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Taylor's (1935) dissipation surrogate reinterpreted

W. David McComb,[a] Arjun Berera, Matthew Salewski, and Samuel Yoffe
School of Physics and Astronomy, University of Edinburgh, James Clerk Maxwell Building, The Kings' Buildings, Edinburgh EH9 3JZ, United Kingdom

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New results from direct numerical simulation of decaying isotropic turbulence show that Taylor's expression for the viscous dissipation rate $\varepsilon = C_\varepsilon U^3/L$ is more appropriately interpreted as a surrogate for the inertial energy flux. As a consequence, the well known dependence of the Taylor prefactor C_ε on Reynolds number $C_\varepsilon(R_L) \to C_{\varepsilon,\infty}$ can be understood as corresponding to the onset of an inertial range. © *2010 American Institute of Physics*. [doi:10.1063/1.3450299]

In this letter, we are concerned with an aspect of the classical theory of turbulence, as associated with the names of Richardson, Kolmogorov, and Taylor. This theory is restricted to homogeneous, isotropic turbulence (HIT) and is a phenomenology based on the Karman–Howarth equation; an exact result derived from the equations of motion and expressing conservation of energy. It deals with the production, inertial transfer, and viscous dissipation of energy, where *energy* means the kinetic energy per unit mass of fluid associated with the fluctuating velocity field $\mathbf{u}(\mathbf{x},t)$. Unfortunately this 60-year-old theory is riven by controversy and unresolved issues (e.g., see the review by Sreenivasan[1]). If turbulence theory is ever to make any progress in practical applications, then these long-standing issues must be resolved. The key fundamental issue which we will consider here is the dependence of the dissipation rate on the Reynolds number.

Apart from the chaotic behavior revealed by flow visualization, the most characteristic feature of turbulence is very high levels of energy dissipation, which are typically one or two orders of magnitude larger than in equivalent laminar flows. Formally, the dissipation rate is defined by the expression

$$\varepsilon = \frac{\nu}{2}\left\langle \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 \right\rangle, \tag{1}$$

where the Cartesian tensor indices i and j take the values of 1, 2, or 3, and the angular brackets $\langle \cdots \rangle$ indicate an average.

In general, it is no small matter to measure all the components of the rate of strain tensor. For this reason, much work on this fundamental aspect of turbulence has been based on Taylor's expression for the dissipation rate, as originally put forward in 1935.[2] In shear flows, this surrogate is typically based on velocity scales such as the friction velocity and on length scales such as the radius of a pipe or width of a jet. However, we shall restrict our attention to HIT, where it is usually written as

$$\varepsilon = C_\varepsilon(R_L) U^3/L, \tag{2}$$

where $C_\varepsilon(R_L)$ is Taylor's dissipation coefficient, L is the integral lengthscale, and U is the root-mean-square velocity.[2] As early as 1953, Batchelor[3] (in the first edition of this book) presented evidence to suggest that C_ε tends to a constant value with increasing Reynolds number. However, the present interest in the subject stems from the seminal paper by Sreenivasan,[4] who established that in grid turbulence, C_ε became constant for Taylor–Reynolds numbers greater than about 50.

As in most aspects of turbulence, the theoretical problem is seen as being very difficult. Few attempts have been made to establish a theoretical relationship between the dissipation rate and the Reynolds number. We are aware of two significant attempts. Lohse[5] used a mean-field closure of the Karman–Howarth equation to obtain an approximate form, whereas Doering and Foias[6] established upper and lower bounds to be satisfied by such a relationship.

In contrast with the theoretical situation, the empirical relationship between the dissipation and the Reynolds number has been extensively studied by a variety of methods, ranging from laboratory experiments through direct numerical simulations (DNSs) to statistical closures. It has been conclusively shown that C_ε is strongly dependent on the Reynolds number at small Reynolds numbers but becomes independent of it as the Reynolds number becomes large. That is, $C_\varepsilon(R_L) \to C_\varepsilon(\infty)$, where $C_\varepsilon(\infty) \equiv C_{\varepsilon,\infty}$ is a constant. The value of the asymptotic constant appears to depend on the initial conditions and a range of values can be found in the literature, with a typical value of around 0.4–0.5.[4–18]

In this letter, we present numerical values of the quantity U^3/L at different Reynolds numbers and compare these to the corresponding inertial flux and viscous dissipation rate. As will be seen, these results strongly suggest that Taylor's expression should be interpreted as a surrogate for the inertial flux rather than for the dissipation. It should perhaps be emphasized that this is a phenomenon associated with *low values of the Reynolds number*. Accordingly, we present results for a range of Taylor–Reynolds number up to about 60. This is very much in accord with existing work in this field. Moreover, our plots of $C_\varepsilon(R_L)$ in Figs. 1–5 are quite typical of results obtained in other investigations. At the same time, all DNSs (both stationary and decaying) are somewhat artificial forms of turbulence and it behooves us to interpret results at low Reynolds numbers with some caution. A

[a]Electronic mail: wdm@ph.ed.ac.uk.

