Nonparametric Bootstrapping of the Reliability Function for Multiple Copies of a Repairable Item Modeled by a Birth Process

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Index Terms

Reliability function, bootstrap, Kaplan-Meier, confidence intervals, censored data

Abstract

Nonparametric bootstrap inference is developed for the reliability function estimated from censored, non-stationary failure time data for multiple copies of repairable items. We assume that each copy has a known, but not necessarily the same, observation period; and upon failure of one copy, design modifications are implemented for all copies operating at that time to prevent further failures arising from the same fault. This implies that, at any point in time, all operating copies will contain the same set of faults. Failures are modeled as a birth process because there is a reduction in the rate of occurrence at each failure. The data structure comprises a mix of deterministic & random censoring mechanisms corresponding to the known observation period of the copy, and the random censoring time of each fault. Hence, bootstrap confidence intervals & regions for the reliability function measure the length of time a fault can remain within the item until realization as failure in one of the copies. Explicit formulae derived for the re-sampling probabilities greatly reduce dependency on Monte-Carlo simulation. Investigations show a small bias arising in re-sampling that can be quantified & corrected. The variability generated by the re-sampling approach approximates the variability in the underlying birth process, and so supports appropriate inference. An illustrative example describes application to a

problem, and discusses the validity of modeling assumptions within industrial practice.

ACRONYMS[1]

pdf	probability density function
i.i.d.	independent, and identically distributed

NOTATION

$\lambda_i^F(t)$	rate of occurrence of failures for all copies at time *t*, given *i* faults have been detected by time *t*
$\lambda(t)$	rate of occurrence of failures for one copy at time *t* having 1 fault within the design
K	number of faults in the design at time 0
$U(t)$	number of copies at risk at time *t*
U_i	number of copies at risk at time of realization of the i^{th} fault
c_i	censored time of the i^{th} copy
t_i	time of the i^{th} fault detection
X_i	bootstrap simulation of the realization of the i^{th} fault
$D(t)$	number of faults realized by time *t*
$D(t+s\|s)$	number of faults realized in the interval $(s,s+t)$
$R(t)$	probability that a particular fault will not be realized on a particular copy in the interval $(0,t)$

[1] The singular and plural of an acronym are always spelled the same.

$R(t+s\|s)$	probability that a particular fault will not be realized on a particular copy in the interval (*s*,*s*+*t*), given it had not been realized in the interval (0,*s*)
$R_F(t+s\|s)$	probability that a particular fault would not be realized within all copies in the interval (*s*,*s*+*t*), given it had not been realized in the interval (0,*s*)
$\hat{R}_{KM}(t+s\|s)$	Kaplan-Meier estimator of $R(t+s\|s)$
$Y_j(t_i)$	number of faults remaining undetected in item *j* prior to time t_i
G(z)	number of faults that have been detected across the fleet by calendar time *z*

ASSUMPTIONS

1. Once a fault is identified within one copy, it is removed from all other copies.
2. The fault removal process does not introduce any new faults.
3. The distinct faults are realized independently of each other.

1. INTRODUCTION

The reliability estimate for a new design can be derived from operational data for items with a similar heritage [1], [2]. Such data can provide information about the operational environment, but must be adapted to account for design changes between generations. An appropriate estimate for the new item should remove the effects of known weaknesses or faults that have been designed out, but include potential faults arising from new features or functions introduced. The former may be achieved by deleting operational database records corresponding to faults removed [3]; however estimating the effects of new features is not trivial. Although processes exist to elicit subjective expert judgment regarding potential faults within new designs [4], little work has been reported about the use of heritage data to estimate when these potential faults may be realized in operation. Evolutionary designs whose failure characteristics change throughout operational life, for example, due to design modifications or upgrades, further challenges such inference.

The primary aim of this research is to develop an efficient non-parametric bootstrap procedure that will provide confidence intervals about the reliability function describing the length of time a fault will remain within an item without resulting in a failure based on censored operational data for items subject to design modifications, and therefore is non-stationary.

It is assumed that the item possesses a fixed, known number of faults, and that when these faults are realized as failures, repair follows with perfect modifications implemented across all copies. The usual Poisson Process models [5], [6] are deemed inadequate because the rate of occurrence of failures decreases with every fault realized & corrected. Moreover, because there are a finite number of faults, once all are corrected, the fault realization process terminates. Therefore, a more suitable counting process describing the fault realization process is a birth process [7], [8], where the realization of a fault results in the reduction of the rate of occurrence of failures across all copies operating at a given time. However, modeling is further complicated because all copies of the item are not observed for the same length of time because each copy can enter operation at different calendar times.

Initially, modeling is restricted to the case where observation of all copies begins at the same calendar time, and so it is assumed that each copy begins observation with the same number of faults. However we shall show this assumption can be relaxed to allow for the case where the copies start observation at different points in calendar time. In either case, the data structures comprise a mixture of deterministic & random censoring mechanisms corresponding to the known observation period of the copy, and the uncertain time at which a fault will be realized as a failure.

Bootstrap procedures [6], [9], [10] are developed to support inference for the reliability function under this two-fold censoring structure because re-sampling techniques provide a useful methodology for constructing nonparametric confidence intervals & regions using Monte-Carlo simulation from the estimated reliability function. Bootstrapping is potentially most useful when the data are obtained from complex sampling schemes, when sampling distributions are difficult to obtain analytically. However, there are three shortcomings to this methodology. Firstly, bootstrapping is computer intensive with the number of simulations required increasing exponentially as the censoring structure increases in complexity. Secondly, incorrect re-sampling plans can result in inconsistent estimates. Thirdly, even for consistent re-sampling plans, the coverage of the confidence intervals is often smaller than specified when the sample size is small.

