New Families of non-Keplerian Orbits: Solar Sail Motion over Cylinders and Spheres

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This paper presents new families of Sun-centered non-Keplerian orbits (NKO), where the motion of a high-performance solar sail is confined to either a cylindrical or spherical surface. These orbits are found by investigating the geometrically constrained sail dynamics and imposing further constraints on the angular velocity and lightness number to generate pure solar sail trajectories. By considering the sail motion in the phase space of the problem, families of new NKOs are identified, and by investigating the oscillating behavior of the orbits, true periodic orbits are found. As extension to the well-known families of displaced NKOs, these three-dimensional NKOs generate a wealth of new solar sail orbits and novel sail applications.

I. Introduction

THE dynamics of two-body solar sail displaced non-Keplerian orbits (NKO) is well-known in the literature [1, 2]: by considering the equations of motion of a solar sail in a rotating frame of reference, equilibrium solutions can be found that correspond to a circular orbit displaced away from the natural Keplerian orbit in an inertial frame. Through the choice for the angular velocity of the rotating frame, families of displaced NKOs can be identified. The dynamics, stability and control of these families have been investigated [3] as well as their abundant applications: to guarantee future geostationary slots for telecommunication, Earth observation and weather satellites the use of NKOs in the form of displaced geostationary orbits has been investigated [4, 5]; NKOs have been suggested to hover above Saturn's ring for high resolution in-situ observations [6]; and highly

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elliptic geocentric NKOs that lie in the ecliptic plane and rotate such that apogee is always located behind the Earth have been proposed for geomagnetic tail investigations [7].

However, as this paper will demonstrate, by moving away from these two-dimensional non-Keplerian orbits, and instead constraining the motion of the solar sail to be over a three-dimensional surface such as a cylinder or sphere, a wealth of new orbits is created that enable a range of novel space applications, including interplanetary communication, astronomical observations, solar physics, and many more.

The required solar sail technology to enable these orbits ranges from near-term to far-term, depending on the out-of-plane displacements considered. However, with advances in solar sail technology through JAXA's successful IKAROS mission [8] and NASA's NanoSail-D2 sail [9] and with more missions scheduled for the next few years, including NASA's Sunjammer mission [10] and The Planetary Society's LightSail mission [11], the far-term solutions get more and more within reach. Furthermore, similar far-term technology has been proposed in previous studies such as those for the Heliopause mission [12] and additional technologies such as perforated sails [13] may in the future enable the far-term solutions presented in this paper.

To introduce these novel cylindrical and spherical orbits, the structure of the paper will be as follows. The paper will start with the cylindrical case: the adopted reference frame is introduced, the equations of motion are derived and the control law required to keep the solar sail on a cylindrical surface are obtained. This control law is a function of the acceleration and for the cylindrical case, two types of solar sail accelerations will be considered: the first type of acceleration assumes only small out-of-plane displacements such that the pitch and cone angles can be assumed to be equal. This will allow for analytical investigations, while the second, true solar sail acceleration only allows for numerical investigations. For both types of acceleration, constraints on the in-plane angular velocity and the sail lightness number are introduced which result into a feasibility map of the cylindrically constrained orbits. The phase spaces of the orbits within this feasibility map allow to structure the obtained orbits into families and Poincaré maps are used to show the periodicity of the orbits. True periodic orbits are subsequently obtained and a set of typical, periodic orbits is presented. Very similar analyses are subsequently performed for the spherical case and at the end of the paper the conclusions will be drawn.

II. Cylindrically constrained NKOs

A. Reference frames and equations of motion

In order to derive the cylindrically constrained NKOs, a cylindrical coordinate frame is adopted with ρ the projected radius, z the out-of-plane displacement and θ the angle measured from the reference x -axis, see Fig.

1. Furthermore, an acceleration of magnitude a is assumed which lies in the (ρ, z) -plane only and is pitched with respect to the (x, y) -plane at an angle α . Under the assumption of a central gravitational force field with a gravitational potential $V = -\frac{\mu}{r}$, the equations of motion are given by:

$$\begin{aligned}\ddot{\rho} - \rho\dot{\theta}^2 &= -\frac{\mu}{r^3}\rho + a \cos \alpha \\ \rho^2\ddot{\theta} + 2\rho\dot{\rho}\dot{\theta} &= 0 \\ \ddot{z} &= -\frac{\mu}{r^3}z + a \sin \alpha\end{aligned}$$

In order for the orbit to be constrained to a cylindrical surface, the following constraint is enforced:

$$\rho = \text{constant} \quad \rightarrow \quad \dot{\rho} = \ddot{\rho} = 0$$

which reduces the equations of motion to:

$$\begin{aligned}-\rho\dot{\theta}^2 &= -\frac{\mu}{r^3}\rho + a \cos \alpha \\ \rho^2\ddot{\theta} &= 0 \\ \ddot{z} &= -\frac{\mu}{r^3}z + a \sin \alpha\end{aligned}$$