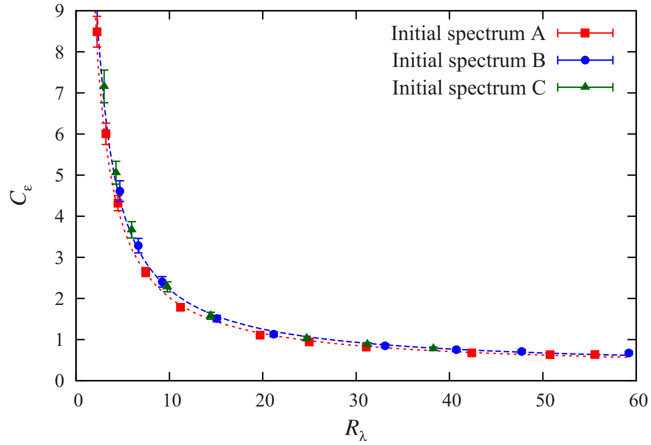

FIG. 1. (Color online) The dimensionless dissipation rate as a function of the Taylor–Reynolds number for the three initial spectra used in the DNS.

detailed discussion of this point will be given in a fuller account of this work to be published in due course.

Our numerical simulations were based on standard pseudospectral methods, with a regular 128^3 lattice, which was adequate for the low values of Reynolds number used in the study. Full dealiasing was performed using truncation of the velocity field.[19] In addition to the usual shell averaging, we also used an ensemble average in order to provide a measure of the statistical error for each parameter. This was based on ten individual realizations, all satisfying a given initial energy spectrum. For a more general discussion of our statistical procedures, see Ref. 20. The data presented here were subject to a standard deviation in the range 2.5%–6%.

We performed DNSs of the Navier–Stokes equation for freely decaying isotropic turbulence, using three different initial spectra. These were generated using the following equation:

$$E(k,0) = C_1 k^{C_2} \exp\{-C_3 k^{C_4}\}, \tag{3}$$

where the parameters $C_1 - C_4$ are given in Table I.

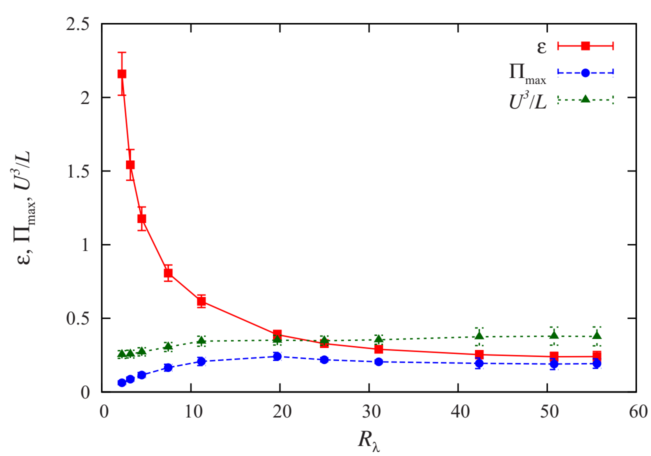

FIG. 2. (Color online) Variation of U^3/L with the Taylor–Reynolds number compared with the peak inertial flux Π_{max} and the viscous dissipation rate ε. Initial spectrum A.