Section 2 describes the censoring structure, and the birth process underpinning the data in detail. Section 3 argues that the usual Kaplan Meier approach to nonparametric inference is biased, and proposes an alternative unbiased estimator of the reliability function based on order statistic arguments. Section 4 proposes a simplified re-sampling procedure to support the bootstrap method, which is much less reliant on Monte-Carlo procedures to determine confidence intervals for the reliability function. Section 5 presents an evaluation of the proposed procedures. An illustrative example is provided in Section 6 along with a discussion of the practical applicability of the approach.

2. CENSORING STRUCTURE, AND BIRTH PROCESS

Figure 1 presents two different data representations. For simplicity, it is assumed there are two copies (labeled 1, and 2) of the repaired item, although in general there is no limit to the number of copies. Each copy is observed for different lengths of (pre-determined) time denoted by c_1, and c_2 respectively. It is assumed that the item contains two faults which are realized as failures at times t_1, and t_2, by copy 1, and 2 respectively. Figure 1a shows a failure history of the 'fleet' of copies by tracking the history of each, while Figure 1b is the corresponding representation of the realization of the faults & censoring times.

It is assumed that each copy of the item is identical with respect to the faults they possess, and the nominal operating environment; and that each copy operates independently of the other. Further, it is assumed that each copy began observation at time 0 with the same K faults, and that once a particular fault is realized it is removed from all copies. Moreover, we assume that distinctly different faults fail independently of one another. We denote the number of copies in operation at time t by $U(t)$, and let $\lambda(t)$ be the rate of occurrence of failures for one copy at time t.

These assumptions are consistent with a birth process with intensity function

$$\lambda_i^F(t) = U(t)\lambda(t)[K-i], \quad K \in \mathbb{N}, \quad i = 0,1,2,\ldots,K \tag{1}$$

which describes the rate of occurrence of failure for the set of all copies, given i faults have been realized.

The resulting probability distribution describing the number of faults realized in the interval $(t,t+s)$ has a Binomial distribution of the form in Equation (2), where $D(t)$ is the number of faults realized in the interval $(0,t)$.

$$\Pr\left[D(t+s)-D(t)=n \middle| D(t)=i\right]=$$
$$\binom{K-i}{n}\left(1-e^{-\int_t^{t+s}U(y)\lambda(y)dy}\right)^n\left(e^{-\int_t^{t+s}U(y)\lambda(y)dy}\right)^{K-i-n},n=0,...,K-i \tag{2}$$

Parametric inference under such censoring can result in optimistic estimates of the reliability function due to the assumptions underlying the probability model [11]. The nonparametric approach to inference provides an alternative. For example, it is trivial to calculate a Kaplan-Meier estimate [12] of the underlying distribution once the data have been converted to a form represented in Figure 1a. However, the construction of confidence intervals is not necessarily straightforward. Standard approaches, such as Greenwood's formula [12], rely on the Central Limit Theorem; and if the required large sample sizes are not achieved, these approaches can result in confidence intervals for the reliability function that exceed 1, or fall below 0.

The use of bootstrapping for obtaining confidence intervals based on the Kaplan-Meier estimate of the reliability function is well documented [13]. However difficulties arise in adequately modeling the censoring structure using such an approach. For simpler data structures, for example, where fault realization times are censored independently of the realizations from other copies, re-sampling directly from the Kaplan-Meier estimate [14] can result in asymptotically incorrect results [15] as opposed to re-sampling directly from the data [16]. Therefore, we can reasonably expect similar problems for the more complex censoring structure of interest in this paper, where the sample size is varying throughout the period of observation because copies are censored at different times. Therefore, we require both means of estimating the reliability function for our scenario before we can develop the bootstrapping procedures.

3. ESTIMATOR OF THE RELIABILITY FUNCTION

We begin by defining the reliability function for the probability that a particular fault is not realized in the interval (0,*t*), assuming there is only one copy of the item. This function provides information about how long a fault can remain within the item if it were left to fail without interference from modifications, and is given by

$$R(t) = e^{-\int_0^t \lambda(u)du} \tag{3}$$

We extend this reasoning to develop an estimator of the reliability function under the assumed censoring structure using the approach of Kaplan Meier. Consider the situation where U_j copies of the item are in operation at time t_j, where t_i represents the time of the i^{th} fault realization. Assume each copy contains *K* faults at the start of operation, because all parts began observation at the same calendar time. If, at time t_i, the i^{th} fault is realized, then *K-i* faults will remain within the item design. Once a fault is exposed in one copy, it is removed from all copies without the addition of another fault into the item design. Hence, an estimator of the conditional reliability function, $R(t|t_{i-1})$, is given by the ratio of the number of faults that will not have been realized by time t_i to the total number of faults that either remain undetected at time t_i or are realized by time t_i

$$\hat{R}_{KM}\left(t_i|t_{i-1}\right) = \frac{U_i\left(K+1-i\right)-1}{\left(K+1-i\right)\times U_i} \tag{4}$$

The estimator of the unconditional reliability function is then the product of the conditional reliability functions

$$\hat{R}_{KM}\left(t_i\right) = \prod_{j=1}^{i} \frac{U_j\left(K-j+1\right)-1}{\left(K-j+1\right)\times U_j} \tag{5}$$

The proposed estimation procedure would be unbiased if the reliability function is compiled from data collected at controlled discrete points selected along the observation period because this would be based on modeling the number of faults observed within any section of the observation period through a Binomial sampling scheme. However, because we propose to estimate the reliability function at each time a fault is realized, this will result in a bias. For example, consider the reliability function conditional on survival to time t_{i-1}, $R(t|t_{i-1})$, where $t<t_i$,. If at time t there are U_i copies being observed that will also be observed at time t_i, then there are $K+1-i$ faults remaining within the design. Denote the probability that a particular fault is not realized by time t, given it has not been realized by time t_{i-1}, as $R_F(t|t_{i-1})$ where the subscript F is used to denote that there are a 'fleet' of copies. $R_F(t|t_{i-1})$ will be the product of the conditional reliability functions for each copy, and so the probability that a fault remains within a copy by time t_i, given it was not realizes by time t_{i-1}, is given by

$$R\left(t_i \middle| t_{i-1}\right) = R_F\left(t_i \middle| t_{i-1}\right)^{\frac{1}{U_i}} \tag{6}$$

