STABILISED APPROXIMATION OF INTERIOR-LAYER SOLUTIONS OF A SINGULARLY PERTURBED SEMILINEAR REACTION-DIFFUSION PROBLEM *

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Abstract. A semilinear reaction-diffusion two-point boundary value problem, whose second-order derivative is multiplied by a small positive parameter ε^2, is considered. It can have multiple solutions. The numerical computation of solutions having interior transition layers is analysed. It is demonstrated that the accurate computation of such solutions is exceptionally difficult.

To address this difficulty, we propose an artificial-diffusion stabilization. For both standard and stabilised finite difference methods on suitable Shishkin meshes, we prove existence and investigate the accuracy of computed solutions by constructing discrete sub- and super-solutions. Convergence results are deduced that depend on the relative sizes of ε and N, where N is the number of mesh intervals. Numerical experiments are given in support of these theoretical results. Practical issues in using Newton's method to compute a discrete solution are discussed.

Key words. semilinear reaction-diffusion problem, interior layer, Shishkin mesh, error estimates

AMS subject classifications. Primary 65L70, Secondary 34E05, 65L10, 65L11, 65L12, 65L50.

1. Introduction.

We are interested in interior-layer solutions of the singularly perturbed semilinear reaction-diffusion boundary-value problem

$$Fu(x) \equiv -\varepsilon^2 u''(x) + b(x,u) = 0 \quad \text{for } x \in (0,1), \tag{1.1a}$$

$$u(0) = g_0, \quad u(1) = g_1, \tag{1.1b}$$

where ε is a small positive parameter, $b \in C^\infty([0,1] \times \mathbb{R})$, and g_0 and g_1 are given constants. Problems of this type arise frequently in the modelling of stationary patterns in biological and chemical phenomena; see [6] and [14, Chapter 2].

The reduced problem of (1.1) is defined by formally setting $\varepsilon = 0$ in (1.1a), viz.,

$$b(x,\varphi) = 0 \quad \text{for } x \in (0,1). \tag{1.2}$$

It is often assumed in the numerical analysis literature that $b_u(x,u) > m > 0$ for all $(x,u) \in (0,1) \times \mathbb{R}$ and some positive constant m; then the reduced problem has a unique solution $\varphi = u_0 \in C^\infty(0,1)$, but this assumption excludes interior layer transitions between distinct reduced solutions that are important in various applications ([5, Section V], [6, Section 2.3]) and form the subject of this paper. Consequently we shall examine (1.1) under weaker local hypotheses, described in Section 3, that permit (1.2) to have more than one solution. No satisfactory numerical method for such problems appears in the literature, but in the present paper we devise and analyse a method that yields an accurate solution of (1.1) by combining a special mesh with a judicious amount of artificial diffusion (cf. [9]).

The structure of the paper is now summarised. We start in Section 2 by discussing the remarkable difficulties that a satisfactory numerical solution of the semilinear problem (1.1) presents owing to the absence of the hypothesis $b_u > 0$. A glimpse of

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these difficulties is given by Figure 2.1, where a standard 3-point difference scheme produces unstable solutions on both an equidistant mesh and an appropriate Shishkin mesh (compare these results with the solutions of the stabilised method that we propose in this paper, which are shown in Figure 2.2). The precise hypotheses that we place on (1.1) are described in Section 3. The numerical methods and a suitable Shishkin mesh are defined in Section 4, where in particular we introduce a stabilised method (4.3) that adds artificial diffusion wherever the mesh size is small compared with ε. In Section 5 our main numerical analysis results are stated: existence and error estimates for both the stabilised and standard numerical methods are established. Then Section 6 is devoted to the proofs of these results. This analysis requires many technical details, some of which resemble results already published in the research literature. We hive off this material to a companion technical report [11] in order to minimise the length of the present paper. Numerical experiments that support our theoretical analysis are presented in Section 7.

REMARK 1.1. *While the analysis and numerical results in this paper are given for the one-dimensional problem* (1.1)*, much of what is here can be generalized to analogues of* (1.1) *posed in higher dimensions; compare the one-dimensional nonlinear problem discussed in [10] and the extension of this work to the two-dimensional case in [8], where a theoretical analysis and numerical results are presented. The one-dimensional analysis is already so complex that the extra notation required to explain it in two dimensions would only obscure the central ideas that we wish to communicate.*

Notation. Throughout the paper, $C, C', \bar{C}$ and $\bar{C}'$, sometimes subscripted, denote generic positive constants that are independent of ε and of the mesh; furthermore, $\bar{C}$ and $\bar{C}'$ are taken sufficiently large where this property is needed. These constants may take different values in different places. Notation such as $f = O(z)$ means $|f| \le Cz$ for some C.

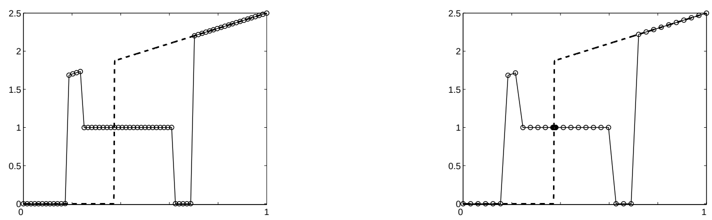

FIG. 2.1. *Unstable computed solutions of standard scheme* (4.2) *versus exact solution (dashed curve) of Example 2.1 with* $\varepsilon = 10^{-3}$*,* $N = 64$ *and initial guess the straight line linking the two boundary values. Left: equidistant mesh. Right: Shishkin mesh (chosen as in Section 7).*

2. Numerical intractability of (1.1)**.** To illustrate the substantial difficulties that the numerical solution of the semilinear problem (1.1) presents when we drop the restrictive assumption $b_u > 0$, we consider an example that is a variant of one appearing in [7].

EXAMPLE 2.1. *Consider (1.1) with* $b(x,u) = u(u-1)(u-x-3/2)(u+x+3/2)$ *and* $g_0 = 0$, $g_1 = 2.5$. *The reduced problem* $b(x,\varphi) = 0$ *has four solutions:* $\varphi_1 = 0$, $\varphi_2 = 1$, $\varphi_3(x) = x + 3/2$, *and* $\varphi_4(x) = -(x + 3/2)$. *An asymptotic analysis [7, 20] shows that if a solution of this problem has an interior layer, then that layer must be centered at a certain point that is* $O(\varepsilon)$ *distant from* $x = 3/8$ *and the solution is approximately equal to* φ_1 *and* φ_3 *respectively to the left and right of the layer.*

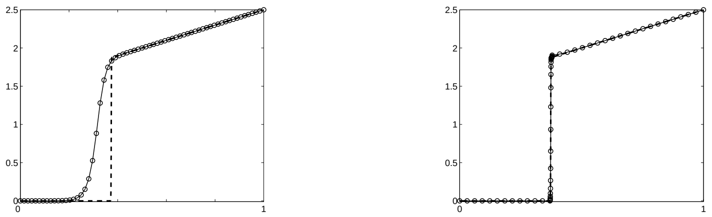

Fig. 2.2. *Computed solutions of stabilised scheme* (4.3) *versus exact solution (dashed curve) of Example 2.1 with* $\varepsilon = 10^{-3}$, $N = 64$, $\hat{C} = 2.5$ *and initial guess the straight line linking the two boundary values. Left: qualitatively correct solution on equidistant mesh. Right: Shishkin mesh (chosen as in Section 7) yields a computed solution with maximal nodal error 5.19e-2.*

A standard 3-point difference scheme—see (4.2) below—on an equidistant mesh and on an appropriate Shishkin mesh, each having N intervals, yielded the unstable solutions shown in Figure 2.1. Here Newton's method, with initial guess the straight line $y = 2.5x$ that joins the given boundary values, was used to solve the discrete system. The observed instability can be easily explained by noting that if $\varepsilon \ll N^{-1}$, then the discretization of the term $\varepsilon^2 u''$ on an equidistant mesh (or the coarse part of the Shishkin mesh) is $O(\varepsilon^2 N^2)$ and so becomes negligible; thus we essentially solve the algebraic equation $b(x, u) = 0$ at each mesh node. But if instead one uses the stabilised method (4.3) that we propose in Section 4, then, irrespective of the choice of initial guess, one obtains the qualitatively correct solution of Figure 2.2 (left) on the same equidistant mesh and moreover the accurate computed solution of Figure 2.2 (right) (with maximum nodal error 5.19e-2) on our Shishkin mesh.

As well as the discrete solution one desires to find, parasitic solutions of the discrete system frequently appear. These may look like solutions of (1.1) but are in fact inaccurate. Figure 2.3 shows some of the phenomena one can encounter. In it, the leftmost diagram shows 3 different solutions computed on the same equidistant mesh. The central diagram, which exhibits solutions computed on 3 different Shishkin meshes, implies that, if one starts from a parasitic solution on a uniform mesh and then uses adaptive mesh refinement, one can converge to a very inaccurate yet plausible computed solution on an adapted mesh. The rightmost diagram reveals a further unpleasant property: for a single Shishkin mesh that is centred on t_0, one can compute multiple discrete solutions each of whose transition layer profiles is shifted by $O(\varepsilon)$. The accuracy of such a shifted solution is only $O(1)$ in the maximum norm.

Alarmingly, every computed solution in Figure 2.3 looks plausible if one has no precise a priori knowledge of the true location of the interior layer. Consequently any one of these solutions might lead the experimentalist to believe that an interior-layer solution of Example 2.1 has been successfully computed—when in fact the discrete solution is only $O(1)$ accurate in the discrete maximum norm.

The inaccuracy of solutions computed on correctly-placed Shishkin meshes in the rightmost diagram of Figure 2.3 will surprise those who view these meshes as a panacea for the computation of layers in the solutions of differential equations. Heuristically, the displacement of the interior layer in solutions of the standard scheme occurs since the discretization of the differential equation may disrupt the mechanism that implicitly puts the interior layer in the correct location. In particular,this mechanism is entirely lost on the coarse mesh when $\varepsilon \ll N^{-1}$, as there one is essentially solving the reduced equation (1.2), which has multiple solutions. Note that switching to the

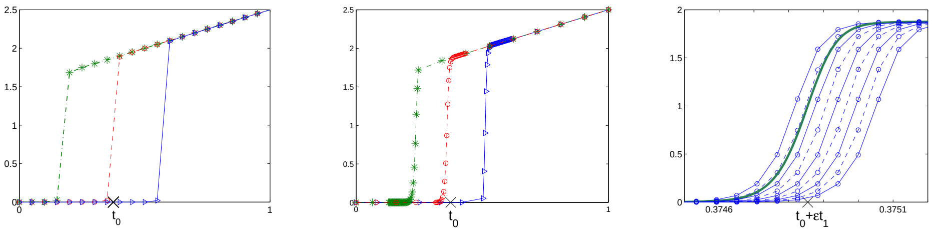

FIG. 2.3. *Parasitic computed solutions of Example 2.1 generated by the standard scheme* (4.2) *(different solutions of the same discrete problem are obtained using different initial guesses of type* $y = 1 + \mathrm{sgn}(x - x_*)$ *with various* x_**). Left: equidistant mesh with* $\varepsilon = 0.01$, $N = 20$. *Centre: solutions computed on Shishkin meshes centred at three different points with* $\varepsilon = 0.01$, $N = 40$. *Right: enlarged multiple solutions of the discrete problem with* $\varepsilon = 10^{-4}$, $N = 64$, *on a Shishkin mesh centred correctly at the point* t_0 *that is defined in* ($A5$)*; the quantity* t_1 *is defined in [11, Lemma 2.4]; the mesh transition parameter is specified by (4.4) with* $C_\tau = 4.2$. *These computed solutions are obtained using shooting to determine the value of* u_m^N *where* $x_m = t_0 + O(\varepsilon)$*; the exact solution is the bold curve.*

stabilised scheme cures entirely the instabilities of both Figure 2.1 (see Figure 2.2) and Figure 2.3(left), but in some cases one may still observe the multiple-solution phenomenon of Figure 2.3(centre, right).

