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RESEARCH ARTICLE

Delay Geometric Brownian Motion in Financial Option Valuation

Xuerong Mao[a] and Sotirios Sabanis[b]*

[a]*Department of Mathematics and Statistics, University of Strathclyde, Glasgow G1 1XT, U.K.*; [b]*Maxwell Institute for Mathematical Sciences and School of Mathematics, University of Edinburgh, Edinburgh EH9 3JZ, U.K.*

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Motivated by influential work on complete stochastic volatility models, such as Hobson and Rogers (1998), we introduce a model driven by a delay geometric Brownian motion (DGBM) which is described by the stochastic delay differential equation $dS(t) = \mu(S(t-\tau))S(t)dt + V(S(t-\tau))S(t)dW(t)$. We show that the equation has a unique positive solution under a very general condition, namely that the volatility function V is a continuous mapping from $\mathbb{R}_+$ to itself. Moreover, we show that the delay effect is not too sensitive to time lag changes. The desirable robustness of the delay effect is demonstrated on several important financial derivatives as well as on the value process of the underlying asset. Finally, we introduce an Euler–Maruyama numerical scheme for our proposed model and show that this numerical method approximates option prices very well. All these features show that the proposed DGBM serves as a rich alternative in modelling financial instruments in a complete–market framework.

Keywords: Stochastic delay differential equations, derivative pricing, Euler–Maruyama, local Lipschitz condition, strong convergence

AMS Subject Classification: 60G44; 91G20; 91G80

1. Introduction

In the continuous time market model of Black and Scholes [4, 23], the price of a risky asset is supposed to be a geometric Brownian motion (GBM). This classical model assumes that continuously compounded returns are normally distributed. The central limit theorem is often invoked as a primary motivation for this assumption. However, it is well documented (see e.g. [9]) and widely accepted that the assumed normality of the returns distribution is violated in both the historical asset price data and the market option prices. As a result, many different types of models such as stochastic volatility [10, 12, 15, 26], jump-diffusion [24], pure jump Lévy processes [2, 17], and various combinations of the aforementioned [5, 6] were created in order to explain results from empirical studies more adequately. However, many of these models often require the presence of a large number of parameters, which makes their calibration computationally expensive, for more details see Carr et al. [6] and the references therein. Moreover, some of these models introduce an additional source of randomness that results in an incomplete market framework, where the uniqueness of preference independent prices for contingent claims is lost.

*Corresponding author. Email: s.sabanis@ed.ac.uk

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All the above models share a common characteristic, they adopt a typical Markovian setting similar to the one employed by Black and Scholes [4]. The rediscovery, in the 1950s, of Bachelier's work and the emergence of eminent scholars like Eugene Fama and Paul Samuelson, who advocated the two closely related hypotheses for assets prices, i.e. efficient markets and random walks have influenced many scholars to adopt such a setting. However, market crashes and well-known phenomena called "market anomalies", e.g. momentum effects which are based on the rate on which investors absorb large volumes of information (e.g. annual accounts for large corporations), have raised serious questions about the accuracy of the statement which claims that "all publicly available information is fully reflected in current asset prices". One then could consider models with past dependency which are Markovian on a higher-dimension, for example see Hobson and Rogers [13].

Furthermore, an important characteristic that any proposed model should have is the reproduction of 'smiles' and 'skews' that are present in options markets. Hobson and Rogers [13] achieved this by proposing a class of volatility models where the instantaneous volatility is expressed in terms of exponentially weighted moments of historic log-prices. Kind, Liptser and Runggaldier [14] also proposed that the instantaneous volatility is expressed in terms of the sample variance of the log-prices over a past interval of fixed length. From a practitioners point of view, it seems more natural to declare this dependency on past data through a similar approach to the one appearing in the latter article since past data are always available in discrete-time setting (e.g., daily, hourly etc).

It seems natural then to consider an approach where volatility can be regarded as a function of the past states $S(t-\tau_1)$, $S(t-\tau_2)$, $\cdots$, $S(t-\tau_n)$, whence the asset price $S(t)$ may obey a stochastic delay differential equation (SDDE)

$$dS(t) = \mu(S(t-\tau_1), \cdots, S(t-\tau_n))S(t)dt + V(S(t-\tau_1), \cdots, S(t-\tau_n))S(t)dW(t). \tag{1.1}$$

As this SDDE evolves also from the classical geometric Brownian motion, we call it the *delay geometric Brownian motion* (DGBM). More carefully, noting that the past states $S(t-\tau_1)$, $S(t-\tau_2)$, $\cdots$, $S(t-\tau_n)$ are a sample of the whole segment $S_t := \{S(u) : u \in [t-\tau, t]\}$, where $\tau = \max_i \tau_i$, the asset price could (in general) obey a stochastic functional differential equation (SFDE)

$$dS(t) = \mu(S_t)S(t)dt + V(S_t)S(t)dW(t), \tag{1.2}$$

with μ and V being functionals from $C([-\tau, T-\tau]; (0,\infty))$ to $(0,\infty)$, where T represents the termination date for our economy. However, given that the SDDE (1.1) could provide a good approximation to the SFDE (1.2) while the former is also much simpler and closer to practitioners' views than the latter, we concentrate our study on the SDDE (1.1).

A recent paper by Arriojas *et al.* [3], where a delay Black–Scholes formula is established through the existence of a unique equivalent martingale measure, also uses an SDDE to model asset price processes. One further observes that there exists a link between the way the risk-neutral measure is obtained in both papers, [3] & [13], that results in a complete and arbitrage market with the correct smiles and skews.

As an example, we present Figures 1 & 2 [1] of simulated implied volatility curves, which are expressed as functions of their exercise prices. Figure 1 presents the case for 3-month European call options which are considered under a model with 1-week

[1] These figures are courtesy of Nairn McWilliams

fixed delay for Tesco, Barclays, Lloyds and Vodafone share prices with nonlinear expressions describing $V(x)$. The strike is quoted as a percentage of the initial value, using closing prices up to the end of Friday 1st July 2011 (with initial values of 401.15, 265.55, 50.81 and 164.5 respectively).

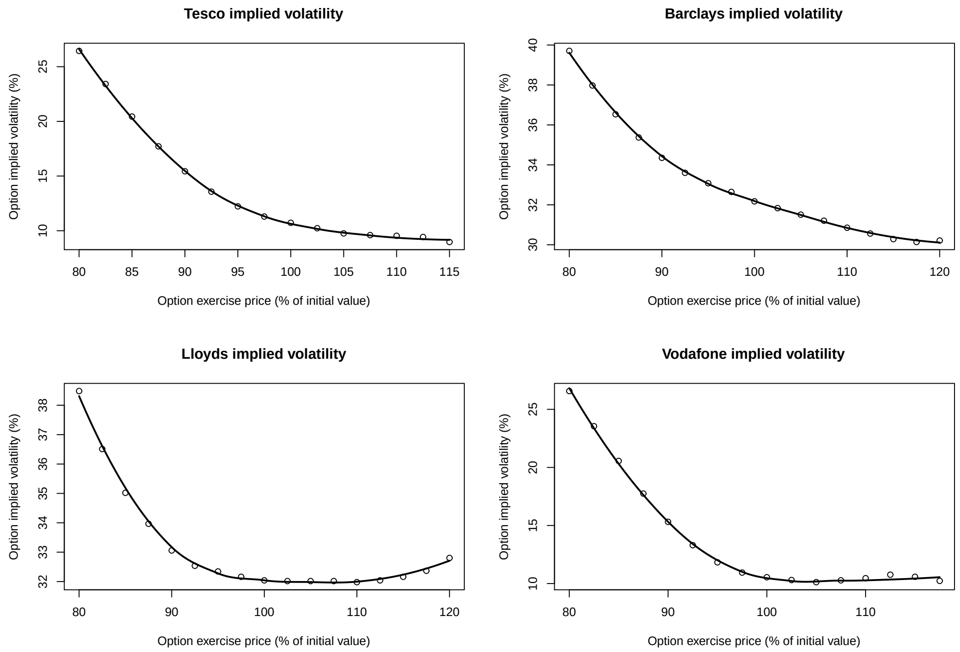

Figure 1. Implied volatility curves of a 1-week fixed delay and 3 months to maturity.

In Figure 2, 1-month European call options are considered under a model with 1-day fixed delay for Tesco, Barclays, Lloyds and Vodafone share prices under the same conditions as before (initial value, strike prices and nonlinear $V(x)$).

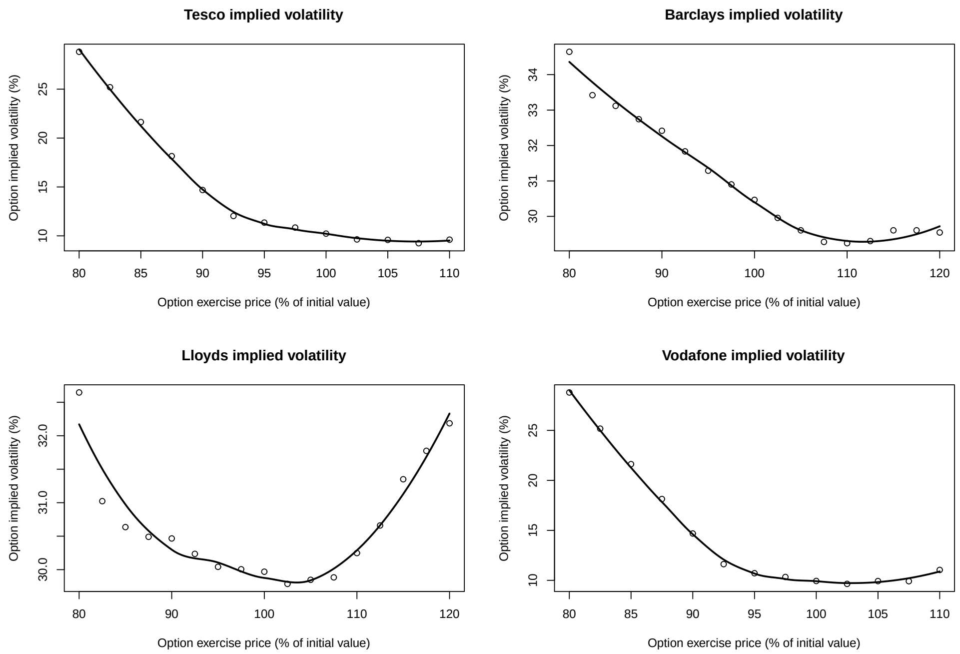

Figure 2. Implied volatility curves of a 1-day fixed delay and a month to maturity.