From these equations it becomes clear that the in-plane angular velocity $\dot{\theta} = \omega$ is constant. Furthermore, using $\bar{\omega}$ to denote the angular velocity of a Keplerian orbit with radius ρ :

$$\bar{\omega} = \sqrt{\frac{\mu}{\rho^3}}$$

Fig. 1 Cylindrical coordinate frame definition.

and using $r = \sqrt{\rho^2 + z^2} = \rho \sqrt{1 + \left(\frac{z}{\rho}\right)^2}$, the equations of motion can be further rewritten as:

$$\begin{aligned} \cos \alpha &= \frac{\rho \bar{\omega}^2}{a} \left(\left(1 + \left(\frac{z}{\rho} \right)^2 \right)^{-\frac{3}{2}} - \left(\frac{\omega}{\bar{\omega}} \right)^2 \right) \\ \ddot{z} &= -z \bar{\omega}^2 \left(1 + \left(\frac{z}{\rho} \right)^2 \right)^{-\frac{3}{2}} + \text{sign}(z) a \sqrt{1 - \cos^2 \alpha} \end{aligned} \quad (1)$$

The first equation in Eq. (1) gives the control law required to stay on a cylindrical surface. Note that, due to the symmetry of the problem, the $\text{sign}(z)$ -term is included to mirror the acceleration in the (x, y) -plane upon crossing.

As mentioned in the introduction of this paper, two different solar sail accelerations will be considered:

1. A true solar sail acceleration law:

$$a = \beta \frac{\mu}{r^2} \cos^2 \left(\alpha - \tan^{-1} \left(\frac{z}{\rho} \right) \right)$$

with β the solar sail lightness number [13] and the term $\alpha - \tan^{-1} \left(\frac{z}{\rho} \right)$ equal to the cone angle, α_c , see

Fig. 1.

2. A solar sail law where the cone angle is approximated by the pitch angle under the assumption that the

out-of-plane displacement is small: $\alpha_c = \alpha - \tan^{-1} \left(\frac{z}{\rho} \right) \approx \alpha$.

$$a = \beta \frac{\mu}{r^2} \cos^2 \alpha$$

The second type of acceleration, for small-displacements, which allows for a range of analytical investigations, will be investigated first.

B. Small displacements solar sail acceleration law

The small-displacements solar sail acceleration law can be written as:

$$a = \beta \frac{\mu}{r^2} \cos^2 \alpha = \frac{\beta \bar{\omega}^2 \rho}{1 + \left(\frac{z}{\rho} \right)^2} \cos^2 \alpha$$

Substitution into Eq. (1) then gives:

$$\cos \alpha = \sqrt[3]{\frac{1}{\beta} \left(\left(1 + \left(\frac{z}{\rho} \right)^2 \right)^{-\frac{3}{2}} - \left(\frac{\omega}{\bar{\omega}} \right)^2 \right) \left(1 + \left(\frac{z}{\rho} \right)^2 \right)} \quad (2)$$

$$\ddot{z} = -z\bar{\omega}^2 \left(1 + \left(\frac{z}{\rho} \right)^2 \right)^{-\frac{3}{2}} + \text{sign}(z) \frac{\beta \bar{\omega}^2 \rho}{1 + \left(\frac{z}{\rho} \right)^2} \cos^2 \alpha \sqrt{1 - \cos^2 \alpha}$$

The most well-known cylindrically constrained orbit is the two-dimensional (2D) NKO as described in Ref. [1, 3, 13]. The sail lightness number required to maintain this 2D NKO can be obtained by setting $\ddot{z} = 0$ and substituting $\cos \alpha$:

$$\beta_{2D} = \frac{\left(\left(\frac{z}{\rho} \right)^2 \left(1 + \left(\frac{z}{\rho} \right)^2 \right)^{-1} + \left(\left(1 + \left(\frac{z}{\rho} \right)^2 \right)^{-\frac{3}{2}} - \left(\frac{\omega}{\bar{\omega}} \right)^2 \right) \left(1 + \left(\frac{z}{\rho} \right)^2 \right)^2 \right)^{\frac{3}{2}}}{\left(\left(1 + \left(\frac{z}{\rho} \right)^2 \right)^{-\frac{3}{2}} - \left(\frac{\omega}{\bar{\omega}} \right)^2 \right) \left(1 + \left(\frac{z}{\rho} \right)^2 \right)^{\frac{4}{3}}} \quad (3)$$

Contours of equal β_{2D} as well as the required direction of the acceleration is provided in Fig. 2a for $\omega = 1$. Note that the results in Fig. 2, as well as all other results for the cylindrical case, are made dimensionless by setting $\rho = 1$ and $\bar{\omega} = 1$. Furthermore, note that the contours differ from the well-known results in Ref. [2] due to the difference in definition for the acceleration: Ref. [2] assumes a true solar sail law, while here the cone angle is approximated by the pitch angle. The correctness of the results in Fig. 2a can thus be verified by calculating the required sail lightness number for a true solar sail, $\beta_{2D,s}$, starting from β_{2D} in Eq. (3):

$$\beta_{2D,s} = \frac{\beta_{2D} \cos^2 \alpha}{\cos^2 \alpha_c} \quad (4)$$

Contours for equal $\beta_{2D,s}$ are provided in Fig. 2b which correspond to the results in Ref. [2] and therefore confirm the correctness of Eq. (3).

