{2\pi\varepsilon}e^{q\hat{\xi}_{[x]}}\Big\{\big[K_0(q\hat{r}_{[x]}) - K_0(q\hat{r}_{[2-x]})\big] - \big[K_0(q\hat{r}_{[-x]}) - K_0(q\hat{r}_{[2+x]})\big]\omega_0(x)\Big\}. \quad (5.3\text{b})$$

Note that $\bar{G}\big|_{\xi=0,1} = 0$ and $\tilde{G}\big|_{x=0,1} = 0$ (the former observation follows from $r_{[x]} = r_{[-x]}$ at $\xi = 0$, and $r_{[x]} = r_{[2-x]}$ and $r_{[-x]} = r_{[2+x]}$ at $\xi = 1$). We shall see shortly (see Lemma 5.1) that $\bar{L}_{\xi\eta}^*\bar{G} \approx L_{\xi\eta}^*G$ and $\tilde{L}_{xy}\tilde{G} \approx L_{xy}G$; in this sense $\bar{G}$ and $\tilde{G}$ give approximations for G.

Rewrite the definitions of $\bar{\mathcal{G}}$ and $\tilde{\mathcal{G}}$ using the notation

$$g_{[x]} := g(x,y;\xi,\eta;q) = \tfrac{1}{2\pi\varepsilon}\, e^{q\hat{\xi}_{[x]}}\, K_0(q\hat{r}_{[x]}), \qquad (5.4\text{a})$$

$$\lambda^{\pm} := e^{2q(1\pm x)/\varepsilon}, \qquad\qquad p := e^{-2qx/\varepsilon}, \qquad (5.4\text{b})$$

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and the observation that

$$\frac{1}{2\pi\varepsilon} e^{q\hat{\xi}_{[x]}} K_0(q\hat{r}_{[d]}) = e^{q(d-x)/\varepsilon} g_{[d]} \qquad \text{for } d = \pm x, 2 \pm x.$$

They yield

$$\bar{\mathcal{G}}(x,y;\xi,\eta;q) = \left[g_{[x]} - p\, g_{[-x]} \right] - \left[\lambda^- g_{[2-x]} - p\, \lambda^+ g_{[2+x]} \right] \omega_1(\xi), \tag{5.5a}$$

$$\tilde{\mathcal{G}}(x,y;\xi,\eta;q) = \left[g_{[x]} - \lambda^- g_{[2-x]} \right] - \left[p\, g_{[-x]} - p\, \lambda^+ g_{[2+x]} \right] \omega_0(x). \tag{5.5b}$$

Note that $\lambda^\pm$ is obtained by replacing x by $2 \pm x$ in the definition (4.18) of λ.

In the next lemma, we estimate the functions

$$\bar{\phi}(x,y;\xi,\eta) = \bar{L}^*_{\xi\eta}\bar{G} - L^*_{\xi\eta}G, \qquad \tilde{\phi}(x,y;\xi,\eta) := \tilde{L}_{xy}\tilde{G} - L_{xy}G. \tag{5.6}$$

Lemma 5.1. *Let $(x,y) \in \Omega = (0,1) \times \mathbb{R}$. Then for the functions $\bar{\phi}$ and $\tilde{\phi}$ of (5.6), one has*

$$\|\bar{\phi}(x,y;\cdot)\|_{1,1;\Omega} + \|\partial_y\bar{\phi}(x,y;\cdot)\|_{1;\Omega} + \|\tilde{\phi}(x,y;\cdot)\|_{1,1;\Omega} \le Ce^{-c_1\alpha/\varepsilon} \le C. \tag{5.7}$$

Furthermore, for $\bar{\phi}$ we also have

$$\bar{\phi}(x,y;\xi,\eta)|_{(\xi,\eta)\in\partial\Omega} = 0. \tag{5.8}$$

Proof. (i) First we prove the desired assertions for $\bar{\phi}$. By (5.2), throughout this part of the proof we set $q = \frac{1}{2}a(x,y) \ge \frac{1}{2}\alpha$. Recall that $\bar{g}$ solves the differential equation (3.1) with the operator $\bar{L}^*_{\xi\eta}$. Comparing the explicit formula for $\bar{g}$ in (3.3) with the notation (5.4a) implies that $\bar{L}^*_{\xi\eta}g_{[d]} = \delta(\xi - d)\delta(\eta - y)$. So, by (2.1), $\bar{L}^*_{\xi\eta}g_{[x]} = L^*_{\xi\eta}G$, and also $\bar{L}^*_{\xi\eta}g_{[d]} = 0$ for $d = -x, 2 \pm x$ and all $(\xi,\eta) \in \Omega$ as $(d,y) \notin \Omega$. Now, by (5.5a), we conclude that $\bar{\phi} = -\bar{L}^*_{\xi\eta}[\omega_1(\xi)\bar{\mathcal{G}}_2]$ where $\bar{\mathcal{G}}_2 := \lambda^- g_{[2-x]} - p\, \lambda^+ g_{[x+2]}$, and $\bar{L}^*_{\xi\eta}\bar{\mathcal{G}}_2 = 0$ for $(\xi,\eta) \in \Omega$.

From these observations, $\bar{\phi} = 2\varepsilon\omega_1'(\xi)\partial_\xi\bar{\mathcal{G}}_2 + [\varepsilon\omega_1''(\xi) - 2q\omega_1'(\xi)]\bar{\mathcal{G}}_2$. The definition (5.1) of ω_1 implies that $\bar{\phi}$ vanishes at $\xi = 0$ and for $\xi \ge \frac{1}{3}$. This implies the desired assertion (5.8). Furthermore, we now get

$$\|\bar{\phi}(x,y;\cdot)\|_{1,1;\Omega} + \|\partial_y\bar{\phi}(x,y;\cdot)\|_{1;\Omega} \le C\big(\|\bar{\mathcal{G}}_2(x,y;\cdot)\|_{2,1;[0,\frac{1}{3}]\times\mathbb{R}} + \|D_y\bar{\mathcal{G}}_2(x,y;\cdot)\|_{1,1;[0,\frac{1}{3}]\times\mathbb{R}}\big).$$

Combining this with the bounds (4.32) for the terms $\lambda^\pm g_{[2\pm x]}$ of $\bar{\mathcal{G}}_2$, and the observation that $|D_y p| \le C|\partial_q p| \le C$ and $\partial_\xi p = \partial_\eta p = 0$, yields our assertions for $\bar{\phi}$ in (5.7).

