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ANALYTICAL PERTURBATIVE METHOD FOR FROZEN ORBITS AROUND THE ASTEROID 433 EROS

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In this paper a method for obtaining initial condition for frozen orbits around fast rotating, highly irregular bodies is presented. Such method is based on a general perturbative theory of motion, for inhomogeneous gravitational fields. Taking into account the terms of the gravitational potential up to an arbitrary order to construct a precise Hamiltonian formulation of the problem, the system is averaged both over the argument of node and the mean anomaly, to reduce its complexity (i.e. the number of degrees of freedom). An approximate system is obtained, which provides an accurate description of the dynamics of the initial system. This can be applied to every celestial body and in particular, can be exploited for constructing a method for finding initial conditions to yield frozen orbits. These orbits can then be used as reference trajectories in missions that require close inspection of asteroids. To this end applications to derive frozen orbits for Eros 433 have been provided which could be of key interest for every observational, discovery mission around this asteroid.

Key words: Asteroid Dynamics · Frozen Orbits · 433 Eros

I. INTRODUCTION

The motion of bodies subject to inhomogeneous gravitational fields is a classical subject of research in the context of celestial mechanics. In recent years this type of research has become important to future planned missions of spacecraft to the moon and other solar system bodies, in addition to asteroid deflection missions [5]. The analysis of spacecraft motion about these bodies is particularly challenging as they typically feature shapes and density distributions more irregular than those of planets [22]. Such irregularities break symmetries and require more complicated analytical expressions for their description, increasing the complexity involved in such studies.

Numerical methods are today widely used to study the trajectories of objects orbiting specific irregular bodies. These methods can be highly computational and require a complete re-design for each different body. Analytical methods, by contrast, can provide a full dynamical picture of the motion around irregular bodies that can be used to search and study particular classes of useful orbits . However, due to the extensive algebraic manipulations required, these methods are usually limited to the consideration of the sole oblateness and ellipticity terms describing the gravitational potential of the of the body, moreover truncating the analytic transformations used to the first degree [20].

In this paper a method for obtaining initial condition for frozen orbits is presented, based on a closed form (i.e. without using series expansion in the eccentricity), analytical, perturbative theory of motion around inhomogeneous bodies, which accounts all the gravity coefficients up to an arbitrary order. Frozen orbits are orbits with no secular perturbations in the inclination, argument of pericenter, and eccentricity [4]. These orbits are periodic, except for the orbital plane of precession, and are therefore called frozen. The Hamiltonian describing the system is firstly found, which takes into account the terms of the gravitational potential up to an arbitrary order. Focussing on the fast rotation case, the system is then averaged both over the argument of node and the mean anomaly, to reduce the number of degrees of freedom, thus yielding to an approximate system arbitrarily close to the original one. This method thus enables to transform the initial non-integrable Hamiltonian into an integrable one plus a negligible, perturbative remainder. The system obtained provides an accurate description of the dynamics of the initial system with the advantage that it only depends on two variables. It can therefore be applied to every celestial body in particular to find initial conditions for frozen orbits. These orbits can then be used as reference trajectories in missions that require close inspection of asteroids. To this end applications to derive frozen orbits for Eros 433 have been provided which could be of key interest for every observational, discovery mission around this asteroid.

II. METHOD

Assuming that the planetary body is in uniform rotation around its axis of greatest inertia, the potential generated by the inhomogeneous gravitational field can be derived in the rotating polar nodal variables [24] convenient for the necessary transformation to Delaunay coordinates. This potential takes into account an arbitrary number of spherical harmonic coefficients, all considered to have the same order, thus providing a dynamical model based on an arbitrarily accurate model of the inhomogeneous body. The usual technique applied previously in the literature i.e. the Delaunay normalization ([10]), cannot be directly applied to a high-order model due to the presence of the argument of node that appears in the Coriolis term. The addition of this term in the Lie derivative prevents the conventional computation of the Lie transform generator ([20]). For fast rotating bodies the Hamiltonian can thus be averaged with respect to the argument of node and, after the transformation to the Delaunay coordinates, averaged again with respect to the mean anomaly. This leads to an integrable Hamiltonian, dependent only on two variables, which provides an arbitrarily accurate approximation of the initial non integrable system and which can be exploited to find initial conditions for frozen orbits. These results have been derived closed form with respect to both the eccentricity and inclination of the probe, i.e. no power series expansion have been performed; therefore, the theory stands for every arbitrary eccentricity and inclination and is not limited to almost-circular orbits.