At time t_{i-1}, there are K-i faults remaining in the item, and there are U_i copies being observed, all of which possess each fault. The distribution of the time to realize the next fault, T_i, can be derived from an order statistic argument [17]. The time to realization of the next fault will be the minimum fault realization time from a sample of $K+1-i$, where each fault realization time is i.i.d. from the distribution with reliability function $R_F(t|t_{i-1})$. The pdf of the time to realize the i^{th} fault, given the (i-1)$^{\text{th}}$ fault was realized at time t_{i-1}, is

$$f_i\left(t\right) = -\left(K+2-i\right)\left[R_F\left(t \middle| t_{i-1}\right)\right]^{K+1-i} dR_F\left(t \middle| t_{i-1}\right), \quad 0<t_{i-1}<t, \quad K \geq i,\ K,i \in \mathbb{N} \tag{7}$$

Therefore, an unbiased estimator of the conditional reliability function at time t_i would be

$$\hat{R}\left(t_i \middle| t_{i-1}\right) = E\left[R_F\left(t_i \middle| t_{i-1}\right)^{\frac{1}{U_i}} \right]$$
$$= -\int_{t_{i-1}}^{\infty} R_F\left(t_i \middle| t_{i-1}\right)^{\frac{1}{U_i}} \left(K+1-i\right) R_F\left(t_i \middle| t_{i-1}\right)^{K+1-i} dR_F\left(t_i \middle| t_{i-1}\right) \quad (8)$$
$$= \frac{K+1-i}{K+1-i+\frac{1}{U_i}}$$

The estimator Equation (8) will always produce an estimate of reliability which is greater than Equation (4) for the following reasons. The difference will decrease as the number of copies increases, or when many faults remain within the item design.

$$\frac{\hat{R}_{KM}\left(t_i \middle| t_{i-1}\right)}{\hat{R}\left(t_i \middle| t_{i-1}\right)} = \frac{\frac{U_i\left(K+1-i\right)-1}{\left(K+1-i\right)U_i}}{\frac{K+1-i}{K+1-i+\frac{1}{U_i}}}$$
$$= \left[1 - \frac{1}{\left(K+1-i\right)U_i}\right]\left[1 + \frac{1}{\left(K+1-i\right)U_i}\right]$$
$$< 1$$

The conditional probabilities in Equation (8) lead naturally to the following estimator for the reliability function

$$\hat{R}\left(t_i\right) = \prod_{j=1}^{i} \frac{K-j+1}{K-j+1+\frac{1}{U_j}} \quad (9)$$

As each of the conditional reliability estimates given by Equation (8) are unbiased & conditionally independent at each fault realization time, then Equation (9) is an unbiased estimator of the underlying reliability function.

4. BOOTSTRAP CONFIDENCE INTERVALS

The bootstrap method of constructing confidence intervals is based on the principle of strong repeated sampling, and is assessed by examining the behavior of the estimates through hypothetical repetitions under the same conditions under which the data were observed [18]. As such, data are simulated from the estimated reliability function subject to both types of censoring: deterministic censoring of the copies, and random censoring at the times at which faults are realized.

Figure 2 illustrates the modeling process. A natural approach to re-sampling would be to simulate a realization time for each fault on each copy using $\hat{R}(t)$. This would require KU_0 simulations. For each simulated realization time, there will be an assessment of whether the fault was realized prior to the censored time of the copy. This would require KU_0 evaluations. For each fault, the earliest time it was observed is recorded, and provides the re-sampled data from which the reliability function can be re-estimated. This process is repeated indefinitely, and allows the variability in estimation to be recorded & used to determine bootstrap confidence intervals. In total, there will be $2MKU_0$ calculations, where M is the number of bootstraps required.

The re-sampling process described in Figure 2 would be simple to code. However it is possible to develop an algorithm for calculating the bootstrap confidence interval requiring fewer simulations with only $2MK$ calculations. This not only reduces computational time, but also supports an explicit representation of the confidence intervals.

To develop the revised algorithm, again we begin by assuming only one copy is being observed for a pre-determined time c_m. Consider the probability distribution describing the time until a particular fault, say j, is realized; and denote this random variable by T_j. This is obtained directly from Equations (8) & (9), noting t_0=0.

$$\Pr\left[T_j = t_{i+1}\right] = \hat{R}(t_i)\left[1 - \hat{R}(t_{i+1}|t_i)\right], \quad c > t_{i+1} > t_i > 0 \tag{10}$$

The probability that a censored time for fault *j* is generated is given by

$$\Pr\left[T_j = c\right] = \hat{R}(c) \tag{11}$$

Because the two components of the right hand side of Equation (10) are conditionally independent, we obtain an unbiased estimate of the probability of detecting a fault at time t_{i+1}.