(ii) Now we prove the desired estimate (5.7) for $\tilde{\phi}$. By (5.2), throughout this part of the proof we set $q = \frac{1}{2}a(\xi,\eta) \ge \frac{1}{2}\alpha$. Comparing the notation (5.4a) with the explicit formula for $\tilde{g}$ in (3.3), we rewrite (3.2) as $\tilde{L}_{xy}g_{[x]} = \delta(x - \xi)\delta(y - \eta)$. So $\tilde{L}_{xy}g_{[x]} = L_{xy}G$, by (2.3). Next, for each value $d = -x, 2 \pm x$ respectively set $s = -\xi, \mp(2 - \xi)$. Now by (3.4), one has $\hat{r}_{[d]} = \sqrt{(s-x)^2 + (\eta-y)^2}/\varepsilon$ so $g(x,y;s,\eta;q) = \frac{1}{2\pi\varepsilon}e^{q(s-x)/\varepsilon}K_0(q\hat{r}_{[d]})$. Note that $\tilde{L}_{xy}g(x,y;s,\eta;q) = \delta(x-s)\delta(y-\eta)$ and none of our three values of s is in $[0,1]$ (i.e. $\delta(s - x) = 0$). Consequently, $\tilde{L}_{xy}[e^{q\hat{\xi}_{[x]}}K_0(q\hat{r}_{[d]})] = 0$ for all $(x,y) \in \Omega$. Comparing (5.3b) and (5.5b), we now conclude that $\tilde{\phi} = -\tilde{L}_{xy}[\omega_0(\xi)\tilde{\mathcal{G}}_2]$ where $\tilde{\mathcal{G}}_2 := p\, g_{[-x]} - p\, \lambda^+ g_{[x+2]}$ and $\tilde{L}_{xy}\tilde{\mathcal{G}}_2 = 0$ for $(x,y) \in \Omega$.

From these observations, $\tilde{\phi} = 2\varepsilon\omega_0'(x)\partial_x\tilde{\mathcal{G}}_2 + [\varepsilon\omega_0''(x) + 2q\omega_0'(x)]\tilde{\mathcal{G}}_2$. As the definition (5.1) of ω_0 implies that $\tilde{\phi}$ vanishes for $x \leq \frac{2}{3}$, we have

$$\|\tilde{\phi}(x, y; \cdot)\|_{1,1;\Omega} \leq C \max_{\substack{(x,y)\in[\frac{2}{3},1]\times\mathbb{R} \\ k = 0,1}} \|\partial_x^k\tilde{\mathcal{G}}_2(x, y; \cdot)\|_{1,1;\Omega}.$$

Here $\tilde{\mathcal{G}}_2$ is smooth and has no singularities for $x \in [\frac{2}{3}, 1]$ (because $\hat{r}_{[2+x]} \geq \hat{r}_{[-x]} \geq \frac{2}{3}\varepsilon^{-1}$ for $x \in [\frac{2}{3}, 1]$). Note that $\|\partial_x^k g_{[-x]}\|_{1,1;\Omega} \leq C\varepsilon^{-2}$, and $\|\partial_x^k(\lambda^+ g_{[2+x]})\|_{1,1;\Omega} \leq C\varepsilon^{-2}$ (these two estimates are similar to the ones in Lemmas 4.1 and 4.3, but easier to deduce as they are not sharp). We combine these two bounds with $|\partial_x^k\partial_\xi^m\partial_\eta^n p| \leq C\varepsilon^{-2}p = C\varepsilon^{-2}e^{-2qx/\varepsilon}$ for $k, m + n \leq 1$. As for $x \geq \frac{2}{3}$ we enjoy the bound $e^{-2qx/\varepsilon} \leq e^{-\frac{2}{3}\alpha/\varepsilon} \leq C\varepsilon^4 e^{-\frac{1}{2}\alpha/\varepsilon}$, the desired estimate for $\tilde{\phi}$ follows. □

Lemma 5.2. *Let the function $R = R(x, y; \xi, \eta)$ be such that $|R| \leq C \min\{\varepsilon\hat{r}_{[x]}, 1\}$. The functions $\bar{G}$ and $\tilde{G}$ of (5.2), (5.5) satisfy*

$$\|\bar{G}(x, y; \cdot)\|_{1;\Omega} + \|\tilde{G}(x, y; \cdot)\|_{1;\Omega} \leq C, \tag{5.9a}$$

$$\|\partial_\xi\bar{G}(x, y; \cdot)\|_{1;\Omega} \leq C(1 + |\ln\varepsilon|), \tag{5.9b}$$

$$\|\partial_\eta\bar{G}(x, y; \cdot)\|_{1;\Omega} \leq C\varepsilon^{-1/2}, \tag{5.9c}$$

$$\|(R\,\partial_\xi\bar{G})(x, y; \cdot)\|_{1;\Omega} + \varepsilon^{1/2}\|(R\,\partial_{\xi\eta}^2\bar{G})(x, y; \cdot)\|_{1;\Omega} \leq C, \tag{5.9d}$$

and for any ball $B(x', y'; \rho)$ of radius ρ centred at any $(x', y') \in [0, 1] \times \mathbb{R}$, one has

$$|\bar{G}(x, y; \cdot)|_{1,1;B(x',y';\rho)\cap\Omega} \leq C\varepsilon^{-1}\rho, \tag{5.9e}$$

while for the ball $B(x, y; \rho)$ of radius ρ centred at (x, y), we have

$$\|\partial_\xi^2\bar{G}(x, y; \cdot)\|_{1;\Omega\backslash B(x,y;\rho)} \leq C\varepsilon^{-1}\ln(2 + \varepsilon/\rho), \tag{5.9f}$$

$$\|\partial_\eta^2\bar{G}(x, y; \cdot)\|_{1;\Omega\backslash B(x,y;\rho)} \leq C\varepsilon^{-1}(\ln(2 + \varepsilon/\rho) + |\ln\varepsilon|). \tag{5.9g}$$

Furthermore, we have

$$\|\partial_y\bar{G}(x, y; \cdot)\|_{1;\Omega} + \|(R\,\partial_{\xi y}^2\bar{G})(x, y; \cdot)\|_{1;\Omega} \leq C\varepsilon^{-1/2}, \tag{5.9h}$$

$$\|\partial_\eta\tilde{G}(x, y; \cdot)\|_{1;\Omega} \leq C\varepsilon^{-1/2}, \tag{5.9i}$$

$$\int_0^1 \left(\|(R\,\partial_{x\eta}^2\tilde{G})(x, y; \cdot)\|_{1;\Omega} + \|\partial_x\tilde{G}(x, y; \cdot)\|_{1;\Omega}\right) dx \leq C\varepsilon^{-1/2}. \tag{5.9j}$$

Proof. First, note that $\hat{r}_{[-x]} \geq \hat{r}_{[x]}$ and $\hat{r}_{[2\pm x]} \geq \hat{r}_{[x]}$ for all $(\xi, \eta) \in \Omega$, therefore

$$|R| \leq C \min\{\varepsilon\hat{r}_{[x]}, \varepsilon\hat{r}_{[-x]}, \varepsilon\hat{r}_{[2-x]}, \varepsilon\hat{r}_{[2+x]}, 1\}. \tag{5.10}$$

Note also that in view of Remark 4.5, all bounds of Lemma 4.1 apply to the components $g_{[\pm x]}$ and all bounds of Lemma 4.3 apply to the components $\lambda^{\pm}g_{[2\pm x]}$ of $\bar{\mathcal{G}}$ and $\tilde{\mathcal{G}}$ in (5.5).