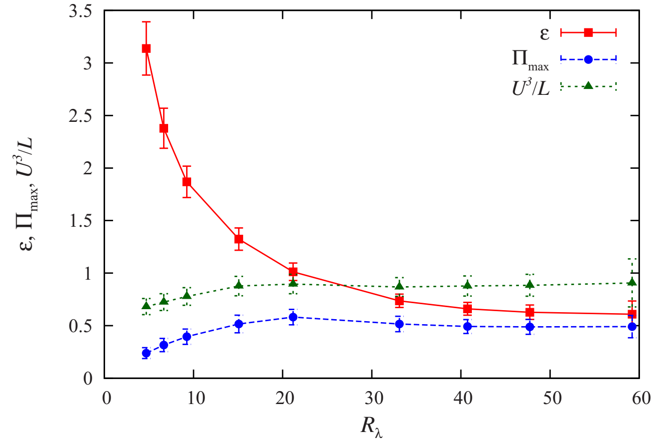

FIG. 3. (Color online) Variation of U^3/L with the Taylor–Reynolds number compared with the peak inertial flux Π_{max} and the viscous dissipation rate ε. Initial spectrum B.

In spectral (k) space, the mean dissipation rate may be evaluated in terms of the energy spectrum $E(k,t)$ and the kinematic viscosity as

$$\varepsilon(t) = \int 2\nu k^2 E(k,t) dk \equiv \int D(k,t) dk, \tag{4}$$

which also defines the dissipation spectrum $D(k,t)$. In all our simulations, the dissipation rate was calculated using this relationship, with the spectral resolution satisfying $k_{\text{max}}/k_D > 1.3$, where k_{max} is the maximum resolved wavenumber and k_D is the Kolmogorov dissipation wavenumber. For the basis of this criterion, see Fig. 2 in Ref. 20.

For each simulation, the initial velocity field is chosen to have Gaussian statistics and, as is well known, has to be allowed time to evolve to produced statistics which are characteristic of turbulence rather than of the initial conditions. When a simulation is started at $t=0$, then all our results presented here (including U, L, and R_λ) are taken at an evolved time $t=t_e$, corresponding to a peak in either the dissipation rate or the inertial flux curves against time. Results for increasing Reynolds number are obtained by running simula-

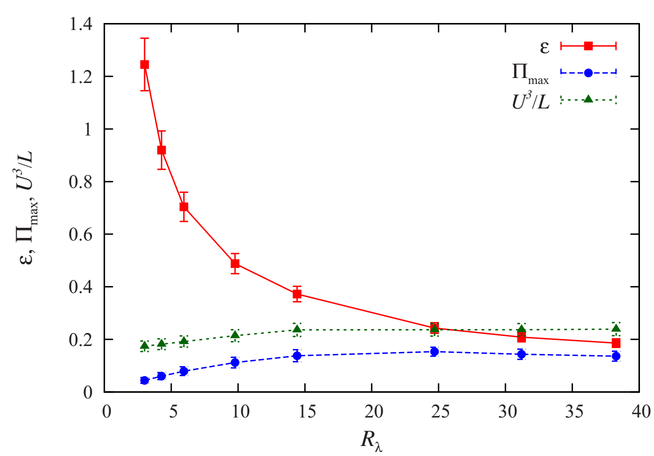

FIG. 4. (Color online) Variation of U^3/L with the Taylor–Reynolds number compared with the peak inertial flux Π_{max} and the viscous dissipation rate ε. Initial spectrum C.

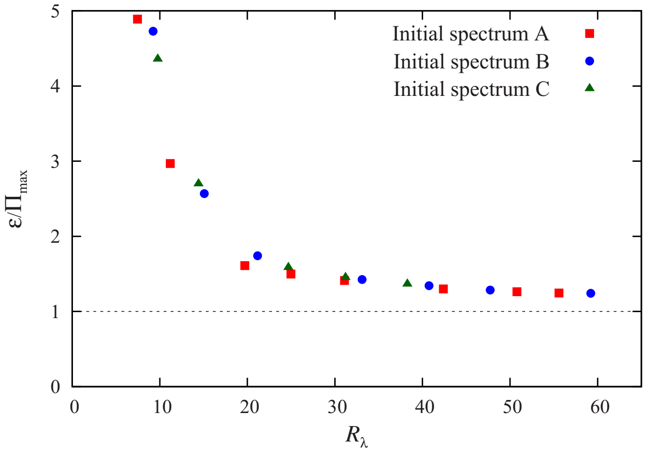

FIG. 5. (Color online) Variation of the ratio of the viscous dissipation rate ε to the peak inertial flux Π_{max} with the Taylor–Reynolds number. Initial spectra A, B, and C.

tions at ever larger initial values of Reynolds number (i.e., at ever smaller values of the kinematic viscosity). Note that the same initial velocity fields are used for each run.