III. THE DYNAMICAL SYSTEM

An inhomogeneous body is considered, which rotates uniformly around its axes of greatest inertia with constant angular velocity $\hat{\omega} = [0, 0, \omega]$.
The total mass of the body is M while $\mathcal{G}$ is the universal gravitational constant. The dynamics is formulated into a reference frame centered in the center of mass of the body and oriented with the "z-axis" parallel to the rotational axes of the asteroid. The frame of reference is rotating with the same velocity of rotation of the asteroid; in such rotating coordinates the Hamiltonian describing the system is:

$$H(\mathbf{x}, \mathbf{X}) = \tfrac{1}{2}(\mathbf{X} \cdot \mathbf{X}) - \hat{\omega}(\mathbf{x} \times \mathbf{X}) + \bar{U}(\mathbf{x}) \qquad (1)$$

where $\mathbf{x},\ \mathbf{X} \in \mathbb{R}^3$ are respectively the position coordinates and conjugate momenta of the spacecraft, while $\bar{U}(\mathbf{x})$ is the perturbing gravitational potential generated by the inhomogeneous rotating body. The equations of motion are:

$$\begin{cases} \dot{\mathbf{x}} = \frac{\partial}{\partial \mathbf{X}} H(\mathbf{x}, \mathbf{X}) \\ \dot{\mathbf{X}} = -\frac{\partial}{\partial \mathbf{x}} H(\mathbf{x}, \mathbf{X}) \end{cases} \qquad (2)$$

It is convenient to express the Hamiltonian and the perturbing potential using the so called nodal-polar variables so that it may easily be transformed to the Delaunay coordinates in the later stage of the methodology. r, θ, and ν (respectively the angular distance of the spacecraft from the line of the ascending node on the orbital plane) and their respective conjugate momenta R, Θ, and N. The transformation required is given in [17], setting $\mathbf{x} = [x, y, z]^T$ and $\mathbf{X} = [X, Y, Z]^T$:

$$\begin{aligned} x &= |\mathbf{r}|(\cos\theta\cos\nu - \sin\theta\cos I\sin\nu) \\ y &= |\mathbf{r}|(\cos\theta\sin\nu + \sin\theta\cos I\cos\nu) \\ z &= |\mathbf{r}|\sin\theta\sin I \\ X &= (R\cos\theta - \tfrac{\Theta}{|\mathbf{r}|}\sin\theta)\cos\nu \\ &\quad -(R\sin\theta + \tfrac{\Theta}{|\mathbf{r}|}\cos\theta)\cos I\sin\nu \\ Y &= (R\cos\theta - \tfrac{\Theta}{|\mathbf{r}|}\sin\theta)\sin\nu \\ &\quad +(R\sin\theta + \tfrac{\Theta}{|\mathbf{r}|}\cos\theta)\cos I\cos\nu \\ Z &= (R\sin\theta + \tfrac{\Theta}{|\mathbf{r}|}\cos\theta)\sin I \end{aligned} \qquad (3)$$

with $N = |\Theta|\cos I$.
In these coordinates the Hamiltonian takes the form:

$$\begin{aligned} H(r, \theta, \nu, R, \Theta, N) &= \tfrac{1}{2}(R^2 + \tfrac{\Theta^2}{r^2}) - \omega N \\ &\quad + \bar{U}(r, \theta, \nu, R, \Theta, N) \end{aligned} \qquad (4)$$