When two or more copies are observed, the model needs to be extended to include the random censoring mechanism. Therefore consider the distribution for the time until fault *j* is first realized across all copies, and denote this random variable by X_j. This distribution can be obtained from Equation (10), adjusting for the varying number of copies being observed, and is given by

$$\Pr\left[X_j = t_i\right] = \left[\prod_{m=1}^{i\text{-}1} \hat{R}(t_m)^{U_{m\text{-}1} - U_m}\right]\left[\hat{R}(t_{i\text{-}1})^{U_i} - \hat{R}(t_i)^{U_i}\right], \quad c > t_{i+1} > t_i > 0 \tag{12}$$

If c_{max} represents the maximum censored time across all copies, then the probability that fault *j* is censored from the fleet data is

$$\Pr\left[X_j = c_{\max}\right] = \left[\prod_{m=1}^{K} \hat{R}(t_m)^{U_{m\text{-}1} - U_m}\right] \tag{13}$$

However, this leads to a bias in re-sampling for the following reasons. It has been argued that the reliability function in Equation (9) is an unbiased estimator of the reliability function at each time of fault realization, t_i. As such, the successive ratios between this estimator & the reliability function form a martingale process [18]. However, the re-sampling proposed in Equation (12) is a power transformation of the estimator in Equation (9), and so, due to the convexity of the transformation, the successive ratios between Equation (12) & the true probability for this process would form a sub-martingale process. Simply, we would have the following relationship

$$E\left[\hat{R}(t_i)^U\right] \geq R(t_i)^U \text{ , } for\ i = 1, \ldots, K$$

An unbiased re-sampling proportion would be obtained by using

$$\Pr\left[X_j = x\right] = \frac{1}{K+1}, \quad where:\ x = t_1, \ldots, t_K, c_{\max} \tag{14}$$

Assuming the realizations of distinctly different faults are independent processes, the bootstrap re-samples for the realizations of the *K* faults can be simulated from a multinomial distribution, with equal proportion assigned to each realization time, or maximum censored time.

Bootstrap re-samples are generated from the data by conditioning on the censoring times of the copies. Because there are *K* faults realized at times t_i (i = 1 to K), then the result from a re-sample will be a vector of fault realization times ($x_1, .., x_k$). If the re-samples can be re-conceptualized as fixed times, where the number of faults assigned to that time as their first realization are randomly selected, then we introduce N_i to represent the number of faults assigned to time t_i for time of first realization. The vector of N_i (i=1 to K+1) has the following multinomial distribution

$$\Pr\left(N_1 = n_1, \ldots, N_K = n_k, N_{K+1} = c_{\max}\right) = \frac{K!}{n_1! \ldots n_K! n_{K+1}!}\left(\frac{1}{K+1}\right)^K$$

$$\sum_{i=1}^{K+1} n_i = K,\ K \in \mathbb{N} \tag{15}$$

Having simulated the bootstrap data from the distribution in Equation (15), the reliability function can be re-assessed. However re-sampling from a discrete distribution in Equation (9) means it is possible that more than one fault can be realized at the same time. Therefore, when estimating the reliability function using the bootstrapped data, it is no longer appropriate to use the approach in Equation (9); but instead, one should employ the usual Kaplan-Meier for the conditional reliability function

$$\begin{aligned}\hat{R}_B\left(t_i|t_{i\text{-}1}\right) &= \frac{U_i\left[K\text{-}D\left(t_{i\text{-}1}\right)\right]-D\left(t_i|t_{i\text{-}1}\right)}{U_i\left[K\text{-}D\left(t_{i\text{-}1}\right)\right]} \\ &= 1-\frac{D\left(t_i|t_{i\text{-}1}\right)}{U_i\left[K\text{-}D\left(t_{i\text{-}1}\right)\right]}\end{aligned} \tag{16}$$

where $D(t_i|t_{i-1})$ represents the number of faults realized at time t_i, and $D(t_{i-1})$ represents the number of copies observed in the interval (0, t_{i-1}]. Thus, the estimates of the reliability function from the bootstrap data are computed from

$$\hat{R}_B\left(t_i\right) = \prod_{j=1}^{i}\left[1-\frac{D\left(t_j|t_{j\text{-}1}\right)}{U_j\left[K\text{-}D\left(t_{j\text{-}1}\right)\right]}\right] \tag{17}$$

5. EVALUATION OF THE BOOTSTRAP

The bootstrap confidence intervals are evaluated by comparing the expectation, and the standard deviation of the conditional reliability function obtained through bootstrapping, with those of the true reliability function.

5.1 Expectation of the Bootstrap Re-samples

From Equation (17), the re-sampled probability assigned to time t_i can be considered as a function of two correlated random variables, $D(t_i|t_{i-1})$ & $D(t_{i-1})$, whose joint distribution is

$$\Pr\left(D\left(t_{i\text{-}1}\right)=d_{i\text{-}1}, D\left(t_i\,|t_{i\text{-}1}\right)=n_i\right) = \frac{K!}{d_{i\text{-}1}!n_i!\left(K\text{-}n_i-d_{i\text{-}1}\right)!}\left(\frac{i\text{-}1}{K+1}\right)^{d_{i\text{-}1}}\left(\frac{1}{K+1}\right)^{n_i}\left(\frac{K+1\text{-}i}{K+1}\right)^{K\text{-}d_{i\text{-}1}-n_i} \tag{18}$$

Repeated samples are taken from a birth process, which can terminate at each of the observed order statistics, t_i. Therefore, from Equation (18), we note that if all faults were realized before time t_i, then the reliability function at time t_i would be indeterminate. To overcome this problem, consider the expectation conditional on there being faults to realize at the given times.