Nevertheless, as our theoretical conclusions and numerical results show, when the Shishkin mesh is placed correctly there is then a qualitatively correct discrete solution that is ε-uniformly accurate outside the layer region. Furthermore, our Theorem 5.1 gives a range of values N that ensure ε-uniform convergence in the entire domain and for which we have not observed the multiple-solution phenomenon of Figure 2.3(right).

3. Hypotheses on the data of the continuous problem. In this section we place hypotheses on the data of (1.1). Assume that the reduced problem (1.2) has three simple roots $\varphi = \varphi_k \in C^\infty[0,1]$ for $k = 0, 1, 2$:

$$(A1) \qquad b(x, \varphi_k(x)) = 0 \qquad \text{for } \ k = 0,1,2 \ \text{ and } \ x \in [0,1]$$

where

$$(A2) \qquad \begin{cases} \varphi_1(x) < \varphi_0(x) < \varphi_2(x) \quad \text{for } x \in [0,1] \\ \text{and there is no other solution of (1.2) between } \varphi_1 \text{ and } \varphi_2. \end{cases}$$

Here and subsequently, numbering such as ($A1$) indicates an *assumption* that holds true throughout the paper. Assume also that

$$(A3) \qquad b_u(x, \varphi_k(x)) > \gamma^2 > 0 \qquad \text{for } \ k = 1,2 \ \text{ and } \ x \in [0,1]$$

but

$$(A4) \qquad b_u(x, \varphi_0) < 0 \qquad \text{for } \ x \in [0,1].$$

Assumption ($A3$) says that $\varphi_1(x)$ and $\varphi_2(x)$ are stable reduced solutions, i.e., one may have a solution u of (1.1) that is very close to either φ_1 or φ_2 on some subdomain of $(0,1)$. Assumption ($A4$) means that the solution $\varphi_0(x)$ is unstable: no solution of (1.1) lies close to φ_0 on any subdomain of (0,1). Under the hypotheses ($A1$)–($A4$), the equation (1.1a) is often described as bistable.

Our further assumption is that the equation $\int_{\varphi_1(x)}^{\varphi_2(x)} b(x,v)\,dv = 0$ has a solution $x = t_0$ such that $\frac{d}{dx}\big[\int_{\varphi_1(x)}^{\varphi_2(x)} b(x,v)\,dv\big]\big|_{x=t_0} \neq 0$, i.e., this root is simple. As in many asymptotic analysis papers, we also assume that the value of this derivative is negative, since this sign corresponds to the Lyapunov stability of an interior-layer solution $u(x)$ of (1.1) that switches from φ_1 to φ_2 when u is regarded as a steady-state solution of the time-dependent parabolic problem $v_t - \varepsilon^2 v_{xx} + b(x,v) = 0$ (see [1, Section 7, Remark 3]; if instead the derivative were positive, this would correspond to Lyapunov stability of an interior-layer solution that switches from φ_2 to φ_1). By Assumption $(A1)$ these hypotheses on the integrals of b are equivalent to the assumptions

$$(A5)\qquad \int_{\varphi_1(t_0)}^{\varphi_2(t_0)} b(t_0,v)\,dv = 0 \qquad \text{and} \qquad \int_{\varphi_1(t_0)}^{\varphi_2(t_0)} b_x(t_0,v)\,dv = -C_I < 0.$$

Similar conditions are assumed in [19, §4.15.4], [20, §2.3.2] and also in [2, 15] for an analogous two-dimensional problem and [4] for a analogous system of equations.

REMARK 3.1. *Assumption $(A2)$ can be relaxed to allow other roots of (1.2) between φ_1 and φ_2 provided that $\int_{\varphi_1(t_0)}^{v} b(t_0,s)\,ds > 0$ for all $v \in (\varphi_1(t_0), \varphi_2(t_0))$. Note that this inequality follows immediately from $(A1)$–$(A5)$ if φ_0 is the only reduced solution between φ_1 and φ_2.*

The solutions φ_1 and φ_2 of (1.2) do not in general satisfy either of the boundary conditions in (1.1b). In order to focus on interior layers, we exclude boundary layers by assuming that

$$(A6)\qquad \varphi_1(0) = g_0, \qquad \varphi_2(1) = g_1, \qquad \varphi_1''(0) = \varphi_2''(1) = 0.$$

Under Assumptions $(A1)$–$(A6)$, the problem (1.1) has a solution that, roughly speaking, lies in the neighbourhood of $\varphi_1(x)$ and $\varphi_2(x)$ for $x \in [0, t_0)$ and $x \in (t_0, 1]$ respectively (see Corollary 6.7). Near $x = t_0$ the solution switches from φ_1 to φ_2, which results in an interior transition layer of width $O(\varepsilon|\ln\varepsilon|)$.

4. Standard and stabilised numerical methods, Shishkin mesh. Here we define our standard and stabilised finite difference methods, and a Shishkin mesh [13, 17, 18] that is tailored to (1.1).

Let N, the number of mesh intervals, be a positive integer. Let the mesh be $0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1$. Set $h_i = x_i - x_{i-1}$ for $i = 1,\dots,N$, and $\hbar_i = (h_i + h_{i+1})/2$ for $i = 1,\dots,N-1$. Throughout the rest of the paper we shall always assume that

$$(4.1)\qquad \varepsilon \le CN^{-1}$$

for some positive constant C. This assumption is reasonable in practical situations: if it were not satisfied then one could apply standard numerical methods to solve (1.1).

A discrete solution of (1.1) on the mesh will be written as $\{u_i^N\}$ or $\{\hat{u}_i^N\}$ for $i = 0,1,\dots,N$. The classical finite difference approximations of $u'(x_{i-1/2})$ and $u''(x_i)$ are defined by

$$Du_i^N := \frac{u_i^N - u_{i-1}^N}{h_i}, \qquad \delta_x^2 u_i^N := \frac{Du_{i+1}^N - Du_i^N}{\hbar_i}.$$

The *standard finite difference scheme* for approximating (1.1) is

$$(4.2)\qquad F^N u_i^N := -\,\varepsilon^2 \delta_x^2 u_i^N + b(x_i, u_i^N) = 0 \quad \text{for } i = 1,\dots,N-1,$$

subject to $u_0^N = g_0$ and $u_N^N = g_1$. This scheme is also generated by the standard mass-lumped piecewise linear finite element method. As it frequently produces unsatisfactory computed solutions—see Section 2—we propose a *stabilised finite difference scheme*

$$\hat{F}^N \hat{u}_i^N := -\frac{\hat{\varepsilon}_{i+1}^2 D\hat{u}_{i+1}^N - \hat{\varepsilon}_i^2 D\hat{u}_i^N}{\hbar_i} + b(x_i, \hat{u}_i^N) = 0 \quad \text{for } i = 1, \ldots, N-1, \tag{4.3}$$

subject to $u_0^N = g_0$ and $u_N^N = g_1$, where $\hat{\varepsilon}_i := \max\{\varepsilon, \hat{C}h_i\}$ for some user-chosen positive constant $\hat{C}$. Compared with (4.2), we observe that if $h_i > \hat{C}^{-1}\varepsilon$ then (4.3) adds artificial diffusion. The stabilised scheme (4.3) can also be generated by a mass-lumped piecewise linear finite element method with artificial diffusion added in a conservative way; in this framework it is easy to generalise (4.3) to higher-dimensional problems.

We now define a *Shishkin mesh* that is appropriate for (1.1). Define the mesh transition point parameter τ by

$$\tau = \tfrac{C_\tau}{\bar{\gamma}}\, \varepsilon \ln N, \tag{4.4}$$

where

$$\bar{\gamma} := \sqrt{\min_{k=1,2} b_u(t_0, \varphi_k(t_0))} \tag{4.5}$$

(so $\bar{\gamma} > \gamma > 0$,where γ appeared in $(A3)$) and C_τ is a user-chosen constant that should be sufficiently large—see Theorems 5.1 and 5.2.

Assume that ε is so small that $\frac{1}{2}t_0 < t_0 - \tau$ and $t_0 + \tau < 1 - \frac{1}{2}(1 - t_0)$. Divide the intervals $[0, t_0 - \tau]$, $[t_0 - \tau, t_0 + \tau]$ and $[t_0 + \tau, 1]$ into N_0, $N/2$ and N_1 equidistant subintervals respectively with $N_0 + N_1 = N/2$ and $N_0 \approx t_0 N/2$, $N_1 \approx (1-t_0)N/2$. In practice one usually has $\tau \ll 1$, so the mesh is relatively fine on $[t_0 - \tau, t_0 + \tau]$. Write h for this fine mesh width; then $h = C\varepsilon N^{-1} \ln N$. On the remainder of $[0, 1]$ one has a coarse mesh of width $O(N^{-1})$.

To solve (1.1) numerically one could use a graded mesh, but we shall confine our attention to the piecewise-equidistant Shishkin mesh as it is easier to analyse; cf. [17].

REMARK 4.1. *Let N be sufficiently large. Then on the above Shishkin mesh, for the discrete operators F^N and $\hat{F}^N$ of (4.2) and (4.3) we have $\hat{F}^N = F^N$ if $|x_i - t_0| < \tau$. Furthermore, $\hat{F}^N - F^N = -(\hat{\varepsilon}_1^2 - \varepsilon^2)\,\delta_x^2$ if $x_i < t_0 - \tau$, and $\hat{F}^N - F^N = -(\hat{\varepsilon}_N^2 - \varepsilon^2)\,\delta_x^2$ if $x_i > t_0 + \tau$.*

5. Existence and accuracy of discrete solutions. Main results. This section states existence results for discrete solutions of the standard difference scheme (4.2) and the stabilised scheme (4.3) near an interior-layer solution u of (1.1). The theorems below deal with two regimes that depend on the relative sizes of ε and N.