where the main or unperturbed term H_0 is the sum of the Keplerian term $\frac{1}{2}(R^2 + \frac{\Theta^2}{r^2})$ and the Coriolis term $H_C = -\omega N$. The perturbing potential, found using the addition formula for non scaled spherical harmonics [14] and Wigner's rotation theorem for non scaled spherical harmonics [25], is:

$$\begin{aligned} \bar{U}(r, \theta, \nu) &= -\tfrac{\mathcal{G}M}{r} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{j=-n}^{n} \left(\frac{\alpha}{r}\right)^n \cdot \\ &\quad \cdot \left(\mathcal{G}^1_{n,m,j}(I)\cos(m\nu - j\theta) + \mathcal{G}^2_{n,m,j}(I)\sin(m\nu - j\theta)\right), \end{aligned} \qquad (5)$$

with:

$$\begin{aligned} \mathcal{G}^1_{n,m,j}(I) &= \bar{\mathcal{G}}_{n,m,j}(I)\left(C_{n,m}\cos\left(\tfrac{\pi}{2}(j+m)\right)\right. \\ &\quad \left. -S_{n,m}\sin\left(\tfrac{\pi}{2}(j+m)\right)\right) \\ \mathcal{G}^2_{n,m,j}(I) &= \bar{\mathcal{G}}_{n,m,j}(I)\left(C_{n,m}\sin\left(\tfrac{\pi}{2}(j+m)\right)\right. \\ &\quad \left. +S_{n,m}\cos\left(\tfrac{\pi}{2}(j+m)\right)\right), \end{aligned} \qquad (6)$$

and

$$\begin{aligned} \bar{\mathcal{G}}_{n,m,j}(I) &= \sum_{\ell=\max\{0,j+m\}}^{\min\{n+m,n+j\}} (-1)^{m+3\ell-j} \cdot \\ &\quad \cdot \tfrac{(n+m)!(n-j)!}{\ell!(n+j-\ell)!(n+m-\ell)!(\ell-m-j)} \cdot \\ &\quad \cdot (-1)^{\frac{n+j}{2}} \tfrac{1}{2^n} \tfrac{(n+j)!}{(\frac{n+j}{2})!(\frac{n-j}{2})!} ((n+j)_{mod_2} - 1) \cdot \\ &\quad \cdot \cos\left(\tfrac{I}{2}\right)^{2n+m+j-2\ell} \sin\left(\tfrac{I}{2}\right)^{2\ell-m-j} \end{aligned} \qquad (7)$$

where x_{mod_y} stands for the value of x modulus y, i.e. the integer reminder of the division of x by y.
The $C_{n,m}$ and $S_{n,m}$ in (6) are called spherical harmonic coefficients, defined as, $\forall 0 \leq m \leq n$:

$$\begin{aligned} C_{n,m} &= \tfrac{(2-\delta_{m,0})}{M} \tfrac{(n-m)!}{(n+m)!} \int_V \left(\frac{r'}{\alpha}\right)^n P_{n,m}(\sin\delta') \cdot \\ &\quad \cdot \cos(m\lambda')\rho(r', \delta', \lambda')dV \\ S_{n,m} &= \tfrac{(2-\delta_{m,0})}{M} \tfrac{(n-m)!}{(n+m)!} \int_V \left(\frac{r'}{\alpha}\right)^n P_{n,m}(\sin\delta') \cdot \\ &\quad \cdot \sin(m\lambda')\rho(r', \delta', \lambda')dV \end{aligned} \qquad (8)$$

where α is a conventionally chosen reference radius, usually taken as the radius of the circumscribing sphere of the small body.
Moreover $\delta_{0,m}$ is the Kronecker delta that gives 1 if $m = 0$, and 0 elsewhere, $P_n^m(x)$ is the associated Legendre function of degree n and order m.
Also $\mathbf{r}' \in \mathbb{R}^3$, $\theta' \in \mathbb{R}$ and $\lambda' \in \mathbb{R}$ are respectively the position, latitude and longitude of the infinitesimal volume element dV in a cartesian frame of reference $O_{x,y,z}$, $\rho(\mathbf{r}', \theta', \lambda')$ is the density of the infinitesimal element of volume and V is the volume of the body.
It must be noted that the series (5) converges only if the condition $\frac{r'}{r} < 1$ is satisfied, which implies that the model is valid only outside the reference sphere.