The expectation of each conditional probability assigned to the fault realization times within the bootstrap can be derived from Equation (18) as

$$E\left[\hat{R}_B\left(t_i \middle| t_{i-1}\right) \middle| D\left(t_{i-1}\right) \le K-1\right]$$
$$= 1 - \frac{\sum_{d_{i-1}=0}^{K-1} \sum_{n_i=0}^{K-d_i} \frac{n_i}{U_i\left[K-d_{i-1}\right]} \frac{K!}{d_{i-1}!\,n_i!\left(K-n_i-d_{i-1}\right)!} \left(\frac{i-1}{K+1}\right)^{d_{i-1}} \left(\frac{1}{K+1}\right)^{n_i} \left(1-\frac{i}{K+1}\right)^{K-d_{i-1}-n_i}}{1-\left(\frac{i-1}{K+1}\right)^K} \qquad (20)$$
$$= 1 - \frac{1}{U_i}\left(\frac{1}{K+2-i}\right)$$

Hence, Equation (16) is clearly a biased estimator of the reliability function at time t_i. There are two obvious approaches to correcting for bias. One is to arithmetically adjust the estimates by adding a corrective term, as shown in Equation (21), and it is denoted by R_{BA}. Alternatively, the conditional estimates can be adjusted multiplicatively, as shown in Equation (22), and it is denoted by R_{BM}. Each approach would produce unbiased estimates, although R_{BM} would also affect the variation, which would increase with time.

$$\hat{R}_{BA}\left(t_i \middle| t_{i-1}\right) = \frac{\left(K+1-i\right)\left[U_i\left(K+2-i\right)+1\right]+\frac{1}{U_i}}{\left(K+2-i\right)\left[U_i\left(K+1-i\right)+1\right]} - \frac{n_i}{U_i\left[K-d_{i-1}\right]} \qquad (21)$$

$$\hat{R}_{BM}\left(t_i \middle| t_{i-1}\right) = \frac{1}{\left(1+\frac{1}{U_i\left(K+1-i\right)}\right)\left(1-\frac{1}{U_i\left(K+2-i\right)}\right)}\left[1-\frac{n_i}{U_i\left[K-d_{i-1}\right]}\right] \qquad (22)$$

Inspection of Equations (21) & (22) indicates that the bias will decrease as the number of copies increases. The bias also increases as a function of t, such that bias is greatest at t_k, the time of the last fault realization.

5.2 Standard Deviation of the Bootstrap

Consider the variability associated with the conditional reliability function, firstly by examining the standard deviation of the underlying stochastic process, and secondly from the bootstrap re-samples before making a comparison.

Assume that immediately after time t_{i-1} there are K-$D(t_{i-1})$ faults remaining, and there are U_i copies being observed. An order statistic argument leads to a closed form solution

$$E\left[R^2\left(t_i \middle| t_{i-1}\right)\middle| D\left(t_1\right),..,D\left(t_{i-1}\right) \le K-1\right] = -\int_{t_{i-1}}^{\infty} R_F\left(t_i \middle| t_{i-1}\right)^{\frac{2}{U_i}} \left(K+1-i\right) R_F\left(t_i \middle| t_{i-1}\right)^{K-i} dR_F\left(t_i \middle| t_{i-1}\right)$$

$$= -\frac{K+1-i}{K+1-i+\frac{2}{U_i}} \int_{t_{i-1}}^{\infty} \left(K+1-i+\frac{2}{U_i}\right) R_F\left(t_i \middle| t_{i-1}\right)^{K-i+\frac{2}{U_i}} dR_F\left(t_i \middle| t_{i-1}\right)$$

$$= \frac{K+1-i}{K+1-i+\frac{2}{U_i}}$$

Similarly, from Equation (8), we have

$$E\left[R\left(t_i \middle| t_{i-1}\right)\middle| D\left(t_1\right),..,D\left(t_{i-1}\right) \le K-1\right] = \frac{K+1-i}{K+1-i+\frac{1}{U_i}}$$

Therefore,

$$\begin{aligned} &Var\left[R\left(t_i \middle| t_{i-1}\right)\middle| D\left(t_1\right),..,D\left(t_{i-1}\right) \le K-1\right] \\ &= E\left[R^2\left(t_i \middle| t_{i-1}\right)\middle| D\left(t_1\right),..,D\left(t_{i-1}\right) \le K-1\right] - \left[E\left[R\left(t_i \middle| t_{i-1}\right)\middle| D\left(t_1\right),..,D\left(t_{i-1}\right) \le K-1\right]\right]^2 \\ &= \frac{K+1-i}{K+1-i+\frac{2}{U_i}} - \left(\frac{K+1-i}{K+1-i+\frac{1}{U_i}}\right)^2 \end{aligned} \qquad (23)$$

and so the standard deviation is

$$\sigma_{R(t_i|t_{i-1})|D(t_1),..,D(t_{i-1})} = \sqrt{\frac{K+1-i}{K+1-i+\dfrac{2}{U_i}} - \left(\frac{K+1-i}{K+1-i+\dfrac{1}{U_i}}\right)^2} \tag{24}$$

The bootstrap conditional reliability functions, are conditionally mutually independent [19], hence we can derive the following expression for the variance.

$$Var\left[\frac{n_i}{U_i\left[K-d_{i-1}\right]}\middle| D(t_1),..,D(t_{i-1})\right] = \left(\frac{K+1-i}{U_i\left(K+2-i\right)}\right)^2 \left[\prod_{j=1}^{i}\left(\frac{1}{(K+1-j)\left(K-D\left(t_{j-1}\right)\right)}+1\right)-1\right]$$

(25)

Therefore, an estimate of the standard deviation of $\hat{R}_{BA}\left(t_i|t_{i-1}\right)$ is given by Equation (26), and for $\hat{R}_{BM}\left(t_i|t_{i-1}\right)$ by Equation (27). These expressions have been obtained by substituting *$D(t_{j-1})$* with *$K+1-j$*, which is the *$E[D(t_{j-1})]$*.

$$\sigma_{\hat{R}_{BA}(t_i|t_{i-1})|D(t_1),..,D(t_{i-1})} = \frac{\left(\dfrac{K+1-i}{K+2-i}\right)\sqrt{\prod_{j=1}^{i}\left(\dfrac{1}{(K+1-j)^2}+1\right)-1}}{U_i} \tag{26}$$

$$\sigma_{\hat{R}_{BM}(t_i|t_{i-1})|D(t_1),..,D(t_{i-1})} = \frac{\left(\dfrac{K+1-i}{K+2-i}\right)\sqrt{\prod_{j=1}^{i}\left(\dfrac{1}{(K+1-j)^2}+1\right)-1}}{\left(1+\dfrac{1}{U_i\left(K+1-i\right)}\right)\left(1-\dfrac{1}{U_i\left(K+2-i\right)}\right)U_i} \tag{27}$$

The standard deviation from both adjusted bootstraps can be compared with the order statistic approach in Equation (24). The calculations were based on fleet sizes (*U*) ranging from 1 to 1000 copies, and the number of faults within the item design (*K*) ranging from 5 to 501. Note that an odd number of faults were selected to simplify the evaluation of the median. We found that the differences between Equations (25) & (26) were negligible; therefore, only the results using Equation (26) are summarized in Table I.