When using a solar sail, it is well-known that the following constraint on the solar sail attitude has to be taken into account since the sail cannot generate a solar sail component in the direction of the Sun:

$$\cos \alpha \approx \cos \alpha_c \geq 0 \quad (5)$$

Fig. 2 Contours of equal lightness number to maintain a 2D NKO for $\omega = 1$. 1) 0.5; 2) 0.8; 3) 0.95; 4) 0.99; 5) 1.0. Left: sail lightness number for a small displacements solar sail acceleration law. Right: sail lightness number for a true solar sail acceleration law.

This constraint is represented in Fig. 2 by thick black lines. The figure shows that, for small displacements, this line is similar for both types of acceleration, but unnecessarily restricts the feasibility region for the small displacements solar sail acceleration law for larger displacements.

By deviating from β_{2D} , either in positive or negative direction, and keeping the control law as given in Eq. (2), additional, and three dimensional, cylindrical orbits can be found. To obtain *feasible* orbits, not only the lower limit on $\cos \alpha$ in Eq. (5) needs to be imposed, but also an upper limit: $0 \leq \cos \alpha \leq 1$. The lower limit, i.e. $\cos \alpha \geq 0$ can be satisfied by defining the following maximum value for the in-plane angular velocity:

$$\left(1 + \left(\frac{z}{\rho}\right)^2\right)^{-\frac{3}{2}} \geq \left(\frac{\omega}{\bar{\omega}}\right)^2 \rightarrow \omega \leq \omega_{\max} = \bar{\omega} \left(1 + \left(\frac{z}{\rho}\right)^2\right)^{-\frac{3}{4}} = f(z) \quad (6)$$

Since $\omega_{\max}$ depends on the value for z , which changes during the orbit, the minimum value for $f(z)$ needs to be found to ensure that the value chosen for ω satisfies Eq. (6) throughout the orbit. The minimum value for $f(z)$ occurs for the maximum value for z , $z_{\max}$, which results in the following final constraint on ω to satisfy $\cos \alpha \geq 0$:

$$\omega \leq \omega_{\max} = \bar{\omega} \left(1 + \left(\frac{z_{\max}}{\rho}\right)^2\right)^{-\frac{3}{4}} \quad (7)$$

Note that, for a 2D NKO, where $z_{\max} = z_0$, Eq. (7) gives the thick black line in Fig. 2a and can also be used to determine the maximum allowable value for ω for a particular value for z_0 . As will be seen later, $z_{\max}$ is

unknown for orbits that lie completely above the 2D NKO and can only be determined through an integration of the equations of motion. However, in case the orbits lies entirely below the 2D NKO, $z_{\max}$ is again equal to z_0 .

The second inequality constraint, $\cos \alpha \leq 1$ can be satisfied through a constraint on β :

$$\beta \geq \beta_{\min} = \left[\left(1 + \left(\frac{z}{\rho} \right)^2 \right)^{\frac{3}{2}} - \left(\frac{\omega}{\bar{\omega}} \right)^2 \right] \left(1 + \left(\frac{z}{\rho} \right)^2 \right) = g(z) \quad (8)$$

A similar reasoning as for the constraint on ω can be applied: since the minimum required value for β depends on the value for z , which changes during the orbit, the maximum value for $g(z)$ needs to be found to ensure that the value chosen for β satisfies Eq. (8) throughout the orbit. The maximum value for $g(z)$ occurs for the minimum value for z , $z_{\min}$. The final constraint to satisfy $\cos \alpha \leq 1$ thus equals:

$$\beta \geq \beta_{\min} = \left[\left(1 + \left(\frac{z_{\min}}{\rho} \right)^2 \right)^{\frac{3}{2}} - \left(\frac{\omega}{\bar{\omega}} \right)^2 \right] \left(1 + \left(\frac{z_{\min}}{\rho} \right)^2 \right) \quad (9)$$

Again, as will be seen later, $z_{\min}$ is unknown for orbits that lie completely below the 2D NKO and can only be determined through an integration of the equations of motion. However, in case the orbits lies entirely above the 2D NKO, $z_{\min}$ is given by z_0 . Note that β_{2D} is always larger than $\beta_{\min}$ as the latter can be written as:

$$\beta_{2D} = \left[\frac{\left(\frac{z_0}{\rho} \right)^2 \left(1 + \left(\frac{z_0}{\rho} \right)^2 \right)^{-1} + \beta_{\min}^2}{\beta_{\min}^{\frac{4}{3}}} \right]^{\frac{3}{2}} = \beta_{\min}^{-2} \left(\frac{z}{\rho} \right)^3 \left(1 + \left(\frac{z}{\rho} \right)^2 \right)^{\frac{3}{2}} + \beta_{\min} \geq \beta_{\min}$$

An overview of the achievable orbits for a particular value of the initial out-of-plane displacement, z_0 , is provided in Fig. 3, which shows the phase spaces for a range of values for ω (along the z -axis) and percentage deviation from β_{2D} , $\Delta\beta$ (through the used color scheme), for the case of $\rho = 1$ and $z_0 = 0.5$. The step size in $\Delta\beta$ between two phase spaces belonging to the same ω -value is 5 percent.