THEOREM 5.1. *Let $\{x_i\}$ be the Shishkin mesh of Section 4 that is defined using (4.4) and (4.5). Set $C' = 4C_\tau/\bar{\gamma}$. For some $\sigma \in [0, 2]$, assume that $c_0\varepsilon \geq (C'N^{-1}\ln N)^{2+\sigma}$; this constant c_0 is used in Lemma 6.10. Let ε be sufficiently small and N sufficiently large.*
(i) If $C_\tau > 2$, then there exists a solution u_i^N of the standard scheme (4.2) such that

$$|u_i^N - u(x_i)| \leq C \begin{cases} (N^{-1}\ln N)^{2-\sigma} & \text{for } |x_i - t_0| < \tau, \\ N^{-2} & \text{for } |x_i - t_0| \geq \tau. \end{cases} \tag{5.1}$$

(ii) If $C_\tau > 3$, then there exists a solution $\hat{u}_i^N$ of the stabilised scheme (4.3) such that

$$|\hat{u}_i^N - u(x_i)| \le C \begin{cases} (N^{-1}\ln N)^{2-\sigma} + N^{-1} & \text{for } |x_i - t_0| < \tau, \\ N^{-1} & \text{for } |x_i - t_0| \ge \tau. \end{cases} \tag{5.2}$$

The next result considers the possibility that the relationship between ε and N is stronger than (4.1). Fix $\lambda \in (0,1)$. The case $\varepsilon \ge CN^{-(4-\lambda)} \ge c_0^{-1}(C'N^{-1}\ln N)^4$ was considered in Theorem 5.1 so now we focus on $\varepsilon \le CN^{-(4-\lambda)}$.

THEOREM 5.2. *Let $\{x_i\}$ be the Shishkin mesh of Section 4 that is defined using (4.4) and (4.5). Fix $\lambda \in (0,1)$. Assume that $\varepsilon \le CN^{-\theta}$ for some $\theta \ge 4-\lambda$ and $C > 0$, and N is sufficiently large independently of ε.*
(i) If $C_\tau > 2$, then there exists a solution u_i^N of the standard scheme (4.2) such that

$$|u_i^N - u(x_i)| \le CN^{-\min\{2,\theta-2\}} \le CN^{-(2-\lambda)} \quad \text{for } |x_i - t_0| \ge \tau. \tag{5.3}$$

(ii) If $C_\tau > 1$, then there exists a solution $\hat{u}_i^N$ of the stabilised scheme (4.3) such that

$$|\hat{u}_i^N - u(x_i)| \le CN^{-1} \quad \text{for } |x_i - t_0| \ge \tau. \tag{5.4}$$

REMARK 5.3. *Consider the use of Newton's method to compute a solution of the standard scheme* (4.2)*. Although Theorems 5.1 and 5.2 guarantee existence of a solution that attains a higher order of convergence, parasitic solutions are numerous and an unsophisticated initial guess in Newton's method will yield an unsatisfactory result (see Section 2). To obtain a higher-order-accurate solution one should use as initial guess a solution of the stabilised scheme* (4.3)*; see Tables 7.3 and 7.4 below.*

REMARK 5.4. *An inspection of the proof of Theorem 5.1 shows that one still obtains existence of a discrete solution satisfying the error estimate (5.1) if the alternative stabilised scheme $-\max\{\varepsilon, \hat{C}\hbar_i\}^2\delta_x^2 u_i^N + b(x_i, u_i^N) = 0$ is used. On an equidistant mesh this scheme is identical with (4.3), while on our Shishkin mesh it differs from (4.3) only at the two transition points $t_0 \pm \tau$. This alternative stabilisation seems somewhat superior to the standard scheme (4.2), but on our Shishkin mesh, some initial guesses in Newton's method produce computed solutions having interior layers outside the interval $[t_0 - \tau, t_0 + \tau]$.*

REMARK 5.5 (ε-uniform accuracy in the layer region via postprocessing). *Theorems 5.1 and 5.2, and the numerical results of Section 7, suggest that as $\varepsilon \to 0$ for any fixed N, the accuracy of discrete solutions of* (4.2) *or* (4.3) *deteriorates inside the layer region $\{x_i : |x_i - t_0| < \tau\}$. To obtain ε-uniform accuracy in the entire domain $[0,1]$ for all $\varepsilon \le CN^{-1}$, one can use a postprocessing technique for smaller values of ε as follows.*

Set $\bar\varepsilon := N^{-2}\ln^3 N$ (or $\bar\varepsilon := CN^{-2}\ln^3 N$ for any positive constant C), and denote by $\bar u$ the corresponding exact solution of (1.1). Note that $\bar\varepsilon$ satisfies, for sufficiently large N, the hypothesis of Theorem 5.1 with $\sigma = 0$. Consequently, if $\{\bar u_i^N\}$ is the computed solution obtained as in part (i) or part (ii) of Theorem 5.1 on the mesh denoted by $\{\bar x_i\}$ with the transition parameter $\tau(\bar\varepsilon) := \frac{C_\tau}{\bar\gamma}\,\bar\varepsilon\ln N$, then $\max_i |\bar u_i^N - \bar u(\bar x_i)|$ is bounded, respectively, by $CN^{-2}\ln^2 N$ or CN^{-1}.

Postprocessing for $\varepsilon \in (0, \bar\varepsilon)$: For $|\bar x_i - t_0| > \tau(\bar\varepsilon)$, set $\tilde x_i := \bar x_i$ and $\tilde u_i^N := \bar u_i^N$; for $|\bar x_i - t_0| \le \tau(\bar\varepsilon)$, set $\tilde x_i := t_0 + \frac{\varepsilon(\bar x_i - t_0)}{\bar\varepsilon}$ and $\tilde u_i^N := \bar u_i^N$ (i.e., in the layer region the computed solution $\{\bar u_i^N\}$ is translated and compressed horizontally). Then the post-processed solution $\{\tilde u_i^N\}$ on the mesh $\{\tilde x_i\}$ satisfies the error estimate

$$\max_i |\tilde u_i^N - u(\tilde x_i)| \le CN^{-2}\ln^4 N + \max_j |\bar u_j^N - \bar u(\bar x_j)| \qquad \text{for } \varepsilon \in (0, \bar\varepsilon). \tag{5.5}$$

To prove this, it suffices to show that $|\bar{u}(\bar{x}_i) - u(\tilde{x}_i)| \le CN^{-2}\ln^4 N$ *for all* i. *Note that [11, Lemma 2.5], combined with Corollary 6.7 below, implies that* $|u(x) - \mathcal{U}(x,\varepsilon)|$ *and* $|\bar{u}(x) - \mathcal{U}(x,\bar{\varepsilon})|$ *are both bounded by* $C(\bar{\varepsilon}\ln N + N^{-2}) \le CN^{-2}\ln^4 N$. *Here the function* $\mathcal{U}$, *which is related to the asymptotic expansion* $u_{\rm as}$ *of (6.3), is defined by* $\mathcal{U}(x,\varepsilon) := V_0(\frac{x-t_0}{\varepsilon};0)$ *for* $|x-t_0| \le \tau(\varepsilon)$, *and* $\mathcal{U}(x,\varepsilon) := u_0(x)$ *for* $|x-t_0| > \tau(\varepsilon)$. *As* $\mathcal{U}(\bar{x}_i,\bar{\varepsilon}) = \mathcal{U}(\tilde{x}_i,\varepsilon)$, *the required bound for* $|\bar{u}(\bar{x}_i) - u(\tilde{x}_i)|$, *and so (5.5) follow.*

6. Analysis of the numerical methods. Proof of Theorems 5.1 and 5.2. Section 6 is devoted to the proofs of Theorems 5.1 and 5.2, i.e., here we establish the existence and accuracy of discrete solutions for problem (1.1). Our analysis is based on the use of sub-solutions and super-solutions for the discrete problems (4.2) and (4.3). While Nefedov [15] discusses continuous sub- and super-solutions for a two-dimensional analogue of problem (1.1), the investigation of discrete sub- and super-solutions is more complicated since one must deal not only with ε but also with the other small parameter N^{-1}.

Throughout this section, let $\{x_i\}$ be the Shishkin mesh of Section 4 that is defined using (4.4) and (4.5). We start by describing briefly asymptotic expansions of that solution of (1.1) that we wish to compute numerically. The expansion of Section 6.1 resembles previously published work [15, 20] so most of the details are diverted to our report [11, Section 2]. In Section 6.2 extra terms are added to this expansion to get sub- and super-solutions of the numerical methods for (1.1). Then in Section 6.3 we estimate the truncation errors of our schemes; these estimates are used in Section 6.4 to establish sufficient conditions for discrete sub- and super-solutions derived from the construction of Section 6.2. Finally, the existence and error estimates of Theorems 5.1 and 5.2—the main results of the paper—are proved in Section 6.5.

6.1. Sketch of asymptotic expansion for the continuous problem. Recall the point $t_0 \in (0,1)$ that was specified in Assumption $(A5)$. Define the stretched variable ξ by

$$\xi := (x - t_0)/\varepsilon.$$

Then a standard calculation shows that the zero-order interior-layer term $V_0(\xi)$ of the asymptotic expansion of u is given by a solution of the following problem:

$$\text{(6.1a)} \qquad -\tfrac{d^2}{d\xi^2}V_0 + b(t_0, V_0) = 0 \quad \text{for } \xi \in \mathbb{R}, \qquad V_0(-\infty) = \varphi_1(t_0), \;\; V_0(\infty) = \varphi_2(t_0).$$

We shall see shortly that (6.1a) has a solution $V_0(\xi)$, but this solution is not unique as $V_0(\xi \pm C)$ is also a solution for any constant C. Once we know that V_0 exists and is a strictly increasing function, consider a specific solution $\hat{V}_0$ of (6.1a) subject to the parametrization

$$\text{(6.1b)} \qquad \hat{V}_0(0) = \varphi_2(t_0).$$

One might expect $u(x) = \varphi_0(t_0)$ to hold at $x = t_0$ and thus the interior layer to be described by $\hat{V}_0(\xi)$. This is not the case, however; in fact $u(x) = \varphi(t_0)$ when $x = t_0 + \varepsilon t_1 + \varepsilon^2 t_2 + \cdots$, and the interior layer is described by $\hat{V}_0(\xi - t_1 - \varepsilon t_2 - \cdots)$. Here $t_1,\ t_2, \ldots$ are independent of ε and can be found when constructing an asymptotic expansion of u; in particular, the values of t_1 and t_2 are specified in the proof of Lemma 6.3 that appears in [11].

In our analysis we take $t = t_0 + \varepsilon t_1 + \varepsilon^2 t_2$, omitting the higher-order terms, and invoke a perturbed version of $\hat{V}_0(\xi - t_1 - \varepsilon t_2)$ defined by

$$\text{(6.1c)} \qquad V_0(\xi; p) = \hat{V}_0(\xi - \bar{t}_1 + p), \qquad \bar{t}_1 = t_1 + \varepsilon t_2.$$

Here the real-valued parameter p satisfies $|p| \le p^*$ for any fixed positive constant p^*, but will typically take very small values.

Lemma 6.1. *For any constant $\bar{t}_1$ and all $|p| \le p^*$, there exists a unique monotone solution $V_0(\xi; p)$ of (6.1). Furthermore, V_0 is in $C^\infty(\mathbb{R})$, and*

$$\chi(\xi; p) := \tfrac{d}{d\xi} V_0(\xi; p) > 0 \qquad \text{for } \xi \in \mathbb{R}. \tag{6.2}$$

For any arbitrarily small but fixed $\lambda \in (0, \bar{\gamma})$, there is a constant C_λ such that $\chi(\xi; p) \le C_\lambda e^{-(\bar{\gamma}-\lambda)|\xi|}$ for $\xi \in \mathbb{R}$, $|p| \le p^*$. There are constants C' and C'' such that for all $|p| \le p^*$ one has $C'\chi \le V_0 - \varphi_1(t_0) \le C''\chi$ for $\xi < 0$ and $C'\chi \le \varphi_2(t_0) - V_0 \le C''\chi$ for $\xi > 0$.

