Analytic perturbative theories in highly inhomogeneous gravitational fields

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Abstract

Orbital motion about irregular bodies is highly nonlinear due to inhomogeneities in the gravitational field. Classical theories of motion close to spheroidal bodies cannot be applied as for inhomogeneous bodies the Keplerian forces do not provide a good approximation of the system dynamics. In this paper a closed form, analytical method for developing the motion of a spacecraft around small bodies is presented, for the so called fast rotating case, which generalize previous results to second order, arbitrary degree, gravitational fields. Through the application of two different Lie transformations, suitable changes of coordinates are found, which reduce the initial non integrable Hamiltonian of the system into an integrable one plus a negligible, perturbative remainder of higher degree. In addition, an explicit analytical formulation for the relegated, first and second order, arbitrary degree Hamiltonian for relatively high altitude motion in any inhomogeneous gravi-

tational field is derived in closed-form. Applications of this algorithm include a method for determining initial conditions for frozen orbits around any irregular body by simply prescribing the desired inclination and eccentricity of the orbit. This method essentially reduces the problem of computing frozen orbits to a problem of solving a 2-D algebraic equation. Results are shown for the asteroid 433-Eros.

Keywords: Asteroids, dynamics, Orbit determination, Asteroids, Celestial mechanics, Irregular satellites

1. Introduction

The motion of bodies subject to non-Keplerian gravitational fields is a classical subject of research in the context of celestial mechanics. In recent years this type of research has become important to future planned missions of spacecraft to the moon and asteroids in addition to asteroid deflection missions such as the European Space Agency's "Don Quijote" concept Carnelli and Gálvez (2006). Research undertaken in this area has studied the effect of the Earth's inhomogeneous gravitational field on the motion of natural and artificial satellites, that is, artificial satellite theory for small and moderate eccentricities Deprit (1970). More recent studies have researched the effects on motion of the inhomogeneous gravitational field of other solar system bodies, including the Moon Abad et al. (2009) and asteroids San-Juan et al. (2004). The analysis of spacecraft motion about these bodies is particularly challenging as they typically feature shapes and density distributions more irregular than those of planets. Such irregularities break symmetries and require more complicated analytical expressions for their description which

17 increases the complexity involved in such studies.

18 Numerical methods are today widely used to study the trajectories of ob-
19 jects orbiting specific irregular bodies Fahnestock and Scheeres (2008) or for
20 finding stability criteria (Lara and Scheeres (2002)). Disadvantages of these
21 methods are that they can be highly computational and require a complete
22 re-design for each different body. Analytical methods, by contrast, have the
23 potential to rapidly identify useful natural motions for general bodies with
24 inhomogeneous gravitational fields. Furthermore, they can provide a full dy-
25 namical picture of the motion around irregular bodies that can be used to
26 search and study particular classes of useful orbits. However, current ana-
27 lytical methods are only used in a limited and semi-numerical way (meaning
28 that analytical expansions constitute the first step in such studies, which are
29 then typically carried out from a numerical standpoint Scheeres et al. (1998)).
30 The main drawbacks of these methods is that their application in the case
31 of highly inhomogeneous bodies requires extensive symbolic computations
32 involving algebraic manipulations, and that they are usually restricted to a
33 certain range of eccentricities due to series convergence. Analytical studies
34 on inhomogeneous gravitational fields have been, so far, limited to low degree
35 gravity fields Palacián (2002), San-Juan et al. (2002), San-Juan et al. (2004),
36 thus restricting the results to a class of bodies for which the dynamics is
37 dominated by a few coefficients (e.g. oblateness or ellipticity).
38 In this paper a closed form (i.e. without using series expansion in the ec-
39 centricity), analytical, perturbative theory of motion around inhomogeneous
40 bodies is presented, generalized to second order, arbitrary degree gravity
41 fields.

42 Using Deprit and Palacián's relegation algorithm (Palacián (1992)) and a
 43 Delaunay normalization, suitable canonical action-angle variables are found,
 44 which reduce the initial non-integrable Hamiltonian into an integrable one
 45 plus a negligible, perturbative remainder.
 46 The method can be used to find useful orbits for space mission applications
 47 such as frozen orbits. Moreover, frozen orbits are orbits with no secular
 48 perturbations in the inclination, argument of pericenter, and eccentricity
 49 (Brouwer (1959)). These orbits are periodic orbits, except for the orbital
 50 plane of precession, and are therefore called frozen. In particular, this paper
 51 extends previous work by:

- 53 • Formulating the inhomogeneous gravitational potential generated by
 54 any inhomogeneous body in polar-nodal coordinates
- 55 • Including arbitrary degree gravitational coefficients, instead of limiting
 56 the study to 2^{nd} degree coefficients
- 57 • Stating the explicit analytical formulation for the closed-form averaged
 58 with respect to the argument of node, second order, arbitrary degree
 59 Hamiltonian of any inhomogeneous gravitational field.
- 60 • Obtaining a resulting Hamiltonian which accounts for the presence of
 61 the angular momentum, in contrast to the trivially integrable Hamil-
 62 tonian of San-Juan et al. (2002) which only accounts for the argument
 63 of node. Again, this previous result was only possible by considering a
 64 Hamiltonian with 2nd degree coefficients.

- 65 • Providing a method for determining initial conditions for frozen orbits
66 around any irregular body by simply prescribing the inclination and
67 eccentricity of the desired orbit.
- 68 • Applying the method to the asteroids 433-Eros, which is the main
69 example studied in.

70 Therefore, the proposed perturbative theory presents a method to derive
71 more accurate descriptions of a spacecraft’s high altitude motion about an
72 asteroid, which enables, for example, one to find precise initial conditions
73 that yield frozen orbits.

74 2. Method

75 Assuming that the planetary body is in uniform rotation around its axis of
76 greatest inertia the potential generated by the inhomogeneous gravitational
77 field can be derived in the rotating polar nodal variables (Whittaker (1917))
78 convenient for the successive transformation to Delaunay coordinates. This
79 potential takes into account an arbitrary number of spherical harmonic co-
80 efficients, all considered to have the same order, thus providing a dynamical
81 model based on an arbitrarily accurate model of the inhomogeneous body.
82 Restricting the analysis to the fast rotation case, i.e. when the angular veloc-
83 ity of rotation the asteroid is higher than the mean motion of the spacecraft,
84 the methodology is then based on the following steps:

- 85 • Relegation of the polar component of the angular momentum N to
86 obtain the relegated nodal variables where the argument of nodes con-
87 jugate momenta is constant along the Hamiltonian flow.

- 88 • Transformation to Delaunay variables to yield a constant total angular
89 momentum in the z-direction
- 90 • Normalization of the Delaunay variables which yields a reduced ordi-
91 nary differential equation in two coordinates; the total angular momen-
92 tum and the argument of pericentre
- 93 • The frozen orbits are identified with the equilibrium points of these
94 equations i.e. where the total angular momentum about the z-axis and
95 the argument of pericentre are constant, therefore the final stage is
96 undertaken by solving a 2-D algebraic equation.

97 The methodology comprises of two different Lie transformations, relega-
98 tion and normalisation, constructed following Deprit and Palacián’s algo-
99 rithm (Palacián (1992)) and Deprit’s method for Lie transformations (Deprit
100 (1969)). The Delaunay normalization (Deprit (1982)), cannot be directly ap-
101 plied to a high-order model due to the presence of the argument of node that
102 appears in the Coriolis term. The addition of this term in the Lie derivative
103 prevents the conventional computation of the Lie transform generator (San-
104 Juan et al. (2002)). However, Deprit and Palacián’s closed form relegation
105 algorithm (Deprit et al. (2001)) can be applied, which “relegates” the action
106 of the argument of node to a negligible remainder. It is shown that, for this
107 model, both relegation and normalization results are equivalent to averaging
108 over the fast angles.

109 3. The dynamical system

110 An inhomogeneous body is considered, which rotates uniformly around
 111 its axes of greatest inertia with constant angular velocity $\hat{\omega} = [0, 0, \omega]$.
 112 The total mass of the body is M while $\mathcal{G}$ is the universal gravitational con-
 113 stant and it is set $\mu = M\mathcal{G}$. The dynamics are formulated into a reference
 114 frame centered in the center of mass of the body and oriented with the “ z -
 115 axis” parallel to the rotational axes of the asteroid. The frame of reference is
 116 rotating with the same velocity of rotation of the asteroid; in such rotating
 117 coordinates the Hamiltonian describing the system is:

$$H(\mathbf{x}, \mathbf{X}) = \frac{1}{2}(\mathbf{X} \cdot \mathbf{X}) - \hat{\omega}(\mathbf{x} \times \mathbf{X}) + \bar{U}(\mathbf{x}) \quad (1)$$

118 where $\mathbf{x}, \mathbf{X} \in \mathbb{R}^3$ are respectively the position coordinates and conjugate
 119 momenta of the spacecraft, while $\bar{U}(\mathbf{x})$ is the perturbing gravitational poten-
 120 tial generated by the inhomogeneous rotating body. The equations of motion
 121 are:

$$\begin{cases} \dot{\mathbf{x}} = \frac{\partial}{\partial \mathbf{X}} H(\mathbf{x}, \mathbf{X}) \\ \dot{\mathbf{X}} = -\frac{\partial}{\partial \mathbf{x}} H(\mathbf{x}, \mathbf{X}) \end{cases} \quad (2)$$