A full explanation of the spherical harmonic coefficients can be found in [14].
However it is important to highlight that, in particular, the equations in (8) imply that:

$$\begin{aligned} &C_{0,0} = 1 \\ &C_{n,0} = \tfrac{1}{M} \int_V \left(\frac{r'}{\alpha}\right)^n P_n(\sin\delta')\rho(r', \delta', \lambda')dV \ \ \forall n > 0 \\ &S_{n,0} = 0 \ \ \forall n \geq 0 \end{aligned} \qquad (9)$$

Moreover, centering the origin of the system of reference at the center of mass it can be demonstrated that the term $C_{1,0} = 0$.
The coefficients $C_{2,0}$ and $C_{2,2}$ express the "ellipticity" and "oblateness" of the body.

IV. AVERAGING THE SYSTEM

For the case $|H_K| < |H_C|$, i.e. the so called fast rotation case. The Hamiltonian in (4) is now firstly averaged over the argument of the node ν in order to "relegate" the action of the polar component of the Angular Momentum, namely to get rid of the angle ν by using a transformation after which the desired perturbation appears (in the new Hamiltonian) with a lower influence. After repeated iterations of the transformation the perturbation should be small enough so that we may neglect it. Then the Delaunay variables are introduced, which lead to the second averaging over the mean anomaly, a so called Delaunay normalization, yielding a reduced ordinary differential equation in two coordinates; the total angular momentum and the argument of pericentre.

Both the averages are Lie transformation, namely one parameter family of mappings $\varphi : (\mathbf{x}', \mathbf{X}', \epsilon) \to (\mathbf{x}, \mathbf{X})$ defined by the solution $\mathbf{x}(\mathbf{x}', \mathbf{X}', \epsilon)$, $\mathbf{X}(\mathbf{x}', \mathbf{X}', \epsilon)$ of the hamiltonian system

$$\begin{array}{l} \frac{dx}{d\epsilon} = \frac{\partial W}{\partial X} \\ \frac{dX}{d\epsilon} = -\frac{\partial W}{\partial x} \end{array} \tag{10}$$

such that $\mathbf{x}(\mathbf{x}', \mathbf{X}', 0) = x'$, $\mathbf{X}(\mathbf{x}', \mathbf{X}', 0) = X'$ which, due to the properties of the Hamiltonian systems, provide a new set of completely canonical variables generated by the transformation $W(\mathbf{x}, \mathbf{X}; \epsilon) = \sum_{s\geq 0} \frac{\epsilon^s}{s!} W_{s+1}(\mathbf{x}, \mathbf{X})$ that maps the Hamiltonian $H(\mathbf{x}, \mathbf{X}; \epsilon) = \sum_{s\geq 0} \frac{\epsilon^s}{s!} H_s(\mathbf{x}, \mathbf{X})$ into an integrable one $K(\mathbf{x}', \mathbf{X}'; \epsilon) = \sum_{s\geq 0} \frac{\epsilon^s}{s!} K_s(\mathbf{x}', \mathbf{X}')$ plus a negligible reminder $R(\mathbf{x}', \mathbf{X}'; \epsilon)$ of smaller influence.