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Asterisk notation. In some parts of this proof, when discussing derivatives of $\bar{\mathcal{G}}$, we shall use the notation $\bar{\mathcal{G}}^*$ prefixed by some differential operator, e.g., $\partial_x \bar{\mathcal{G}}^*$. This will mean that the differential operator is applied only to the terms of the type $g_{[d\pm x]}$, e.g., $\partial_x \bar{\mathcal{G}}^*$ is obtained by replacing each of the four terms $g_{[d\pm x]}$ in the definition (5.5a) of $\bar{\mathcal{G}}$ by $\partial_x g_{[d\pm x]}$ respectively.

(a) The first desired estimate (5.9a) follows from the bound (4.2a) for $g_{[\pm x]}$ and the bound (4.19a) for $\lambda^\pm g_{[2\pm x]}$ combined with $|p| \leq 1$ and $|\omega_{0,1}| \leq 1$ (in fact, the bound for $\bar{G}$ can obtained by imitating the proof of Lemma 2.1).

(b)(c)(d) Rewrite (5.5a) as

$$\bar{\mathcal{G}} = \bar{\mathcal{G}}_1 - \omega_1(\xi)\bar{\mathcal{G}}_2, \quad \text{where} \quad \bar{\mathcal{G}}_1 := g_{[x]} - p\, g_{[-x]}, \quad \bar{\mathcal{G}}_2 := \lambda^- g_{[2-x]} - p\, \lambda^+ g_{[2+x]}.$$

As $q = \frac{1}{2}a(x,y)$ in $\bar{G}$ (i.e. p and $\lambda^\pm$ in $\bar{G}$ do not involve ξ, η), one gets

$$\partial_\xi \bar{G} = \partial_\xi \bar{\mathcal{G}}^* - \omega_1'(\xi)\bar{\mathcal{G}}_2, \qquad \partial_\eta \bar{G} = \partial_\eta \bar{\mathcal{G}}^*, \qquad \partial_{\xi\eta}^2 \bar{G} = \partial_{\xi\eta}^2 \bar{\mathcal{G}}^* - \omega_1'(\xi)\partial_\eta \bar{\mathcal{G}}_2^*.$$

Now the desired estimate (5.9b) follows from the bound (4.2b) for $\partial_\xi g_{[\pm x]}$, the bound (4.19b) for $\lambda^\pm \partial_\xi g_{[2\pm x]}$, and the bound (4.19a) for $\lambda^\pm g_{[2\pm x]}$. Similarly, (5.9c) follows from the bound (4.2c) for $\partial_\eta g_{[\pm x]}$, and the bound (4.19c) for $\lambda^\pm \partial_\eta g_{[2\pm x]}$.

The next desired estimate (5.9d) is deduced using

$$|R\,\partial_\xi \bar{G}| \leq |R\,\partial_\xi \bar{\mathcal{G}}_1^*| + C|\partial_\xi \bar{\mathcal{G}}_2^*| + C|\bar{\mathcal{G}}_2|, \qquad |R\,\partial_{\xi\eta}^2 \bar{G}| \leq |R\,\partial_{\xi\eta}^2 \bar{\mathcal{G}}^*| + C|\partial_\eta \bar{\mathcal{G}}_2^*|.$$

Here, in view of (5.10), the term $R\,\partial_\xi \bar{\mathcal{G}}_1^*$ is estimated using the bound (4.2d) for $\varepsilon\hat{r}_{[\pm x]}\partial_\xi g_{[\pm x]}$, while the term $R\,\partial_{\xi\eta}^2 \bar{\mathcal{G}}^*$ is estimated using the bound (4.2e) for $\varepsilon\hat{r}_{[\pm x]}\partial_{\xi\eta}^2 g_{[\pm x]}$ and the bound (4.19c) for $\lambda^\pm \varepsilon\hat{r}_{[2\pm x]}\partial_{\xi\eta}^2 g_{[2\pm x]}$. The remaining terms $\partial_\xi \bar{\mathcal{G}}_2^*$, $\bar{\mathcal{G}}_2$ and $\partial_\eta \bar{\mathcal{G}}_2^*$ appear in $\partial_\xi \bar{G}$ and $\partial_\eta \bar{G}$, so have been bounded when obtaining (5.9b), (5.9c).

(e) The next assertion (5.9e) is proved similarly to (5.9b) and (5.9c), only using the bound (4.2f) for $g_{[\pm x]}$ and the bound (4.19e) for $\lambda^\pm g_{[2\pm x]}$.

(f)(g) As $q = \frac{1}{2}a(x,y)$ in $\bar{G}$, then $\partial_\xi^2 \bar{G} = \partial_\xi^2 \bar{\mathcal{G}}^*$ and $\partial_\eta^2 \bar{G} = \partial_\eta^2 \bar{\mathcal{G}}^*$, and the assertions (5.9f) and (5.9g) immediately follow from the bounds (4.2g) and (4.2h) for $\partial_\xi^2 g_{[\pm x]}$ and $\partial_\eta^2 g_{[\pm x]}$, respectively, combined with the bound (4.19f) for $\lambda^\pm \partial_\xi^2 g_{[2\pm x]}$ and $\lambda^\pm \partial_\eta^2 g_{[2\pm x]}$.

(h) We again have $q = \frac{1}{2}a(x,y)$ in $\bar{G}$, so using the operator D_y of (4.1), one gets

$$\partial_y \bar{G} = D_y\big[g_{[x]} - p\, g_{[-x]}\big]^* - \omega_1(\xi)\big[D_y(\lambda^- g_{[2-x]}) - p\, D_y(\lambda^+ g_{[2+x]})\big]$$
$$- \tfrac{1}{2}\partial_y a(x,y) \cdot \partial_q p \cdot \big[g_{[-x]} - \omega_1(\xi)\lambda^+ g_{[2+x]}\big],$$

where $|\partial_q p| \leq C$ by (5.4b) (and we used the previously defined notation $*$). Now, $\partial_y \bar{G}$ is estimated using the bound (4.3b) for $D_y g_{[\pm x]}$ and the bound (4.20a) for $D_y(\lambda^\pm g_{[2\pm x]})$. For the term $g_{[-x]}$ in $\partial_y \bar{G}$ we use the bound (4.2a), and for the term $\lambda^+ g_{[2+x]}$ the bound (4.19a). Consequently, one gets the desired bound (5.9h) for $D_y \bar{G}^*$.