One then could compare these figures with well-known results about implied volatility shapes so as to verify that the right shape has been achieved in all cases. Note also that different companies (i.e. shares), delay parameters and times to maturity have been used in the above simulations producing similar/comparable results which, in the opinion of the authors, demonstrate the robustness of the delay approach. Moreover, one could claim that these findings reinforce the point that such a modelling approach is in agreement with empirical data.

Moreover, very recent theoretical findings by Liang et al [16] demonstrate that backward stochastic differential equations (BSDEs) can be reformulated as ordinary functional differential equations (OFDEs) on certain path spaces. This will certainly have a significant impact on the use of OFDEs, or more general SFDEs, in financial modelling, especially when one considers the extensive use of BSDEs in this area since the publication of the seminal paper by Pardoux and Peng [25].

The main aim of this paper is to call for further attention into the possibility of modelling asset prices via SDDEs or SFDEs. In order to persuade the reader, we show here that the SDDE (1.1) has many important properties which are desired when modelling a financial quantity. In this paper we will show:

- The proposed SDDE (called DGBM) has a unique nonnegative (or even strictly positive) solution under relatively weak conditions imposed on its volatility function.
- The aforementioned solution has finite probability expectation (at any time $t \geq 0$) which is essential for pricing various contingent claims in a well defined framework.
- This pricing is done under a unique equivalent martingale measure which guarantees an arbitrage-free and complete market.
- Smiles and skews are present and in agreement with empirical evidence.
- The delay effect is not too sensitive to time lag changes. Small changes for the time lag τ have an analogous small impact on the values of the underlying asset $S(t)$ and its associated options.
- The pricing of contingent claims under the DGBM approach is computable numerically if not analytically.

However, we would not occupy ourselves here with the task of showing which type of volatility functions $V(\cdot)$ may be appropriate for modelling purposes. Nonetheless, we will highlight the fact that the conditions imposed on it are very weak so that *a wide class of volatility functions may be used to fit a wide range of financial quantities.*

Finally, for reasons of notational simplicity and elegance, the following simpler SDDE

$$dS(t) = \mu(S(t-\tau))S(t)dt + V(S(t-\tau))S(t)dW(t),$$

is considered here, although all the results presented henceforth can be obtained for the case where equation (1.1) holds.

2. The Delay Geometric Brownian Motion

Throughout this paper, unless otherwise specified, we will employ the following notation. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $W(t)$, $t \geq 0$, be a scalar Brownian motion defined on the above probability space. If x, y are real numbers, then $x \vee y$ denotes the maximum of x and y, and $x \wedge y$ denotes the minimum of x and y. For $\tau > 0$, $C([-\tau, 0]; (0, \infty))$ denotes the space of all continuous functions $\xi : [-\tau, 0] \to (0, \infty)$ with the norm $\|\xi\| = \sup_{-\tau \le u \le 0} \xi(u)$. Also, $\mathbb{R}_+ = [0, \infty)$ and $C(\mathbb{R}_+; \mathbb{R}_+)$ denotes the space of all nonnegative continuous functions defined on $\mathbb{R}_+$. Moreover, $\emptyset$ denotes the empty set and we set $\inf \emptyset = \infty$. For a set A, its indication function is denoted by I_A.

2.1 A Complete Market Model with a Smile

Let the asset price process S be governed by the following delay geometric Brownian motion

$$dS(t) = \mu(S(t-\tau))S(t)dt + V(S(t-\tau))S(t)dW(t) \tag{2.1}$$

on $t \ge 0$ with initial data $S(u) = \xi(u)$ on $u \in [-\tau, 0]$. Here τ is a positive constant, $r > 0$ is the risk-free interest rate and $W(t)$ is a scalar Brownian motion, and the initial data $\xi := \{\xi(u) : u \in [-\tau, 0]\} \in C([-\tau, 0]; (0, \infty))$ while the functions μ and V are in $C(\mathbb{R}_+; \mathbb{R}_+)$.

Furthermore, let

$$Z(t) := \ln(e^{-rt}S(t)), \qquad \text{for every } t \ge 0,$$

denote the log-price of the discounted asset. One immediately observes that

$$dZ(t) = [\mu(S(t-\tau)) - r - \frac{1}{2}V^2(S(t-\tau))]dt + V(S(t-\tau))dW(t), \tag{2.2}$$

for every $t \ge 0$. Therefore, changing the measure to achieve no-arbitrage leads to the choice

$$\theta(x) = \frac{1}{2}V(x) + \frac{\mu(x) - r - \frac{1}{2}V^2(x)}{V(x)} = \frac{\mu(x) - r}{V(x)}$$

It is here where one can observe a link between Arriojas *et al.* [3] and Hobson & Rogers [13] since both approaches require the same structure for the unique (in each case) equivalent martingale measure

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} := \exp\Big(-\frac{1}{2}\int_0^T \theta^2(X(u))du - \int_0^T \theta(X(u))dW(u)\Big)$$

Then, under $\tilde{\mathbb{P}}$, the asset is given by

$$dS(t) = rS(t)dt + V(S(t-\tau))S(t)d\tilde{W}(t)$$

where $\tilde{W}(t)$, $t \ge 0$, which is defined through $\tilde{W}(t) := W(t) + \int_0^t \theta(X(s))ds$, is a scalar Brownian motion in $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \tilde{\mathbb{P}})$. As a result, the discounted asset price process $e^{-rt}S(\mathrm{t})$ is a martingale and the model is arbitrage-free. The uniqueness of the equivalent martingale measure guarantees that the market is complete and therefore appropriate hedging strategies can also be obtain, for more details see Arriojas *et al.* [3]. Hobson & Rogers [13] also observed that their model produces the right smiles and skews in the resulting implied volatility plots as in Figure 1.

Again for reasons of notational simplicity, we will drop the explicit dependence on the risk-neutral measure $\tilde{\mathbb{P}}$ for all relevant calculations in all subsequent sections. Therefore, although we will work with the 'risk-neutral' probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \tilde{\mathbb{P}})$, we will avoid the use of the $\tilde{}$ notation.

2.2 Properties of the Delay Geometric Brownian Motion

Consider the delay geometric Brownian motion (DGBM) described by a stochastic delay differential equation (SDDE)

$$dS(t) = rS(t)dt + V(S(t-\tau))S(t)dW(t) \tag{2.3}$$

In general, for a stochastic delay differential equation to have a unique global solution, both of its shift and diffusion coefficient are required to be locally Lipschitz continuous and to obey the linear growth condition (see e.g. [18, 19, 22]). If we would apply this general theory to the DGBM (2.3), we could have been forced to assume that the volatility function V is locally Lipschitz continuous and is bounded. However, the following theorem shows that the SDDE (2.3) has a unique global positive solution without any additional condition.

Theorem 2.1: *The SDDE (2.3) has a unique global positive solution $x(t)$ on $t \geq 0$, which can be computed step by step as follows: for $k = 0, 1, 2, \cdots$ and $t \in [k\tau, (k+1)\tau]$,*

$$S(t) = S(k\tau)e^{r(t-k\tau)-\frac{1}{2}\int_{k\tau}^{t} V^2(S(u-\tau))du+\int_{k\tau}^{t} V(S(u-\tau))dW(u)}. \tag{2.4}$$

Moreover,

$$\mathbb{E}S(t) = \xi(0)e^{rt}, \tag{2.5}$$

for every $t \geq 0$.

Proof: If it is restricted on $t \in [0, \tau]$, the SDDE (2.3) becomes the following linear SDE

$$dS(t) = rS(t)dt + V(\xi(t-\tau))S(t)dW(t).$$

It is well known that this SDE has the unique explicit solution

$$S(t) = \xi(0)\exp\Big(rt - \frac{1}{2}\int_0^t V^2(\xi(u-\tau))du + \int_0^t V(\xi(u-\tau))dW(u)\Big).$$

That is, (2.4) holds for $k = 0$. Given that $S(t)$ is now known on $t \in [0, \tau]$, we may restrict the SDDE (2.3) on $t \in [\tau, 2\tau]$ so that it becomes the linear SDE

$$dS(t) = rS(t)dt + V(S(t-\tau))S(t)dW(t).$$

It has the explicit solution

$$S(t) = S(\tau)\exp\Big(r(t-\tau) - \frac{1}{2}\int_\tau^t V^2(S(u-\tau))du + \int_\tau^t V(S(u-\tau))dW(u)\Big),$$

where both integrals are well defined as $S(t)$ is a continuous stochastic process on $t \in [0, \tau]$. This shows that (2.4) holds for $k = 1$. Repeating this procedure we see that (2.4) holds for all $k \geq 0$.