The maximum difference increases as the number of faults increases, while the median difference, and the smallest difference both decrease. The number of copies has a greater impact on the differences, whereby an increase in the number of copies by a factor of 10 approximately decreases the difference by a factor of 10. In summary, for situations where at least 10 copies are being observed, the differences in standard deviations were small, and to all extents & purposes, negligible.

6. ILLUSTRATIVE EXAMPLE

This example is motivated by the development of complex electronic equipment for aerospace systems. These data have been desensitized, but the key messages associated with the application of the method are representative of actual experiences. The item of equipment being developed was a variant of earlier designs for which there was accumulated operating experience in similar environments. Elicitation of engineering judgment was conducted to assess the potential faults within the new design [20] based on the processes discussed in [4] and historical data provided the duration of operating time until a fault is detected within a copy of the item.

For an earlier generation of the design, 15 faults were realized over a period of 2 years. There were 20 copies in-service, of which half were censored after the first year of operation, and the remainder after the second year. Each copy was exposed to approximately 6000 operating hours per annum. Once a fault was realized in operation, a modification was implemented across the fleet; this was assumed to occur instantaneously. The estimated reliability function was calculated using Equation (6), and is illustrated for the first year of operation in Figure 3, with 95% bootstrap confidence intervals as well as the true 95% confidence intervals obtained through the order statistic approach. Due to the high number of censored faults, the increments on the reliability function are small, and at the end of the observation period, there is still a high chance a fault would remain within the item without resulting in a failure.

We used Monte Carlo methods to simulate the number of faults exposed at each of the fault realization times. At each time, the 2.5th and 97.5th percentiles were identified to provide the 95% bootstrap point-wise confidence intervals. For the first year, there is very little difference between the two sets of confidence intervals. Computations for the order statistic confidence intervals for the second year are more challenging because the fleet size changes. The largest deviation between the two sets of confidence intervals occurs about 7000 hours with the difference on the lower bounds being 0.00592.

The preferred methodology for constructing confidence intervals prior to the development of the procedures presented in this paper would have been based on the use of Greenwood's formula. For this example, as expected, these approximate intervals are consistently wider than the bootstrap point-wise, and the order statistic confidence intervals, although the difference is not statistically large.

There are two main advantages for using bootstrap rather than analytical solutions. First, as the number of copies changes throughout the observation period, the calculations required to derive confidence intervals increases substantially. Second, the bootstrap approach easily supports the determination of a confidence region for the reliability function. For example, Figure 4 illustrates the 95% confidence region, together with the point estimate, of the reliability function. This region is bounded by the two curves that contain 95% of the bootstrap reliability functions. Figure 4 shows that the point estimate is very close to the lower bound of the confidence region near the end of the process. This is due to the termination of the birth processes prior to the end of the observation period. Finally, an important characteristic of these birth processes is that they do not possess independent increments; therefore the usefulness of point-wise confidence intervals is limited.

6.1 Discussion

Assuming all copies begin observation at the same time, contain the same number of faults, which when identified in any one copy are corrected, instantly & perfectly, across all copies is unrealistic. For example, typical problems giving rise to sequences of failure times for multiple copies of repaired items include aircraft fleet reliability monitoring, warranty analysis of consumer goods such as mobile phones, and plant-wide analysis of common components. In many cases, it may be that the copies build up over calendar time giving rise to different exposure times, and different numbers of inherent faults at any age. Our approach is adaptable to such situations.

For example, if we have fleet data where the entry into service dates vary for each copy, then this not only affects the exposure of faults to operating conditions but some younger copies may be released into service with fewer faults than older copies due to modifications implemented prior to their release. However, we assume that the realization of faults is i.i.d. for each fault. As such, Equation (14) is a valid approach for simulating the first realization of each fault, but Equation (16) requires correction.

The first necessary amendment is to record the calendar time of the realization of each fault. Denote the calendar time of the realization of fault i by z_i. The first stage of the bootstrapping is to simulate the operational time of the first realization of each fault, then simulate not only an operational time but also a calendar time using

Equation (28). The operational time, t_i, is associated with the copy that realized the fault, and the index *i* is assigned to the i^{th} smallest operational time.

$$\Pr\left[\tilde{X}_j = (t_i, z_i)\right] = \frac{1}{K+1}, \quad where:\ i = 1,..,K+1 \quad t_{K+1} = c_{\max} \quad z_{K+1} = \infty \tag{28}$$

From Equation (28), the number of faults that exist within each copy is evaluated as a function of operating time; however, because copies enter service at different calendar times, the number of faults per copy may differ. Denoting the number of faults realized across the fleet by calendar time *z* as *G(z)*, and the entry into service calendar time of copy *j* by s_j, then the number of faults in copy *j* after t_i operating hours is denoted by $Y_j(t_i)$, and expressed as

$$Y_j(t_i) = \begin{cases} 0, & if\ c_j \le t_i \\ K + 1 - G(s_j + t_i), & if\ c_j > t_i \end{cases}$$

The conditional reliability function is estimated by

$$\hat{R}_B(t_i | t_{i-1}) = 1 - \frac{D(t_i | t_{i-1})}{\sum_{j=1}^{U} Y_j(t_i)} \tag{29}$$

where $D(t_i | t_{i-1})$ is the number of faults realized at time t_i through the bootstrap simulation. The overall reliability function is calculated as usual by evaluating the products of the conditional reliability estimates.