To estimate $R\,\partial_{\xi y}^2 \bar{G}$, a calculation shows that

$$\partial_{\xi y}^2 \bar{G} = (D_y \partial_\xi)\big[g_{[x]} - p\, g_{[-x]}\big]^* - \omega_1(\xi)\big[D_y(\lambda^- \partial_\xi g_{[2-x]}) - p\, D_y(\lambda^+ \partial_\xi g_{[2+x]})\big]$$
$$- \tfrac{1}{2}\partial_y a(x,y) \cdot \partial_q p \cdot \big[\partial_\xi g_{[-x]} - \omega_1(\xi)\lambda^+ \partial_\xi g_{[2+x]}\big] - \omega_1'(\xi)\partial_y \bar{\mathcal{G}}_2,$$

where $\bar{G}_2 := \bar{\mathcal{G}}_2\big|_{q=a(x,y)/2}$. The assertion (5.9h) for $R\,\partial_{\xi y}^2 \bar{G}$ is now deduced as follows. In view of (5.10), we employ the bound (4.3c) for the terms $\varepsilon\hat{r}_{[\pm x]} D_y \partial_\xi g_{[\pm x]}$ and the bound (4.20b) for the terms $\varepsilon\hat{r}_{[2\pm x]} D_y(\lambda^\pm \partial_\xi g_{[2\pm x]})$. For the remaining terms (that appear in the second line) we use $|R| \leq C$ and $|\partial_q p| \leq C$. Then we combine the bound (4.2b) for $\partial_\xi g_{[-x]}$ and the bound (4.19b) for $\lambda^+ \partial_\xi g_{[2+x]}$. The term $\partial_y \bar{G}_2$ is a part of $\partial_y \bar{G}$, which was estimated above, so for $\partial_y \bar{G}_2$ we have the same bound as for $\partial_y \bar{G}$ in (5.9h). This observation completes the proof of the bound for $R\,\partial_{\xi y}^2 \bar{G}$ in (5.9h).

(i)(j) We now proceed to estimating derivatives of $\tilde{G}$, so $q = \frac{1}{2}a(\xi,\eta)$ in this part of the proof. Let $\tilde{\mathcal{G}}^\pm := g_{[\pm x]} - \lambda^\mp g_{[2\mp x]}$. Then (5.5b), (5.4b) imply that $\tilde{G} = \tilde{\mathcal{G}}^+ - p_0\tilde{\mathcal{G}}^-$, where $p_0 := \omega_0(x)p = \omega_0(x)e^{-2qx/\varepsilon}$. Note that

$$D_\eta p_0 = \tfrac{1}{2}\partial_\eta a(\xi,\eta)\cdot(-2x/\varepsilon)\,p_0, \qquad \partial_x p_0 = [\omega_0'(x) - (2q/\varepsilon)\omega_0(x)]e^{-2qx/\varepsilon}.$$

Combining this with $|(-2x/\varepsilon)\,p_0| \leq Ce^{-qx/\varepsilon}$ and $q \geq \frac{1}{2}\alpha$ yields

$$|D_\eta p_0| \leq C, \qquad \int_0^1 \big(|\partial_x p_0| + |D_\eta \partial_x p_0|\big)dx \leq \int_0^1 \big(C\varepsilon^{-1} e^{-\frac{1}{2}\alpha x/\varepsilon}\big)dx \leq C. \qquad (5.11)$$

Furthermore, we claim that

$$\|\tilde{\mathcal{G}}^-\|_{1;\Omega} \leq C, \qquad \|\partial_x \tilde{\mathcal{G}}^\pm\|_{1;\Omega} \leq C(1 + |\ln\varepsilon|), \qquad \|D_\eta \tilde{\mathcal{G}}^\pm\|_{1;\Omega} \leq C\varepsilon^{-1/2}. \qquad (5.12)$$

Here the first estimate follows from the bounds (4.2a), (4.19a) for the terms $g_{[-x]}$ and $\lambda^+ g_{[2+x]}$. The estimate for $\partial_x \tilde{\mathcal{G}}^\pm$ in (5.12) follows from the bound (4.3a) for $\partial_x g_{[\pm x]}$ and the bound (4.20a) for $\partial_x(\lambda^\pm g_{[2\pm x]})$. Similarly, the estimate for $D_\eta \tilde{\mathcal{G}}^\pm$ in (5.12) is obtained using the bound (4.3b) for $D_\eta g_{[\pm x]}$ and the bound (4.20a) for $D_\eta(\lambda^\pm g_{[2\pm x]})$.

Next, a calculation shows that

$$\partial_\eta \tilde{G} = D_\eta \tilde{\mathcal{G}}^+ - p_0\, D_\eta \tilde{\mathcal{G}}^- - D_\eta p_0 \cdot \tilde{\mathcal{G}}^-, \qquad \partial_x \tilde{G} = \partial_x \tilde{\mathcal{G}}^+ - p_0\, \partial_x \tilde{\mathcal{G}}^- - \partial_x p_0 \cdot \tilde{\mathcal{G}}^-.$$

Combining these with (5.11), (5.12) yields (5.9i) and the bound for $\partial_x \tilde{G}$ in (5.9j).