In Fig. 3, black dots are used on a gray transparent surface to represent the 2D NKO from which the cylindrically constrained orbits bifurcate. As indicated below Eq. (7), a maximum allowable value for ω exists for the 2D NKO. For $z_0 = 0.5$, this value is $\omega_{\max} = 0.846$ and is indicated in Fig. 3 with a red dot.

Fig. 3 Phase spaces for cylindrically constrained NKOs with a small-displacements solar sail acceleration law and for $\rho = 1$ and $z_0 = 0.5$. The colors indicate the percentage deviation from β_{2D} with a step size of 5% (negative deviation to the left of the gray transparent plane; positive deviation to the right of the gray transparent plane).

The phase spaces to the right of the transparent surface are created by *increasing* β with respect to β_{2D} , while phase spaces to the left of this surface are generated by *decreasing* β with respect to β_{2D} . Inspecting the phase spaces it becomes clear that, for $\beta > \beta_{2D}$, the spacecraft will move in a confined band *above* z_0 (“north orbits”) such that $z_0 \leq z \leq z_{\max}$, while for $\beta < \beta_{2D}$, the spacecraft will move in a confined band *below* z_0 (“south orbits”) such that $z_{\min} \leq z \leq z_0$. For some cases, β can be decreased to such extent that the phase space crosses the $z = 0$ plane, after which the $\text{sign}(z)$ -term in Eq. (2) mirrors the acceleration vector in the (x, y) -plane and the orbit immediately changes into an orbit that oscillates around the (x, y) -plane, i.e. $-z_0 \leq z \leq z_0$. Decreasing β further does not change this behaviour, i.e. the orbit will always oscillate around the (x, y) -plane and will not go beyond $-z_0$.

Additional observations that can be made from Fig. 3 are as follows: firstly, the limitations introduced by the constraints in Eqs. (7) and (9) are clear both in terms of the possible deviations from β_{2D} and the attainable values for ω . This is further illustrated in Fig. 4, which shows the feasibility region in terms of achievable

values for ω and allowable percentage deviation from β_{2D} for the same case as presented in Fig. 3, i.e. $\rho = 1$ and $z_0 = 0.5$. Furthermore, for a particular value for ω , the larger $\Delta\beta$, the wider the band covered on the cylinder. Comparing the phase spaces between different values for ω shows that, the larger ω , the narrower is the band for a particular value for β . This means that for large values for ω combined with large deviations from β_{2D} , the sail does not move away much from the 2D NKO. For south orbits (i.e. $\beta < \beta_{2D}$), this can result in significant savings in the lightness number required compared to a 2D NKO while only introducing a slight penalty on the constancy in z -direction, providing a potential alternative to the 2D NKO.

Since Fig. 3 and Fig. 4 only present the cylindrically orbits in an abstract way, Fig. 5 is included to show some of the types of trajectories that are possible. In counterclockwise direction and starting from the top left plot in Fig. 5a, the following cases are represented: an orbit oscillating around the (x, y) -plane, an orbit that creates a band south of the corresponding 2D NKO, and finally two orbits that create a band north of the corresponding 2D NKO. From the plots, especially the last one, it becomes clear that some of the trajectories are (quasi-) periodic, which is confirmed by the Poincaré maps of these trajectories in Fig. 5b, which shows the state-vector after each full revolution. Although the required lightness numbers for the orbits in Fig. 5 are significant, it must be noted that these can be reduced substantially by decreasing the out-of-plane displacement. However, for illustration purposes and clear interpretation of the results, an out-of-plane displacement of 0.5 will be used throughout the paper. Furthermore, the introduction of the paper already hinted at ways to achieve these high solar sail performances, e.g. through the use of perforated sail.

Fig. 4 Feasibility region (in white) in terms of in-plane angular velocity, ω , and percentage deviation from β_{2D} , $\Delta\beta$, for cylindrically constrained NKOs with a small-displacements solar sail acceleration law and for $\rho = 1$ and $z_0 = 0.5$.

Fig. 5 Examples of cylindrically constrained NKOs with a small-displacement solar sail acceleration law for $\rho = 1$, $z_0 = 0.5$ and $\omega = 0.2$ and 20 orbital revolutions. a) Trajectories. b) Poincaré maps with the revolution number increasing from dark to light colors.

Fig. 6 Periodicity of the set of feasible cylindrically constrained NKOs with a small displacements solar sail acceleration law and for $\rho = 1$ and $z_0 = 0.5$, where orbits with a periodicity of up to 30 orbital revolutions are included and orbits with periods of up to 15 are indicated with a numerical value. The smaller the marker and the lighter the color, the shorter the period.