Proof. In view of $(A1)$–$(A5)$, these properties follow from the proof of [3, Lemma 2.1] or a slight extension of the proof of [10, Lemma 2.1] using phase-plane analysis. ∎

One then continues the asymptotic analysis of (1.1) along the lines of [20, Section 2.3.2] and [15, Section 3], but with the modification that one expands about the point t_0 instead of about the point $t = t_0 + \varepsilon t_1 + \varepsilon^2 t_2 + \cdots$ (which is unknown a priori); this will aid the numerical analysis because our layer-adapted mesh will be centred on the known point t_0. Details of this asymptotic construction are given in [11, Section 2]. Essentially, two asymptotic expansions are constructed separately on $[0, t_0]$ and $[t_0, 1-t_0]$ and then matched at $x = t_0$. One arrives finally at the expansion

$$u_{\mathrm{as}}(x; p) := u_0(x) + \varepsilon^2 u_2(x) + v_0(\xi; p) + \varepsilon v_1(\xi; p) + \varepsilon^2 v_2(\xi; p), \tag{6.3}$$

where the smooth component $u_0 + \varepsilon^2 u_2$ combines the functions

$$u_0(x) := \begin{cases} \varphi_1(x), & x \in [0, t_0), \\ \varphi_2(x), & x \in (t_0, 1], \end{cases} \qquad u_2(x) := u_0''(x)/b_u(x, u_0(x)),$$

while the boundary-layer component, whose properties will be described in a moment, is $v_0 + \varepsilon v_1 + \varepsilon^2 v_2$. We use the auxiliary functions

$$B(x, s) := b(x, u_0(x) + s) \qquad \text{and} \qquad \hat{t}_0 = \hat{t}_0(x) := \begin{cases} t_0^- & \text{if } x \in [0, t_0) \\ t_0^+ & \text{if } x \in (t_0, 1] \end{cases}$$

(thus for example $u_0(\hat{t}_0) = \varphi_1(t_0)$ for $x < t_0$). Now $v_0(\xi) = v_0(\xi; p)$ is defined by

$$-\tfrac{d^2}{d\xi^2} v_0 + B(\hat{t}_0, v_0) = 0, \quad v_0(0^\pm) = V_0(0; p) - u_0(t_0^\pm), \quad v_0(\pm\infty) = 0. \tag{6.4}$$

Comparing this with (6.1a), we see that

$$v_0(\xi; p) = V_0(\xi; p) - u_0(\hat{t}_0).$$

Higher-order interior-layer components $v_j(\xi) = v_1(\xi; p)$ for $j = 1, 2$ are defined by

$$[-\tfrac{d^2}{d\xi^2} + B_s(\hat{t}_0, v_0)]\, v_j = \psi_j(\xi), \quad v_1(0) = v_j(\pm\infty) = 0, \quad v_2(0^\pm) = -u_2(t_0^\pm), \tag{6.5}$$

where $\psi_1(\xi) := -\xi B_x(\hat{t}_0, v_0)$, while ψ_2 is similar, but has a more complicated structure that is described in [11, (2.13)]. The functions v_1 and v_2 depend on p since they are defined using $v_0(\xi; p)$. Note that v_0 and v_2 have a discontinuity at $\xi = 0$, but

$u_0 + v_0 = [u_0(x) - u_0(\hat{t}_0)] + V_0$ and $u_2 + v_2$ are continuous at $x = t_0$. Thus $u_{\rm as}(x;p)$ is continuous for $x \in [0,1]$.

LEMMA 6.2. *For any constants t_1 and t_2 in (6.1c), there exist solutions v_0, v_1 and v_2 of the problems (6.4), (6.5). The function v_0 satisfies*

$$(\operatorname{sgn}\xi)\cdot v_0(\xi) > 0 \qquad \text{and} \qquad |v_0(\xi)| \le C''\chi(\xi) \qquad \text{for } \xi \in \mathbb{R}\setminus\{0\}. \tag{6.6}$$

Furthermore, assuming that $|t_1| + |t_2| \le C$ and $|p| \le p^*$, for any arbitrarily small but fixed $\lambda \in (0, \bar{\gamma})$, there is a constant C_λ such that

$$\left|\tfrac{d^k}{d\xi^k} v_j\right| \le C_\lambda e^{-(\bar{\gamma}-\lambda)|\xi|} \qquad \text{for } \xi \in \mathbb{R}\setminus\{0\},\ \ j = 0,1,2,\ \ k = 0,\ldots,6. \tag{6.7}$$

Proof. We outline the proof; for details see [11, Lemma 2.3]. First, Lemma 6.1 yields existence of the function v_0, the properties (6.6), and the bound (6.7) for $j = 0$, $k = 0,1$. The remaining assertions are derived on noting that v_1, v_2 and $\frac{d^k}{d\xi^k} v_j$, for $j = 0,1,2$, all satisfy linear differential equations that share the same operator $[-\frac{d^2}{d\xi^2} + B_s(\hat{t}_0, v_0)]$ —one can then use an explicit solution formula obtained by variation of parameters since the function χ of (6.2) satisfies $[-\frac{d^2}{d\xi^2} + B_s(\hat{t}_0, v_0)]\chi = 0$; see [3, Lemma 2.2]. ∎

Given any suitable function $v(x)$, define the functional

$$\Phi[v(\cdot)] := \varepsilon \tfrac{dv}{dx}\Big|_{x=t_0^-} - \varepsilon \tfrac{dv}{dx}\Big|_{x=t_0^+}.$$

The main result of this section now follows.

LEMMA 6.3. *For the asymptotic expansion $u_{\rm as}(x;p)$ from (6.3) we have*

$$F u_{\rm as}(x;p) = O(\varepsilon^3) \qquad \text{for } x \in (0,1)\setminus\{t_0\}. \tag{6.8a}$$

Furthermore, there exist values of t_1 and t_2 in (6.1c), independent of ε and p, and positive constants C_1, C_2 and $\varepsilon^* = \varepsilon^*(p^*)$ such that for all $\varepsilon \le \varepsilon^*$ and $0 < |p| \le p^*$ we have

$$(\operatorname{sgn} p)\cdot \Phi[u_{\rm as}(\cdot;p)] \ge C_1\varepsilon|p| - C_2\varepsilon^3. \tag{6.8b}$$

Proof. We sketch the detailed proof that is given in [11, Lemma 2.4]: the relation (6.8a) is a standard outcome of the method of asymptotic expansions that was applied to generate the terms in (6.3), while to establish (6.8b) one uses $\Phi[u_0] = \varepsilon[\varphi_1'(t_0) - \varphi_2'(t_0)]$, $\Phi[v_0] = \Phi[V_0] = 0$, $\Phi[\varepsilon^2 u_2] = O(\varepsilon^3)$, and $\Phi[\varepsilon v_1 + \varepsilon^2 v_2]$ is computed using explicit representations of v_1 and v_2 described in the proof of Lemma 6.2. ∎

Note that Lemma 6.3 implies that there exists $|p_0| \le \varepsilon^2 \frac{C_2}{C_1}$ such that $u_{\rm as}(x;p_0) \in C^2[0,1]$ so $F u_{\rm as}(x;p_0) = O(\varepsilon^3)$ for all $x \in (0,1)$, which is a standard property of an asymptotic expansion.

6.2. Perturbed asymptotic expansion, sub- and super-solutions. For the numerical analysis that comes later, we modify the asymptotic expansion $u_{\rm as}(x;p)$ of (6.3): set

$$\beta(x) = \beta(x;p,p',\hat{h}) := u_{\rm as}(x;p) + p'\,[v_*(\xi;p) + C_0] + \hat{h}^2\, z(\xi;p). \tag{6.9}$$

The function β is a small perturbation of $u_{\rm as}$ when the parameters p' and $\hat{h}$ are small. The parameter $\hat{h}$ is related to the mesh used and the component $\hat{h}^2\, z(\xi;p)$ is added to compensate for the principal part of the truncation error produced when our finite difference operator is applied to $u_{\rm as}(x,t)$. The component $p'[v_*(\xi;p)+C_0]$ is added to ensure that $(\operatorname{sgn} p')\cdot F(u_{\rm as}+p'[v_*+C_0])\geq 0$.

The functions $v_*(\xi)=v_*(\xi;p)$ and $z(\xi)=z(\xi;p)$ used in (6.9) are defined by

$$\text{(6.10)}\qquad [-\tfrac{d^2}{d\xi^2}+B_s(\hat{t}_0,v_0)]\,v_*=|v_0| \qquad \text{for } \xi\in\mathbb{R}\setminus\{0\},\quad v_*(0)=v_*(\pm\infty)=0,$$

$$\text{(6.11)}\qquad [-\tfrac{d^2}{d\xi^2}+B_s(\hat{t}_0,v_0)]\;z\;=\tfrac{1}{12}\tfrac{d^4}{d\xi^4}V_0 \quad \text{for } \xi\in\mathbb{R}, \qquad z(0)=z(\pm\infty)=0.$$

These functions depend on p since they are defined using $v_0(\xi;p)$ and $V_0(\xi;p)$.