To establish the estimate for $R\,\partial_{x\eta}^2 \tilde{G}$ in (5.9j), note that

$$\partial_{x\eta}^2 \tilde{G} = D_\eta \partial_x \tilde{\mathcal{G}}^+ - p_0 \cdot D_\eta \partial_x \tilde{\mathcal{G}}^- - \partial_x p_0 \cdot D_\eta \tilde{\mathcal{G}}^- - \partial_\eta p_0 \cdot \partial_x \tilde{\mathcal{G}}^- - D_\eta \partial_x p_0 \cdot \tilde{\mathcal{G}}^-.$$

In view of (5.10), (5.11) and (5.12), it now suffices to show that $\|R\, D_\eta \partial_x \tilde{\mathcal{G}}^\pm\|_{1;\Omega} \leq C\varepsilon^{-1/2}$. This latter estimate follows from the bound (4.3c) for the terms $\varepsilon\hat{r}_{[\pm x]} D_\eta \partial_x g_{[\pm x]}$ and the bound (4.20b) for the terms $\varepsilon\hat{r}_{[\pm x]} D_\eta \partial_x(\lambda^\pm g_{[2\pm x]})$. This completes the proof of (5.9j). □

5.2 Approximations $\bar{G}$ and $\tilde{G}$ for the domain $\Omega = (0,1)^2$

We now define approximations, denoted by $\bar{G}_\square$ and $\tilde{G}_\square$, for our original square domain $\Omega = (0,1)^2$. For this, we use the approximations $\bar{G}$ and $\tilde{G}$ of (5.2), (5.3) for the domain $(0,1)\times\mathbb{R}$ and again employ the method of images with an inclusion of the cut-off functions of (5.1) as follows:

$$\bar{G}_\square(x,y;\xi,\eta) := \bar{G}(x,y;\xi,\eta) - \omega_0(\eta)\,\bar{G}(x,y;\xi,-\eta) - \omega_1(\eta)\,\bar{G}(x,y;\xi,2-\eta), \qquad (5.13a)$$

$$\tilde{G}_\square(x,y;\xi,\eta) := \tilde{G}(x,y;\xi,\eta) - \omega_0(y)\,\tilde{G}(x,-y;\xi,\eta) - \omega_1(y)\,\tilde{G}(x,2-y;\xi,\eta). \qquad (5.13b)$$

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Then $\bar{G}_\square\big|_{\xi=0,1} = 0$ and $\tilde{G}_\square\big|_{x=0,1} = 0$ (as this is valid for $\bar{G}$ and $\tilde{G}$, respectively), and furthermore, by (5.1), we have $\bar{G}_\square\big|_{\eta=0,1} = 0$ and $\tilde{G}_\square\big|_{y=0,1} = 0$.

Remark 5.3. *Lemmas 5.1 and 5.2 of the previous section remain valid if Ω is understood as $(0,1)^2$, and $\bar{G}$ and $\tilde{G}$ are replaced by $\bar{G}_\square$ and $\tilde{G}_\square$, respectively, in the definition (5.6) of $\bar{\phi}$ and $\tilde{\phi}$ and in the lemma statements.*

This is shown by imitating the proofs of these two lemmas. We leave out the details and only note that the application of the method of images in the η- $(y$-$)$ direction is relatively straightforward as an inspection of (3.4) shows that in this direction, the fundamental solution g is symmetric and exponentially decaying away from the singular point.

As $\bar{G}_\square$ and $\tilde{G}_\square$ in the domain $\Omega = (0,1)^2$ enjoy the same properties as $\bar{G}$ and $\tilde{G}$ in the domain $(0,1) \times \mathbb{R}$, we shall sometimes skip the subscript $\square$ when there is no ambiguity.

6 Green's function for the original problem in $\Omega = (0,1)^2$. Proof of Theorem 2.2

We are now ready to establish our main result, Theorem 2.2, for the original variable-coefficient problem (1.1) in the domain $\Omega = (0,1)^2$. In Section 5, we have already obtained various bounds for the approximations $\tilde{G}_\square$ and $\bar{G}_\square$ of G in $\Omega = (0,1)^2$. So now we consider the two functions $\tilde{v}$ and $\bar{v}$ given by

$$\tilde{v}(x,y;\xi,\eta) := [G - \tilde{G}_\square](x,y;\xi,\eta), \qquad \bar{v}(x,y;\xi,\eta) = [G - \bar{G}_\square](x,y;\xi,\eta).$$

Throughout this section, we shall skip the subscript $\square$ as we always deal with the domain $\Omega = (0,1)^2$.

Note that, by (5.6), we have $L_{xy}\tilde{v} = L_{xy}[G - \tilde{G}] = [\tilde{L}_{xy} - L_{xy}]\tilde{G} - \tilde{\phi}$, and similarly $L^*_{\xi\eta}\bar{v} = L^*_{\xi\eta}[G - \bar{G}] = [\bar{L}^*_{\xi\eta} - L^*_{\xi\eta}]\bar{G} - \bar{\phi}$. Consequently, the functions $\tilde{v}$ and $\bar{v}$ are solutions of the following problems:

$$L_{xy}\tilde{v}(x,y;\xi,\eta) = \tilde{h}(x,y;\xi,\eta) \text{ for } (x,y) \in \Omega, \quad \tilde{v}(x,y;\xi,\eta) = 0 \text{ for } (x,y) \in \partial\Omega, \quad (6.1a)$$

$$L^*_{\xi\eta}\bar{v}(x,y;\xi,\eta) = \bar{h}(x,y;\xi,\eta) \text{ for } (\xi,\eta) \in \Omega, \quad \bar{v}(x,y;\xi,\eta) = 0 \text{ for } (\xi,\eta) \in \partial\Omega. \quad (6.1b)$$

Here the right-hand sides are given by

$$\tilde{h}(x,y;\xi,\eta) := \partial_x\{R\tilde{G}\}(x,y;\xi,\eta) - b(x,y)\tilde{G}(x,y;\xi,\eta) - \tilde{\phi}(x,y;\xi,\eta), \qquad (6.2a)$$

$$\bar{h}(x,y;\xi,\eta) := \{R\partial_\xi\bar{G}\}(x,y;\xi,\eta) - b(\xi,\eta)\bar{G}(x,y;\xi,\eta) - \bar{\phi}(x,y;\xi,\eta), \qquad (6.2b)$$

where

$$R(x,y;\xi,\eta) := a(x,y) - a(\xi,\eta), \qquad \text{so } |R| \le C \min\{\varepsilon\hat{r}_{[x]}, 1\}. \qquad (6.3)$$

Applying the solution representation formulas (2.2) and (2.4) to problems (6.1a) and (6.1b), respectively, one gets

$$\tilde{v}(x,y;\xi,\eta) = \iint_\Omega G(x,y;s,t)\,\tilde{h}(s,t;\xi,\eta)\,ds\,dt, \qquad (6.4a)$$

$$\bar{v}(x,y;\xi,\eta) = \iint_\Omega G(s,t;\xi,\eta)\,\bar{h}(x,y;s,t)\,ds\,dt. \qquad (6.4b)$$

We now proceed to the proof of Theorem 2.2.

Proof. (i) First we establish (2.6b). Note that, by the bounds (5.9i) and (5.9h) for $\partial_\eta \tilde{G}$ and $\partial_y \bar{G}$, respectively, it suffices to show that $\|\partial_\eta \tilde{v}(x,y;\cdot)\|_{1;\Omega} + \|\partial_y \bar{v}(x,y;\cdot)\|_{1;\Omega} \le C\varepsilon^{-1/2}$.