Furthermore, let us consider the discounted asset price process $M := \{M(t)\}_{t \geq 0}$, where $M(t) := e^{-rt}S(t)$ for all $t \geq 0$. As a result

$$M(t) = M(k\tau) \exp\Big(-\tfrac{1}{2} \int_{k\tau}^{t} V^2(M(u-\tau))du + \int_{k\tau}^{t} V(M(u-\tau))dW(u)\Big) \quad (2.6)$$

which satisfies the following SDDE

$$dM(t) = V(M(t-\tau))M(t)dW(t) \quad (2.7)$$

on $t \geq 0$ with initial data $M(u) = e^{-ru}\xi(u)$ on $u \in [-\tau, 0]$. In other words,

$$M(t) = M(0) + \int_0^t V(M(u-\tau))M(u)dW(u)$$

and due to the continuity of paths for S (and consequently for M), we obtain that

$$\int_0^t V^2(M(u-\tau))M^2(u)du < \infty \quad \text{(a.s.) for every } t \geq 0,$$

which implies of course that $L := \{L(t)\}_{t \geq 0}$, where

$$L(t) := \int_0^t V(M(u-\tau))M(u)dW(u),$$

is a (positive) local martingale and thus a supermartingale.

One, then, further observes that M is a (true) martingale since for every $t \geq 0$ there exists a positive integer $k = k(t)$ such that $t \in [k\tau, (k+1)\tau]$,

$$\begin{aligned}
\mathbb{E}|M(t)| =& \mathbb{E}\Big(M(k\tau) \exp\Big(-\tfrac{1}{2} \int_{k\tau}^{t} V^2(M(u-\tau))du + \int_{k\tau}^{t} V(M(u-\tau))dW(u)\Big)\Big) \\
=& \mathbb{E}\Big(M(k\tau)\mathbb{E}\Big(\exp\Big(-\tfrac{1}{2} \int_{k\tau}^{t} V^2(M(u-\tau))du \\
&+ \int_{k\tau}^{t} V(M(u-\tau))dW(u)\Big) | \mathcal{F}_{k\tau}\Big)\Big) \\
=& \mathbb{E}M(k\tau) = \mathbb{E}(\mathbb{E}(\ldots \mathbb{E}(M(k\tau)|\mathcal{F}_{(k-1)\tau}) \ldots |\mathcal{F}_{\tau})) = M(0) = \xi(0) < \infty
\end{aligned}$$

and

$$\mathbb{E}(M_t|\mathcal{F}_s) = M_s \quad \text{(by using again nested conditional expectations)}$$

for every $0 \leq s \leq t < \infty$. As a result L is also a (true) martingale, and thus

$$\mathbb{E}\Big(\int_0^t V(S(u-\tau))S(u)dW(u)\Big) = 0, \quad \text{for every } t \geq 0.$$

Assertion (2.5) follows from above and equation (2.3).

□

Remark 2.2 The above theorem shows that the local Lipschitz condition on V is unnecessary. This idea was developed in [20] and used recently by [3]. However, we will need the local Lipschitz condition in the next sections when we study the sensitivity of the time lag. We still do not know if the results in the next section hold without the local Lipschitz condition.

Assertion (2.5) guarantees that, under the DGBM approach, the price functions of various options are well-defined. A typical example is the price of a European call option (at $t = 0$ with exercise price E and expiry date T) which is given by

$$C = e^{-rT}\mathbb{E}(S(T) - E)^+$$

and which is well-defined. However, for some more complicated options and their associated mathematical analysis, it is useful for the solution of equation (2.3) to obey, for example

$$\mathbb{E}\Big(\sup_{0\le t\le T} S(t)\Big) < \infty \quad \forall T > 0.$$

A sufficient condition for this is the following assumption.

Assumption 2.3 The volatility function V is bounded by a positive constant K, namely

$$V(x) \le K \quad \forall x \ge 0. \tag{2.8}$$

In this case, it is known (see e.g. [20, Theorem 4.1 on page 158]) that for any $p \ge 1$,

$$\mathbb{E}\Big(\sup_{0\le t\le T} S^p(t)\Big) < \infty \quad \forall T > 0. \tag{2.9}$$

But the following theorem gives more precise estimations.

Theorem 2.4: *Let Assumption 2.3 hold and $p \ge 1$. Then*

$$\mathbb{E}S^p(t) \le \xi(0)e^{p[r+0.5(p-1)K^2]t} \tag{2.10}$$

for any $t \ge 0$ and

$$\mathbb{E}\Big(\sup_{0\le t\le T} S^p(t)\Big) \le \xi^p(0)\Big(2 + \frac{9p^2K^2}{p[r+0.5(p-1)K^2]}\Big)e^{p[r+0.5(p-1)K^2]T} \tag{2.11}$$

for any $T \ge 0$.

Proof: Let $\lambda = p[r + 0.5(p-1)K^2]$. By the Itô formula,

$$\begin{aligned} d[e^{-\lambda t}S^p(t)] &= e^{-\lambda t}\Big(-\lambda S^p(t)dt + pS^{p-1}(t)dS(t) + \frac{1}{2}p(p-1)V^2(S(t-\tau))S^p(t)dt\Big) \\ &= e^{-\lambda t}S^p(t)\big[-\lambda + p\mu + \tfrac{1}{2}p(p-1)V^2(S(t-\tau))\big]dt \\ &\quad + pe^{-\lambda t}V(S(t-\tau))S^p(t)dW(t). \end{aligned}$$

Hence

$$
\begin{aligned}
e^{-\lambda t}S^p(t) &= \xi^p(0) + \int_0^t e^{-\lambda u}S^p(u)\big[-\lambda + p\mu + \tfrac{1}{2}p(p-1)V^2(S(u-\tau))\big]dt \\
&\quad + \int_0^t pe^{-\lambda u}V(S(u-\tau))S^p(u)dW(u) \\
&\le \xi^p(0) + \int_0^t pe^{-\lambda u}V(S(u-\tau))S^p(u)dW(u).
\end{aligned}
$$

Due to the known property (2.9) we can take the expectation on both sides to obtain

$$e^{-\lambda t}\mathbb{E}S^p(t) \le \xi^p(0)$$

which yields assertion (2.10). To show (2.11) we compute, by the Itô formula again, that

$$
\begin{aligned}
S^p(t) &= \xi^p(0) + \int_0^t \big[p\mu + \tfrac{1}{2}p(p-1)V^2(S(t-\tau))\big]S^p(u)du \\
&\quad + \int_0^t pV(S(u-\tau))S^p(u)dW(u) \\
&\le \xi^p(0) + \int_0^t \lambda S^p(u)du + \int_0^t pV(S(u-\tau))S^p(u)dW(u).
\end{aligned}
$$

Hence

$$\mathbb{E}\Big(\sup_{0\le t\le T} S^p(t)\Big) \le \xi^p(0) + \mathbb{E}\int_0^T \lambda S^p(u)du + \mathbb{E}\Big(\sup_{0\le t\le T}\int_0^t pV(S(u-\tau))S^p(u)dW(u)\Big).$$

But, by the well-known Burkholder–Davis–Gundy inequality (see e.g. [20, 22]),

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le t\le T}\int_0^t pV(S(u-\tau))S^p(u)dW(u)\Big) \\
&\le 3\mathbb{E}\Big[\Big(\int_0^T p^2V^2(S(u-\tau))S^{2p}(u)du\Big)^{\frac{1}{2}}\Big] \\
&\le 3pK\mathbb{E}\Big\{\Big(\Big[\sup_{0\le u\le T} S^p(u)\Big]\int_0^T S^p(u)du\Big)^{\frac{1}{2}}\Big\} \\
&\le \tfrac{1}{2}\mathbb{E}\Big[\sup_{0\le u\le T} S^p(u)\Big] + 4.5p^2K^2\mathbb{E}\int_0^T S^p(u)du.
\end{aligned}
$$

Thus

$$\mathbb{E}\Big(\sup_{0\le t\le T} S^p(t)\Big) \le 2\xi^p(0) + (2\lambda + 9p^2K^2)\int_0^T \mathbb{E}S^p(u)du.$$

But, by (2.10),

$$\int_0^T \mathbb{E}S^p(u)du \le \int_0^T \xi^p(0)e^{\lambda u}du = \frac{\xi^p(0)}{\lambda}(e^{\lambda T}-1).$$

Therefore

$$\mathbb{E}\Big(\sup_{0\le t\le T} S^p(t)\Big) \le \xi^p(0)\Big(2+\frac{9p^2K^2}{\lambda}\Big)e^{\lambda T}$$

which is the required assertion (2.11).