True periodic orbits can be found numerically by integrating the equations of motion and truncating the integration when the orbit returns back to the initial conditions in terms of z - and $\dot{z}$ -values. The time at which this occurs is expressed as a fraction of a full revolution and as such gives an insight in the periodicity. The results are presented in Fig. 6, which are obtained through a fine grid search within the feasibility region. Only orbits with a period of less than 30 revolutions are provided and those with a period of up to 15 revolutions are indicated with the actual period. Furthermore, the smaller and darker the marker, the larger the period. The figure shows that a combination of ω and $\Delta\beta$ exists for which the orbital period is one revolution. This orbit, together with some additional periodic orbits are illustrated in Fig. 7 together with their corresponding Poincaré maps.

Fig. 7 Periodic cylindrically constrained NKOs for a small-displacement solar sail acceleration law for $\rho = 1$ and $z_0 = 0.5$. a) Trajectories. b) Poincaré maps with the revolution number increasing from dark to light colors.

C. Solar sail acceleration law

When replacing the small displacements solar sail acceleration law with the true solar sail acceleration law, the acceleration to be used in Eq. (1) is:

$$a = \beta \frac{\mu}{r^2} \cos^2 \left(\alpha - \tan^{-1} \left(\frac{\rho}{z} \right) \right) = \beta \frac{\bar{\omega}^2 \rho}{1 + \left(\frac{z}{\rho} \right)^2} \cos^2 \left(\alpha - \tan^{-1} \left(\frac{z}{\rho} \right) \right)$$

The required control law for the pitch angle α to remain on a cylindrical surface can then be derived from the first equation in Eq. (1) and is implicitly given through:

$$\cos^2 \left(\alpha - \tan^{-1} \left(\frac{z}{\rho} \right) \right) \cos \alpha = \frac{1}{\beta} \left(1 + \left(\frac{z}{\rho} \right)^2 \right) \left(\left(1 + \left(\frac{z}{\rho} \right)^2 \right)^{-\frac{3}{2}} - \left(\frac{\omega}{\bar{\omega}} \right)^2 \right) \quad (10)$$

To solve Eq. (10) for $\cos \alpha$, the roots of the following sixth order polynomial need to be found:

$$\begin{aligned} & (1 + 2c_1^2 + c_1^4)x^6 + (-2c_1^2 - 2c_1^4)x^4 + (-2c_2 + 2c_1^4c_2)x^3 + c_1^4x^2 + \\ & (-2c_1^2c_2 - 2c_1^4c_2)x + (c_2^2 + 2c_1^2c_2^2 + c_1^4c_2^2) = 0 \end{aligned} \quad (11)$$

with

$$c_1 = \frac{z}{\rho}, \quad c_2 = \frac{1}{\beta} \left(\left(1 + \left(\frac{z}{\rho} \right)^2 \right)^{-\frac{3}{2}} - \left(\frac{\omega}{\bar{\omega}} \right)^2 \right) \left(1 + \left(\frac{z}{\rho} \right)^2 \right)$$

and

$$x = \cos \alpha$$

Although this polynomial has six roots, it appeared that only one of those six roots is a real solution as well as a true solution to Eq. (10). Since no explicit expression exists for $\cos \alpha$, also no explicit expression can be found for β_{2D} from Eq. (1). However, from the literature it is known that β_{2D} is given through [13]:

$$\beta_{2D} = \left(1 + \left(\frac{z}{\rho} \right)^2 \right)^{\frac{1}{2}} \frac{\left(\left(\frac{z}{\rho} \right)^2 + \left(1 - \left(\frac{\omega}{\bar{\omega}} \right)^2 \right)^2 \right)^{\frac{3}{2}}}{\left(\left(\frac{z}{\rho} \right)^2 + \left(1 - \left(\frac{\omega}{\bar{\omega}} \right)^2 \right) \right)^2} \quad (12)$$

The lack in an expression for $\cos \alpha$ furthermore implies that also the constraints on the in-plane angular velocity and allowed deviation from β_{2D} cannot be derived. They can only be enforced within the integration of the equations of motion as soon as the constraint $0 \leq \cos \alpha_s \leq 1$ is violated. When doing so, the phase spaces as depicted in Fig. 8 can be obtained for $\rho = 1$ and $z_0 = 0.5$. Comparing Fig. 8 with Fig. 3 immediately shows the effect of the real solar sail acceleration law as it significantly limits the orbits that are achievable both in terms of the in-plane angular velocity and the deviation from β_{2D} . This reduction in the feasibility region becomes even clearer from Fig. 9. However, the behavior of the orbits is still the same, i.e. in terms of north- and south orbits for increasing and decreasing β with respect to β_{2D} , respectively, and the switch to orbits that oscillate around the (x, y) -plane for a large enough reduction in β . Furthermore, again a fairly large region for negative values of $\Delta\beta$ exists, which again suggests the potential of the cylindrical orbits for decreasing the sail technology

requirements with respect to a 2D NKO at the cost of a slight penalty on the constancy in z -direction. This can among others be seen from the results for the periodicity analysis which are presented in Fig. 10 and Fig. 11. For $\omega = 0.725$ an almost 10 percent reduction in the sail lightness number provides an orbit that has a period of 4 revolutions while almost coinciding with the original 2D NKO, see the top right figure in Fig. 11a.