Applying ∂_η to (6.4a) and ∂_y to (6.4b), we arrive at

$$\partial_\eta \tilde{v}(x,y;\xi,\eta) = \iint_\Omega G(x,y;s,t)\,\partial_\eta \tilde{h}(s,t;\xi,\eta)\,ds\,dt,$$

$$\partial_y \bar{v}(x,y;\xi,\eta) = \iint_\Omega G(s,t;\xi,\eta)\,\partial_y \bar{h}(x,y;s,t)\,ds\,dt.$$

From this, a calculation shows that

$$\|\partial_\eta \tilde{v}(x,y;\cdot)\|_{1;\Omega} \le \left(\sup_{s\in[0,1]}\int_{\mathbb{R}} |G(x,y;s,t)|\,dt\right) \cdot \int_0^1 \left(\sup_{t\in\mathbb{R}} \|\partial_\eta \tilde{h}(s,t;\cdot)\|_{1;\Omega}\right) ds,$$

$$\|\partial_y \bar{v}(x,y;\cdot)\|_{1;\Omega} \le \left(\sup_{(s,t)\in\Omega} \|G(s,t;\cdot)\|_{1;\Omega}.\right) \cdot \|\partial_y \bar{h}(x,y;\cdot)\|_{1;\Omega}.$$

So, in view of (2.5), to prove (2.6b), it remains to show that

$$\int_0^1 \left(\sup_{y\in\mathbb{R}} \|\partial_\eta \tilde{h}(x,y;\cdot)\|_{1;\Omega}\right) dx \le C\varepsilon^{-1/2}, \qquad \|\partial_y \bar{h}(x,y;\cdot)\|_{1;\Omega} \le C\varepsilon^{-1/2}.$$

These two bounds follows from the definitions (6.2), (6.3) of $\tilde{h}$ and $\bar{h}$, which imply that

$$|\partial_\eta \tilde{h}(x,y;\xi,\eta)| \le |R\,\partial_{x\eta}^2 \tilde{G}| + C\big(|\partial_x \tilde{G}| + |\partial_\eta \tilde{G}|\big) + |\partial_\eta \tilde{\phi}|,$$

$$|\partial_y \bar{h}(x,y;\xi,\eta)| \le |R\,\partial_{\xi y}^2 \bar{G}| + C\big(|\partial_\xi \bar{G}| + |\partial_y \bar{G}|\big) + |\partial_y \bar{\phi}|,$$

combined with the bounds (5.7) for $\bar{\phi}$, $\tilde{\phi}$, the bounds (5.9i), (5.9j) for $\tilde{G}$ and the bounds (5.9b), (5.9h) for $\bar{G}$. Thus we have shown (2.6b).

(ii) Next we proceed to obtaining the assertions (2.6a), (2.6d) and (2.6e). We claim that to get these two bounds, it suffices to show that

$$\mathcal{V} := \sup_{(x,y)\in\Omega} \|\partial_\eta^2 \bar{v}(x,y;\cdot)\|_{1;\Omega} \le C(\varepsilon^{-1} + \varepsilon^{-1/2}\mathcal{W}), \tag{6.5a}$$

$$\mathcal{W} := \sup_{(x,y)\in\Omega} \|\partial_\xi G(x,y;\cdot)\|_{1;\Omega} \le C(1 + |\ln\varepsilon| + \varepsilon\mathcal{V}), \tag{6.5b}$$

$$\sup_{(x,y)\in\Omega} \|\partial_\xi^2 \bar{v}(x,y;\cdot)\|_{1;\Omega} \le C\varepsilon^{-1}(1 + \varepsilon\mathcal{V}). \tag{6.5c}$$

Indeed, there is a sufficiently small constant c_* such that for $\varepsilon \le c_*$, combining the bounds (6.5a), (6.5b), one gets $\mathcal{W} \le C(1 + |\ln\varepsilon|)$, which is identical with (2.6a). Then (6.5a) implies that $\mathcal{V} \le C\varepsilon^{-1}$, which, combined with (5.9g), yields (2.6e). Finally, $\mathcal{V} \le C\varepsilon^{-1}$ combined with (6.5c) and then (5.9f) yields (2.6d).

In the simpler non-singularly-perturbed case of $\varepsilon > c_*$, by imitating part (i) of this proof, one obtains $\mathcal{W} \le C_1$, where C_1 depends on c_*. Combining this bound with (6.5a) and (6.5c), we again get (2.6a), (2.6d) and (2.6e).

We shall obtain (6.5a) in part (iii), and (6.5b) with (6.5c) in part (iv) below.

(iii) To get (6.5a), let $\bar{V} := \partial_\eta^2 \bar{v}$. The problem (6.1b) for $\bar{v}$ implies that

$$L^*_{\xi\eta} \bar{V}(x,y;\xi,\eta) = \bar{H}(x,y;\xi,\eta) \text{ for } (\xi,\eta) \in \Omega, \quad \bar{V}(x,y;\xi,\eta) = 0 \text{ for } (\xi,\eta) \in \partial\Omega. \quad (6.6)$$

The homogeneous boundary conditions $\partial_\eta^2 \bar{v}\big|_{\xi=0,1} = 0$ in (6.6) immediately follow from $\bar{v}\big|_{\xi=0,1} = 0$. The homogeneous boundary conditions on the boundary edges $\eta = 0,1$ are obtained as follows. As $\bar{v}\big|_{\eta=0,1} = 0$ so $\partial_\xi \bar{v}\big|_{\eta=0,1} = \partial_\xi^2 \bar{v}\big|_{\eta=0,1} = 0$. Combining this with $\bar{h}\big|_{\eta=0,1} = 0$ (for which, in view of Remark 5.3, we used (5.8)) and the differential equation for $\bar{v}$ at $\eta = 0,1$, one finally gets $\partial_\eta^2 \bar{v}\big|_{\eta=0,1} = 0$.