□

Remark 2.5 One could improve the above estimate and obtain a smaller bound given by

$$\mathbb{E}\Big(\sup_{0\le t\le T} S^p(t)\Big) \le \xi^p(0)(\frac{p}{p-1})^p e^{\lambda T},$$

where $\lambda = p[r+\frac{p-1}{2}K^2]$, provided that $r \le K^2/2$ (which is expected in real world applications) and $p \ge 2$. To see this, one observes that under Assumption 2.3, one obtains for a $t \in [k\tau, (k+1)\tau] \subset [0,T]$, where k is some positive integer,

$$\mathbb{E}S^p(t) \le \xi(0)e^{p[r+0.5(p-1)K^2]t}$$

(as proven in Theorem 2.4). Moreover, one observes that

$$\mathbb{E}\Big((\sup_{0\le t\le T} M(t))^p\Big) \le (\frac{p}{p-1})^p\mathbb{E}M^p(T) \qquad \text{(Doob's inequality)}$$

and

$$e^{-rpT}\mathbb{E}\Big(\sup_{0\le t\le T} S^p(t)\Big) \le \mathbb{E}\Big(\sup_{0\le t\le T} M^p(t)\Big) = \mathbb{E}\Big((\sup_{0\le t\le T} M(t))^p\Big)$$

due to the fact that e^{-rpt} is a strictly decreasing function of t. Thus,

$$\mathbb{E}\Big(\sup_{0\le t\le T} S^p(t)\Big) \le (\frac{p}{p-1})^p\mathbb{E}S^p(T) \le (\frac{p}{p-1})^p\xi^p(0)e^{p[r+\frac{p-1}{2}K^2]T}$$

Finally, for $r \le K^2/2$ and $p \ge 2$

$$2+\frac{9p^2K^2}{p[r+0.5(p-1)K^2]} = 2+9\frac{p}{p-1}\frac{K^2}{\frac{r}{p-1}+0.5K^2} \ge 2+9\frac{p}{p-1}\frac{K^2}{0.5K^2+0.5K^2} \ge 11$$

whereas

$$(1+\frac{1}{p-1})^{p-1} \le e \quad \Rightarrow \quad (\frac{p}{p-1})^p \le e(1+\frac{1}{p-1}) \le 2e$$

and thus

$$(\frac{p}{p-1})^p < 2+\frac{9p^2K^2}{p[r+0.5(p-1)K^2]}.$$

3. Delay Effect on European Options

Recall that the motivations for us to introduce the DGBM (2.3) is the estimation of volatility using the past asset price(s). We observe that there is a time lag τ when we estimate the volatility. It is very important to know whether the time lag τ is sensitive in the sense that a little change of τ will have a significant effect on the underlying asset price and its associated option price. If this is the case, then the time lag needs to be controlled tightly in practice; otherwise the delay effect is robust. In this section we shall show the robustness of delay effect on European options.

To make our problem more clearly, let us assume that one holds a European call option at $t = 0$ on the underlying asset price with the exercise price E at the expiry date T. Originally, the holder thinks the underlying asset price follows the DGBM (2.3) so the price of the European call option at $t = 0$ is

$$C_\tau = e^{-rT}\mathbb{E}(S(T) - E)^+. \tag{3.1}$$

On second thought, the holder may wonder that if the volatility at time t is estimated by the corresponding option price at time $t - \bar\tau$, instead of $t - \tau$, then the underlying asset price could follow an alternative DGBM

$$d\bar S(t) = r\bar S(t)dt + V(\bar S(t - \bar\tau))\bar S(t)dW(t), \tag{3.2}$$

whence the price of the European call option at $t = 0$ could be

$$C_{\bar\tau} = e^{-rT}\mathbb{E}(\bar S(T) - E)^+. \tag{3.3}$$

If the difference between C_τ and $C_{\bar\tau}$ is small when the difference between τ and $\bar\tau$ is small, then the holder can simply choose either (2.3) or (3.2) as the equation for the underlying asset price; otherwise the holder has to control the time delay tightly.

Without loss of any generality, we may assume that $\bar\tau < \tau$. Note that the underlying asset prices before time 0 should be the same for both $S(t)$ and $\bar S(t)$. Recalling that the underlying asset prices for the period $t \in [-\tau, 0]$ are known as $\{\xi(t) : t \in [-\tau, 0]\}$, we observe that the initial data for equation (3.2) should be $\bar S(t) = \xi(t)$ on $t \in [-\bar\tau, 0]$.

The difference $C_\tau - C_{\bar\tau}$ is due to the difference of the two time lags, namely $\tau - \bar\tau$. Note that

$$\begin{aligned} |C_\tau - C_{\bar\tau}| &\le e^{-rT}\mathbb{E}|(S(T) - E)^+ - (\bar S(T) - E)^+| \\ &\le e^{-rT}\mathbb{E}|S(T) - \bar S(T)|. \end{aligned} \tag{3.4}$$

Hence, if we can show

$$\lim_{\tau - \bar\tau \to 0} \mathbb{E}|S(T) - \bar S(T)| = 0,$$

then

$$\lim_{\tau - \bar\tau \to 0} |C_\tau - \bar C_{\bar\tau}| = 0.$$

This shows the continuity of the European call option price on the time lag. For this purpose we need to impose a local Lipschitz condition on the volatility function.

Assumption 3.1 The volatility function V is locally Lipschitz continuous. That is, for each $R > 0$, there is a $K_R > 0$ such that

$$|V(x) - V(\bar{x})| \le K_R|x - \bar{x}| \quad \forall x, \bar{x} \in [0, R].$$

Let us first establish two lemmas.

Lemma 3.2: *Let $R \ge \|\xi\|$ and define the stopping time*

$$\rho_R = \inf\{t \ge 0 : S(t) > R\}.$$

Let θ be a stopping time such that $0 \le \theta \le \rho_R$. Then, for any $0 \le u < v < \infty$,

$$\mathbb{E}|S(v \wedge \theta) - S(u \wedge \theta)|^2 \le 2R^2(v - u)[r^2(v - u) + \bar{K}_R^2],$$

where $\bar{K}_R = \max_{0 \le x \le R} V(x)$.

Proof: It follows from (2.3) that

$$S(v \wedge \theta) - S(u \wedge \theta) = \int_{u\wedge\theta}^{v\wedge\theta} rS(t)dt + \int_{u\wedge\theta}^{v\wedge\theta} V(S(t - \tau))S(t)dW(t).$$

Hence

$$\begin{aligned}\mathbb{E}|S(v \wedge \theta) - S(u \wedge \theta)|^2 &\le 2\mathbb{E}|\int_{u\wedge\theta}^{v\wedge\theta} rS(t)dt|^2 + 2\mathbb{E}\int_{u\wedge\theta}^{v\wedge\theta} [V(S(t - \tau))S(t)]^2dt \\ &\le 2r^2R^2(v - u)^2 + 2R^2\bar{K}_R^2(v - u)\end{aligned}$$

as required. □

Lemma 3.3: *Let Assumption 3.1 hold. Let $R \ge \|\xi\|$ and $\tau - \bar{\tau} \le 1$. Define the stopping times*

$$\rho_R = \inf\{t \ge 0 : S(t) > R\} \quad \text{and} \quad \bar{\rho}_R = \inf\{t \ge 0 : \bar{S}(t) > R\}$$

and set $\theta_R = \rho_R \wedge \bar{\rho}_R$. Define

$$\delta(\tau - \bar{\tau}) = \sup\{|\xi(u) - \xi(v)| : u, v \in [-\tau, 0], \; |u - v| \le \tau - \bar{\tau}\}.$$

Then, for any $T > 0$,

$$\mathbb{E}\Big(\sup_{0\le t\le T} |S(t \wedge \theta_R) - \bar{S}(t \wedge \theta_R)|\Big) \le c_R\sqrt{T}e^{c_RT}(\delta(\tau - \bar{\tau}) + \sqrt{\tau - \bar{\tau}}), \tag{3.5}$$

where c_R is a positive constant independent of T and $\tau - \bar{\tau}$. In particular,

$$\lim_{\tau-\bar{\tau}\to 0} \mathbb{E}|S(T \wedge \theta_R) - \bar{S}(T \wedge \theta_R)| = 0. \tag{3.6}$$

Proof: As $\xi(u)$ is continuous on $u \in [-\tau, 0]$, it must be uniformly continuous. Thus $\delta(\tau - \bar{\tau}) < \infty$ and $\lim_{\tau-\bar{\tau}\to 0} \delta(\tau - \bar{\tau}) = 0$. So (3.6) follows from (3.5). Our proof is therefore complete if we can show (3.5).

Set $Z(t) = S(t \wedge \theta_R) - \bar{S}(t \wedge \theta_R)$ for $t \geq 0$. It follows from (2.3) and (3.2) that

$$\begin{aligned} Z(t) = & \int_0^{t\wedge\theta_R} r(S(u) - \bar{S}(u))du \\ & + \int_0^{t\wedge\theta_R} \big[V(S(u-\tau))S(u) - V(\bar{S}(u-\bar{\tau}))\bar{S}(u)\big]dW(u) \\ = & \int_0^{t\wedge\theta_R} rZ(u)du + \int_0^{t\wedge\theta_R} \big[V(S(u-\tau))S(u) - V(\bar{S}(u-\bar{\tau}))S(u)\big]dW(u) \\ & + \int_0^{t\wedge\theta_R} \big[V(\bar{S}(u-\bar{\tau}))S(u) - V(\bar{S}(u-\bar{\tau}))\bar{S}(u)\big]dW(u). \end{aligned}$$

Hence

$$\mathbb{E}\Big(\sup_{0\leq t\leq T} |Z(t)|\Big) \leq J_1 + J_2 + J_3, \tag{3.7}$$

where

$$\begin{aligned} J_1 &= \mathbb{E}\Big(\sup_{0\leq t\leq T} \Big|\int_0^{t\wedge\theta_R} rZ(u)du\Big|\Big), \\ J_2 &= \mathbb{E}\Big(\sup_{0\leq t\leq T} \Big|\int_0^{t\wedge\theta_R} \big[V(S(u-\tau))S(u) - V(\bar{S}(u-\bar{\tau}))S(u)\big]dW(u)\Big|\Big), \\ J_3 &= \mathbb{E}\Big(\sup_{0\leq t\leq T} \Big|\int_0^{t\wedge\theta_R} \big[V(\bar{S}(u-\bar{\tau}))S(u) - V(\bar{S}(u-\bar{\tau}))\bar{S}(u)\big]dW(u)\Big|\Big). \end{aligned}$$

Compute

$$J_1 \leq \mathbb{E}\int_0^{T\wedge\theta_R} r|Z(u)|du \leq \mathbb{E}\int_0^T r|Z(u)|du = \int_0^T r\mathbb{E}|Z(u)|du. \tag{3.8}$$

In what follows, c_R denotes a positive constant dependent on R etc. but independent of T and $\tau - \bar{\tau}$ while it many change line by line. Compute, by the Burkholder–