Fig. 8 Phase spaces for cylindrically constrained NKOs with a solar sail acceleration law and for $\rho = 1$ and $z_0 = 0.5$. The colors indicate the percentage deviation from β_{2D} with a step size of 1% (negative deviation to the left of the gray transparent plane; positive deviation to the right of the gray transparent plane).

Fig. 9 Feasibility region (in white) in terms of in-plane angular velocity, ω , and deviation from β_{2D} for cylindrically constrained NKOs with a solar sail acceleration law and for $\rho = 1$ and $z_0 = 0.5$.

Fig. 10 Periodicity of the set of feasible cylindrically constrained NKOs with a solar sail acceleration law and for $\rho = 1$ and $z_0 = 0.5$, where orbits with a periodicity of up to 30 orbital revolutions are included and orbits with periods of up to 15 are indicated with a numerical value. The smaller the marker and the lighter the color, the shorter the period.

Fig. 11 Periodic cylindrically constrained NKOs for a solar sail acceleration law for $\rho = 1$ and $z_0 = 0.5$.

a) Trajectories. b) Poincaré maps with the revolution number increasing from dark to light colors.

III. Spherically constrained NKOs

In order to obtain spherically constrained NKOs, a spherical cylindrical coordinate frame is adopted with r the projected radius, θ the in-plane angle between the x -axis and the projected radius and ϕ the out-of-plane angle, see Fig. 12. Again, assuming an acceleration in the (r, ϕ) -plane only, pitched at an angle α (which for the spherical orbits is defined with respect to the radius rather than with respect to the projected radius), the equations of motion for a central gravitational force field are given by:

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 \cos^2 \phi - r\dot{\phi}^2 &= -\frac{\mu}{r^2} + a \cos \alpha \\ r \cos \phi \ddot{\theta} + 2\dot{r}\dot{\theta} \cos \phi - 2r\dot{\theta}\dot{\phi} \sin \phi &= 0 \\ r\ddot{\phi} + 2\dot{r}\dot{\phi} + r\dot{\theta}^2 \sin \phi \cos \phi &= a \sin \alpha \end{aligned} \quad (13)$$

Fig. 12 Spherical coordinate frame definition.

The geometrical constraint required to restrict the motion to a spherical surface is:

$$r = \text{constant} \quad \rightarrow \quad \dot{r} = \ddot{r} = 0$$

Substituting these conditions into the equations of motion, Eq. (13) reduces to:

$$\begin{aligned} -\dot{\theta}^2 \cos^2 \phi - \dot{\phi}^2 &= -\frac{\mu}{r^3} + \frac{a}{r} \cos \alpha \\ \ddot{\theta} &= 2\dot{\theta}\dot{\phi} \tan \phi \\ \ddot{\phi} + \dot{\theta}^2 \sin \phi \cos \phi &= \frac{a}{r} \sin \alpha \end{aligned} \quad (14)$$

Note that, contrary to the cylindrical case, the in-plane angular velocity, $\dot{\theta}$, is not constant and can therefore not be replaced by the notation ω as done for the cylindrical case. Furthermore, the parameter $\bar{\omega}$ will now denote the angular velocity of a Keplerian orbit with radius r :

$$\bar{\omega} = \sqrt{\frac{\mu}{r^3}}$$

Finally, since for the spherical case the pitch angle α is identical to the cone angle, only a true solar sail acceleration law will be considered as this will immediately allow for some analytical investigations:

$$a = \beta \frac{\mu}{r^2} \cos^2 \alpha = \beta \bar{\omega}^2 r \cos^2 \alpha \quad (15)$$

Substituting the above, the equations of motion in Eq. (14) can be further reduced to:

$$\begin{aligned} \cos \alpha &= \sqrt[3]{\frac{1}{\beta} \left(1 - \frac{1}{\bar{\omega}^2} (\dot{\theta}^2 \cos^2 \phi + \dot{\phi}^2) \right)} \\ \ddot{\theta} &= 2\dot{\theta}\dot{\phi} \tan \phi \\ \ddot{\phi} &= -\dot{\theta}^2 \sin \phi \cos \phi + \text{sign}(\phi) \beta \bar{\omega}^2 \cos^2 \alpha \sqrt{1 - \cos^2 \alpha} \end{aligned} \quad (16)$$

As for the cylindrical case, the analysis for the spherically constrained NKOs will start from the 2D NKO, which can be derived from Eq. (16) by setting $\phi = \text{constant} \rightarrow \dot{\phi} = \ddot{\phi} = 0$, which gives:

$$\beta_{2D} = \left(\frac{\left(\sin \phi \cos \phi \left(\frac{\omega}{\bar{\omega}} \right)^2 \right)^2 + \left(1 - \cos^2 \phi \left(\frac{\omega}{\bar{\omega}} \right)^2 \right)^2}{\left(1 - \cos^2 \phi \left(\frac{\omega}{\bar{\omega}} \right)^2 \right)^{\frac{4}{3}}} \right)^{\frac{3}{2}} \quad (17)$$

Since in this case the definition for the acceleration is the same as used in Ref. [2], the contours of equal β_{2D} are exactly the same as those in Ref. [2] and as in the right figure of Fig. 2.