For the right-hand side $\bar{H}$ in (6.6), a calculation shows that $\bar{H} = \bar{H}(x,y;\xi,\eta) = \partial_\eta \bar{h}_1 + \bar{h}_2$ with $\bar{h}_k(x,y;\xi,\eta)$, for $k = 1,2$, defined by

$$\bar{h}_1 := \partial_\eta \bar{h} - 2\partial_\eta a(\xi,\eta) \cdot \partial_\xi \bar{v}, \qquad \bar{h}_2 := \partial_\eta^2 a(\xi,\eta) \cdot \partial_\xi \bar{v} - 2\partial_\eta b(\xi,\eta) \cdot \partial_\eta \bar{v} - \partial_\eta^2 b(\xi,\eta) \cdot \bar{v},$$

Here we used $\partial_\eta^2[a\,\partial_\xi \bar{v}] = a\,\partial_\xi \bar{V} + 2\partial_\eta a\,\partial_{\xi\eta}^2 \bar{v} + \partial_\eta^2 a\,\partial_\xi \bar{v} = a\,\partial_\xi \bar{V} + \partial_\eta[2\,\partial_\eta a\,\partial_\xi \bar{v}] - \partial_\eta^2 a\,\partial_\xi \bar{v}$ and $\partial_\eta^2[b\bar{v}] = b\bar{V} + 2\,\partial_\eta b\,\partial_\eta \bar{v} + \partial_\eta^2 b\,\bar{v}$. (Note that $\bar{H}$ is understood in the sense of distributions; see Remark 6.1 below.)

Now, applying the solution representation formula (2.4) to problem (6.6), and then integrating the term with $\bar{h}_1$ by parts, yields

$$\bar{V}(x,y;\xi,\eta) = \iint_\Omega \left[-\partial_t G(s,t;\xi,\eta)\,\bar{h}_1(x,y;s,t) + G(s,t;\xi,\eta)\,\bar{h}_2(x,y;s,t) \right] ds\,dt$$

(for the validity of the above integration by parts we again refer to Remark 6.1). As (2.6b) implies $\sup_{(s,t)\in\Omega} \|\partial_t G(s,t;\cdot)\| \le C\varepsilon^{-1/2}$, while (2.5) implies $\sup_{(s,t)\in\Omega} \|G(s,t;\cdot)\| \le C$, imitating the argument used in part (i) of this proof yields

$$\|\partial_\eta^2 \bar{v}(x,y;\cdot)\|_{1;\Omega} = \|\bar{V}(x,y;\cdot)\|_{1;\Omega} \le C\left(\varepsilon^{-1/2}\|\bar{h}_1(x,y;\cdot)\|_{1;\Omega} + \|\bar{h}_2(x,y;\cdot)\|_{1;\Omega} \right).$$

So to get our assertion (6.5a), it remains to show that $\|\bar{h}_{1,2}(x,y;\cdot)\|_{1;\Omega} \le C(\varepsilon^{-1/2} + \mathcal{W})$. To check this latter bound, note that $|\bar{h}_1| + |\bar{h}_2| \le C(|\partial_\eta \bar{h}| + |\partial_\xi \bar{v}| + |\partial_\eta \bar{v}| + |\bar{v}|)$. Note also that

$$\|\bar{v}(x,y;\cdot)\|_{1,1;\Omega} \le C(\varepsilon^{-1/2} + \mathcal{W}) + \|\bar{G}(x,y;\cdot)\|_{1,1;\Omega},$$

where we employed $\bar{v} = G - \bar{G}$ and then the bounds (2.5), (2.6b) and the definition (6.5b) of $\mathcal{W}$ for G. Combining these two observations with

$$|\partial_\eta \bar{h}(x,y;\xi,\eta)| \le |R\,\partial_{\xi\eta}^2 \bar{G}| + C\left(|\partial_\xi \bar{G}| + |\partial_\eta \bar{G}| + |\bar{G}| \right) + |\partial_\eta \bar{\phi}|$$

(where we used (6.2b), (6.3)), and then with the bounds (5.9a)–(5.9d) for $\bar{G}$, and the bound (5.7) for $\bar{\phi}$, one gets the required estimate for $\|\bar{h}_{1,2}(x,y;\cdot)\|_{1;\Omega}$. Thus (6.5a) is established.

(iv) To prove (6.5b) and (6.5c), rewrite the problem (6.1b) as a two-point boundary-value problem, in which x, y and η appear as parameters, as follows

$$\left[-\varepsilon \partial_\xi^2 + a(\xi,\eta)\partial_\xi \right] \bar{v}(x,y;\xi,\eta) = \bar{\bar{h}}(x,y;\xi,\eta) \text{ for } \xi \in (0,1), \quad \bar{v}(x,y;\xi,\eta)\big|_{\xi=0,1} = 0, \quad (6.7)$$

where

$$\bar{\bar{h}}(x, y; \xi, \eta) := \bar{h}(x, y; \xi, \eta) + \varepsilon\, \partial_\eta^2 \bar{v}(x, y; \xi, \eta) - b(\xi, \eta)\, \bar{v}(x, y; \xi, \eta). \qquad (6.8)$$

Consequently, one can represent $\bar{v}$ via the Green's function $\Gamma = \Gamma(\xi, \eta; s)$ of the one-dimensional operator $[-\varepsilon\partial_\xi^2 + a(\xi, \eta)\partial_\xi]$. Note that Γ, for any fixed η and s, satisfies the equation $[-\varepsilon\partial_\xi^2 + a(\xi, \eta)\partial_\xi]\Gamma(\xi, \eta; s) = \delta(\xi - s)$ and the boundary conditions $\Gamma(\xi, \eta; s)\big|_{\xi=0,1} = 0$. Note also that

$$\int_0^1 |\partial_\xi \Gamma(\xi, \eta; s)|\,d\xi \le 2\alpha^{-1} \qquad (6.9)$$

[2, Lemma 2.3]; see also [19, (I.1.18)], [17, (3.10b) and Section 3.4.1.1].
   The solution representation for $\bar{v}$ via Γ is given by

$$\bar{v}(x, y; \xi, \eta) = \int_0^1 \Gamma(\xi, \eta; s)\, \bar{\bar{h}}(x, y; s, \eta)\, ds.$$

Applying ∂_ξ to this representation yields

$$\|\partial_\xi \bar{v}(x, y; \cdot)\|_{1;\Omega} \le \left(\sup_{(s,\eta)\in\Omega} \int_0^1 |\partial_\xi \Gamma(\xi, \eta; s)|\,d\xi \right) \cdot \left\| \bar{\bar{h}}(x, y; \cdot) \right\|_{1;\Omega}.$$

In view of (6.9), we now have $\|\partial_\xi \bar{v}\|_{1;\Omega} \le 2\alpha^{-1}\|\bar{\bar{h}}\|_{1;\Omega}$. Note that the differential equation (6.7) for $\bar{v}$ implies that $\varepsilon\|\partial_\xi^2 \bar{v}\|_{1;\Omega} \le C(\|\partial_\xi \bar{v}\|_{1;\Omega} + \|\bar{\bar{h}}\|_{1;\Omega})$. So, furthermore, we get

$$\|\partial_\xi \bar{v}\|_{1;\Omega} + \varepsilon\|\partial_\xi^2 \bar{v}\|_{1;\Omega} \le C\|\bar{\bar{h}}\|_{1;\Omega}.$$