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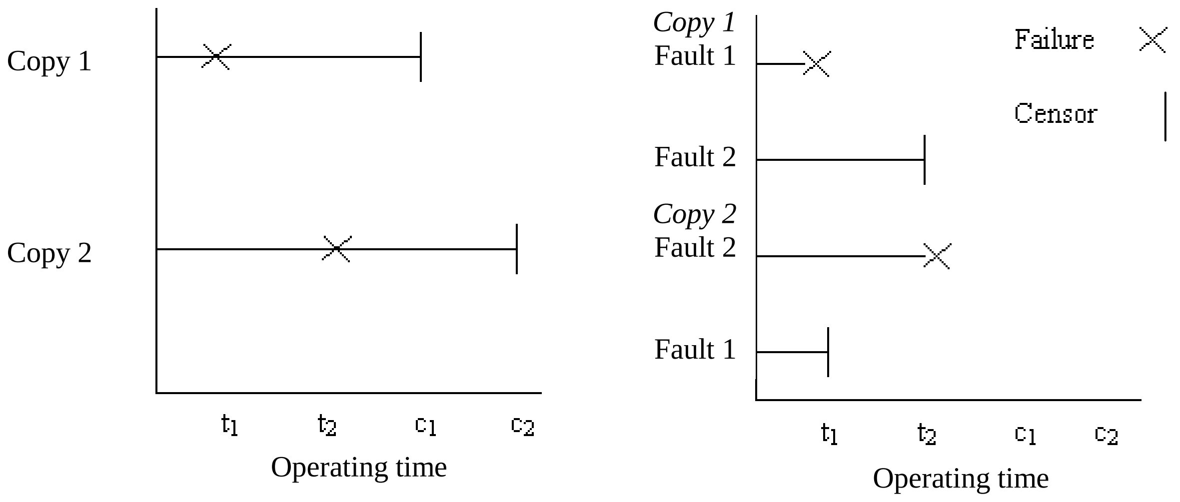

a) Data recorded by copy tracking. b) Data recorded by fault tracking.

Figure 1: Representations of operational data with two censoring mechanisms.

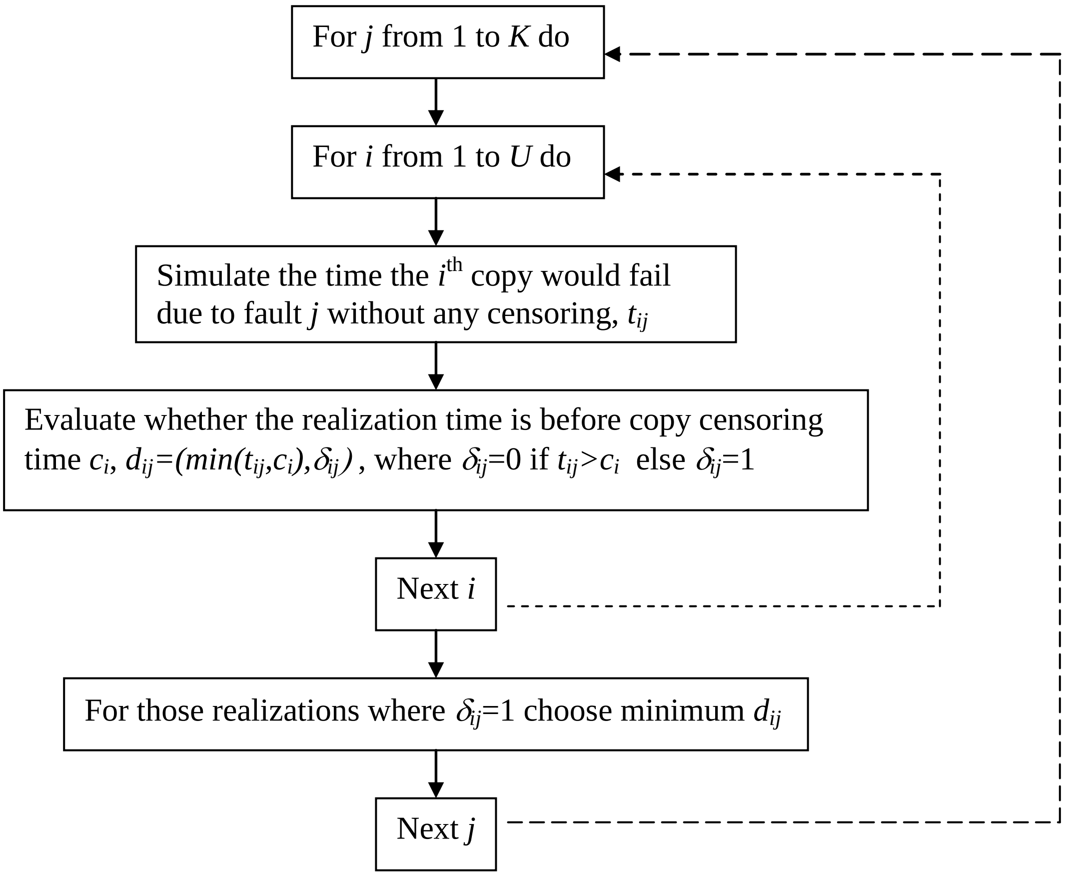

Figure 2: Computationally intensive bootstrapping procedure.