Lemma 6.4. *Assume that $|p|\leq p^*$ for some positive constant p^*. Then there exist solutions v_* and z of problems (6.10) and (6.11) respectively, and for any arbitrarily small but fixed $\lambda\in(0,\bar\gamma)$, there is a constant C_λ such that*

$$\text{(6.12)}\qquad v_*\geq 0,\quad \bigl|\tfrac{d^k}{d\xi^k}v_*\bigr|+\bigl|\tfrac{d^k}{d\xi^k}z\bigr|\leq C_\lambda e^{-(\bar\gamma-\lambda)|\xi|}\quad \textit{for } \xi\in\mathbb{R}\setminus\{0\},\ \ k=0,\dots,4.$$

Furthermore, there exist positive constants C_1, C_2, C_3 and $\varepsilon^*=\varepsilon^*(p^*)$ such that for all $\varepsilon\leq\varepsilon^*$ and $0<|p|\leq p^*$ we have

$$\text{(6.13)}\qquad (\operatorname{sgn} p)\cdot\Phi[\beta(x;p,p',\hat{h})]\geq C_1\varepsilon|p|-C_2\varepsilon^3-C_3|p'|.$$

Proof. A detailed proof is given in [11, Lemma 3.1]. In this, like the proof of Lemma 6.2, we use explicit solution formulas for v_* and z that are derived from a particular solution χ of the corresponding homogeneous equation. One thereby obtains $\Phi[v_*]\geq -C_3$ and the crucial identity $\Phi[z]=-\frac{1}{\chi(0)}\int_{-\infty}^{\infty}(\frac{1}{12}\frac{d^4}{d\xi^4}V_0)\chi\,d\xi=0$. □

Lemma 6.5. *There exist positive constants C_0, C_4, p'^* and ε^* such that for all $x\in(0,1)\setminus\{t_0\}$, $\varepsilon\leq\varepsilon^*$, $|p|\leq p^*$, $0<|p'|\leq p'^*$, the function β of (6.9) satisfies*

$$\text{(6.14)}\qquad (\operatorname{sgn} p')\cdot[F\beta-\tfrac{\hat{h}^2}{12}\tfrac{d^4}{d\xi^4}V_0]\geq \tfrac12 C_0|p'|\gamma^2-C_4(\varepsilon^3+\varepsilon\hat{h}^2+\hat{h}^4).$$

Proof. Imitate the analysis of [10, Lemma 3.2]; for details see [11, Lemma 3.2]. □

Lemma 6.6. *Let $p\geq 0$, $p'=C'\varepsilon p$ for some positive constant C', and $\hat{h}^2\leq C\varepsilon^\mu$ for some fixed $\mu\in(0,1]$. Then there exists $\varepsilon^*=\varepsilon^*(C',\mu)$ such that for the function β from (6.9) we have*

$$\text{(6.15)}\qquad \beta(x;-p,-p',\hat{h})\leq\beta(x;p,p',\hat{h})\qquad \textit{for } x\in[0,1],\ \ \varepsilon\leq\varepsilon^*,\ \ |p|\leq p^*.$$

Furthermore, for any arbitrarily small but fixed $\lambda\in(0,\bar\gamma)$, there is a constant C_λ such that $u_{\rm as}$ from (6.3) satisfies

$$\text{(6.16)}\qquad |\beta(x;\pm p,\pm p',\hat{h})-u_{\rm as}(x;0)|\leq C_\lambda\,(|p|+\hat{h}^2)\,e^{-(\bar\gamma-\lambda)|\xi|}+C\varepsilon|p|.$$

Proof. This result is obtained using the exponential decay of $\frac{\partial}{\partial p}v_0=\chi>0$ and $\frac{\partial}{\partial p}v_j$, $\frac{\partial}{\partial p}v_*$ $\frac{\partial}{\partial p}z$; see [11, Lemma 3.3] for further details. □

The above properties of $\beta=\beta(x;p,p',\hat{h})$ will be used in Section 6.4 to construct discrete sub- and super-solutions. First, we prove the following auxiliary result:

COROLLARY 6.7. *There is $\varepsilon^* > 0$ such that for all $\varepsilon \le \varepsilon^*$ there exists a solution u of problem (1.1) that satisfies*

$$|u(x) - u_{\rm as}(x;0)| \le C\varepsilon^2 \quad \text{for } x \in [0,1]. \tag{6.17}$$

Proof. Let $\bar p \ge C_2\varepsilon^2/(2C_1)$ and $\bar p' = C_1\varepsilon\bar p/(2C_3)$. By (6.13) we then get $\pm\Phi[\beta(\cdot;\pm\bar p,\pm\bar p',0)] \ge 0$. Choose $\bar p = O(\varepsilon^2)$ so large that $\frac12 C_0\bar p'\gamma^2 \ge C_4\varepsilon^3$; then by (6.14) one obtains $\pm F\beta(x;\pm\bar p,\pm\bar p',0) \ge 0$ for $x \in [0,1]\setminus\{t_0\}$. Furthermore, by (6.15), we have $\beta(x;-\bar p,-\bar p',0) \le \beta(x;\bar p,\bar p',0)$ for $x \in [0,1]$. Thus $\beta(x;-\bar p,-\bar p',0)$ and $\beta(x;\bar p,\bar p',0)$ are ordered sub- and super-solutions for the equation (1.1a). In view of $(A6)$, one gets $\pm\beta(x;\pm\bar p,\pm\bar p',0)\big|_{x=0,1} \ge g_{0,1}$, so $\beta(x;-\bar p,-\bar p',0)$ and $\beta(x;\bar p,\bar p',0)$ are sub- and super-solutions for the entire problem (1.1). It follows [16, Corollary 7.1 of Chapter 1] that there exists a solution u of (1.1) that lies between these sub- and super-solutions. By (6.16), this solution lies in an $O(\bar p) = O(\varepsilon^2)$ neighbourhood of $u_{\rm as}(x;0)$. ◻

A similar result can be found in [15] but for a slightly different asymptotic expansion.

6.3. Truncation error. We first examine the truncation error of the discrete operators F^N and $\hat F^N$ of (4.2) and (4.3) applied on our Shishkin mesh to a particular case of β from (6.9). Set

$$\beta_*(x) = \beta_*(x;p) = \beta(x;p,p',\hat h), \text{ where } \hat h := h/\varepsilon = CN^{-1}\ln N,\ p' := \varepsilon\frac{C_1}{2C_3}p, \tag{6.18}$$

where p is such that $|p| \le p^*$ for some fixed positive constant p^*, and

$$I_i := \begin{cases} 1 & \text{for } |x_i - t_0| \ge \tau, \\ 0 & \text{for } |x_i - t_0| < \tau. \end{cases} \tag{6.19}$$

LEMMA 6.8. *Let $C_\tau > 2$. Fix p with $|p| \le p^*$. Then*

$$F^N\beta_*(x_i) - F\beta_*(x_i) = -\tfrac{\hat h^2}{12}\tfrac{d^4}{d\xi^4}V_0(\xi_i) + O(\varepsilon\hat h^2 + \hat h^4 + N^{-2}I_i) \quad \text{for } x_i \ne t_0, \tag{6.20a}$$

where $\xi_i = (x_i - t_0)/\varepsilon$, and

$$F^N\beta_*(t_0) - \tfrac12[F\beta_*(t_0^-) + F\beta_*(t_0^+)] = -\tfrac{\hat h^2}{12}\tfrac{d^4}{d\xi^4}V_0(0) + \hat h^{-1}\big\{\Phi[\beta] + O(\varepsilon\hat h^2 + \hat h^4)\big\}. \tag{6.20b}$$

Furthermore,

$$|(\hat F^N - F^N)\,\beta_*(x_i)| \le CN^{-1}I_i. \tag{6.20c}$$

Proof. Throughout this proof we write β for β_*.

(a) Clearly $F^N v_i - Fv(x_i) = -\varepsilon^2[\delta_x^2 v_i - \frac{d^2}{dx^2}v(x_i)] =: r_i[v]$ for any suitable function $v(x)$. Thus we need to estimate $r_i[\beta]$. Let $x_i \ne t_0$. First, one has $|r[u_0 + \varepsilon^2 u_2]| \le C\varepsilon^2 N^{-1} \le C\varepsilon\hat h^2$, where we used (4.1). In the exponential-decay estimates (6.7) and (6.12), choose λ sufficiently small so that $C_\tau(1-\lambda/\bar\gamma) \ge 2$ and thus $e^{-(\bar\gamma-\lambda)\tau/\varepsilon} \le N^{-2}$ by (4.4). Then by imitating the truncation error analysis of [10, Lemma 3.3], we get $|r[v_{1,2}]| + |r[v_*]| + |r[z]| \le CN^{-2}\ln^2 N \le C\hat h^2$. Recalling the definition (6.9) of β and noting that (6.18) implies $|p'| \le C\varepsilon$, we now have $|r[\beta - v_0]| \le C(\varepsilon + p' + \hat{h}^2)\hat{h}^2 \le C(\varepsilon\hat{h}^2 + \hat{h}^4)$, so we will be done if we show that

$$(6.21) \qquad r_i[v_0] = r_i[V_0] = -\tfrac{\hat{h}^2}{12}\tfrac{d^4}{d\xi^4}V_0(\xi_i) + O(\hat{h}^4 + N^{-2}I_i).$$

For all $|x_i - t_0| < \tau$, the relationship (6.21) follows from a standard truncation error analysis using $\varepsilon^2\frac{h^2}{12}\frac{d^4}{dx^4}V_0 = \frac{\hat{h}^2}{12}\frac{d^4}{d\xi^4}V_0$ and $\varepsilon^2 h^4 \frac{d^6}{dx^6}V_0 = \hat{h}^4\frac{d^6}{d\xi^6}V_0 = O(\hat{h}^4)$; while if $|x_i - t_0| \ge \tau$, then by (6.7) the above choice of λ yields $|r[v_0]| \le 2\varepsilon^2|\frac{d^2}{dx^2}v_0| \le CN^{-2} \le C\hat{h}^2$ and $\left|\frac{\hat{h}^2}{12}\frac{d^4}{d\xi^4}V_0(\xi_i)\right| \le C\hat{h}^2N^{-2} \le C\hat{h}^4$, so we again get (6.21). Thus (6.20a) is established.

(b) At $x_i = t_0$, one again has (6.21) and it remains to estimate the truncation error $r[\tilde{\beta}]$ for the function $\tilde{\beta}(x) := \beta(x) - V_0(\xi)$. This function is continuous but has a discontinuous derivative at $x = t_0$. Note that

$$(6.22) \qquad \tilde{\beta}(t_0 \pm h) = \tilde{\beta}(t_0) \pm h\tfrac{d}{dx}\tilde{\beta}(t_0^\pm) + \tfrac{h^2}{2}\tfrac{d^2}{dx^2}\tilde{\beta}(t_0^\pm) \pm \tfrac{h^3}{6}\tfrac{d^3}{dx^3}\tilde{\beta}(\hat{x}^\pm)$$

for some $|\hat{x}^\pm - t_0| \le h$. Now $\frac{d^3}{dx^3}\tilde{\beta} = \frac{d^3}{dx^3}[u_0 + \varepsilon^2 u_2] + \varepsilon^{-3}\frac{d^3}{d\xi^3}[\varepsilon v_1 + \varepsilon^2 v_2 + p'v_* + \hat{h}^2 z]$, so $|\frac{d^3}{dx^3}\tilde{\beta}| \le C\varepsilon^{-3}(\varepsilon + p' + \hat{h}^2) \le C\varepsilon^{-3}(\varepsilon + \hat{h}^2)$. A calculation using (6.22) gives

$$-\varepsilon^2\delta_x^2\tilde{\beta}\big|_{t_0} = \tfrac{\varepsilon}{h}\Phi[\tilde{\beta}] - \tfrac{1}{2}\varepsilon^2[\tfrac{d^2}{dx^2}\tilde{\beta}(t_0^-) + \tfrac{d^2}{dx^2}\tilde{\beta}(t_0^+)] + \hat{h}\,O(\varepsilon + \hat{h}^2).$$

Combining this with (6.21) at $x_i = t_0$, where $\xi_i = 0$, and noting that $\varepsilon/h = \hat{h}^{-1}$, $\Phi[\tilde{\beta}] = \Phi[\beta]$, and $\frac{d^2}{dx^2}V_0$ is continuous at $x = t_0$, we get

$$-\varepsilon^2\delta_x^2\beta\big|_{t_0} = \hat{h}^{-1}\Phi[\beta] - \tfrac{1}{2}\varepsilon^2[\tfrac{d^2}{dx^2}\beta(t_0^-) + \tfrac{d^2}{dx^2}\beta(t_0^+)] - \tfrac{\hat{h}^2}{12}\tfrac{d^4}{d\xi^4}V_0(0) + \hat{h}^{-1}O(\varepsilon\hat{h}^2 + \hat{h}^4),$$

which yields (6.20b).