122 It is convenient to express the Hamiltonian and the perturbing potential
 123 using the so called nodal-polar variables so that it may easily be transformed
 124 to the Delaunay coordinates in the later stage of the methodology. The
 125 six nodal-polar coordinates are r , θ , and ν (respectively the distance of the
 126 spacecraft from the body, its angular distance from the line of the ascending
 127 node and the argument of node) and their corresponding conjugate momenta
 128 R , Θ , and N . The transformation required is given in Palacián (2002),

129 setting $\mathbf{x} = [x, y, z]^T$ and $\mathbf{X} = [X, Y, Z]^T$:

$$\begin{aligned}
x &= r(\cos \theta \cos \nu - \sin \theta \cos I \sin \nu) \\
y &= r(\cos \theta \sin \nu + \sin \theta \cos I \cos \nu) \\
z &= r \sin \theta \sin I \\
X &= (R \cos \theta - \frac{\Theta}{r} \sin \theta) \cos \nu - (R \sin \theta + \frac{\Theta}{r} \cos \theta) \cos I \sin \nu \\
Y &= (R \cos \theta - \frac{\Theta}{r} \sin \theta) \sin \nu + (R \sin \theta + \frac{\Theta}{r} \cos \theta) \cos I \cos \nu \\
Z &= (R \sin \theta + \frac{\Theta}{r} \cos \theta) \sin I
\end{aligned} \tag{3}$$

130 with $N = |\Theta| \cos I$.

131 In these coordinates the Hamiltonian takes the form:

$$H(r, \theta, \nu, R, \Theta, N) = \frac{1}{2}(R^2 + \frac{\Theta^2}{r^2}) - \omega N + \bar{U}(r, \theta, \nu, -, \Theta, N) \tag{4}$$

132 where the perturbing potential, found using Wigner's rotation theorem (Wigner,
133 1959) and the addition formula for non scaled spherical harmonics (Hofmann-
134 Wellenhof et al., 1967), is:

$$\begin{aligned}
\bar{U}(r, \theta, \nu, -, \Theta, N) &= - \sum_{n=0}^{\infty} \sum_{m=0}^n \sum_{j=-n}^n \sum_{t=\max\{0, j+m\}}^{\min\{n+m, n+j\}} \text{ci}^{2n+m+j-2t} \text{si}^{2t-m-j} \cdot \\
&\quad \cdot \frac{1}{r^{n+1}} (\mathcal{A}_{n,m,j,t} \cos(m\nu - j\theta) + \mathcal{B}_{n,m,j,t} \sin(m\nu - j\theta)),
\end{aligned} \tag{5}$$

135 where

$$\begin{aligned}
\text{ci} &:= \text{ci}(N, \Theta) = \cos\left(\frac{I}{2}\right) = \sqrt{\frac{1+\cos I}{2}} = \sqrt{\frac{1+\frac{N}{\Theta}}{2}} \\
\text{si} &:= \text{si}(N, \Theta) = \sin\left(\frac{I}{2}\right) = \sqrt{\frac{1-\cos I}{2}} = \sqrt{\frac{1-\frac{N}{\Theta}}{2}}
\end{aligned} \tag{6}$$

136 with

$$\begin{aligned}
\mathcal{A}_{n,m,j,t} &= \bar{\mathcal{G}}_{n,m,j,t} (C_{n,m} \cos(\frac{\pi}{2}(j+m)) - S_{n,m} \sin(\frac{\pi}{2}(j+m))) \\
\mathcal{B}_{n,m,j,t} &= \bar{\mathcal{G}}_{n,m,j,t} (C_{n,m} \sin(\frac{\pi}{2}(j+m)) + S_{n,m} \cos(\frac{\pi}{2}(j+m))),
\end{aligned} \tag{7}$$

137 and

$$\bar{\mathcal{G}}_{n,m,j,t} = (-1)^{m+3t-j+1} \mu \alpha^n \frac{(n+m)!(n-j)!}{t!(n+j-t)!(n+m-t)!(t-m-j)!} (-1)^{\frac{n+j}{2}} \frac{1}{2^n} \frac{(n+j)!}{\left(\frac{n+j}{2}\right)! \left(\frac{n-j}{2}\right)!} \cdot$$

$$\cdot ((n+j)_{\text{mod}_2} - 1). \quad (8)$$

138 In these α is a conventionally chosen reference radius, usually taken as the
 139 radius of the circumscribing sphere of the small body and x_{mod_y} stands for
 140 the value of x modulus y , i.e. the integer remainder of the division of x by y .
 141 The $C_{n,m}$ and $S_{n,m}$ in (17) are called spherical harmonic coefficients, defined
 142 as, $\forall 0 \leq m \leq n$:

$$C_{n,m} = \frac{(2-\delta_{m,0})}{M} \frac{(n-m)!}{(n+m)!} \int_V \left(\frac{r'}{\alpha}\right)^n P_{n,m}(\sin \delta') \cos(m\lambda') \rho(r', \delta', \lambda') dV$$

$$S_{n,m} = \frac{(2-\delta_{m,0})}{M} \frac{(n-m)!}{(n+m)!} \int_V \left(\frac{r'}{\alpha}\right)^n P_{n,m}(\sin \delta') \sin(m\lambda') \rho(r', \delta', \lambda') dV \quad (9)$$

143 Where $\delta_{0,m}$ is the Kronecker delta that gives 1 if $m = 0$, and 0 elsewhere,
 144 $P_n^m(x)$ is the associated Legendre function of degree n and order m .
 145 Moreover $r' \in (0; \infty)$, $\theta' \in [-\pi; \pi)$ and $\lambda' \in [0; 2\pi)$ are respectively the po-
 146 sition, latitude and longitude of the infinitesimal volume element dV in a
 147 cartesian frame of reference $O_{x,y,z}$, $\rho(r', \theta', \lambda')$ is the density of the infinitesi-
 148 mal element of volume and V is the volume of the body.

149 Note that, in order to obtain formula (16) the gravitational potential

$$U(r) = -\frac{\mathcal{G}}{r} \int_V \frac{\rho(r')}{\sqrt{1 - 2\frac{r'}{r} \cos \psi + \left(\frac{r'}{r}\right)^2}} dV \quad (10)$$

150 where $\cos \psi = \sin \delta \sin \delta' + \cos \delta \cos \delta' \cos(\lambda - \lambda')$, has been developed in
 151 terms of Legendre Polynomials using $(1 - 2WZ + Z^2)^{-1/2} = \sum_{n=0}^{\infty} Z^n P_n(W)$.

152 Thus it has been obtained that

$$U(r) = -\frac{\mathcal{G}}{r} \int_V \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos \psi) \rho(r') dV, \quad (11)$$

153 which converges only if the condition $\frac{r'}{r} < 1$ is satisfied, which implies that
 154 the model is valid only outside the reference sphere.

155 A full explanation of the spherical harmonic coefficients can be found in
 156 Hofmann-Wellenhof et al. (1967). However it is important to highlight that
 157 equations in (9) imply:

$$\begin{aligned} C_{0,0} &= 1 \\ C_{n,0} &= \frac{1}{M} \int_V \left(\frac{r'}{\alpha}\right)^n P_n(\sin \delta') \rho(r', \delta', \lambda') dV \quad \forall n > 0 \end{aligned} \quad (12)$$

$$S_{n,0} = 0 \quad \forall n \geq 0$$

158 Moreover, centering the origin of the system of reference at the center of
 159 mass it can be demonstrated that the term $C_{1,0} = 0$.

160 The coefficients $C_{2,0}$ and $C_{2,2}$ express the “ellipticity” and “oblateness” of
 161 the body.

162 **4. The relegation of the polar component of the angular momen-** 163 **tum N**

164 In the context of artificial satellite theory, in general, one needs to order
 165 the terms of the Hamiltonian H according to an asymptotic expansion in
 166 order to build a perturbation theory. The usual way to arrange the Hamil-
 167 tonian for the cases in which the angular velocity of the asteroid is higher
 168 than the mean motion of the spacecraft (which holds, for example, for fast

169 rotating bodies or for relatively high altitudes) is here followed (see Segerman
 170 and Coffey (2000)). It consists in placing the full unperturbed part at ze-
 171 roth order and distribute the perturbation at first and second orders. The
 172 dominant (unperturbed) part of the Hamilton function is set to be the sum
 173 of the two-body Hamiltonian H_K and the Coriolis term H_C . The perturbing
 174 potential takes into account an arbitrary number of spherical harmonic coef-
 175 ficients, distributed as first or second orders perturbations, depending on the
 176 harmonics of the specific asteroid studied, thus providing a dynamical model
 177 based on an arbitrarily accurate model of the inhomogeneous body.
 178 The flows associated to the two components of the unperturbed Hamiltonian
 179 are used to relegate the whole system first and then to put it into normal
 180 form by means of symplectic transformations.