Davis–Gundy inequality and Assumption 3.1,

$$\begin{aligned}
J_2 &\le \sqrt{32}\mathbb{E}\Big(\Big[\int_0^{T\wedge\theta_R} |V(S(u-\tau)) - V(\bar{S}(u-\bar{\tau}))|^2 S^2(u)du\Big]^{\frac{1}{2}}\Big) \\
&\le c_R\mathbb{E}\Big(\Big[\int_0^{T\wedge\theta_R} |S(u-\tau) - \bar{S}(u-\bar{\tau})|^2 du\Big]^{\frac{1}{2}}\Big) \\
&\le c_R\mathbb{E}\Big(\Big[\int_0^{T\wedge\theta_R} |S(u-\tau) - S(u-\bar{\tau})|^2 + |S(u-\bar{\tau}) - \bar{S}(u-\bar{\tau})|^2 du\Big]^{\frac{1}{2}}\Big) \\
&\le c_R\mathbb{E}\Big(\Big[\int_0^{T} |S((u-\tau)\wedge\theta_R) - S((u-\bar{\tau})\wedge\theta_R)|^2 du\Big]^{\frac{1}{2}}\Big) \\
&+ c_R\mathbb{E}\Big(\Big[\int_0^{T} |Z(u-\bar{\tau})|^2 du\Big]^{\frac{1}{2}}\Big) \\
&\le c_R\Big[\int_0^{T} \mathbb{E}|S((u-\tau)\wedge\theta_R) - S((u-\bar{\tau})\wedge\theta_R)|^2 du\Big]^{\frac{1}{2}} \\
&+ c_R\mathbb{E}\Big(\Big[\int_0^{T} |Z(u-\bar{\tau})|^2 du\Big]^{\frac{1}{2}}\Big). \qquad (3.9)
\end{aligned}$$

But, for $u \in [0, \bar{\tau}]$,

$$\mathbb{E}|S((u-\tau)\wedge\theta_R) - S((u-\bar{\tau})\wedge\theta_R)|^2 = |\xi(u-\tau) - \xi(u-\bar{\tau})|^2 \le \delta^2(\tau-\bar{\tau});$$

while, by Lemma 3.2, for $u \in (\bar{\tau}, \tau]$,

$$\begin{aligned}
&\mathbb{E}|S((u-\tau)\wedge\theta_R) - S((u-\bar{\tau})\wedge\theta_R)|^2 \\
&\le 2|\xi(u-\tau) - \xi(0)|^2 + 2\mathbb{E}|S((u-\bar{\tau})\wedge\theta_R) - S(0)|^2 \\
&\le 2\delta^2(\tau-\bar{\tau}) + c_R(\tau-\bar{\tau});
\end{aligned}$$

and for $u > \tau$,

$$\mathbb{E}|S((u-\tau)\wedge\theta_R) - S((u-\bar{\tau})\wedge\theta_R)|^2 \le c_R(\tau-\bar{\tau}).$$

Hence

$$\Big[\int_0^{T} \mathbb{E}|S((u-\tau)\wedge\theta_R) - S((u-\bar{\tau})\wedge\theta_R)|^2 du\Big]^{\frac{1}{2}} \le c_R\sqrt{T}(\delta(\tau-\bar{\tau}) + \sqrt{\tau-\bar{\tau}}).$$

Moreover,

$$\begin{aligned}
&c_R\mathbb{E}\Big(\Big[\int_0^{T} |Z(u-\bar{\tau})|^2 du\Big]^{\frac{1}{2}}\Big) \le c_R\mathbb{E}\Big(\Big[\int_0^{T} |Z(u)|^2 du\Big]^{\frac{1}{2}}\Big) \\
&\le c_R\mathbb{E}\Big(\Big[\big(\sup_{0\le u\le T}|Z(u)|\big)\int_0^{T} |Z(u)|du\Big]^{\frac{1}{2}}\Big) \\
&\le c_R\int_0^{T} \mathbb{E}|Z(u)|du + \frac{1}{4}\mathbb{E}\Big(\sup_{0\le u\le T}|Z(u)|\Big).
\end{aligned}$$

Substituting these into (3.9) yields

$$J_2 \leq c_R\sqrt{T}(\delta(\tau-\bar{\tau})+\sqrt{\tau-\bar{\tau}})+c_R\int_0^T \mathbb{E}|Z(u)|du+\frac{1}{4}\mathbb{E}\Big(\sup_{0\leq u\leq T}|Z(u)|\Big). \quad (3.10)$$

Similarly, we can estimate

$$J_3 \leq c_R\int_0^T \mathbb{E}|Z(u)|du+\frac{1}{4}\mathbb{E}\Big(\sup_{0\leq u\leq T}|Z(u)|\Big). \quad (3.11)$$

Substituting (3.8), (3.10) and (3.11) into (3.7) we obtain

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\leq t\leq T}|Z(t)|\Big) &\leq c_R\sqrt{T}(\delta(\tau-\bar{\tau})+\sqrt{\tau-\bar{\tau}})+c_R\int_0^T \mathbb{E}|Z(u)|du\\ &\leq c_R\sqrt{T}(\delta(\tau-\bar{\tau})+\sqrt{\tau-\bar{\tau}})+c_R\int_0^T \mathbb{E}\Big(\sup_{0\leq t\leq u}|Z(t)|\Big)du.\end{aligned}$$

Since this holds for any $T\geq 0$, the Gronwall inequality implies

$$\mathbb{E}\Big(\sup_{0\leq t\leq T}|Z(t)|\Big) \leq c_R\sqrt{T}e^{c_R T}(\delta(\tau-\bar{\tau})+\sqrt{\tau-\bar{\tau}}).$$

as required. □

It is now easy to show the following theorem.

Theorem 3.4: *Under Assumption 3.1, with the definitions of (3.1) and (3.3), we have*

$$\lim_{\tau-\bar{\tau}\to 0}|C_\tau - C_{\bar{\tau}}| = 0. \quad (3.12)$$

Proof: Equation (3.4) implies that it is sufficient to show

$$\lim_{\tau-\bar{\tau}\to 0}\mathbb{E}|S(T)-\bar{S}(T)| = 0.$$

For any sufficiently large R, let θ_R be the stopping time as defined in Lemma 3.3. Then, one observes that

$$\begin{aligned}\mathbb{E}|S(T)-\bar{S}(T)| &= \mathbb{E}\big(|S(T)-\bar{S}(T)|I_{\{\theta_R>T\}}\big)+\mathbb{E}\big(|S(T)-\bar{S}(T)|I_{\{\theta_R\leq T\}}\big)\\ &\leq \mathbb{E}|S(T\wedge\theta_R)-\bar{S}(T\wedge\theta_R)|+\mathbb{E}\big([S(T)+\bar{S}(T)]I_{\{\theta_R\leq T\}}\big). \quad (3.13)\end{aligned}$$

and also

$$\mathbb{E}S(T\wedge\rho_R) \geq \mathbb{E}\Big(S(T\wedge\rho_R)I_{\{\rho_R\leq T\}}\Big) = R\,\mathbb{P}(\rho_R\leq T) \quad (3.14)$$

which yields (in view of Theorem 2.1) that

$$\mathbb{P}(\theta_R\leq T)\leq \mathbb{P}(\rho_R\leq T)+\mathbb{P}(\bar{\rho}_R\leq T)\leq \frac{2\xi(0)e^{rT}}{R}\to 0 \text{ as } R\to\infty,$$

while

$$\mathbb{E}[S(T) + \bar{S}(T)] \le 2\xi(0)e^{rT}.$$

Hence, by the classical dominated convergence theorem,

$$\lim_{R\to\infty} \mathbb{E}\big([S(T) + \bar{S}(T)]I_{\{\theta_R \le T\}}\big) = 0.$$

Given any $\varepsilon > 0$, we can then find a sufficiently large R for

$$\mathbb{E}\big([S(T) + \bar{S}(T)]I_{\{\theta_R \le T\}}\big) < \tfrac{1}{2}\varepsilon.$$

For this R, by Lemma 3.3, we can find a $\delta_1 > 0$ sufficiently small such that if $\tau - \bar{\tau} < \delta_1$,

$$\mathbb{E}|S(T \wedge \theta_R) - \bar{S}(T \wedge \theta_R)| \le \tfrac{1}{2}\varepsilon.$$

As a result,

$$\mathbb{E}|S(T) - \bar{S}(T)| < \varepsilon$$

whenever $\tau - \bar{\tau} < \delta_1$. This means

$$\lim_{\tau-\bar{\tau}\to 0} \mathbb{E}|S(T) - \bar{S}(T)| = 0$$

and the desired assertion (3.12) follows. □

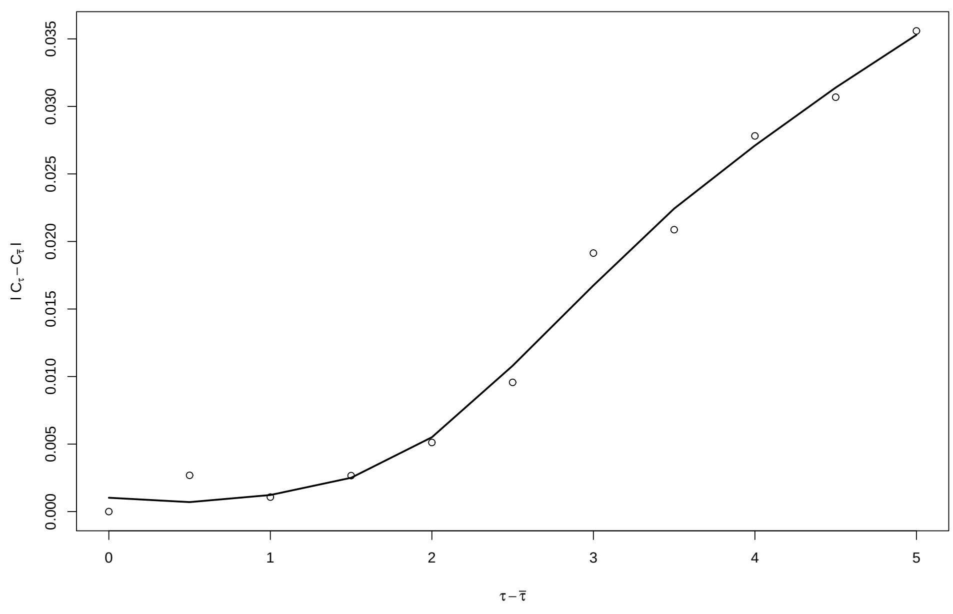

Figure 3. Call: Implied volatility surfaces of a 1-week fixed delay.