(c) Combining Remark 4.1 with the observations that $\hat{\varepsilon}_i^2 \le CN^{-2}$ and $|\delta_x^2\beta| \le C$ for $|x_i - t_0| > \tau$, we immediately get $|(\hat{F}^N - F^N)\,\beta(x_i)| \le CN^{-2}I_i$ for $|x_i - t_0| \ne \tau$. Now let $x_n = t_0 - \tau$ so $|(\hat{F}^N - F^N)\,\beta(x_n)| \le \hbar_n^{-1}(\hat{\varepsilon}_n^2 - \varepsilon^2)|D\beta(x_n)| \le CN^{-1}$, where we used $\hbar_n^{-1} \le CN$, $\hat{\varepsilon}_n^2 - \varepsilon^2 \le CN^{-2}$ and $|D\beta(x_n)| \le C$. Getting a similar bound at $x_i = t_0 + \tau$ completes the proof of (6.20c). ◻

To deal with the terms $N^{-2}I_i$ and $N^{-1}I_i$ in the above truncation error estimates, we need the following lemma.

Lemma 6.9. *Let N be sufficiently large and $p'' > 0$ be sufficiently small, independently of ε. Let $C'_\tau < C_\tau$. Then there exists w_i such that $0 \le w_i \le 1$ and for all $|p| \le p^*$ the function $\beta_*(x;p)$ from (6.18) satisfies $\pm\{F^N[\beta_*(x_i;p) \pm p''w_i] - F^N\beta_*(x_i;p)\} \ge p''(\gamma^2 I_i - CN^{-C'_\tau})$. Furthermore, this estimate holds true with F^N replaced by $\hat{F}^N$.*

Proof. Choose λ sufficiently small so that $C_\tau(1-\lambda/\bar{\gamma}) \ge C'_\tau$. We claim that there are a sufficiently small c' and a sufficiently large $\bar{C}'$ such that for all $|p| \le p^*$ we have

$$(6.23) \qquad b_u(x, \beta_*(x) + s) \ge \begin{cases} \gamma^2 & \text{for} \quad \varepsilon\bar{C}' \le |x - t_0|, \qquad |s| \le c', \\ (\bar{\gamma} - \lambda/2)^2 & \text{for} \quad \varepsilon\bar{C}' \le |x - t_0| < \tau, \quad |s| \le c'. \end{cases}$$

Indeed, for $x < t_0$ (the other case is similar), one has $b_u(x, \beta_*(x) + s) = b_u(x, \varphi_1(x)) + O(\beta_* - \varphi_1) + O(s)$, where the term $O(s)$ can be made as small as needed by choosing c' sufficiently small, while $\beta_*(x) = \varphi_1(x) + v_0 + O(N^{-1})$ so the term $O(\beta_* - \varphi_1)$ can be made as small needed by choosing $\bar{C}'$ and N sufficiently large. Combining this observation with Assumption $(A3)$ for $b_u(x,\varphi_1(x))$ yields the first bound of (6.23). Next, for $\varepsilon\bar{C}' \le t_0 - x < \tau$, one has $b_u(x,\varphi_1(x)) = b_u(t_0,\varphi_1(t_0)) + O(\tau) \ge \bar{\gamma}^2 - C\varepsilon \ln N$ by (4.5), so by virtue of (4.1) we get the second bound of (6.23) for sufficiently large N.

Define

$$w_i := \begin{cases} 1 & \text{for} \quad |x_i - t_0| \ge \tau, \\ \omega(x_i)\, e^{-(\bar{\gamma}-\lambda)(\tau - |x_i - t_0|)/\varepsilon} & \text{for} \quad |x_i - t_0| < \tau, \end{cases}$$

where $\omega(x)$ is a smooth cut-off function that takes values in $[0,1]$, equals 1 for $|x-t_0| \ge 2\varepsilon\bar{C}'$ and vanishes for $|x - t_0| \le \varepsilon\bar{C}'$. As $0 \le w_i \le 1$, choosing $0 < p'' \le c'$, we apply the standard linearisation and then invoke (6.23) to get

$$\begin{aligned} &F^N[\beta_*(x_i) + p''w_i] - F^N\beta_*(x_i) \\ &\qquad \ge p'' \begin{cases} -\varepsilon^2\,\delta_x^2 w_i + \gamma^2 w_i & \text{for} \quad \tau \le |x_i - t_0|, \\ -\varepsilon^2\,\delta_x^2 w_i + (\bar{\gamma} - \lambda/2)^2 w_i & \text{for} \quad \varepsilon\bar{C}' \le |x_i - t_0| < \tau, \\ 0 & \text{for} \quad |x_i - t_0| < \varepsilon\bar{C}', \end{cases} \\ &\qquad \ge p'' \begin{cases} \gamma^2 & \text{for } |x_i - t_0| \ge \tau, \\ 0 & \text{for } |x_i - t_0| \in [0,\tau)\backslash(\varepsilon\bar{C}',\, 2\varepsilon\bar{C}'), \\ -CN^{-C'_\tau} & \text{for } |x_i - t_0| \in (\varepsilon\bar{C}',\, 2\varepsilon\bar{C}'). \end{cases} \end{aligned} \tag{6.24}$$

Here we used $-\delta_x^2 w_i \ge 0$ for $|x_i - t_0| = \tau$ and $\varepsilon|\frac{d}{dx}\omega| + \varepsilon^2|\frac{d^2}{dx^2}\omega| \le C$ combined with $e^{-(\bar{\gamma}-\lambda)(\tau - |x-t_0|)/\varepsilon} \le CN^{-C'_\tau}$ for $|x - t_0| \in (\varepsilon\bar{C}' - h,\, 2\varepsilon\bar{C}' + h)$. (The final estimate here follows from the choice of λ earlier, which implies that $e^{-(\bar{\gamma}-\lambda)\tau/\varepsilon} \le N^{-C'_\tau}$.) Now, in view of the definition (6.19) of I_i, (6.24) yields the desired result for $F^N[\beta_*(x_i) + p''w_i] - F^N\beta_*(x_i)$.

The assertions for $F^N[\beta_*(x_i) - p''w_i] - F^N\beta_*(x_i)$ and $\hat{F}^N[\beta_*(x_i) \pm p''w_i] - \hat{F}^N\beta_*(x_i)$ are obtained similarly. In particular, when carrying out the analysis for $\hat{F}^N$, we use Remark 4.1 and the following two observations: $\delta_x^2 w_i = 0$ for $|x_i - t_0| > \tau$ and the stabilised discretization of $\varepsilon^2\frac{d^2}{dx^2}$ in $\hat{F}^N$, when applied to w_i, simply yields $\varepsilon^2\delta_x^2 w_i$ at $|x_i - t_0| = \tau$ (e.g., at $x_n = t_0 - \tau$ we have $\hbar_n^{-1}[\varepsilon^2 D w_{n+1} - \hat{\varepsilon}_n^2 \cdot 0] = \varepsilon^2\delta_x^2 w_n$). Hence (6.24) with F^N replaced by $\hat{F}^N$ remains true. ◻

6.4. Sufficient conditions for discrete sub-solutions and super-solutions. Combining Lemmas 6.8 and 6.9 with the bounds for β that were obtained in Section 6.2, we now establish sufficient conditions for $\beta_*(x;\pm p) \pm p''w_i$ to be sub- and super-solutions of the discrete equations (4.2) and (4.3).

LEMMA 6.10. *Let $C_\tau > 2$. There exist sufficiently large positive constants $\bar{C}$, $\bar{C}'$ and sufficiently small positive constants $\varepsilon^* = \varepsilon^*(p^*)$, $c_0 = c_0(p^*)$ and $c_1 = c_1(p^*)$ such that if $\varepsilon \le \varepsilon^*$, $N \ge c_1^{-1}$ and $\hat{h}^4 \le c_0\varepsilon$, then $\bar{p} := \bar{C}(\varepsilon^2 + \hat{h}^2 + \hat{h}^4/\varepsilon) \le p^*$, and $\beta_*(x;p)$ from (6.18) with $p = \pm\bar{p}$ satisfies*

$$F^N[\beta_*(x_i;-\bar{p}) - \bar{C}'N^{-2}w_i] \le 0 \le F^N[\beta_*(x_i;\bar{p}) + \bar{C}'N^{-2}w_i] \tag{6.25}$$

for $i = 1,\dots,N-1$, where $0 \le w_i \le 1$.

Furthermore, if $C_\tau > 3$, then for $i = 1,\dots,N-1$ we have

$$\hat{F}^N[\beta_*(x_i;-\bar{p}) - \bar{C}'N^{-1}w_i] \le 0 \le \hat{F}^N[\beta_*(x_i;\bar{p}) + \bar{C}'N^{-1}w_i] \tag{6.26}$$

Proof. By (6.18), $\operatorname{sgn} p = \operatorname{sgn} p'$. Combining (6.20a) with (6.14) yields

$$(6.27a)\quad (\operatorname{sgn} p)\cdot F^N\beta_*(x_i;p) \ge \tfrac{1}{2}C_0|p'|\gamma^2 - C_4'(\varepsilon^3+\varepsilon\hat h^2+\hat h^4+N^{-2}I_i) \quad \text{for } x_i \ne t_0,$$

where $C_4' > C_4$ and takes into account the term $O(\varepsilon\hat h^2+\hat h^4+N^{-2}I_i)$ in (6.20a). Similarly, at $x_i = t_0$, using (6.20b) and (6.14), we get

$$(\operatorname{sgn} p)\cdot F^N\beta_*(t_0;p) \ge \tfrac{1}{2}C_0|p'|\gamma^2 - C_4(\varepsilon^3+\varepsilon\hat h^2+\hat h^4)+\hat h^{-1}\bigl\{(\operatorname{sgn} p)\cdot\Phi[\beta]+O(\varepsilon\hat h^2+\hat h^4)\bigr\}.$$

Combining this inequality with (6.13)—in which, by (6.18), we have $C_1\varepsilon p - C_3 p' = \frac{1}{2}C_1\varepsilon p$—we arrive at

$$(6.27b)\qquad \begin{aligned}(\operatorname{sgn} p)\cdot F^N\beta_*(t_0;p) \ge{}& \tfrac{1}{2}C_0|p'|\gamma^2 - C_4'(\varepsilon^3+\varepsilon\hat h^2+\hat h^4)\\ &+\tfrac{1}{\hat h}\bigl\{\tfrac{1}{2}C_1\varepsilon|p| - C_2\varepsilon^3 - C_3'(\varepsilon\hat h^2+\hat h^4)\bigr\}\end{aligned}$$

for some C_3' and $C_4' > C_4$. Applying Lemma 6.9 with $p'' := \bar C' N^{-2} \ge (C_4'/\gamma^2)N^{-2}$ and $C_\tau' := 2 < C_\tau$, one gets

$$(6.28)\qquad \pm\bigl\{F^N[\beta_*(x_i)\pm\bar C'N^{-2}w_i] - F^N\beta_*(x_i)\bigr\} \ge C_4'N^{-2}I_i - C_5N^{-4}$$

for some $C_5 > 0$. The desired assertion (6.25) now follows from (6.27) combined with (6.28) provided that there exist $\bar p \in (0,p^*]$ and $\bar p' := \varepsilon\frac{C_1}{2C_3}\bar p$ that satisfy

$$\tfrac{1}{2}C_0\bar p'\gamma^2 \ge C_4'(\varepsilon^3+\varepsilon\hat h^2+\hat h^4)+C_5N^{-4} \quad\text{and}\quad \tfrac{1}{2}C_1\varepsilon\bar p \ge C_2\varepsilon^3 + C_3'(\varepsilon\hat h^2+\hat h^4).$$

As $N^{-4} \le C\hat h^2$, we choose $\bar p := \bar C(\varepsilon^2+\hat h^2+\hat h^4/\varepsilon)$ for some sufficiently large $\bar C$, and then impose the conditions $\varepsilon \le \varepsilon^*$, $N \ge c_1^{-1}$ and $\hat h^4 \le c_0\varepsilon$ so that $\bar C(\varepsilon^2+\hat h^2+\hat h^4/\varepsilon) \le p^*$.