181 The Hamiltonian in (4) is therefore rearranged as:

$$H = H_0 + \epsilon H_1 + \frac{\epsilon^2}{2} H_2 + O(\epsilon^3), \quad (13)$$

182 where ϵ is merely an ordering dimensionless parameter, which will be decided
 183 later on for the applications, and

$$\begin{aligned}
 H_0 &= H_K + H_C \\
 H_1 &= U^{(1)}(r, \theta, \nu, -, \Theta, N) \\
 H_2 &= U^{(2)}(r, \theta, \nu, -, \Theta, N)
 \end{aligned} \quad (14)$$

184 where:

$$\begin{aligned}
 H_K &:= \frac{1}{2} \left(R^2 + \frac{\Theta^2}{r^2} \right) - \frac{\mu}{r} \\
 H_C &= -\omega N,
 \end{aligned} \quad (15)$$

185 and, for $s = 1, 2$

$$\begin{aligned}
 U^{(s)}(r, \theta, \nu, -, \Theta, N) = & -\frac{s!}{\epsilon^s} \sum_{n=1}^{\infty} \sum_{m=0}^n \sum_{j=-n}^n \sum_{t=\max\{0, j+m\}}^{\min\{n+m, n+j\}} \text{ci}^{2n+m+j-2t} \text{si}^{2t-m-j} \cdot \\
 & \cdot \frac{1}{r^{n+1}} \left(\mathcal{A}_{n,m,j,t}^{(s)} \cos(m\nu - j\theta) + \mathcal{B}_{n,m,j,t}^{(s)} \sin(m\nu - j\theta) \right),
 \end{aligned} \tag{16}$$

186 with:

$$\mathcal{A}_{n,m,j,t}^{(s)} = \bar{\mathcal{G}}_{n,m,j,t} \left(C_{n,m}^{(s)} \cos\left(\frac{\pi}{2}(j+m)\right) - S_{n,m}^{(s)} \sin\left(\frac{\pi}{2}(j+m)\right) \right) \tag{17}$$

$$\mathcal{B}_{n,m,j,t}^{(s)} = \bar{\mathcal{G}}_{n,m,j,t} \left(C_{n,m}^{(s)} \sin\left(\frac{\pi}{2}(j+m)\right) + S_{n,m}^{(s)} \cos\left(\frac{\pi}{2}(j+m)\right) \right),$$

187 with:

$$\begin{aligned}
 C_{n,m}^{(s)} &= \begin{cases} C_{n,m} & \text{if the term containing } C_{n,m} \text{ is } \sim O(\epsilon^q) \\ 0 & \text{otherwise} \end{cases} \\
 S_{n,m}^{(s)} &= \begin{cases} S_{n,m} & \text{if the term containing } S_{n,m} \text{ is of } \sim O(\epsilon^q) \\ 0 & \text{otherwise} \end{cases}
 \end{aligned} \tag{18}$$

188 Again ci and si as in (6) and $\bar{\mathcal{G}}_{n,m,j,t}$ as in (8).

189

190 Now, considering the case $|H_K| < |H_C|$, two different Lie transforma-
 191 tions are performed: the relegation of the polar component of the angular
 192 momentum N first and the Delaunay normalisation.

Definition 1. A Lie transformation ϕ is a one-parameter family of mappings $\phi : (y, Y; \epsilon) \rightarrow (x, X)$, defined by the solution $x(y, Y; \epsilon)$ and $X(y, Y; \epsilon)$ of the Hamiltonian system

$$\begin{cases} \frac{dx}{d\epsilon} = \frac{\partial W}{\partial X} \\ \frac{dX}{d\epsilon} = -\frac{\partial W}{\partial x} \end{cases}$$

with initial conditions $x(y, Y; 0) = y$ and $X(y, Y; 0) = Y$, and where the function

$$W(x, X; \epsilon) = \sum_{s \geq 0} \frac{\epsilon^s}{s!} W_{s+1}(x, X)$$

193 is the generator of the transformation.

Due to the properties of the Hamiltonian systems, the Lie transformation ϕ is a completely canonical transformation that maps a Hamiltonian

$$H(x, X; \epsilon) = \sum_{s \geq 0} \frac{\epsilon^s}{s!} H_s(x, X)$$

onto an equivalent Hamiltonian K of the form

$$K(y, Y; \epsilon) = \sum_{s \geq 0} \frac{\epsilon^s}{s!} K_s(y, Y; 0).$$

194 found by solving a series of homological equations:

$$[H_0; W_s] + \tilde{H}_s = K_s \quad \forall s \geq 1 \quad (19)$$

195 where the symbol $[;]$ stands for the Poisson Brackets. In equation (19) the
 196 element $\tilde{H}_s$ collects the terms from the previous orders (see Deprit (1969)
 197 and Palacián (2002)). The relegation and the normalization algorithms (see
 198 Deprit et al. (2001) and Deprit (1982) respectively) are two different methods
 199 of solving such homological equations. In particular, the relegation maps the
 200 Hamiltonian (13) into an equivalent one of the form:

$$K = K_0 + \sum_{s \geq 1} \frac{\epsilon^s}{s!} K_s = \sum_{s \geq 0} \frac{\epsilon^s}{s!} \left(\sum_{j=0}^p K_{s,p} + R_s \right) \quad (20)$$

201 with $K_0 = H_0(y, Y)$ and the coefficients $K_{s,p} \in \ker(\mathcal{L}_{H_C})$, where $\mathcal{L}_{H_C}$ is the

202 Lie derivative with respect to the Coriolis term¹.
 203 In contrast with normalization, the term K_s may not belong to $\ker(\mathcal{L}_{H_C})$
 204 due to the presence of the residual R_s . In this resulting Hamiltonian the
 205 terms containing the variable ν will only appear in the remainder R_s . More-
 206 over, for every order s of the Hamiltonian, the algorithm iterated $p^{(s)}$ times
 207 (depending on the choice of the small parameter ϵ), progressively diminish-
 208 ing the importance of the remainder R_s , such that after $p^{(s)}$ times it results
 209 $R_s \sim O(\epsilon^3)$.
 210 As a result the truncated system

$$K = \sum_{s \geq 0} \frac{\epsilon^s}{s!} \sum_{j=0}^p K_{s,p}, \quad (21)$$

211 is obtained, which represents an approximation of the starting Hamiltonian
 212 independent from ν and admits H_C as an integral.
 213 In this section, in order to keep the generality of the analysis, the relegation
 214 is performed to the second order, arbitrary number of iterations $p^{(s)}$. In
 215 the applications section, once the parameter ϵ will be fixed, the number of
 216 iterations necessary to relegate the terms of the Hamiltonian containing ν to
 217 orders $\sim O(\epsilon^3)$ will therefore be estimated.

¹Let $\mathcal{L}_W$ be the Lie derivative induced by the function W , then $\mathcal{L}_W$ which maps any function $f(X, x)$ into its Poisson Bracket with W , namely $f(X, x) \mapsto [f; W]$.

It must be noted that $\mathcal{L}_{H_C} H_K = 0$ and that $\mathcal{L}_{H_C}$ is semi-simple over a Poisson algebra of functions P .

218 *4.1. Algorithm*

219 The general relegation algorithm is briefly described here before the ap-
 220 plication to the problem. For each homological equation ($\forall s \geq 1$):

$$[H_0; W_s] + \tilde{H}_s = K_s \quad (22)$$

221 considering that, as $\mathcal{L}_{H_C}$ is semi-simple, there $\exists K_{s,0}, W_{s,0} \in P$ s.t.

$$\begin{cases} \tilde{H}_s = K_{s,0} + [W_{s,0}; H_C] \\ K_{s,0} \in \text{Ker}(\mathcal{L}_{H_C}). \end{cases} \quad (23)$$

222

223

224 Therefore (22) becomes:

$$[H_0; W_s] + [W_{s,0}; H_C] = K_s - K_{s,0}. \quad (24)$$

225 Thus, setting $W_s = W_{s,0}^* + W_{s,0}$, (24) yields:

$$[H_0; W_s^*] + [H_0 - H_C; W_{s,0}] = K_s - K_{s,0}. \quad (25)$$

226 The algorithm continues re-invoking $p^{(s)}$ -times the semi-simplicity of $\mathcal{L}_{H_C}$,

227 and finding $\forall 1 \leq p \leq p^{(s)} \quad K_{s,p}, W_{s,p} \in P$ s.t.

$$\begin{cases} [H_0 - H_C; W_{s,p-1}] = K_{s,p} + [W_{s,p}; H_C] \\ K_{s,p} \in \text{Ker}(\mathcal{L}_{H_C}) \end{cases} \quad (26)$$

228

229 and setting $p^{(s)}$ -times $\forall 1 \leq p \leq p^{(s)} \quad W_{s,p-1} = W_{s,p}^* + W_{s,p}$.

230

231 Finally the algorithm ends at a certain iteration $p^{(s)}$ setting $W_{s,p^{(s)}}^* = 0$ and
 232 obtaining (25) to become:

$$K_s = \sum_{p=0}^{p^{(s)}} (K_{s,p}) + R_s \quad (27)$$

233 with $R_s := [H_0 - H_C; W_{s,p^{(s)}}]$.

234

235 Although the procedure is general, in view of the applications, only the
 236 first two homological equations will here be considered and explicitly solved.