As $G = \bar{v} + \bar{G}$ and we have the bound (5.9b) for $\partial_\xi \bar{G}$, to obtain the desired bounds (6.5b) and (6.5c), it now remains to show that $\|\bar{\bar{h}}(x, y; \cdot)\|_{1;\Omega} \le C + \varepsilon\mathcal{V}$. Furthermore, the definitions (6.8) of $\bar{\bar{h}}$ and (6.5a) of $\mathcal{V}$, imply that it suffices to prove the two estimates

$$\|\bar{v}(x, y; \cdot)\|_{1;\Omega} \le C, \qquad \|\bar{h}(x, y; \cdot)\|_{1;\Omega} \le C. \qquad (6.10)$$

The first of them follows from $\bar{v} = G - \bar{G}$ combined with (2.5) and (5.9a). The second is obtained from the definition (6.2b) of $\bar{h}$ using (5.9h) for $\|R\,\partial_\xi \bar{G}\|_{1;\Omega}$, (5.9a) for $\|\bar{G}\|_{1;\Omega}$ and (5.7) for $\|\bar{\phi}\|_{1;\Omega}$. This completes the proof of (6.5b) and (6.5c), and thus of (2.6a), (2.6d) and (2.6e).
   (v) We now focus on the remaining assertion (2.6d). Rewrite the problem (6.1b) as

$$[-\varepsilon(\partial_\xi^2 + \partial_\eta^2) + 1]\,\bar{v}(x, y; \xi, \eta) = \bar{h}_0(x, y; \xi, \eta) \quad \text{for } (\xi, \eta) \in \Omega, \qquad \bar{v}(x, y; \xi, \eta)\big|_{\partial\Omega} = 0,$$

where

$$\bar{h}_0(x, y; \xi, \eta) := \bar{h}(x, y; \xi, \eta) - a(\xi, \eta)\,\partial_\xi \bar{v}(x, y; \xi, \eta) + [1 + b(\xi, \eta)]\,\bar{v}(x, y; \xi, \eta). \qquad (6.11)$$

We shall represent $\bar{v}$ via the Green's function $\Psi = \Psi(s, t; \xi, \eta)$ of the two-dimensional self-adjoint operator $[-\varepsilon(\partial_\xi^2 + \partial_\eta^2) + 1]$. Note that Ψ, for any fixed (s, t), satisfies the equation $[-\varepsilon(\partial_\xi^2 + \partial_\eta^2) + 1]\Psi(s, t; \xi, \eta) = \delta(\xi - s)\delta(\eta - t)$, and also the boundary conditions $\Psi(s, t; \xi, \eta)\big|_{(\xi,\eta)\in\partial\Omega} = 0$. Furthermore, for any ball $B(x', y'; \rho)$ of radius ρ centred at any (x', y'), we cite the estimate [15, (3.5b)]

$$|\Psi(s, t; \cdot)|_{1,1;B(x',y';\rho)\cap\Omega} \le C\varepsilon^{-1}\rho. \tag{6.12}$$

The solution representation for $\bar{v}$ via Ψ is given by

$$\bar{v}(x, y; \xi, \eta) = \iint_\Omega \Psi(s, t; \xi, \eta)\, \bar{h}_0(x, y; s, t)\, ds\, dt.$$

Applying ∂_ξ and ∂_η to this representation yields

$$|\bar{v}(x, y; \cdot)|_{1,1;B(x',y';\rho)\cap\Omega} \le \left(\sup_{(s,t)\in\Omega} |\Psi(s, t; \cdot)|_{1,1;B(x',y';\rho)\cap\Omega} \right) \cdot \|\bar{h}_0(x, y; \cdot)\|_{1;\Omega}. \tag{6.13}$$

To estimate $\|\bar{h}_0\|_{1;\Omega}$, recall that it was shown in part (iv) of this proof that $\|\partial_\xi\bar{v}\|_{1;\Omega} \le 2\alpha^{-1}\|\bar{\bar{h}}\|_{1;\Omega}$ and $\|\bar{\bar{h}}(x, y; \cdot)\|_{1;\Omega} \le C + \varepsilon\mathcal{V}$, and in part (ii) that $\mathcal{V} \le C\varepsilon^{-1}$. Consequently $\|\partial_\xi\bar{v}\|_{1;\Omega} \le C$. Combining this with (6.11) and (6.10) yields $\|\bar{h}_0\|_{1;\Omega} \le C$. In view of (6.13) and (6.12), we now get $|\bar{v}|_{1,1;B(x',y';\rho)\cap\Omega} \le C\varepsilon^{-1}\rho$, which, combined with (5.9e), immediately gives the final desired bound (2.6c). $\qquad\square$

Remark 6.1. *Note that the term $\partial_\eta^2\bar{h}$ in the right-hand side $\bar{H}$ of (6.6) has such a singularity at $(\xi, \eta) = (x, y)$ that it is not absolutely integrable on Ω. So the function $\bar{H}$ and the differential equation in (6.6) are understood in the sense of distributions [9, Chapters 1, 3]. In particular $\partial_\eta^2\bar{h}$ is a generalised η-derivative of the regular function $\partial_\eta\bar{h}$.*

7   Generalisations

To generalise our results to more than two dimensions, one needs to employ an n-dimensional version of the fundamental solution g of (3.4), that will be denoted by g_n. Let $x = (x_1, x_2, \ldots, x_n)$ and $(\xi_1, \xi_2, \ldots, \xi_n)$ be in $\mathbb{R}^n$, and consider an n-dimensional version of problem (1.1) posed in the box domain $\Omega = (0, 1)^n$, with an x_1-direction of convection. The corresponding constant-coefficient operator is $-\varepsilon\triangle_x - (2q)\,\partial_{x_1}$ (compare with the two-dimensional operator $\tilde{L}_{xy}$ of (3.2)), where $\triangle_x := \sum_{i=1}^n \partial_{x_i}^2$ is the standard n-dimensional Laplacian. For this operator a calculation yields the fundamental solutions

$$g_3(x, \xi) = \frac{1}{4\pi\varepsilon}\, r^{-1}\, e^{q(\xi_1 - x_1 - r)/\varepsilon}, \qquad g_n(x, \xi) = \frac{1}{(2\pi\varepsilon)^{n/2}}\, e^{q(\xi_1 - x_1)/\varepsilon} K_{n/2-1}(qr/\varepsilon),$$

where $r = |x - \xi|$, and $K_{n/2-1}$ is the modified Bessel function of second kind of (half-integer) order $n/2 - 1$.

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