Figure 3 [1] provides a graphical representation of the sensitivity of a European

[1]This figure is courtesy of Nairn McWilliams

call option as $\tau - \bar{\tau}$ tends to 0. Real world data (Tesco share prices) are used under a model with 1-week fixed delay. The strike price has been set as 95% of the initial value ($S(0) = 401.15$).

In [3], a closed form expression is produced for pricing European call options. One could in principle prove Theorem 3.4 by a more direct argument using the aforementioned formula. However, we decided to use the above methodology in order to be consistent with the approach taken in proofs of theorems that follow.

Although Theorem 3.4 shows that C_τ and $C_{\bar{\tau}}$ are close to each other when $\tau - \bar{\tau}$ is sufficiently small, it does not give an explicit estimate on the difference (or the error) in terms of $\tau - \bar{\tau}$ etc. From this point of view, the following theorem is more useful in practice.

Theorem 3.5 : *Let Assumptions 2.3 and 3.1 hold. Let R be any sufficiently large number such that $R > \|\xi\|$. Then, with the definitions of (3.1) and (3.3), we have*

$$|C_\tau - C_{\bar{\tau}}| \le e^{-rT}\Big(c_R T e^{c_R T}(\delta(\tau - \bar{\tau}) + \sqrt{\tau - \bar{\tau}}) + \frac{2\xi^{1.5}(0)e^{[1.5r+0.5K^2]T}}{\sqrt{R}}\Big), \quad (3.15)$$

where c_R is the number described in Lemma 3.3.

Proof: Using the notations as before, we compute, for any $\bar{\varepsilon} > 0$,

$$\mathbb{E}\big([S(T) + \bar{S}(T)]I_{\{\theta_R \le T\}}\big) \le \bar{\varepsilon}\,\mathbb{E}[S(T) + \bar{S}(T)]^2 + \frac{1}{4\bar{\varepsilon}}\,\mathbb{P}(\theta_R \le T).$$

Hence, by Theorem 2.4 and (3.14),

$$\mathbb{E}\big([S(T) + \bar{S}(T)]I_{\{\theta_R \le T\}}\big) \le 2\bar{\varepsilon}\xi^2(0)e^{[2r+K^2]T} + \frac{\xi(0)e^{rT}}{2\bar{\varepsilon}R}.$$

Choosing $\bar{\varepsilon} = e^{-0.5(r+K^2)T}/(2\sqrt{R\xi(0)})$ yields

$$\mathbb{E}\big([S(T) + \bar{S}(T)]I_{\{\theta_R \le T\}}\big) \le \frac{2\xi^{1.5}(0)e^{[1.5r+0.5K^2]T}}{\sqrt{R}}.$$

Substituting this and (3.5) into (3.13), we get

$$\mathbb{E}|S(T) - \bar{S}(T)| \le c_R T e^{c_R T}(\delta(\tau - \bar{\tau}) + \sqrt{\tau - \bar{\tau}}) + \frac{2\xi^{1.5}(0)e^{[1.5r+0.5K^2]T}}{\sqrt{R}}. \quad (3.16)$$

This, together with (3.4), gives the required assertion (3.15). □

In practice, for any given $\varepsilon > 0$, one can choose R large enough for

$$\frac{2\xi^{1.5}(0)e^{[0.5r+0.5K^2]T}}{\sqrt{R}} < 0.5\varepsilon,$$

and then further choose $\tau - \bar{\tau}$ sufficiently small for

$$c_R T e^{(c_R - r)T}(\delta(\tau - \bar{\tau}) + \sqrt{\tau - \bar{\tau}}) < 0.5\varepsilon$$

to give $|C_\tau - C_{\bar{\tau}}| < \varepsilon$.

In Theorem 3.5 we impose Assumption 2.3, i.e. the boundedness of the volatility function. This may be restrictive sometimes. However, for the European put options, we can still control the error explicitly without the boundedness of the volatility function. Let us assume that one holds a European put option at $t = 0$ on the underlying asset price with the exercise price E at the expiry date T. According to equation (2.3) or (3.2) that the underlying asset price follows, the price of the European put option at $t = 0$ is

$$P_\tau = e^{-rT}\mathbb{E}(E - S(T))^+ \quad \text{or} \quad P_{\bar\tau} = e^{-rT}\mathbb{E}(E - \bar S(T))^+, \tag{3.17}$$

respectively.

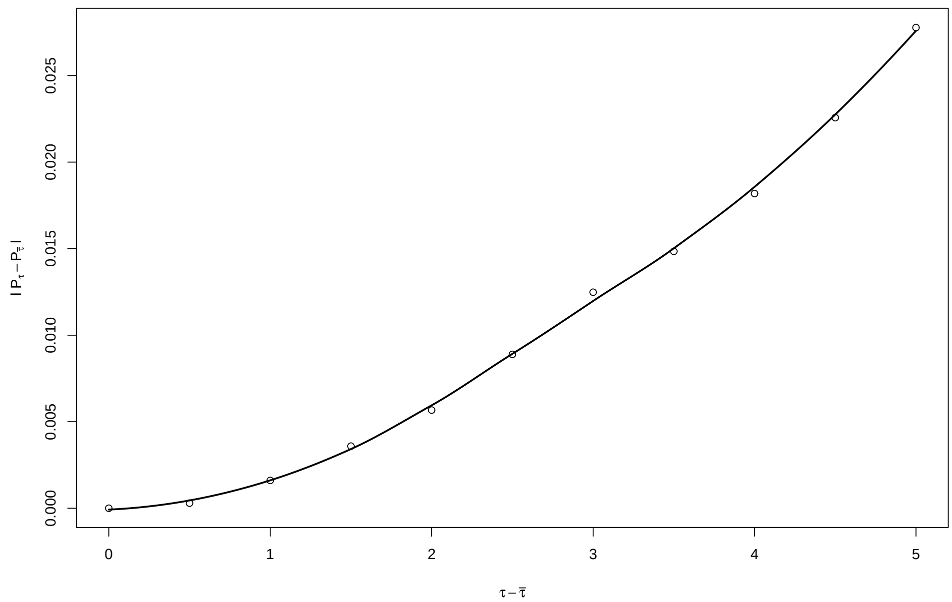

Figure 4. Put: Implied volatility surfaces of a 1-week fixed delay.

Figure 4 [1] provides a graphical representation of the sensitivity of a European put option as $\tau - \bar\tau$ tends to 0 using the same parameters as in Figure 3.

Theorem 3.6 : *Let Assumption 3.1 hold and let R be any sufficiently large number such that $R > \|\xi\|$. Then, with the definitions of (3.17), we have*

$$|P_\tau - P_{\bar\tau}| \le c_R T e^{(c_R - r)T}(\delta(\tau - \bar\tau) + \sqrt{\tau - \bar\tau}) + \frac{2E\xi(0)}{R}, \tag{3.18}$$

where c_R is the number described in Lemma 3.3. In particular, we have

$$\lim_{\tau - \bar\tau \to 0} |P_\tau - P_{\bar\tau}| = 0. \tag{3.19}$$

[1]This figure is courtesy of Nairn McWilliams

Proof: We still use the same notation as before. Compute

$$
\begin{aligned}
|P_\tau - P_{\bar\tau}| &\le e^{-rT}\mathbb{E}|(E - S(T))^+ - (E - \bar S(T))^+| \\
&\le e^{-rT}\Big[\mathbb{E}\big(|(E - S(T))^+ - (E - \bar S(T))^+|\, I_{\{\theta_R > T\}}\big) \\
&\qquad + \mathbb{E}\big(|(E - S(T))^+ - (E - \bar S(T))^+|\, I_{\{\theta_R \le T\}}\big)\Big] \\
&\le e^{-rT}\Big[\mathbb{E}\big(|S(T) - \bar S(T)|\, I_{\{\theta_R > T\}}\big) + E\,\mathbb{P}(\theta_R \le T)\Big] \\
&\le e^{-rT}\Big[\mathbb{E}|S(T\wedge\theta_T) - \bar S(T\wedge\theta_R)| + E\big(\mathbb{P}(\rho_R \le T) + \mathbb{P}(\bar\rho_R \le T)\big)\Big] \qquad (3.20)
\end{aligned}
$$

By Lemma 3.3 and (3.14), we obtain assertion (3.18). Letting $\tau - \bar\tau \to 0$ in (3.18) yields

$$\limsup_{\tau-\bar\tau\to 0} |P_\tau - P_{\bar\tau}| \le \frac{2E\xi(0)}{R}.$$

As this holds for any sufficiently large R, assertion (3.19) must hold. □

4. Delay Effect on Lookback Options

Let us proceed now with the case where we examine the impact of the delay effect on lookback options. If $S(t)$ is governed by equation (2.3), then the payoff of such an option is given by $S(T) - \min_{0\le t\le T} S(t)$ and thus its price, which is denoted by L_τ, is derived by calculating the corresponding discounted expected payoff

$$L_\tau = e^{-rT}\mathbb{E}\Big(S(T) - \min_{0\le t\le T} S(t)\Big). \qquad (4.1)$$