237 4.2. Results

238 Following the procedure just described and Deprit (1969), for the first order
 239 $s = 1$ of the Hamiltonian (13), we have that:

$$\tilde{H}_{1,0} = H_1 \quad (28)$$

240 therefore, after the first iteration $p = 1$, it results:

$$\begin{aligned} K_{1,0} = -\frac{1}{\epsilon} \sum_{n=1}^{\infty} \sum_{j=-n}^n \sum_{t=\max\{0,j\}}^{\min\{n,n+j\}} \text{ci}^{2n+j-2t} \text{si}^{2t-j} \frac{1}{r^{n+1}} \left(\mathcal{A}_{n,0,j,t}^{(1)} \cos(-j\theta) \right. \\ \left. + \mathcal{B}_{n,0,j,t}^{(1)} \sin(-j\theta) \right). \end{aligned} \quad (29)$$

241 Moreover

$$\begin{aligned} W_{1,0} &= -\frac{1}{\omega} \int (H_1 - K_{1,0}) d\nu \\ &= - \left(\frac{1}{\epsilon} \sum_{n=1}^{\infty} \sum_{m=1}^n \sum_{j=-n}^n \sum_{t=\max\{0,j+m\}}^{\min\{n+m,n+j\}} \text{ci}^{2n+m+j-2t} \text{si}^{2t-m-j} \left(-\frac{1}{m\omega} \right) \frac{1}{r^{n+1}} \right. \\ &\quad \cdot \left(\mathcal{A}_{n,m,j,t}^{(1)} \sin(m\nu - j\theta) + \mathcal{B}_{n,m,j,t}^{(1)} (-\cos(m\nu - j\theta)) \right) \Big), \end{aligned} \quad (30)$$

242 and

$$\begin{aligned}
[H_K, W_{1,0}] &= R \frac{\partial W_{1,0}}{\partial r} + \frac{\Theta}{r^2} \frac{\partial W_{1,0}}{\partial \theta} - \left(\frac{\Theta^2}{r^3} - \frac{MG}{r^2} \right) \frac{\partial W_{1,0}}{\partial R} \\
&= -\frac{1}{\epsilon} \sum_{n=1}^{\infty} \sum_{m=1}^n \sum_{j=-n}^n \sum_{t=\max\{0, j+m\}}^{\min\{n+m, n+j\}} \text{ci}^{2n+m+j-2t} \text{si}^{2t-m-j} \left(-\frac{1}{m\omega} \right) \left(-\frac{R}{r} \right) \cdot \\
&\quad \cdot (-(n+1)) \frac{1}{r^{n+1}} \left(\mathcal{A}_{n,m,j,t}^{(1)} \sin(m\nu - j\theta) + \mathcal{B}_{n,m,j,t}^{(1)} (-\cos(m\nu - j\theta)) \right) \\
&- \frac{1}{\epsilon} \sum_{n=1}^{\infty} \sum_{m=1}^n \sum_{j=-n}^n \sum_{t=\max\{0, j+m\}}^{\min\{n+m, n+j\}} \text{ci}^{2n+m+j-2t} \text{si}^{2t-m-j} \left(-\frac{1}{m\omega} \right) \left(\frac{j\Theta}{r^2} \right) \frac{1}{r^{n+1}} \cdot \\
&\quad \cdot \left(\mathcal{A}_{n,m,j,t}^{(1)} \cos(m\nu - j\theta) + \mathcal{B}_{n,m,j,t}^{(1)} \sin(m\nu - j\theta) \right), \tag{31}
\end{aligned}$$

243 Then the algorithm is iterated $\forall 1 < p \leq p^{(s)}$, where at each iteration it
244 results:

$$K_{1,p} = 0 \tag{32}$$

245 Calling $p_{O_{max}} = 2\lfloor \frac{p-1}{2} \rfloor + 1$, $p_{E_{max}} = 2\lfloor \frac{p}{2} \rfloor$, and:

$$\begin{aligned}
\mathcal{S}(\hat{k}, k^*) &= \sum_{k=\hat{k}}^{k^*} a_k \\
\mathcal{S}_E(\hat{k}, k^*) &= \sum_{\substack{k=\hat{k}, \\ k \text{ even}}}^{k^*} a_k, \quad \mathcal{S}'_E(\hat{k}, k^*) = \sum_{\substack{k=\hat{k}, \\ k \text{ even}}}^{k^*} a'_k, \quad \mathcal{S}''_E(\hat{k}, k^*) = \sum_{\substack{k=\hat{k}, \\ k \text{ even}}}^{k^*} a''_k \\
\mathcal{S}_O(\hat{k}, k^*) &= \sum_{\substack{k=\hat{k}, \\ k \text{ odd}}}^{k^*} a_k, \quad \mathcal{S}'_O(\hat{k}, k^*) = \sum_{\substack{k=\hat{k}, \\ k \text{ odd}}}^{k^*} a'_k, \quad \mathcal{S}''_O(\hat{k}, k^*) = \sum_{\substack{k=\hat{k}, \\ k \text{ odd}}}^{k^*} a''_k
\end{aligned} \tag{33}$$

246 Also, calling:

$$\mathcal{D} := (-1)^{p-\mathcal{S}(1,p)} \binom{p-\mathcal{S}(2,p)}{p-\mathcal{S}(1,p)} \frac{(n+p-\mathcal{S}(1,p))!}{(n+a_1)!} \tag{34}$$

247 and $\forall k \text{ odd}$

$$\begin{aligned}
\mathcal{O}_k := & \left(\binom{a_{p_{Omax}}}{a'_{p_{Omax}}} \dots \binom{a_5}{a'_5} \binom{a_3}{a'_3} \right) \left(\binom{a_{p_{Omax}} - a'_{p_{Omax}}}{a''_{p_{Omax}}} \dots \binom{a_5 - a'_5}{a''_5} \binom{a_3 - a'_3}{a''_3} \right) \\
& \left(\frac{(a_1 + n + p + 2\mathcal{S}_E(2, k-1) - \mathcal{S}(k, p) + a_k - a''_k + \mathcal{S}'_O(3, k-2) - \mathcal{S}''_O(3, k-2))!}{(a_1 + n + p + 2\mathcal{S}_E(2, k-1) - \mathcal{S}(k, p) + a'_k + \mathcal{S}'_O(3, k-2) - \mathcal{S}''_O(3, k-2))!} \right) \\
& \left(\frac{(a_1 + n + p + \mathcal{S}_E(2, k-1) - \mathcal{S}(k, p) + a_k + \mathcal{S}_O(3, k-2) - \mathcal{S}''_O(3, k-2))!}{(a_1 + n + p + \mathcal{S}_E(2, k-1) - \mathcal{S}(k, p) + a_k - a''_k + \mathcal{S}_O(3, k-2) - \mathcal{S}''_O(3, k-2))!} \right)
\end{aligned} \tag{35}$$

248 while $\forall k$ even

$$\begin{aligned}
\mathcal{E}_2 &:= \left(\frac{(p - \mathcal{S}(1, p))!}{(p - \mathcal{S}(1, p) - a_2)!} \right) \\
\mathcal{E}_k &:= \left(\frac{(p - a_1 - 2\mathcal{S}_E(2, k-2) - \mathcal{S}(k, p) - \mathcal{S}'_O(3, k-1) + a_k + 1)!}{(p - a_1 - 2\mathcal{S}_E(2, k-2) - \mathcal{S}(k, p) - \mathcal{S}'_O(3, k-1) - 1)!} \right) \quad \forall k \geq 4, \text{ } k \text{ even}
\end{aligned} \tag{36}$$

and

$$[H_K, W_{1,p}] = -\frac{1}{\epsilon} \sum_{n=1}^{\infty} \sum_{m=1}^n \sum_{j=-n}^n \sum_{t=\max\{0, j+m\}}^{\min\{n+m, n+j\}} \text{Ci}^{2n+m+j-2t} \text{Si}^{2t-m-j}$$

$$\sum_{a_{p+1}=0}^1 \left(\sum_{a_p=1-\delta_{a_{p+1},0}}^{\max\{p+1-(p-1),0\}} \dots \sum_{a_3=1-\delta_{a_4,0}}^{\max\{p+1-\mathcal{S}(4,p+1)-2,0\}} \sum_{a_2=1-\delta_{a_3,0}}^{\max\{p+1-\mathcal{S}(3,p+1)-1,0\}} \right)$$

$$\left(\sum_{a_1=0}^{\max\{p+1-\mathcal{S}(2,p+1),0\}} \mathcal{D}^* \left(\sum_{a'_{p_{Omax}+1}=0}^{a_{p_{Omax}+1}} \dots \sum_{a'_5}^{a_5} \sum_{a'_3}^{a_3} \left(-\frac{1}{m\omega} \right)^{p+1} \right. \right.$$

$$\left(\sum_{a''_{p_{Omax}+1}=0}^{a_{p_{Omax}+1}-a'_{p_{Omax}+1}} \dots \sum_{a''_5=0}^{a_5-a'_5} \sum_{a''_3=0}^{a_3-a'_3} \right)$$

$$\left(\mathcal{O}_{p_{Omax}+1}^* \cdot \dots \cdot \mathcal{O}_5^* \cdot \mathcal{O}_3^*\right) \left(\mathcal{E}_{p_{Emax}+1}^* \cdot \dots \cdot \mathcal{E}_4^* \mathcal{E}_2^*\right)$$

$$\left(\frac{1}{r}\right)^{3(\mathcal{S}_O(3,p_{Omax}+1)-\mathcal{S}'_O(3,p_{Omax}+1))}$$

$$\left(-\frac{1}{r}\right)^{p+1-a_1-\mathcal{S}_E(2,p_{Emax}+1)-\mathcal{S}'_O(3,p_{Omax}+1)}$$