In order for the control law to be feasible, the constraint $0 \leq \cos \alpha \leq 1$ needs to be imposed on the first equation in Eq. (16). Since it was found that the minimum and maximum values for the term $1 - \frac{1}{\bar{\omega}^2} (\dot{\theta}^2 \cos^2 \phi + \dot{\phi}^2)$ occur at the maximum and minimum value for z (or equivalently for ϕ), respectively,

the following constraints on the in-plane angular velocity and sail lightness number can be derived:

$$\dot{\theta} \Big|_{\phi=\phi_{\max}} \leq \dot{\theta}_{\max} \Big|_{\phi=\phi_{\max}} = \sqrt{\frac{\bar{\omega}^2 - \dot{\phi}^2}{\cos^2 \phi}} \Big|_{\phi=\phi_{\max}} \quad (18)$$

$$\beta \geq \beta_{\min} = 1 - \frac{1}{\bar{\omega}^2} (\dot{\theta}^2 \cos^2 \phi + \dot{\phi}^2) \Big|_{\phi=\phi_{\min}} \quad (19)$$

As for the cylindrically constrained orbits, spherically constrained orbits can be found by deviating from β_{2D} , either in positive or negative direction, while keeping the control law as given in Eq. (16). By increasing

β with respect to β_{2D} , the spacecraft will move in a confined band above ϕ_0 (“north orbits”) such that $\phi_0 = \phi_{\min}$ while $\phi_{\max}$ is unknown. The constraint in Eq. (18) can therefore only be determined through an integration of the equations of motion, while the constraint in Eq. (19) can be rewritten as:

$$\beta \geq \beta_{\min} = 1 - \left(\frac{\dot{\theta}_0}{\bar{\omega}} \right)^2 \cos^2 \phi_0 \quad (20)$$

Contrary, for the case of decreasing β with respect to β_{2D} , the spacecraft will move in a confined band below ϕ_0 (“south orbits”) such that $\phi_0 = \phi_{\max}$, while $\phi_{\min}$ is unknown. The constraint in Eq. (19) can therefore only be determined through an integration of the equations of motion, while the constraint in Eq. (18) can be rewritten as:

$$\dot{\theta}_0 \leq \dot{\theta}_{0,\max} = \frac{\bar{\omega}}{\cos \phi_0} \quad (21)$$

A set of feasible orbits for the case of $r = 1$ and $z_0 = 0.5$ (i.e. $\phi_0 = 0.5236$ rad) is shown in Fig. 13. Note that, for the spherically constrained orbits, the results in Fig. 13 and all subsequent figures are made dimensionless by setting $r = 1$ and $\bar{\omega} = 1$. The behavior of the orbits in the phase space is very similar to the behavior seen in Fig. 3 and Fig. 8 for the cylindrical case: the bands above and below the 2D NKO are clearly visible, where a larger value for the in-plane angular velocity leads to tighter bands, i.e. a smaller deviation from ϕ_0 for a particular percentage deviation from β_{2D} . Furthermore, also the orbits oscillating around the (x, y) -axis are once again clear for large enough negative values of $\Delta\beta$.

The full feasibility region in terms of $\dot{\theta}_0$ and $\Delta\beta$ is shown in Fig. 14. Although the figure suggests that β can be increased unlimitedly, this is not the case. For large enough values for $\Delta\beta$ the constraint in Eq. (18) will eventually be violated. However, this only occurs for extremely large values for $\Delta\beta$, which are not considered reasonable and are therefore not included in Fig. 14.

As for the cylindrically constrained orbits, a periodicity analysis has been performed for the spherically constrained orbits. The results are shown in Fig. 15 with some characteristic orbits provided in Fig. 16. The orbits in Fig. 16a show, in clockwise direction and starting from the top left figure, a north orbit that covers a narrow band on the spherical surface; a north orbit that covers a much wider band and closely passes to the spherical poles; a south orbit; and a south orbit that oscillates around the (x, y) -plane. When considering applications for these types of orbits, one can imagine that the second orbit would be very suitable for solar polar observation, especially if the out-of-plane displacement is increased further such that the solar sail spends even

more time directly above the solar poles, while the fourth orbit, which covers a significant part of the spherical surface would be suitable for three-dimensional imaging of the Sun.

Fig. 13 Phase spaces for spherically constrained NKOs with a solar sail acceleration law and for $r = 1$ and $z_0 = 0.5$ (i.e. $\phi_0 = 0.5236$ rad). The colors indicate the percentage deviation from β_{2D} with a step size of 5% (negative deviation to the left of the gray transparent plane; positive deviation to the right of the gray transparent plane).

Fig. 14 Feasibility region (in white) in terms of the initial in-plane angular velocity, $\dot{\theta}_0$, and deviation from β_{2D} for spherically constrained NKOs with a solar sail law and for $r = 1$ and $z_0 = 0.5$ (i.e. $\phi_0 = 0.5236$ rad).