The relation (6.26) is established in a similar manner invoking (6.20c) and choosing $p'' := \bar C'N^{-1}$ and $C_\tau' = 3 < C_\tau$ in Lemma 6.9. □

6.5. Proofs of Theorems 5.1 and 5.2. We are finally ready to establish the main results of the paper.

Proof of Theorem 5.1. (i) First, $c_0\varepsilon \ge \hat h^{2+\sigma} \ge \hat h^4$. By Lemma 6.10, we choose $\bar p = \bar C(\varepsilon^2+\hat h^2+\hat h^4/\varepsilon) \le C(N^{-1}\ln N)^{2-\sigma}$ such that (6.25) holds true with $0 \le w_i \le 1$. Furthermore, applying Lemma 6.6 with $\mu = 1/2$ yields $\beta_*(x_i;-\bar p) \le \beta_*(x_i;\bar p)$, while Assumption $(A6)$ implies that $\pm\beta_*(x;\pm\bar p)\big|_{x=0,1} \ge g_{0,1}$. These observations imply that $\beta_*(x_i;-\bar p) - \bar C'N^{-2}w_i$ and $\beta_*(x_i;\bar p)+\bar C'N^{-2}w_i$ are ordered sub- and super-solutions for the discrete problem (4.2). Hence, invoking [10, Lemma 3.1] (see also [12]), we conclude that there exists a solution $\{u_i^N\}$ of (4.2) such that $\beta_*(x_i;-\bar p)-\bar C'N^{-2}w_i \le u_i^N \le \beta_*(x_i;\bar p)+\bar C'N^{-2}w_i$.

But combining (6.16), (6.17) and (4.1) shows that for all i one has $|\beta_*(x_i;\pm\bar p) - u(x_i)| \le C\bar p e^{-(\bar\gamma-\lambda)|\xi|} + C(\varepsilon\bar p + N^{-2})$. It now follows that $|u_i^N - u(x_i)| \le C\bar p e^{-(\bar\gamma-\lambda)|\xi|} + C(\varepsilon\bar p+N^{-2})$ for all i. This implies $|u_i^N - u(x_i)| \le C(\bar p+N^{-2}) \le C(N^{-1}\ln N)^{2-\sigma}$ for $|x_i - t_0| < \tau$. For $|x_i - t_0| \ge \tau$, by choosing λ as in the proof of Lemma 6.8 one obtains $e^{-(\bar\gamma-\lambda)|\xi|} \le N^{-2}$ and $\varepsilon\bar p = \bar C(\varepsilon^3+\varepsilon\hat h^2+\hat h^4) \le CN^{-2}$, which imply $|u_i^N - u(x)| \le CN^{-2}$. Thus we have obtained the error estimate (5.1).

(ii) The existence of $\hat u_i^N$ and the error estimate (5.2) are established in a similar manner, but using $\beta_*(x_i;\pm\bar p)\pm\bar C'N^{-1}w_i$ as discrete sub- and super-solutions and applying (6.26) instead of (6.25). □

Proof of Theorem 5.2. Here we use simpler sub- and super-solutions based on the function

$$\alpha(x_i;p;\mathcal{I}_1) := \begin{cases} \varphi_1(x_i)+p & \text{for } i\in\mathcal{I}_1, \\ \varphi_2(x_i)+p & \text{for } i\in\mathcal{I}_2=\{0,\ldots,N\}\setminus\mathcal{I}_1. \end{cases}$$

(i) Set $\bar p := \bar C N^{-\min\{2,\theta-2\}}$. We claim that for sufficiently large $\bar C$, the functions $\tilde\alpha = \alpha(x_i;-\bar p;\{x_i<t_0+\tau\})$ and $\hat\alpha = \alpha(x_i;\bar p;\{x_i\le t_0-\tau\})$ are ordered discrete sub- and super-solutions for (4.2). Indeed, for $x_n = t_0-\tau$ we have $-\varepsilon^2\delta_x^2\hat\alpha_{n+1}\ge 0$ and also $\varepsilon^2|\delta_x^2\hat\alpha_n| \le C\varepsilon^2\hbar_n^{-1}[1+h^{-1}] \le CN^{-(\theta-2)}$ as $\hbar_n^{-1}=O(N)$ and $\varepsilon^2/h = \varepsilon/\hat h \le CN^{-(\theta-1)}$. Note also that $\varepsilon^2|\delta_x^2\hat\alpha| \le C\varepsilon^2 \le CN^{-2}$ at the other mesh points. Combining this with $b(x,\hat\alpha)\ge \frac12\gamma^2\bar p$ yields $F^N\hat\alpha\ge 0$. Similarly, $F^N\tilde\alpha\le 0$. Consequently there exists a discrete solution $\{u_i^N\}$ of (4.2) such that $\tilde\alpha \le u_i^N \le \hat\alpha$ for all i. But as $C_\tau > 2$, for $|x_i-t_0|>\tau$ one gets (writing α for $\tilde\alpha$ and $\hat\alpha$) $\alpha = u_0(x)+O(\bar p) = u(x)+O(\varepsilon^2+N^{-2}+\bar p) = u(x)+O(N^{-\min\{2,\theta-2\}})$, which immediately yields (5.3).

(ii) The existence of $\{\hat u_i^N\}$ and the error estimate (5.4) are established in a similar manner by showing that $\tilde\alpha$ and $\hat\alpha$, but with $\bar p := \bar C N^{-1}$, are discrete sub- and super-solutions for (4.3). In particular, we use three observations: first, the discretization of $\varepsilon^2\frac{d^2}{dx^2}$ in $\hat F^N$ applied to $\hat\alpha$ at $t_0\pm\tau$ is bounded by CN^{-1}, e.g., at $x_n=t_0-\tau$ it is bounded by $C\hbar_n^{-1}[\varepsilon^2/h+\hat\varepsilon_n^2]\le CN^{-1}$ as $\hbar_n^{-1}=O(N)$, $\varepsilon^2/h=\varepsilon/\hat h\le CN^{-2}$ and $\hat\varepsilon_n^2 = O(N^{-2})$; second, the same quantity at x_{n+1} is clearly non-negative; third, in view of Remark 4.1 and $\hat\varepsilon_i^2\le CN^{-2}$, the discretization of $\varepsilon^2\frac{d^2}{dx^2}$ in $\hat F^N$ applied to $\hat\alpha$ at the other mesh points is bounded by CN^{-2}. Consequently, as in part (i) of the proof, there exists a discrete solution $\{\hat u_i^N\}$ of (4.3) such that $\tilde\alpha\le\hat u_i^N\le\hat\alpha$ for all i. But $C_\tau>1$ implies that $u_0(x)=u(x)+O(\varepsilon^2+N^{-1})$ for $|x_i-t_0|>\tau$, which leads to the error estimate (5.4). ◻

REMARK 6.11. *An inspection of the proof of Theorem 5.2 shows that its statement remains valid on other layer-adapted meshes, such as the Bakhvalov mesh, whose description can be found, e.g., in [8, 9, 17]. But it would be more difficult to establish a version of Theorem 5.1 on layer-adapted meshes other than the Shishkin mesh, because dealing with the principal part of the intricate truncation error estimated in Lemma 6.8 would require an alteration of the statement of this lemma and consequently some modification of the right-hand side in problem* (6.11) *would be needed.*

7. Numerical results. In this section we present numerical results for the interior-layer solution of Example 2.1 displayed in Figure 2.2. It is not feasible to construct a truly representative test problem whose exact solution is known, since the location of the interior layer in such a problem must depend on ε in a very complicated way, as the analysis of Section 6.1 reveals. Thus for Example 2.1 we computed reference solutions by using a Shishkin mesh with $N=2^{15}$ (centred about $t:=t_0+\varepsilon t_1$ with $t_1\approx -1.45$ defined in [11, Lemma 2.4]) and combining the standard discretization (4.2) on the intervals $[0,t]$ and $[t,1]$ with shooting in the value $U_n = 1+O(\varepsilon)$ at $x_n=t$.

All our results below are for the Shishkin mesh of Section 4 centred about t_0, with the transition parameter $\tau := \min\{(3.25\,\varepsilon/\bar\gamma)\ln(N/4),\, t_0/2\}$. Thus in (4.4) we have set $C_\tau := 3.25$, replaced $\ln N$ by $\ln(N/4)$, and also required $\tau\le t_0/2$; all our theoretical conclusions still remain valid.

The discrete nonlinear systems of equations were solved by *Newton's method*, modified by the constraint that the iterates remain non-negative. The initial guess

TABLE 7.1
Maximum nodal errors for the stabilised scheme (4.3) with $\hat{C} = 2.5$.