$$R^{p+1-a_1-2\mathcal{S}_E(2,p_{Emax}+1)-\mathcal{S}'_O(3,p_{Omax}+1)}$$

$$\left(\frac{j\Theta}{r^2}\right)^{a_1+\mathcal{S}'_O(3,p_{Omax}+1)}\left(\frac{-\Theta^2+r\mu}{r^3}\right)^{\mathcal{S}_E(2,p_{Emax}+1)-\mathcal{S}_O(3,p_{Omax}+1)+\mathcal{S}'_O(3,p_{Omax}+1)}$$

$$\Theta^{2(\mathcal{S}_O(3,p_{Omax}+1)-\mathcal{S}'_O(3,p_{Omax}+1)-\mathcal{S}''_O(3,p_{Omax}+1))}(-r\mu)^{\mathcal{S}''_O(3,p_{Omax}+1)}\frac{1}{r^{n+1}}$$

$$\begin{aligned}
& \left(\mathcal{A}_{n,m,j,t}^{(1)} \left(\cos(m\nu - j\theta) \cos\left(\frac{\pi}{2}(-(p+1) + a_1 + \mathcal{S}'_O(3, p_{O_{max}}))\right) \right. \right. \\
& \quad \left. \left. - \sin(m\nu - j\theta) \sin\left(\frac{\pi}{2}(-(p+1) + a_1 + \mathcal{S}'_O(3, p_{O_{max}}))\right) \right) \right. \\
& \quad \left. + \mathcal{B}_{n,m,j,t}^{(1)} \left(\sin(m\nu - j\theta) \cos\left(\frac{\pi}{2}(-(p+1) + a_1 + \mathcal{S}'_O(3, p_{O_{max}}))\right) \right. \right. \\
& \quad \left. \left. + \cos(m\nu - j\theta) \sin\left(\frac{\pi}{2}(-(p+1) + a_1 + \mathcal{S}'_O(3, p_{O_{max}}))\right) \right) \right) \right) \right) \quad (38)
\end{aligned}$$

where $\mathcal{D}^*$, $\mathcal{O}_t^*$ and $\mathcal{E}_t^*$ are like the one in (34), (35) and (36) respectively with $p+1$ instead of p .

As

$$[H_K; \cdot] = \left[\frac{1}{2} \left(R^2 + \frac{\Theta^2}{r^2} \right) - \frac{\mu}{r}; \cdot \right] = R \frac{\partial}{\partial r} + \frac{\Theta}{r^2} \frac{\partial}{\partial \theta} - \left(\frac{\Theta^2}{r^3} - \frac{\mu}{r^2} \right) \frac{\partial}{\partial R} \quad (39)$$

at each step p the term $[H_K, W_{1,p}]$ is the sum functions that have the same order of the preceding $[H_K, W_{1,p-1}]$ but multiplied by $\frac{R}{\omega r}$, $\frac{\Theta}{\omega r^2}$ or $\frac{\Theta^2 + r}{\omega R r^3}$. Following Segerman and Coffey (2000), as $R \sim \frac{\Theta}{r}$ and as, at order zero, for the two-body problem, $\Theta \sim r^2 \dot{\theta}$, for a satellite period greater than the rotational period of the asteroid (i.e. $\dot{\theta} < \omega$), and therefore these coefficients are less than unity over an orbit $\sim \frac{\dot{\theta}}{\omega} < 1$. Therefore, at each step of the relegation, the transformation process reduces the magnitude of the terms of the perturbing potential which contain the angle ν . Thus, after fixing the parameter ϵ , the number of iteration $p^{(1)}$ is fixed such that $[H_K, W_{1,p^{(1)}}] \sim O(\epsilon^3)$.

The relegation of the first order is ended setting:

$$\begin{aligned}
W_1 &:= \sum_{p=0}^{p^{(1)}} W_{1,p} \\
R_1 &:= [H_K, W_{1,p^{(1)}}] \\
K_1 &:= \sum_{p=0}^{p^{(1)}} K_{1,p} + R_1 = K_{1,0} + R_1
\end{aligned} \quad (40)$$

To pass to the second order $s = 2$, the evaluation of $\tilde{H}_{2,0} = H_2 + 2[H_1, W_1] + [[H_0, W_1], W_1]$ is first required, from which the expression for

265 $K_{2,0}$ is derived (see the Electronic Supplementary Material). In analogy with
 266 the first order, it results $K_{2,p} = 0 \quad \forall p \geq 1$.
 267 The relegation of the second order is ended setting:

$$\begin{aligned} W_2 &:= \sum_{p=0}^{p^{(2)}} W_{2,p} \\ R_2 &:= [H_K, W_{2,p^{(2)}}] \\ K_2 &:= \sum_{p=0}^{p^{(2)}} K_{2,p} + R_2 = K_{2,0} + R_2 \end{aligned} \tag{41}$$

268 where $p^{(2)}$ is chosen such that $[H_K, W_{2,p^{(2)}}] \sim O(\epsilon^3)$ which is $p^{(2)} = \lfloor \frac{p^{(1)}+1}{2} \rfloor$.

269

270 The resulting Hamiltonian $K = K_0 + \epsilon K_1 + \frac{\epsilon^2}{2} K_2$ is completely equivalent
 271 to the one in (13). However, as the terms R_s , $s = 1, 2$ are of order $\sim \epsilon^3$,
 272 a truncated system is considered in which such terms have been neglected.
 273 Setting:

$$\begin{aligned} \tilde{K}_0 &:= K_0 \\ \tilde{K}_1 &:= \sum_{p=0}^{p^{(1)}} K_{1,p} = K_{1,0} \\ \tilde{K}_2 &:= \sum_{p=0}^{p^{(2)}} K_{2,p} = K_{2,0} \end{aligned} \tag{42}$$

274 the truncated system is described by the Hamiltonian:

$$\tilde{K} = \tilde{K}_0 + \epsilon \tilde{K}_1 + \frac{\epsilon^2}{2} \tilde{K}_2 \tag{43}$$

275 where, to simplify notation, the $\sim$ will be ignored. This Hamiltonian is equiv-
 276 alent to the one in the main problem of the artificial satellite, in which the
 277 argument of node ν is cyclic, which implies that the coriolis term $-\omega N$ is
 278 constant and can therefore be dropped from the Hamiltonian. A closed form
 279 Delaunay normalization can now be performed, for a further reduction of the
 280 degrees of freedom, thus yielding an integrable Hamiltonian.

281

282 It must be noted that, in complete analogy with the procedure adopted so
 283 far, the explicit formulation for every higher order $s \geq 2$ could be obtained.

284 5. Delaunay Normalization

285 In order to perform the Delaunay normalisation the Hamiltonian is trans-
 286 formed from the relegated Whittaker variables to the Delaunay coordinates.
 287 The Delaunay coordinates are symplectic action-angle variables
 288 (L, G, H, ℓ, g, h) , where the angles ℓ , g and h are conjugate to the actions
 289 L , G and H respectively. Among the angle variables ℓ is the *mean anomaly*
 290 measured from the pericenter, g is the argument of the pericenter while h is
 291 the argument of the node. For the actions instead L is related to the major
 292 semi-axis, a , by $L = \sqrt{\mu a}$, G is the *total angular momentum* of the spacecraft
 293 with respect to the Asteroid (in the inertial frame), related to the eccentricity
 294 and the variable L by $e = \sqrt{1 - \frac{G^2}{L^2}}$, and H is the z -component of the total
 295 angular momentum, i.e. $H = G \cos I$.

296 The relation between the *True anomaly* and the *Eccentric anomaly* u is de-
 297 fined as $\tan(\frac{f}{2}) = \sqrt{\frac{1+e}{1-e}} \tan(\frac{u}{2})$, which, in particular, implies $r = a(1 -$
 298 $e \cos u) = a \frac{1-e^2}{1+e \cos f}$.

299 Moreover, by Section 3, we know that $N = G \cos I \Rightarrow H = G \cos I$ and
 300 $R = \frac{\mu e \sin f}{G}$.

301

302 The relegated Hamiltonian (43) in the Delaunay coordinates takes the
 303 form:

$$J = J_0 + \epsilon J_1 + \frac{\epsilon^2}{2} J_2 \quad (44)$$

304 with:

$$\begin{aligned}
J_0 &= -\frac{(\mathcal{G}M)^2}{2L^2} \\
J_1 &= -\frac{1}{\epsilon} \sum_{n=1}^{\infty} \sum_{j=-n}^n \sum_{t=\max\{0,j\}}^{\min\{n,n+j\}} \text{ci}^{2n+j-2t} \text{si}^{2t-j} \left(\frac{(1 + e \cos f)}{(a(1 - e^2))} \right)^{n+1} \\
&\quad \left(\mathcal{A}_{n,0,j,t}^{(1)} \cos(-j(f+g)) + \mathcal{B}_{n,0,j,t}^{(1)} \sin(-j(f+g)) \right).
\end{aligned} \tag{45}$$

305 with

$$\begin{aligned}
\text{ci} &= \sqrt{\frac{1+\frac{H}{G}}{2}} \\
\text{si} &= \sqrt{\frac{1-\frac{H}{G}}{2}}
\end{aligned} \tag{46}$$

306 For brevity of exposition the expression for J_2 will not be explicit written in
307 this paper.

308

309 5.1. The Normalization algorithm

310 The closed form normalization algorithm (Deprit (1982)) is here adopted,
311 which, instead of using the expansions of r and f in powers of the eccentric-
312 ity, changes the independent variable from time to the true anomaly f .