Fig. 15 Periodicity of the set of feasible spherically constrained NKOs with a solar sail acceleration law and for $\rho = 1$ and $z_0 = 0.5$, where orbits with a periodicity of up to 20 orbital revolutions are included and orbits with periods of up to 10 are indicated with a numerical value. The smaller the marker and the lighter the color, the shorter the period.

Fig. 16 Periodic spherically constrained NKOs for a solar sail acceleration law for $\rho = 1$ and $z_0 = 0.5$. a) Trajectories. b) Poincaré maps with the revolution number increasing from dark to light colors.

IV. Conclusions

In this paper, new families of non-Keplerian solar sail orbits have been introduced where the sail motion is restricted to a cylindrical or spherical surface. The feasibility region in terms of acceleration magnitude and in-plane angular velocity has been established and true periodic orbits have been found. The range of possible orbits include both those that cover only a narrow band of the cylindrical or spherical surface and those that cover a significant portion of that surface. With applications in planetary observation, interplanetary communication, astronomical observations and solar physics, the potential of these novel orbits is clear. For example, the latter can significantly benefit from the spherically constrained orbits as they enable orbits high above the solar poles for solar polar observations as well as orbits that cover a significant part of the spherical surface for obtaining a full three dimensional picture of the features and structures of the Sun.

Finally, the results indicate that a reduction in the required propulsion system performance can be achieved through the use of these 3D rather than 2D NKOs, at the cost of only a minor deviation from the original 2D NKO, increasing the potential of these cylindrically and spherically constrained orbits even further.

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References

- [1]. McInnes, C. R. "The Existence And Stability Of Families Of Displaced Two-Body Orbits," *Celestial Mechanics and Dynamical Astronomy* Vol. 67, No. 2, 1997, pp. 167-180. Doi: 10.1023/A:1008280609889
- [2]. McInnes, C., and Simmons, J. F. L. "Solar Sail Halo Orbits. Part I - Heliocentric Case," *Journal of Spacecraft and Rockets* Vol. 29, No. 4, 1992, pp. 466-471. Doi: 10.2514/3.25487
- [3]. McInnes, C. R. "Dynamics, Stability, and Control of Displaced Non-Keplerian Orbits," *Journal of Guidance, Control, and Dynamics* Vol. 21, No. 5, 1998, pp. 799-805. Doi: 10.2514/2.4309
- [4]. Heiligers, J., Ceriotti, M., McInnes, C. R., and Biggs, J. D. "Displaced Geostationary Orbit Design Using Hybrid Sail Propulsion," *Journal of Guidance, Control, and Dynamics* Vol. 34, No. 6, 2011, pp. 1852-1866. Doi: 10.2514/1.53807
- [5]. Baig, S., and McInnes, C. R. "Light-Levitated Geostationary Cylindrical Orbits Are Feasible," *Journal of Guidance, Control, and Dynamics* Vol. 33, No. 3, 2010, pp. 782-793. Doi: 10.2514/1.46681
- [6]. Spilker, T. R. "Saturn Ring Observer," *Acta Astronautica* Vol. 52, 2003, pp. 259-265.
- [7]. Macdonald, M., McInnes, C., Alexander, D., and Sandman, A. "GeoSail: Exploring the Magnetosphere Using a Low-cost Solar Sail," *Acta Astronautica* Vol. 59, 2006, pp. 757-767. Doi: 10.1016/j.actaastro.2005.07.023
- [8]. Tsuda, Y., Mori, O., Funase, R., Sawada, H., Yamamoto, T., Saiki, T., Endo, T., and Kawaguchi, J. "Flight Status of IKAROS Deep Space Solar Sail Demonstrator," *Acta Astronautica* Vol. 69, No. 9-10, 2011, pp. 833-840. Doi: 10.1016/j.actaastro.2011.06.005
- [9]. Johnson, L., Whorton, M., Heaton, A., Pinson, R., Laue, G., and Adams, C. "NanoSail-D: A Solar Sail Demonstration Mission," *Acta Astronautica* Vol. 68, 2011, pp. 571-575. Doi: 10.1016/j.actaastro.2010.02.008
- [10]. L'Garde Inc. "Sunjammer." <http://www.lgarde.com/programs/space-propulsion/sunjammer>, Accessed 8 May 2013.
- [11]. Bidy, C., and Svitek, T. "LightSail-1 Solar Sail Design and Qualification," *Proceedings of the 41st Aerospace Mechanisms Symposium*. Pasadena, CA, 2012.
- [12]. Leipold, M., Fichtner, H., Heber, B., Groepper, P., Lascar, S., Burger, F., Eiden, M., Niederstadt, T., Sickinger, C., Herbeck, L., Dachwald, B., and Seboldt, W. "Heliopause Explorer - A Sailcraft Mission to the Outer Boundaries of the Solar System," *Acta Astronautica* Vol. 59, 2006, pp. 785-796. Doi: 10.1016/j.actaastro.2005.07.024
- [13]. McInnes, C. R. *Solar Sailing: Technology, Dynamics and Mission Applications*. Berlin: Springer-Praxis Books in Astronautical Engineering, Springer-Verlag, 1999.