Similarly, if the asset price follows the DGBM (3.2), then the price of the lookback option is given by

$$L_{\bar\tau} = e^{-rT}\mathbb{E}\Big(\bar S(T) - \min_{0\le t\le T} \bar S(t)\Big). \qquad (4.2)$$

Theorem 4.1: *Under Assumption 3.1, with the definitions of (4.1) and (4.2), we have*

$$\lim_{\tau-\bar\tau\to 0} |L_\tau - L_{\bar\tau}| = 0. \qquad (4.3)$$

Proof: It is easy to see that

$$|L_\tau - L_{\bar\tau}| \le e^{-rT}\Big(\mathbb{E}|S(T) - \bar S(T)| + \mathbb{E}\big|\min_{0\le t\le T} S(t) - \min_{0\le t\le T} \bar S(t)\big|\Big).$$

But from the proof of Theorem 3.4 we know that

$$\lim_{\tau-\bar\tau\to 0} \mathbb{E}|S(T) - \bar S(T)| = 0.$$

We therefore only need to show

$$\lim_{\tau-\bar\tau\to 0}\mathbb{E}\big|\min_{0\le t\le T}S(t)-\min_{0\le t\le T}\bar S(t)\big|=0 \tag{4.4}$$

in order to complete the proof. Let R be sufficiently large and θ_R be the stopping time as defined in Lemma 3.3. Write

$$\begin{aligned}\mathbb{E}\big|\min_{0\le t\le T}S(t)-\min_{0\le t\le T}\bar S(t)\big| &= \mathbb{E}\Big(\big|\min_{0\le t\le T}S(t)-\min_{0\le t\le T}\bar S(t)\big|\,I_{\{\theta_R>T\}}\Big)\\ &\quad+\mathbb{E}\Big(\big|\min_{0\le t\le T}S(t)-\min_{0\le t\le T}\bar S(t)\big|\,I_{\{\theta_R\le T\}}\Big). \end{aligned}\tag{4.5}$$

Noting that we always have

$$\big|\min_{0\le t\le T}S(t)-\min_{0\le t\le T}\bar S(t)\big|\le \xi(0),$$

we compute, by (3.14),

$$\mathbb{E}\Big(\big|\min_{0\le t\le T}S(t)-\min_{0\le t\le T}\bar S(t)\big|\,I_{\{\theta_R\le T\}}\Big)\le \xi(0)\mathbb{P}(\theta_R\le T)\le \frac{2\xi^2(0)e^{rT}}{R}. \tag{4.6}$$

On the other hand,

$$\mathbb{E}\Big(\big|\min_{0\le t\le T}S(t)-\min_{0\le t\le T}\bar S(t)\big|\,I_{\{\theta_R>T\}}\Big)\le \mathbb{E}\big|\min_{0\le t\le T}S(t\wedge\theta_R)-\min_{0\le t\le T}\bar S(t\wedge\theta_R)\big|. \tag{4.7}$$

Note that for any $t\in[0,T]$,

$$\begin{aligned}S(t\wedge\theta_R)&=S(t\wedge\theta_R)-\bar S(t\wedge\theta_R)+\bar S(t\wedge\theta_R)\\ &\le \max_{0\le t\le T}|S(t\wedge\theta_R)-\bar S(t\wedge\theta_R)|+\bar S(t\wedge\theta_R),\end{aligned}$$

whence

$$\min_{0\le t\le T}S(t\wedge\theta_R)\le \max_{0\le t\le T}|S(t\wedge\theta_R)-\bar S(t\wedge\theta_R)|+\min_{0\le t\le T}\bar S(t\wedge\theta_R).$$

This yields

$$\min_{0\le t\le T}S(t\wedge\theta_R)-\min_{0\le t\le T}\bar S(t\wedge\theta_R)\le \max_{0\le t\le T}|S(t\wedge\theta_R)-\bar S(t\wedge\theta_R)|.$$

Similarly,

$$\min_{0\le t\le T}\bar S(t\wedge\theta_R)-\min_{0\le t\le T}S(t\wedge\theta_R)\le \max_{0\le t\le T}|S(t\wedge\theta_R)-\bar S(t\wedge\theta_R)|.$$

We hence have

$$|\min_{0\le t\le T}S(t\wedge\theta_R)-\min_{0\le t\le T}\bar S(t\wedge\theta_R)|\le \max_{0\le t\le T}|S(t\wedge\theta_R)-\bar S(t\wedge\theta_R)|.$$

Substituting this into (4.5), and using Lemma 3.3, then gives

$$\mathbb{E}\Big(\big|\min_{0\le t\le T}S(t)-\min_{0\le t\le T}\bar S(t)\big|\,I_{\{\theta_R>T\}}\Big)\le c_RTe^{c_RT}(\delta(\tau-\bar\tau)+\sqrt{\tau-\bar\tau}). \tag{4.8}$$

Putting (4.6) and (4.8) into (4.5) yields

$$\mathbb{E}\big|\min_{0\le t\le T} S(t) - \min_{0\le t\le T} \bar{S}(t)\big| \le c_R T e^{c_R T}(\delta(\tau-\bar{\tau}) + \sqrt{\tau-\bar{\tau}}) + \frac{2\xi^2(0)e^{rT}}{R}.$$

Letting $\tau - \bar{\tau} \to 0$ and then $R \to \infty$ we obtain (4.4) as required. $\square$

5. Delay Effect on Barrier Options

Let us now consider a barrier option under the DGBM (2.3). That is, consider an up-and-out call option, which, at expiry time T, pays the European value with the exercise price E if $S(t)$ never exceeded a given fixed barrier, B, and pays zero otherwise. Hence, the expected payoff at expiry time T is

$$\mathbb{E}\Big((S(T)-E)^+ I_{\{0\le S(t)\le B,\ 0\le t\le T\}}\Big).$$

Accordingly, the price of the barrier option at $t=0$ is

$$B_\tau = e^{-rT}\,\mathbb{E}\Big((S(T)-E)^+ I_{\{0\le S(t)\le B,\ 0\le t\le T\}}\Big). \tag{5.1}$$

Alternatively, if the asset price obeys the DGBM (3.2), the option price is

$$B_{\bar{\tau}} = e^{-rT}\,\mathbb{E}\Big((\bar{S}(T)-E)^+ I_{\{0\le \bar{S}(t)\le B,\ 0\le t\le T\}}\Big). \tag{5.2}$$

Theorem 5.1: *Under Assumption 3.1, with the definitions of (5.1) and (5.2), we have*

$$\lim_{\tau-\bar{\tau}\to 0} |B_\tau - B_{\bar{\tau}}| = 0. \tag{5.3}$$

Proof: Let $A := \{0\le S(t)\le B,\ 0\le t\le T\}$ and $\bar{A} := \{0\le \bar{S}(t)\le B,\ 0\le t\le T\}$. Making use of the inequality

$$|(S(T)-E)^+ - (\bar{S}(T)-E)^+| \le |S(T)-\bar{S}(T)|,$$

we have

$$\begin{aligned}
|B_\tau - B_{\bar{\tau}}| &\le e^{-rT}\mathbb{E}\left|(S(T)-E)^+ I_A - (\bar{S}(T)-E)^+ I_{\bar{A}}\right| \\
&\le e^{-rT}\Big[\mathbb{E}\left(\left|(S(T)-E)^+ - (\bar{S}(T)-E)^+\right| I_{A\cap\bar{A}}\right) \\
&\qquad + \mathbb{E}\left((S(T)-E)^+ I_{A\cap\bar{A}^c}\right) + \mathbb{E}\left((\bar{S}(T)-E)^+ I_{A^c\cap\bar{A}}\right)\Big] \\
&\le e^{-rT}\Big[\mathbb{E}\left(|S(T)-\bar{S}(T)| I_{A\cap\bar{A}}\right) + (B-E)\mathbb{P}(A\cap\bar{A}^c) \\
&\qquad + (B-E)\mathbb{P}(A^c\cap\bar{A})\Big].
\end{aligned}$$

Choose any $R \ge B \vee \|\xi\|$ and let θ_R be the stopping time as defined in Lemma 3.3.