313

314 **Definition 2.**

315 A formal series $K'(y, Y, \epsilon) = \sum_{s=0}^{\infty} \frac{\epsilon^s}{s!} K'_s(y, Y)$ is said to be in Delaunay normal
316 form if the Lie derivative $L_{K'_0} K'$ is zero, that is $[K'_s, K'_0] = 0 \quad \forall s \geq 0$.

317

318

In our case, as $K'_0 = J_0 = -\frac{(\mathcal{G}M)^2}{2L^2}$, the Lie derivative

$$L_{K'_0}(\cdot) = \frac{(\mathcal{G}M)^2}{L^3} \frac{\partial(\cdot)}{\partial \ell}$$

therefore the new Hamiltonian (44) will be in normal form if and only if

$$\frac{\partial K'_1}{\partial \ell} = 0 \quad \text{and} \quad \frac{\partial K'_2}{\partial \ell} = 0$$

319

320 Note that, as for the relegation for the angle ν , the normalization degenerates

321 into an average over the mean anomaly ℓ . Moreover it will be used that:

$$\frac{df}{d\ell} = \frac{a^2 \sqrt{1-e^2}}{r^2}. \quad (47)$$

322

323

324 5.2. Results

325 The explicit formula for the normalized J_1 is:

$$\begin{aligned} K'_1 = & -\frac{1}{\epsilon} \sum_{n=1}^{\infty} \sum_{t=\max\{0,j\}}^{\min\{n,n+j\}} \text{ci}^{2n+j-2t} \text{si}^{2t-j} \\ & \frac{\sqrt{1-e^2}}{a^{n+1}(1-e^2)^n} \left(\sum_{k=0}^{n-1} \binom{n-1}{k} e^k \mathcal{A}_{n,0,j,t}^{(1)} \frac{(k-1)!!}{k!!} (k+1)_{\text{mod}_2} + \right. \\ & + 2(n+1)_{\text{mod}_2} \sum_{j=1}^n \sum_{k=0}^{n-1} \sum_{q=0}^{\lfloor \frac{j}{2} \rfloor} \sum_{v=0}^q \binom{n-1}{k} \binom{j}{2q} \binom{q}{v} (-1)^{q+v} e^k \mathcal{A}_{n,0,j,t}^{(1)} \cdot \\ & \cdot \cos(gj) \frac{((j-2q+k+2v)-1)!!}{(j-2q+k+2v)!!} ((j-2q+k+2v)+1)_{\text{mod}_2} \\ & - 2(n)_{\text{mod}_2} \sum_{j=1}^n \sum_{k=0}^{n-1} \sum_{q=0}^{\lfloor \frac{j}{2} \rfloor} \sum_{v=0}^q \binom{n-1}{k} \binom{j}{2q} \binom{q}{v} (-1)^{q+v} e^k \mathcal{B}_{n,0,j,t}^{(1)} \cdot \\ & \cdot \sin(gj) \frac{((j-2q+k+2v)-1)!!}{(j-2q+k+2v)!!} ((j-2q+k+2v)+1)_{\text{mod}_2} \Big) \end{aligned} \quad (48)$$

326 obtained using that, $\forall 1 \leq j \leq n$:

$$\begin{aligned} \text{if } n \text{ even } \mathcal{A}_{n,0,j}^{(1)} &= \mathcal{A}_{n,0,-j}^{(1)} \\ \text{if } n \text{ odd } \mathcal{A}_{n,0,j}^{(1)} &= -\mathcal{A}_{n,0,-j}^{(1)} \end{aligned} \quad (49)$$

327 and

$$\begin{aligned} \text{if } n \text{ even } \mathcal{B}_{n,0,j}^{(1)} &= \mathcal{B}_{n,0,-j}^{(1)} \\ \text{if } n \text{ odd } \mathcal{B}_{n,0,j}^{(1)} &= -\mathcal{B}_{n,0,-j}^{(1)} \end{aligned} \quad (50)$$

328 The first order generating function is obtained by:

$$W'_1 = \int \frac{L^3}{(\mu)^2} \left(J_1 - \frac{1}{2\pi} \int_0^{2\pi} J_1 d\ell \right) d\ell \quad (51)$$

329 Finally the normalised J_2 , namely

$$K'_2 = \frac{1}{2\pi} \int_0^{2\pi} (J_2 + 2[J_1, W'_1] + [[J_0, W'_1], W'_1]) d\ell \quad (52)$$

330 and its corresponding generating function

$$W'_2 = \int \frac{L^3}{(\mu)^2} (J_2 - K'_2) d\ell \quad (53)$$

331 have been evaluated, using integration by parts, with the aid the software
332 Mathematica.

333 As a result $K' = K'_0 + \epsilon K'_1 + \frac{\epsilon^2}{2} K'_2$ is obtained which is the analytical for-
334 mulation for the closed-form averaged (with respect to both the argument
335 of node and the mean anomaly), second order, arbitrary degree Hamilto-
336 nian of any inhomogeneous gravitational field of a body uniformly rotating
337 around its main axes of inertia for the case $|H_K| < |H_C|$. This two degree
338 of freedom, integrable Hamiltonian approximates the initial system, and can
339 now be applied to every inhomogeneous body in order to determine possible
340 orbits useful for scientific observation missions such as frozen orbits.

341

342 6. Applications

343 The Hamiltonian obtained is of the form: $K'(L, G, H, -, g, -)$ thus the
 344 equations of motion are:

$$\begin{aligned}
 \ell'(t) &= \frac{\partial K'}{\partial L} \\
 g'(t) &= \frac{\partial K'}{\partial G} \\
 h'(t) &= \frac{\partial K'}{\partial H} \\
 L'(t) &= 0 \\
 G'(t) &= -\frac{\partial K'}{\partial g} \\
 H'(t) &= 0,
 \end{aligned} \tag{54}$$

345 which can be derived by (48) and (52) where L and H are constants and all
 346 the other motions will only depend on $G(t)$ and $g(t)$.

347

348 **Definition 3.** (*Frozen orbit*)

349 *A frozen orbit is an orbit in which the Inclination, the Eccentricity and the*
 350 *Argument of pericenter remains constant during the motion.*

351 This in particular implies that such an orbit is then perfectly periodic except
 352 for the orbital plane precession.

353 A frozen orbit it thus described by the system:

$$\begin{aligned}
 \dot{e} &= \frac{d}{dt} \frac{\sqrt{L^2 - G^2}}{L} = 0 \\
 \dot{I} &= \frac{d}{dt} \arccos \frac{H}{G} = 0 \\
 \dot{g} &= 0.
 \end{aligned} \tag{55}$$

354 For the properties of the Lie transformations, the “normalized” eccentricity,
 355 inclination and argument of pericenter are related to their relative “real”
 356 equivalents by the generator of the transformation (see Deprit (1969)), and

357 can thus be interpreted as a perturbed version of their real correspondents.

358 In the normalized variables (54), the system (55) is equivalent to:

$$\begin{aligned}\dot{G} &= 0 \\ \dot{g} &= 0.\end{aligned}\tag{56}$$

359 Thus fixing normalized eccentricity e and inclination I for the desired nor-
360 malized frozen orbit, and solving the system gives:

$$\begin{aligned}\dot{G} &= 0 \\ \dot{g} &= 0 \\ e &= \frac{\sqrt{L^2 - G^2}}{L} \\ I &= \arccos \frac{H}{G},\end{aligned}\tag{57}$$

361 and the initial conditions (L_0, G_0, H_0, g_0) for normalized frozen orbits can
362 be found.

363 Moreover, as this all procedure is valid for the case $|H_K| < |H_C|$ such initial
364 conditions must satisfy:

$$\omega H_0 > \frac{\mu^2}{2L_0^2}\tag{58}$$

365 and also

$$0 < |H_0| < G_0 < L_0\tag{59}$$

366 These resulting initial conditions can transformed back to the initial system
367 describing the full dynamics (see (13)) by the inverse of the generating func-
368 tions (Deprit (1969)), to generate an initial guess for frozen orbits around
369 any inhomogeneous body.

370 7. Conclusions

371 Setting the desired eccentricity and inclination it is thus possible to deter-
372 mine initial conditions which lead to frozen orbits in the truncated system.

Such initial conditions are used to approximate the solutions for the secular motion of the satellite in the real system thus showing a good agreement between the approximated and the real dynamics.

An example of the application of the method is shown for the asteroid 433-Eros, a highly irregular, elongated, Near Earth Asteroid, which is the main example used in the literature, for which the spherical harmonic coefficients up to the 15th order and degree (i.e. 272 coefficients) are listed in the Appendix A.

The physical properties of this asteroid are summarized in the table (1).

	Mass Kg	Rotational velocity <i>rad/s</i>	Reference Radius <i>Km</i>
433-Eros	6.6904×10^{15}	3.31182×10^{-4}	16

Table 1: Physical properties of 433-Eros

In inverse analogy with Palacián (2002) we would like to take $\epsilon \sim \frac{\mu^2}{\omega L_0^3}$. Considering the resulting frozen orbits to be at an altitude high enough to satisfy the condition $|H_K| > |H_C|$, and trying to include an high number of spherical harmonic coefficients in the model, in the example shown the ordering parameter ϵ is set to be $\epsilon = 10^{-2}$ (i.e. semimajor axes $\sim 300km$, $p^{(1)} = 2$, $p^{(2)} = 2$ s.t. $R_1 \sim \left(\frac{\dot{\theta}}{\omega}\right)^{p^{(1)}+1} \sim 10^{-6}$). For this example the numerical estimation of the terms containing 433-Eros' spherical Harmonics up to order and degree 15, leads to the distribution of the $C_{n,m}$, $S_{n,m}$ between the $C_{n,m}^{(1)}$, $C_{n,m}^{(2)}$ and the $S_{n,m}^{(1)}$, $S_{n,m}^{(2)}$ respectively.