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$	$N = 2048$
2^{-4}	8.88e-3	2.24e-3	5.63e-4	1.41e-4	3.52e-5	8.79e-6
2^{-5}	1.48e-2	5.39e-3	1.34e-3	3.34e-4	8.34e-5	2.08e-5
2^{-6}	1.72e-2	5.76e-3	1.99e-3	6.74e-4	2.20e-4	6.94e-5
2^{-7}	2.20e-2	9.65e-3	3.44e-3	6.81e-4	2.22e-4	6.98e-5
2^{-8}	2.45e-2	1.17e-2	5.19e-3	1.91e-3	2.47e-4	7.01e-5
2^{-9}	3.39e-2	1.28e-2	6.08e-3	2.71e-3	1.01e-3	1.50e-4
2^{-10}	5.27e-2	1.33e-2	6.53e-3	3.11e-3	1.39e-3	5.23e-4
2^{-11}	8.41e-2	1.36e-2	6.76e-3	3.31e-3	1.58e-3	7.05e-4
2^{-12}	1.32e-1	1.37e-2	6.87e-3	3.41e-3	1.67e-3	7.94e-4
2^{-13}	1.98e-1	1.38e-2	6.93e-3	3.46e-3	1.71e-3	8.38e-4
2^{-14}	2.80e-1	1.38e-2	6.96e-3	3.48e-3	1.74e-3	8.59e-4
2^{-15}	3.82e-1	2.01e-2	6.97e-3	3.50e-3	1.75e-3	8.70e-4
2^{-16}	4.97e-1	3.40e-2	6.98e-3	3.50e-3	1.75e-3	8.75e-4
2^{-17}	6.05e-1	5.82e-2	6.98e-3	3.51e-3	1.76e-3	8.78e-4
2^{-18}	6.99e-1	9.70e-2	6.98e-3	3.51e-3	1.76e-3	8.79e-4
2^{-19}	7.90e-1	1.54e-1	6.98e-3	3.51e-3	1.76e-3	8.80e-4
2^{-20}	8.52e-1	2.30e-1	7.24e-3	3.51e-3	1.76e-3	8.80e-4
2^{-21}	8.87e-1	3.22e-1	1.24e-2	3.51e-3	1.76e-3	8.80e-4
2^{-22}	9.08e-1	4.22e-1	2.24e-2	3.51e-3	1.76e-3	8.81e-4
2^{-23}	9.21e-1	5.25e-1	4.05e-2	3.51e-3	1.76e-3	8.81e-4
2^{-24}	9.28e-1	6.27e-1	7.10e-2	3.51e-3	1.76e-3	8.81e-4
2^{-25}	9.32e-1	7.09e-1	1.18e-1	3.51e-3	1.76e-3	8.81e-4
2^{-26}	9.35e-1	7.66e-1	1.82e-1	4.59e-3	1.76e-3	8.81e-4
2^{-27}	9.36e-1	8.01e-1	2.65e-1	7.93e-3	1.76e-3	8.81e-4
2^{-28}	9.36e-1	8.20e-1	3.57e-1	1.61e-2	1.76e-3	8.81e-4
2^{-29}	9.36e-1	8.30e-1	4.55e-1	3.26e-2	1.76e-3	8.81e-4
2^{-30}	9.36e-1	8.42e-1	5.42e-1	1.23e-1	1.76e-3	8.81e-4
2^{-31}	9.37e-1	8.32e-1	5.56e-1	3.63e-1	1.76e-3	2.37e-3
2^{-32}	9.37e-1	8.32e-1	9.55e-1	3.01e-1	3.30e-2	1.27e-2

was (unless otherwise specified) the straight line $y = 2.5x$ that joins the two boundary values. The iterations were terminated when two conditions were both satisfied: two successive iterates differ pointwise by at most 10^{-8} and $|(F^N u^N)_i| \leq 10^{-8}$ for all i.

Tables 7.1 and 7.2 show the maximum nodal errors for the stabilised scheme (4.3), in which we used the *stabilisation parameter* $\hat{C} := 2.5$ so that $\hat{C}^2 \geq -\min b_u(x, v) \approx 2.5^2$, where the minimum is taken over the set $\{v \in [\varphi_1(x), \varphi_2(x)], x \in [0, 1]\}$. (Note that with this choice, the diagonal entries in the Jacobian matrix for the stabilised scheme (4.3) are always positive.) Table 7.1 shows that, for each fixed ε, the errors typically decrease as N increases. But if N is fixed, the errors deteriorate as ε decreases, which agrees with the error estimate (5.2) (this deterioration is partly due to the fact that the multiple-solution phenomenon exhibited in the rightmost diagram of Figure 2.3 also occurs for (4.3) when ε becomes too small relative to N^{-1}). From the practical point of view, the errors are quite small provided ε is not too small (while our error estimate (5.2) shows their dependence on N). In Table 7.2 we observe that, in agreement with the error estimates (5.2) and (5.4), the errors outside the layer region stabilise as ε becomes smaller, and first-order accurate uniformly in ε.

We have observed that the stabilized scheme is robust with respect to the initial guess in the sense that all smooth positive initial guesses that we have tried, including $y = 1 - x$, $y = 1$, $y = x + 1.5$, $y = 4$, $y = \sin(\pi x)$ and $y = e^x$, yielded errors almost identical with those in Tables 7.1 and 7.2 where the initial guess was $y = 2.5x$. Furthermore, similar tables of errors were produced when using some discontinuous initial guesses such as $y = 1 + \operatorname{sgn}(x - 0.5)$ and $y = e^x - \operatorname{sgn}(x - 0.5)$; for others, the errors were identical whenever Newton's method converged.

We have described in Section 2 how the standard scheme (4.2) yields completely unsatisfactory solutions if we start with an unsophisticated initial guess for Newton's method. But if we take the initial guess equal to the computed solution of the stabilised scheme (4.3), which is already close to the exact solution, then the resulting errors presented in Tables 7.3 and 7.4 show improved accuracy both in the layer region (unless $\varepsilon \ll N^{-1}$) and outside it when compared with Tables 7.1 and 7.2, and agree with the error estimates (5.1) and (5.3).

TABLE 7.2

Maximum nodal errors for the stabilised scheme (4.3) with $\hat{C} = 2.5$ *on* $[0, t_0 - \tau] \cup [t_0 + \tau, 1]$ *(i.e., outside the layer region).*

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$	$N = 2048$
2^{-4}	2.80e-3	7.14e-4	1.79e-4	4.49e-5	1.12e-5	2.80e-6
2^{-5}	8.46e-3	4.45e-6	1.12e-6	2.81e-7	7.02e-8	1.75e-8
2^{-6}	1.72e-2	5.76e-3	1.28e-4	8.20e-9	7.97e-10	5.64e-11
2^{-7}	2.20e-2	9.65e-3	3.44e-3	3.25e-4	3.27e-9	6.98e-11
2^{-8}	2.45e-2	1.17e-2	5.19e-3	1.91e-3	2.47e-4	4.76e-10
2^{-9}	2.58e-2	1.28e-2	6.08e-3	2.71e-3	1.01e-3	1.50e-4
2^{-10}	2.65e-2	1.33e-2	6.53e-3	3.11e-3	1.39e-3	5.23e-4
2^{-11}	2.68e-2	1.36e-2	6.76e-3	3.31e-3	1.58e-3	7.05e-4
2^{-12}	2.70e-2	1.37e-2	6.87e-3	3.41e-3	1.67e-3	7.94e-4
2^{-13}	2.71e-2	1.38e-2	6.93e-3	3.46e-3	1.71e-3	8.38e-4
2^{-14}	2.71e-2	1.38e-2	6.96e-3	3.48e-3	1.74e-3	8.59e-4
2^{-15}	2.71e-2	1.38e-2	6.97e-3	3.50e-3	1.75e-3	8.70e-4
2^{-16}	2.72e-2	1.38e-2	6.98e-3	3.50e-3	1.75e-3	8.75e-4
2^{-17}	2.72e-2	1.38e-2	6.98e-3	3.51e-3	1.76e-3	8.78e-4
2^{-18}	2.72e-2	1.38e-2	6.98e-3	3.51e-3	1.76e-3	8.79e-4
$2^{-19} \dots 2^{-21}$	2.72e-2	1.38e-2	6.98e-3	3.51e-3	1.76e-3	8.80e-4
$2^{-22} \dots 2^{-32}$	2.72e-2	1.38e-2	6.99e-3	3.51e-3	1.76e-3	8.81e-4

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TABLE 7.3
Maximum nodal errors for the standard scheme (4.2): solutions of the stabilised scheme (4.3) were used as the initial guess for Newton's method.

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$	$N = 2048$
2^{-4}	8.88e-3	2.24e-3	5.63e-4	1.41e-4	3.52e-5	8.79e-6
2^{-5}	1.43e-2	5.39e-3	1.34e-3	3.34e-4	8.34e-5	2.08e-5
2^{-6}	1.56e-2	5.53e-3	1.99e-3	6.74e-4	2.20e-4	6.94e-5
2^{-7}	1.81e-2	5.60e-3	2.01e-3	6.81e-4	2.22e-4	6.98e-5
2^{-8}	2.29e-2	5.67e-3	2.03e-3	6.85e-4	2.23e-4	7.01e-5
2^{-9}	3.22e-2	5.78e-3	2.03e-3	6.89e-4	2.24e-4	7.03e-5
2^{-10}	5.12e-2	6.02e-3	2.04e-3	6.90e-4	2.24e-4	7.04e-5
2^{-11}	8.31e-2	6.50e-3	2.05e-3	6.90e-4	2.25e-4	7.05e-5
2^{-12}	1.32e-1	7.45e-3	2.06e-3	6.91e-4	2.25e-4	7.05e-5
2^{-13}	1.98e-1	9.33e-3	2.08e-3	6.91e-4	2.25e-4	7.05e-5
2^{-14}	2.81e-1	1.30e-2	2.12e-3	6.92e-4	2.25e-4	7.06e-5
2^{-15}	3.85e-1	2.01e-2	2.19e-3	6.92e-4	2.25e-4	7.04e-5
2^{-16}	5.04e-1	3.41e-2	2.35e-3	6.94e-4	2.25e-4	7.05e-5
2^{-17}	6.18e-1	5.84e-2	2.65e-3	6.97e-4	2.24e-4	6.95e-5
2^{-18}	7.32e-1	9.75e-2	3.32e-3	7.07e-4	2.28e-4	7.32e-5
2^{-19}	8.49e-1	1.55e-1	4.62e-3	7.17e-4	2.24e-4	6.98e-5
2^{-20}	9.59e-1	2.32e-1	7.29e-3	7.46e-4	2.26e-4	7.08e-5
2^{-21}	1.09e+0	3.27e-1	1.25e-2	8.01e-4	2.27e-4	7.12e-5
2^{-22}	1.13e+0	4.31e-1	2.26e-2	9.11e-4	2.28e-4	7.25e-5
2^{-23}	1.39e+0	5.45e-1	4.09e-2	1.14e-3	2.31e-4	7.06e-5
2^{-24}	1.35e+0	6.67e-1	7.18e-2	1.61e-3	2.36e-4	7.20e-5
2^{-25}	1.42e+0	7.83e-1	1.19e-1	2.56e-3	2.48e-4	7.28e-5
2^{-26}	1.29e+0	9.01e-1	1.85e-1	4.67e-3	3.43e-4	6.96e-5
2^{-27}	1.30e+0	1.01e+0	2.72e-1	8.10e-3	8.42e-4	8.39e-5
2^{-28}	1.30e+0	1.11e+0	3.71e-1	1.64e-2	1.75e-4	7.15e-5
2^{-29}	1.30e+0	1.21e+0	4.82e-1	3.32e-2	5.46e-4	7.75e-5
2^{-30}	1.61e+0	1.35e+0	5.96e-1	1.25e-1	8.17e-4	8.54e-4
2^{-31}	1.43e+0	1.51e+0	5.84e-1	3.62e-1	1.56e-3	2.32e-3
2^{-32}	1.30e+0	1.26e+0	1.10e+0	3.01e-1	3.29e-2	1.27e-2

TABLE 7.4
Maximum nodal errors for the standard scheme (4.2) on $[0, t_0 - \tau] \cup [t_0 + \tau, 1]$ *(i.e., outside the layer region): solutions of the stabilised scheme (4.3) were used as the initial guess for Newton's method.*

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$	$N = 2048$
$\max_{2^{-k}, k=4\ldots32}$	3.12e-3	7.14e-4	1.79e-4	4.49e-5	1.12e-5	2.80e-6

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