Then

$$\mathbb{E}\big(|S(T)-\bar{S}(T)|I_{A\cap\bar{A}}\big) \le \mathbb{E}\big(|S(T\wedge\theta_R)-\bar{S}(T\wedge\theta_R)|I_{A\cap\bar{A}}\big) \le \mathbb{E}|S(T\wedge\theta_R)-\bar{S}(T\wedge\theta_R)|.$$

From Lemma 3.3, we have

$$\lim_{\tau-\bar{\tau}\to 0}\mathbb{E}\big(|S(T)-\bar{S}(T)|I_{A\cap\bar{A}}\big)=0.$$

Hence, our proof is complete if we can show that

$$\lim_{\tau-\bar{\tau}\to 0}\mathbb{P}(A\cap\bar{A}^c)=0 \tag{5.4}$$

and

$$\lim_{\tau-\bar{\tau}\to 0}\mathbb{P}(A^c\cap\bar{A})=0. \tag{5.5}$$

For any sufficiently small h, we have

$$\begin{aligned} A &= \{\sup_{0\le t\le T} S(t)\le B\} \\ &= \{\sup_{0\le t\le T} S(t)\le B-h\}\cup\{B-h<\sup_{0\le t\le T} S(t)\le B\} \\ &=: A_1\cup A_2. \end{aligned}$$

Hence,

$$A\cap\bar{A}^c=(A_1\cap\bar{A}^c)\cup(A_2\cap\bar{A}^c)\subseteq\Big\{\sup_{0\le t\le T}|S(t)-\bar{S}(t)|\ge h\Big\}\cup A_2.$$

So,

$$\mathbb{P}(A\cap\bar{A}^c)\le\mathbb{P}\big(\sup_{0\le t\le T}|S(t)-\bar{S}(t)|\ge h\big)+\mathbb{P}(A_2).$$

Now, for any $\varepsilon>0$, we may choose h so small that

$$\mathbb{P}(A_2)<\frac{\varepsilon}{3}.$$

Moreover,

$$\mathbb{P}\big(\sup_{0\le t\le T}|S(t)-\bar{S}(t)|\ge h\big)\le\mathbb{P}\big(\sup_{0\le t\le T}|S(t)-\bar{S}(t)|\ge h,\ \theta_R>T\big)+\mathbb{P}(\theta_R\le T).$$

By (3.14), we can choose R sufficiently large for

$$\mathbb{P}(\theta_R\le T)\le\frac{2\xi(0)e^{rT}}{R}<\frac{\varepsilon}{3}.$$

For this chosen R,

$$\mathbb{P}\big(\sup_{0\le t\le T}|S(t)-\bar{S}(t)|\ge h,\ \theta_R>T\big)\le \mathbb{P}\big(\sup_{0\le t\le T}|S(t\wedge\theta_R)-\bar{S}(t\wedge\theta_R)|\ge h\big)$$
$$\le \frac{1}{h}\,\mathbb{E}\big(\sup_{0\le t\le T} S(t\wedge\theta_R)-\bar{S}(t\wedge\theta_R)|\big).$$

By Lemma 3.3, we can choose $\tau-\bar{\tau}$ sufficiently small for

$$\mathbb{P}\big(\sup_{0\le t\le T}|S(t)-\bar{S}(t)|\ge h,\ \theta_R>T\big)<\frac{\varepsilon}{3}.$$

Summarizing the above arguments, we see that $\mathbb{P}(A\cup\bar{A}^c)<\varepsilon$ whenever $\tau-\bar{\tau}$ is sufficiently small. This confirms (5.4). Similarly, we can show (5.5). The proof is therefore complete. □

All the above findings, where the robustness of the delay effect on various option valuations under certain conditions is established, provide additional support on the suitability of the DGBM approach. The next section tackles the problem of pricing contingent claims under this approach in the absence of explicit pricing formulae.

6. Euler–Maruyama Approximation

Theorem 2.1 states that the DGBM $S(t)$ can be computed explicitly step by step, since conditionally (on time-lagged information) one obtains a lognormally distributed random variable for each t. Nevertheless, the (unconditional) probability distribution of $S(t)$ is not known when $t>\tau$. It is therefore difficult to compute even simple vanilla options, not to mention more complicated path-dependent options such as lookback and barrier options.

An Euler–Maruyama numerical scheme is a well-established method to overcome the aforementioned problem. There are numerous examples in the literature where authors discretize SDDEs, typically with an Euler-type scheme (see e.g. [20–22]) so as to study their properties. Moreover, one can observe immediately that such an approach will allow the use of Monte Carlo simulations so as to compute the expected value of either a function of $S(T)$ or a functional of $\{S(t):0\le t\le T\}$, and thus obtain the expected payoff of an option (see e.g. [8, 11, 27]).

In order to define the EM approximate solution to the DGBM (2.3), let us first extend the definition of the volatility function V from $\mathbb{R}_+$ to the whole $\mathbb{R}$ by setting $V(x)=V(0)$ for $x<0$. The advantages of such an extension are obvious. It does not have any effect on the solution of the DGBM (2.3), since the solution is always positive, and it also preserves the local Lipschitz continuity (resp. boundedness of the volatility function) if Assumption 2.3 (resp. 3.1) holds. More importantly, it enable us to define the EM approximate solution to the DGBM (2.3) following the author's earlier paper [21]. Let the time-step size $\Delta t\in(0,1)$ be a fraction of τ, that is $\Delta t=\tau/N$ for some sufficiently large integer N. The discrete EM approximate solution is defined as follows: Set $\overline{s}_k=\xi(k\Delta t)$ for $k=-N,-(N-1),\cdots,1,0$ and form

$$\overline{s}_k=\overline{s}_{k-1}[1+r\Delta t+V(\overline{s}_{k-1-N})\Delta W_k],\quad k=1,2,\cdots, \tag{6.1}$$

where $\Delta W_k=W(k\Delta t)-W((k-1)\Delta t)$. To define the continuous extension, we introduce the step process

$$\bar{s}(t) = \sum_{k=-N}^{\infty} \bar{s}_k \, I_{[k\Delta t,(k+1)\Delta t)}(t), \quad t \in [-\tau, \infty). \tag{6.2}$$

The continuous EM approximate solution is then defined by setting $s(t) = \xi(t)$ for $t \in [-\tau, 0]$ and forming

$$s(t) = \xi(0) + \int_0^t r\bar{s}(u)du + \int_0^t V(\bar{s}(u-\tau))\bar{s}(u)dW(u), \quad t \geq 0. \tag{6.3}$$

It is easy to see that $s(k\Delta t) = \bar{s}(k\Delta t) = \bar{s}_k$ for all $k = -N, -(N-1), \cdots$. To use the strong convergence result established in [21], let us impose one more condition.

Assumption 6.1 The initial data ξ is Hölder continuous with order $\gamma \in (0, \frac{1}{2}]$, that is

$$\sup_{-\tau \leq u < v \leq 0} \frac{|\xi(v) - \xi(u)|}{(v-u)^\gamma} < \infty.$$

The following theorem follows directly from [21, Theorem 2.1 and Lemma 3.2].

Theorem 6.2: *Under Assumptions 2.3, 3.1 and 6.1, the continuous approximate solution (6.3) will converge to the true solution of the DGBM (2.3) in the sense*

$$\lim_{\Delta t \to 0} \mathbb{E}\Big(\sup_{0 \leq t \leq T} |S(t) - s(t)|^2 \Big) = 0, \quad \forall T \geq 0. \tag{6.4}$$

Moreover, the step process (6.2) and the continuous approximate solution (6.3) obey

$$\lim_{\Delta t \to 0} \Big(\sup_{0 \leq t \leq T} \mathbb{E}|\bar{s}(t) - s(t)|^2 \Big) = 0, \quad \forall T \geq 0. \tag{6.5}$$

Proof: The result can be obtained directly from [21] (and is omitted here so as to save some pages) □

Based on the strong convergence properties described in this theorem, we can show that the expected payoff from the numerical method converges to the correct expected payoff as $\Delta t \to 0$ for various options. For example, for a European call option, it is straightforward to show

$$\lim_{\Delta t \to 0} |\mathbb{E}(S(T) - E)^+ - \mathbb{E}(\bar{s}(T) - E)^+| = 0.$$

Note that using the step function $\bar{s}(T)$ in the above is equivalent to using the discrete solution (6.1). Hence, for a sufficiently small Δt, $e^{-rT}\mathbb{E}(\bar{s}(T) - E)^+$ gives a nice approximation to the European call option price $e^{-rT}\mathbb{E}(S(T) - E)^+$.

Another example (where also the aforementioned numerical method can be applied) is the case of an up-and-out call option. Let us recall that this option has the same payoff, at expiry T, as its European counterpart if $S(t)$ does not exceed the fixed barrier B, for any $t \in [0, T]$, and pays zero otherwise.

Suppose that the expected payoff of an up-and-out call option is computed via a Monte Carlo simulation based on the method (6.2). Then, one obtains the following result:

Theorem 6.3 : *For the DGBM (2.3) and numerical method (6.2), define*

$$\Gamma := \mathbb{E}\left[(S(T)-E)^{+}\mathbf{1}_{\{0\le S(t)\le B,\ 0\le t\le T\}}\right], \tag{6.6}$$

$$\bar{\Gamma}_{\Delta t} := \mathbb{E}\left[(\overline{s}(T)-E)^{+}\mathbf{1}_{\{0\le \overline{s}(t)\le B,\ 0\le t\le T\}}\right], \tag{6.7}$$

where the exercise price, E, and barrier, B, are constant. If Assumptions 2.3, 3.1 and 6.1 hold, then

$$\lim_{\Delta t\to 0}|\Gamma-\bar{\Gamma}_{\Delta t}|=0.$$

This theorem can be proved in the same way as [11, Theorem 5.1] was proved. It is possible to remove Assumptions 2.3 and 6.1 but the proof is very technical so we will present it elsewhere due to the page limit here.

7. Summary

The DGBM market model, which is described by an SDDE, is examined in this paper as an alternative approach to modelling the evolution of asset prices. A number of points are presented that demonstrate the suitability of the aforementioned approach:

- The DGBM (2.3) has a unique positive solution with a finite expected value for any $t \geq 0$ that leads to a complete market framework. As a result, one could choose from a wide class of continuous volatility functions so as to fit empirical data to equation (2.3) and price various contingent claims.
- All the results from Sections 3 to 5 hold under the mild assumption that the volatility function V is locally Lipschitz continuous, except of course from Theorem 3.5 where the boundedness of the volatility function is also needed. These results reveal that the time-delay effect is robust.
- Although the DGBM can be computed explicitly step by step, the same is not true for option pricing formulae under this approach. However, an Euler–Maruyama numerical scheme is presented here that allows the implementation of Monte Carlo simulation techniques so as to closely approximate true option prices.

Finally, although Assumptions 2.3 and 6.1 are imposed for the numerical analysis in Section 6, it is possible to remove these conditions. The reason for having these Assumptions here is to obtain the strong convergence result, which is presented by Theorem 6.2. The proof for the convergence without the above Assumptions is very technical and it will be presented elsewhere.

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