392 For the result shown below it will thus be fixed that:

$$\begin{aligned}
C_{n,m}^{(1)} &= \begin{cases} C_{n,m} \text{ if } (n, m) \in \{(0, 0), (2, 0), (2, 2)\} \\ 0 \text{ otherwise} \end{cases} \\
C_{n,m}^{(2)} &= \begin{cases} C_{n,m} \text{ if } (n, m) \in \{(3, 1), (3, 3), (4, 0), (4, 2), (4, 4), (5, 1), \\ (5, 3), (5, 5), (6, 2), (6, 6)\} \\ 0 \text{ otherwise} \end{cases} \\
S_{n,m}^{(1)} &= \begin{cases} S_{n,m} \text{ if } (n, m) \in \{(2, 2)\} \\ 0 \text{ otherwise} \end{cases} \\
S_{n,m}^{(2)} &= \begin{cases} S_{n,m} \text{ if } (n, m) \in \{(3, 1), (3, 3), (4, 2), (4, 4), (5, 3), (5, 5)\} \\ 0 \text{ otherwise} \end{cases}
\end{aligned} \tag{60}$$

393 For illustration purposes the initial eccentricity has been set to $E_0 =$
394 0.5 , the inclination to $I_0 = 1.1$ and argument of pericenter to $g_0 = -\frac{\pi}{2}$,
395 yielding to the initial conditions f_0 , h_0 , L_0 , G_0 , and H_0 for the (relegated
396 and normalized) frozen orbit collected in Table (2) for 433-Eros. In the last
397 row of the table, the initial semimajor axes a_0 of the resulting orbits has also
398 been recorded.

$I_0(rad)$	1.1
E_0	0.5
g_0	$\frac{\pi}{2}$
h_0	π
f_0	π
G_0	315633
L_0	364462
H_0	143170
$a_0(km)$	297.493

Table 2: 433-Eros: initial conditions for frozen orbits

399 The initial conditions found with this method are transformed back by
 400 canonic transformations inverse to the relegating and normalizing transfor-
 401 mations of coordinates found in the paper, leading to approximated initial
 402 conditions for frozen orbits in the full model. The integration of such sys-
 403 tem shows a good agreement of the dynamics between the approximated
 404 and the full system, namely the resulting orbits for the full system result
 405 to be good approximations of frozen orbits. The resulting orbit for 433-
 406 Eros, for the example in Table (2), is shown below, in the cartesian inertial
 407 frame of reference centered in the center of mass of the inhomogeneous body
 408 (unit of measure km). The order of magnitude of the oscillation of inclina-
 409 tion and eccentricity around their initial value is $\Delta eccentricity \sim O(10^{-6})$,
 410 $\Delta eccentricity \sim O(10^{-2})deg$ for at least 20 years.

411

Figure 1: The resulting frozen orbit for $E_0 = 0.5$, $I_0 = 1.1$ and $g_0 = -\frac{\pi}{2}$ for 5 years

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492 **Appendix A: 433-Eros spherical harmonics**

493 The un-normalized spherical harmonic coefficients of 433-Eros are here listed.
494 This coefficients are the harmonic coefficients gravity solution NEAR15A, a
495 15th degree and order model obtained from radiometric tracking (Doppler
496 and range data) and landmark tracking of the NEAR spacecraft in orbit
497 about Eros. The gravity model includes data from the entire mission begin-
498 ning with orbit insertion on Feb. 14, 2000 and ending with the first descent
499 maneuver for landing on Feb. 12, 2001

$C_{0,0}$	1
$C_{1,0}$	0
$C_{1,1}$	0
$C_{2,0}$	-1.65899×10^{-1}
$C_{2,1}$	-2.11454×10^{-6}
$C_{2,2}$	5.31886×10^{-2}
$C_{3,0}$	-5.29244×10^{-3}
$C_{3,1}$	4.38548×10^{-3}
$C_{3,2}$	6.0659×10^{-4}
$C_{3,3}$	-1.4525×10^{-3}
$C_{4,0}$	5.48636×10^{-2}
$C_{4,1}$	-9.52013×10^{-5}
$C_{4,2}$	-3.90614×10^{-3}
$C_{4,3}$	-1.79405×10^{-5}
$C_{4,4}$	3.68808×10^{-4}
$C_{5,0}$	3.09067×10^{-3}
$C_{5,1}$	-2.36787×10^{-3}
$C_{5,2}$	-1.26781×10^{-4}
$C_{5,3}$	1.51169×10^{-4}
$C_{5,4}$	3.86908×10^{-6}
$C_{5,5}$	-2.51307×10^{-5}
$C_{6,0}$	-2.53848×10^{-2}
$C_{6,1}$	-1.91651×10^{-5}
$C_{6,2}$	8.13891×10^{-4}
$C_{6,3}$	5.9664×10^{-6}
$C_{6,4}$	-2.13764×10^{-5}
$C_{6,5}$	-3.93777×10^{-7}
$C_{6,6}$	1.18484×10^{-6}

$C_{7,0}$	-2.50016×10^{-3}
$C_{7,1}$	1.26047×10^{-3}
$C_{7,2}$	3.82038×10^{-5}
$C_{7,3}$	-3.48143×10^{-5}
$C_{7,4}$	-5.15671×10^{-7}
$C_{7,5}$	1.33563×10^{-6}
$C_{7,6}$	2.25518×10^{-9}
$C_{7,7}$	-1.25528×10^{-7}
$C_{8,0}$	1.53478×10^{-2}
$C_{8,1}$	-3.43765×10^{-5}
$C_{8,2}$	-2.57667×10^{-4}
$C_{8,3}$	-3.12096×10^{-6}
$C_{8,4}$	3.61153×10^{-6}
$C_{8,5}$	8.73471×10^{-8}
$C_{8,6}$	-7.09764×10^{-8}
$C_{8,7}$	-9.71194×10^{-10}
$C_{8,8}$	2.89016×10^{-9}
$C_{9,0}$	1.12427×10^{-3}
$C_{9,1}$	-4.97634×10^{-4}
$C_{9,2}$	-2.57824×10^{-5}
$C_{9,3}$	1.07011×10^{-5}
$C_{9,4}$	-4.14388×10^{-7}
$C_{9,5}$	-1.7556×10^{-7}
$C_{9,6}$	-3.11553×10^{-9}
$C_{9,7}$	5.83725×10^{-9}
$C_{9,8}$	1.43792×10^{-10}
$C_{9,9}$	-2.52185×10^{-10}
$C_{10,0}$	-2.23924×10^{-3}

$C_{10,1}$	-3.65977×10^{-4}
$C_{10,2}$	8.59725×10^{-5}
$C_{10,3}$	2.44668×10^{-6}
$C_{10,4}$	-2.12904×10^{-8}
$C_{10,5}$	-3.91544×10^{-8}
$C_{10,6}$	1.06018×10^{-8}
$C_{10,7}$	6.6781×10^{-10}
$C_{10,8}$	-1.03388×10^{-10}
$C_{10,9}$	-2.93031×10^{-11}
$C_{10,10}$	4.93363×10^{-12}
$C_{11,0}$	1.04666×10^{-2}
$C_{11,1}$	3.72982×10^{-4}
$C_{11,2}$	3.37686×10^{-6}
$C_{11,3}$	1.80367×10^{-6}
$C_{11,4}$	-5.5386×10^{-7}
$C_{11,5}$	7.57115×10^{-8}
$C_{11,6}$	2.19576×10^{-9}
$C_{11,7}$	5.19815×10^{-11}
$C_{11,8}$	3.75133×10^{-11}
$C_{11,9}$	1.74028×10^{-11}
$C_{11,10}$	-2.76742×10^{-13}
$C_{11,11}$	-2.57971×10^{-13}
$C_{12,0}$	1.71922×10^{-3}
$C_{12,1}$	3.7954×10^{-4}
$C_{12,2}$	1.55553×10^{-4}
$C_{12,3}$	6.86842×10^{-6}
$C_{12,4}$	2.99064×10^{-7}
$C_{12,5}$	-8.38626×10^{-8}