In Fig. 1, we show the typical variation of the dimensionless dissipation rate C_ε with the Taylor–Reynolds number R_λ for all three initial spectra. In each case, the asymptotic value of the dimensionless dissipation is about 0.4. In order to understand this characteristic behavior of the Taylor prefactor, we introduce the spectral energy flux as follows. The well known spectral energy balance equation[3] takes the form

$$\frac{\partial}{\partial t}E(k,t) = T(k,t) - D(k,t), \quad (5)$$

where the transfer spectrum $T(k,t)$ depends on the triple correlation of the velocities. We may next define the energy transfer rate, or flux, through mode κ as

$$\Pi(\kappa,t) = \int_\kappa^\infty dj T(j,t) = -\int_0^\kappa dj T(j,t). \quad (6)$$

As is well known, the peak value of the inertial flux corresponds to the wavenumber where the transfer spectrum passes through zero. If we denote this wavenumber by k_c, then the peak flux can be calculated from

$$\Pi_{max}(t) = \Pi(k_c,t) \quad \text{where} \quad T(k_c,t) = 0. \quad (7)$$

In Figs. 2–4, we plot the expression U^3/L as a function of the Taylor–Reynolds number for each of the initial spectra. In each case, we compare it with both the dissipation rate and the maximum value of the inertial flux. In each case, it can be seen that U^3/L closely follows the peak flux from the lowest Reynolds numbers to the highest. In contrast, the dissipation rate is very much larger than the other two quantities at low Reynolds numbers and only approaches them as it approaches its own asymptotic value.

Qualitatively, therefore, it seems that U^3/L and the peak flux behave in a similar fashion whereas the behavior of the dissipation rate is quite different. Accordingly, from inspection of Figs. 2–4, we conclude that Taylor's expression may more plausibly be regarded as a surrogate for the peak inertial transfer rather than for the viscous dissipation.

With that interpretation, the asymptotic behavior of the dissipation rate can be readily understood in terms of the usual phenomenology. As is well known, the peak inertial flux increases with increasing Reynolds number and tends toward the value of the dissipation rate asymptotically. That is,

$$\Pi(k_c,t) = \Pi_{max}(t) \sim \varepsilon(t), \quad (8)$$

for sufficiently large values of the Reynolds number. This behavior is well established and recent high-resolution numerical simulations show the effect very clearly.[21] At that stage, we have the beginning of an inertial range, and thereafter the dissipation rate is controlled by the nonlinear term, rather than by the viscosity, as the Reynolds number increases.

We may illustrate this by plotting the ratio of the dissipation rate to the peak flux as a function of Reynolds number for each of the initial spectra (see Fig. 5). Qualitatively, the plots resemble those of Fig. 1 but in this case they all asymptote *toward* unity. It is worth noting that the asymptotic process can only be toward unity as this is decaying turbulence. In stationary turbulence the asymptote *is* unity, whereas it has been shown[22] that for freely decaying turbulence, the presence of the term $\partial E/\partial t$ means that the peak flux can never actually be equal to the dissipation rate. This point has also been noted elsewhere,[23] along with a discussion of the competing roles of inertial transfer and dissipation.

Overall, if we have to choose between the peak flux and the dissipation, when we are interpreting Taylor's expression (and there are no other candidates), then we choose the peak flux. This view, although in our opinion is fully justified by the numerical evidence presented here, is also supported by a recent theoretical analysis based on the Karman–Howarth equation.[22]

To sum up, our results indicate that U^3/L is a more appropriate measure of peak inertial flux than of dissipation. It can, of course, still be used in practice to nondimensionalize the dissipation rate. We do not suggest that the maximum value of the inertial flux can replace the Taylor expression as a practical method of estimating the dissipation. However, our more correct interpretation of the basic physics gives us a better understanding of the Richardson–Kolmogorov–Taylor phenomenology. The asymptotic behavior of the dissipation rate can be seen as corresponding to the onset of an inertial range when the dissipation rate becomes controlled by the inertial transfer rate. Lastly, we note that Taylor's

TABLE I. Initial energy spectra parameter values for use in the numerical computations. These parameters are substituted into Eq. (3) to generate the required initial spectrum.

Initial spectra	C_1	C_2	C_3	C_4
Spectrum A	0.0017	4	0.08	2
Spectrum B	0.08	2	0.0824	2
Spectrum C	0.0319	2	0.08	2

expression can justifiably still be used as a surrogate for dissipation at higher values of the Reynolds number, where dissipation is equal to the inertial transfer rate.

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