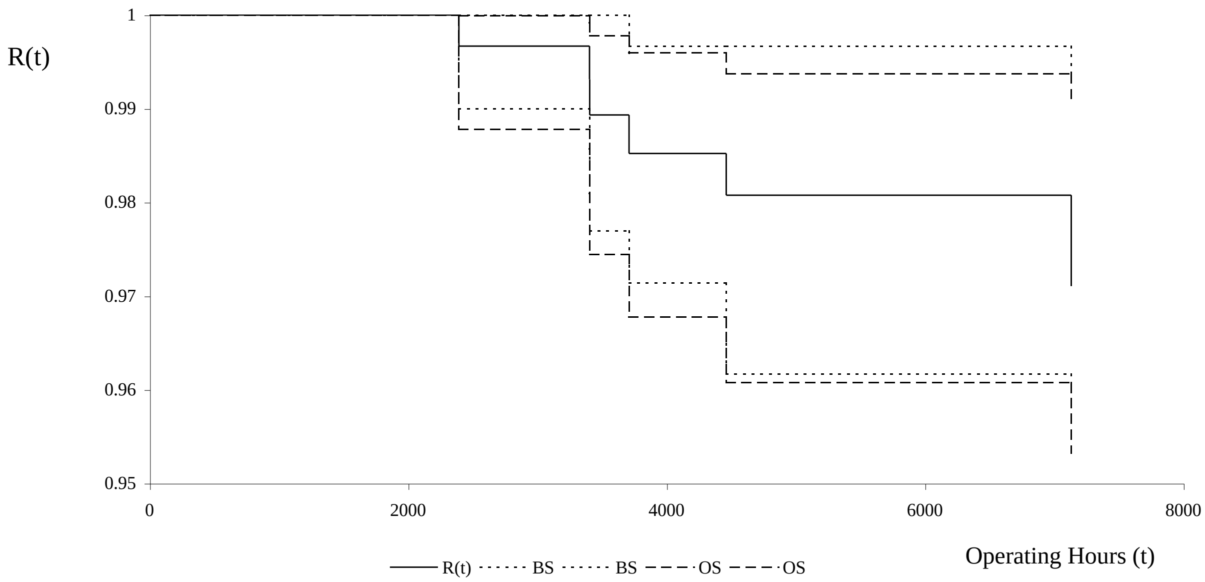

Figure 3: Comparison of point-wise bootstrap (BS), and true (OS) 95% confidence intervals for reliability function for year 1 data.

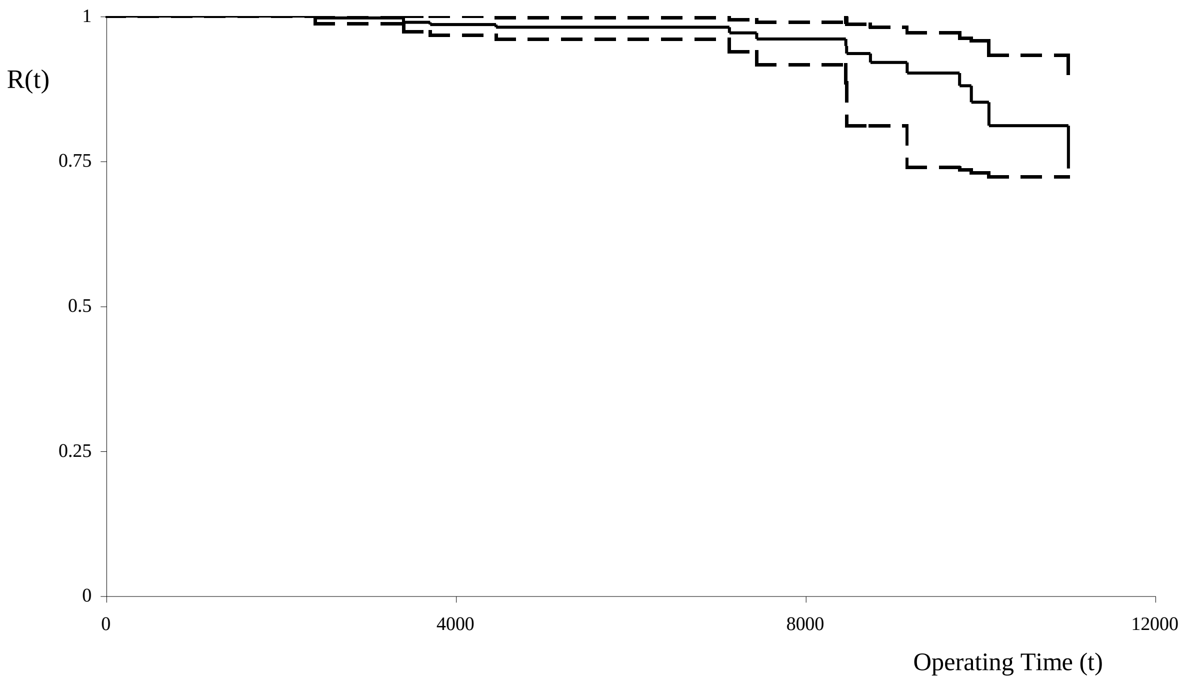

Figure 4: 95% confidence region for reliability function with point estimate of R(t) for data from both years.

Table I: Bootstrap minus order statistic standard deviations of conditional reliability.

a) Maximum Difference.

*K**U*	1	10	100	1000
5	0.4306	0.0043	0.0003	0.000025
51	0.5184	0.0199	0.0019	0.000192
501	0.5281	0.0217	0.0021	0.000210

b) Median Difference.

*K**U*	1	10	100	1000
5	0.1643	0.0020	-0.0001	-0.00001
51	0.1004	0.0094	0.0009	0.00009
501	0.0406	0.0040	0.0004	0.00004

c) Minimum Difference.

*K**U*	1	10	100	1000
5	0.025800	-0.01416	-0.00260	-0.00028
51	0.000400	-0.00580	-0.00180	-0.00019
501	0.000004	-0.00480	-0.00168	-0.00018

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