$C_{12,6}$	4.07023×10^{-9}
$C_{12,7}$	-1.60746×10^{-10}
$C_{12,8}$	1.86662×10^{-11}
$C_{12,9}$	-4.62122×10^{-12}
$C_{12,10}$	-5.72445×10^{-13}
$C_{12,11}$	2.02689×10^{-14}
$C_{12,12}$	8.12551×10^{-15}
$C_{13,0}$	2.75545×10^{-2}
$C_{13,1}$	-2.9199×10^{-3}
$C_{13,2}$	-2.02593×10^{-6}
$C_{13,3}$	6.84023×10^{-6}
$C_{13,4}$	3.23691×10^{-7}
$C_{13,5}$	-3.54904×10^{-8}
$C_{13,6}$	2.59498×10^{-10}
$C_{13,7}$	4.03437×10^{-11}
$C_{13,8}$	-1.37277×10^{-11}
$C_{13,9}$	-7.31327×10^{-13}
$C_{13,10}$	-7.27471×10^{-14}
$C_{13,11}$	2.30772×10^{-14}
$C_{13,12}$	2.58196×10^{-16}
$C_{13,13}$	-2.18667×10^{-16}
$C_{14,0}$	-1.53377×10^{-2}
$C_{14,1}$	7.66068×10^{-4}
$C_{14,2}$	2.96292×10^{-4}
$C_{14,3}$	5.32869×10^{-6}
$C_{14,4}$	-5.87731×10^{-7}
$C_{14,5}$	-5.31799×10^{-8}
$C_{14,6}$	2.30096×10^{-9}

$C_{14,7}$	-7.86604×10^{-11}
$C_{14,8}$	-7.59718×10^{-12}
$C_{14,9}$	-1.0193×10^{-13}
$C_{14,10}$	5.29761×10^{-14}
$C_{14,11}$	7.4175×10^{-15}
$C_{14,12}$	-9.24318×10^{-16}
$C_{14,13}$	-2.2948×10^{-17}
$C_{14,14}$	1.81628×10^{-17}
$C_{15,0}$	2.06404×10^{-2}
$C_{15,1}$	-2.65164×10^{-3}
$C_{15,2}$	9.46812×10^{-6}
$C_{15,3}$	-3.69445×10^{-6}
$C_{15,4}$	3.18757×10^{-7}
$C_{15,5}$	-2.84101×10^{-8}
$C_{15,6}$	-1.9038×10^{-10}
$C_{15,7}$	-6.24463×10^{-11}
$C_{15,8}$	-1.06965×10^{-11}
$C_{15,9}$	-2.61478×10^{-13}
$C_{15,10}$	5.55852×10^{-16}
$C_{15,11}$	-1.64954×10^{-15}
$C_{15,12}$	4.81127×10^{-17}
$C_{15,13}$	2.5553×10^{-17}
$C_{15,14}$	5.60796×10^{-19}
$C_{15,15}$	-5.49434×10^{-19}

$S_{0,0}$	0
$S_{1,0}$	0
$S_{1,1}$	0
$S_{2,0}$	0
$S_{2,1}$	-1.80744×10^{-7}
$S_{2,2}$	-1.81446×10^{-2}
$S_{3,0}$	0
$S_{3,1}$	3.63836×10^{-3}
$S_{3,2}$	-2.40395×10^{-4}
$S_{3,3}$	-1.68328×10^{-3}
$S_{4,0}$	0
$S_{4,1}$	1.29913×10^{-4}
$S_{4,2}$	1.0351×10^{-3}
$S_{4,3}$	-7.12399×10^{-6}
$S_{4,4}$	-1.92384×10^{-4}
$S_{5,0}$	0
$S_{5,1}$	-1.04273×10^{-3}
$S_{5,2}$	6.17062×10^{-5}
$S_{5,3}$	1.16925×10^{-4}
$S_{5,4}$	-5.43531×10^{-6}
$S_{5,5}$	-1.43782×10^{-5}
$S_{6,0}$	0
$S_{6,1}$	-9.74106×10^{-5}
$S_{6,2}$	-1.48126×10^{-4}
$S_{6,3}$	1.56395×10^{-6}
$S_{6,4}$	1.56395×10^{-6}
$S_{6,5}$	3.86799×10^{-8}
$S_{6,6}$	-3.73278×10^{-7}

$S_{7,0}$	0
$S_{7,1}$	5.15445×10^{-4}
$S_{7,2}$	-1.97429×10^{-5}
$S_{7,3}$	-2.02322×10^{-5}
$S_{7,4}$	6.94006×10^{-7}
$S_{7,5}$	6.72634×10^{-7}
$S_{7,6}$	-3.44172×10^{-8}
$S_{7,7}$	-4.07766×10^{-8}
$S_{8,0}$	0
$S_{8,1}$	-1.24043×10^{-5}
$S_{8,2}$	2.30047×10^{-6}
$S_{8,3}$	-3.22691×10^{-7}
$S_{8,4}$	-6.27617×10^{-7}
$S_{8,5}$	-1.85513×10^{-8}
$S_{8,6}$	6.66802×10^{-10}
$S_{8,7}$	-3.57144×10^{-10}
$S_{8,8}$	1.74786×10^{-9}
$S_{9,0}$	0
$S_{9,1}$	-8.17618×10^{-5}
$S_{9,2}$	-1.31237×10^{-5}
$S_{9,3}$	7.54724×10^{-6}
$S_{9,4}$	-2.35188×10^{-7}
$S_{9,5}$	-1.00222×10^{-7}
$S_{9,6}$	1.12056×10^{-9}
$S_{9,7}$	1.6534×10^{-9}
$S_{9,8}$	-2.32921×10^{-11}
$S_{9,9}$	-5.56697×10^{-11}
$S_{10,0}$	0

$S_{10,1}$	6.94286×10^{-4}
$S_{10,2}$	-4.56443×10^{-5}
$S_{10,3}$	2.62557×10^{-6}
$S_{10,4}$	-4.14985×10^{-7}
$S_{10,5}$	-5.74199×10^{-8}
$S_{10,6}$	6.45742×10^{-9}
$S_{10,7}$	-7.47668×10^{-10}
$S_{10,8}$	-4.99191×10^{-12}
$S_{10,9}$	9.74982×10^{-13}
$S_{10,10}$	5.59573×10^{-12}
$S_{11,0}$	0
$S_{11,1}$	-8.17892×10^{-4}
$S_{11,2}$	-6.92074×10^{-5}
$S_{11,3}$	-1.13881×10^{-6}
$S_{11,4}$	-4.84678×10^{-7}
$S_{11,5}$	8.37324×10^{-8}
$S_{11,6}$	-1.09462×10^{-9}
$S_{11,7}$	-2.46115×10^{-10}
$S_{11,8}$	-2.79264×10^{-11}
$S_{11,9}$	9.02775×10^{-12}
$S_{11,10}$	1.31812×10^{-13}
$S_{11,11}$	-1.94565×10^{-13}
$S_{12,0}$	0
$S_{12,1}$	1.50676×10^{-3}
$S_{12,2}$	9.64141×10^{-5}
$S_{12,3}$	2.73675×10^{-6}
$S_{12,4}$	-2.39721×10^{-7}
$S_{12,5}$	-3.08972×10^{-8}

$S_{12,6}$	9.18684×10^{-9}
$S_{12,7}$	-5.56246×10^{-10}
$S_{12,8}$	5.98262×10^{-12}
$S_{12,9}$	1.23035×10^{-13}
$S_{12,10}$	-7.24925×10^{-13}
$S_{12,11}$	1.701×10^{-14}
$S_{12,12}$	1.63895×10^{-14}
$S_{13,0}$	0
$S_{13,1}$	-1.24564×10^{-3}
$S_{13,2}$	1.54632×10^{-5}
$S_{13,3}$	-6.74004×10^{-7}
$S_{13,4}$	-1.19607×10^{-6}
$S_{13,5}$	6.26074×10^{-9}
$S_{13,6}$	-1.26688×10^{-10}
$S_{13,7}$	-7.5178×10^{-13}
$S_{13,8}$	-1.60844×10^{-11}
$S_{13,9}$	-9.10394×10^{-14}
$S_{13,10}$	-7.19669×10^{-15}
$S_{13,11}$	-5.20369×10^{-15}
$S_{13,12}$	-1.01803×10^{-16}
$S_{13,13}$	-4.21829×10^{-16}
$S_{14,0}$	0
$S_{14,1}$	8.65044×10^{-4}
$S_{14,2}$	1.51562×10^{-4}
$S_{14,3}$	4.31479×10^{-7}
$S_{14,4}$	1.77234×10^{-7}
$S_{14,5}$	-1.76094×10^{-9}
$S_{14,6}$	4.30073×10^{-9}

$S_{14,7}$	-2.43475×10^{-10}
$S_{14,8}$	-1.42072×10^{-11}
$S_{14,9}$	4.1348×10^{-13}
$S_{14,10}$	8.33334×10^{-15}
$S_{14,11}$	6.89565×10^{-16}
$S_{14,12}$	-3.88959×10^{-16}
$S_{14,13}$	3.71979×10^{-18}
$S_{14,14}$	2.08219×10^{-17}
$S_{15,0}$	0
$S_{15,1}$	-6.5828×10^{-5}
$S_{15,2}$	9.63909×10^{-5}
$S_{15,3}$	9.90187×10^{-7}
$S_{15,4}$	-7.56365×10^{-7}
$S_{15,5}$	-3.05489×10^{-8}
$S_{15,6}$	-2.45565×10^{-10}
$S_{15,7}$	-1.12172×10^{-11}
$S_{15,8}$	2.66204×10^{-12}
$S_{15,9}$	-2.21231×10^{-14}
$S_{15,10}$	-7.67107×10^{-15}
$S_{15,11}$	-2.08224×10^{-15}
$S_{15,12}$	4.21957×10^{-17}
$S_{15,13}$	1.21087×10^{-17}
$S_{15,14}$	-3.91552×10^{-19}
$S_{15,15}$	-4.94421×10^{-19}