Following Deprit's method for the Lie transformations [7] both the transformations are carried on by solving a series of Homologic equations[1]:

$$[H_0; W_s] + \tilde{H}_s = K_s \quad \forall s \geq 1 \tag{11}$$

where the terms $\tilde{H}_s$ are found by:

$$\begin{array}{l} \tilde{H}_1 := H_1 \\ \tilde{H}_s = \tilde{H}_{s-1,1} + [\tilde{H}_{s-1,0} + [H_0; W_{s-1}]; W_1] \;\; \forall s \geq 2 \end{array} \tag{12}$$

with

$$\begin{cases} \tilde{H}_{1,0} := \tilde{H}_1 := H_1 \\ \tilde{H}_{i,s-i} = H_s + \sum_{j=0}^{s-2} \binom{s-i}{j} [H_{s-j-i}; W_{j+1}] \\ if \;\; i = 1, \; s \geq 2 \\ \tilde{H}_{i,s-i} = \tilde{H}_{i-1,s-i+1} + \sum_{j=0}^{s-i} \binom{s-i}{j} \Big[\tilde{H}_{i-1,s-j-i} \\ \qquad +[H_0; W_{s-j-1}]; W_{j+1} \Big] \\ if \;\; i \neq 1, \; s \geq 2 \end{cases} \tag{13}$$

IV.I RELEGATION OF THE ARGUMENT OF NODE

As suggested in [20], we now want to apply the Relegation algorithm to the Hamiltonian (4) with the potential truncated to the terms of order $\sim \frac{1}{r^{n_{max}+1}}$ (included), therefore we set:

$$\begin{array}{l} H_K := \frac{1}{2}(R^2 + \frac{\Theta^2}{r^2}) - \frac{M\mathcal{G}}{r} \\ G = -\omega N, \end{array} \tag{14}$$

obtaining that: $H = H_0 + \sum_{s=0}^{\infty} \frac{\epsilon^s}{s!} H_s$ with

$$\begin{array}{l} H_0 = H_K + G \\ H_1 = U_{nmax} \\ H_j = 0 \;\; \forall j \geq 2 \end{array} \tag{15}$$

and

$$U_{nmax} := -\frac{\mathcal{G}M}{r\epsilon} \sum_{n=1}^{n_{max}} \left(\frac{\alpha}{r}\right)^n \sum_{m=0}^{n} \sum_{j=-n}^{n} \big(\mathcal{G}^1_{n,m,j}(I) \cdot \\ \cdot \cos(m\nu - j\theta) + +\mathcal{G}^2_{n,m,j}(I) \sin(m\nu - j\theta)\big) \tag{16}$$

and $\mathcal{G}^1_{n,m,j}(I)$, $\mathcal{G}^2_{n,m,j}(I)$ as in (6). In counter analogy with [18] parameter ϵ is set to be equal to $\frac{\mu^2}{\omega L_0^3}$.

It is first noticed that

$$[\,\cdot\,; G] = [\,\cdot\,; -\omega N] = -\omega \frac{\partial \cdot}{\partial \nu} \tag{17}$$

and that

$$\begin{array}{l} [H_K;\,\cdot\,] = [\frac{1}{2}(R^2 + \frac{\Theta^2}{r^2}) - \frac{M\mathcal{G}}{r};\,\cdot\,] \\ \qquad = R\frac{\partial \cdot}{\partial r} + \frac{\Theta}{r^2}\frac{\partial \cdot}{\partial \theta} - (\frac{\Theta^2}{r^3} - \frac{M\mathcal{G}}{r^2})\frac{\partial \cdot}{\partial R} \end{array} \tag{18}$$

The first homologic equation is set recalling that, by (12), $\tilde{H}_1 = H_1$.
Therefore:

$$[H_0; W_1] + U_{nmax} = K_1 \tag{19}$$

* * * * *

First Relegation iteration:

The first iteration consists in finding $K_{1,0}$ and $W_{1,0}$ such that

$$\begin{cases} U_{nmax} = K_{1,0} + [W_{1,0}; G] \\ K_{1,0} \in Ker(L_G). \end{cases} \tag{20}$$

That is

$$\begin{cases} U_{nmax} = K_{1,0} - \omega \frac{\partial W_{1,0}}{\partial \nu} \\ \frac{\partial K_{1,0}}{\partial \nu} = 0. \end{cases} \tag{21}$$

Thus $K_{1,0}$ is the collection of all the terms of U_{nmax} such that their derivative with respect to ν is zero, namely the collection of all the terms of U_{nmax} which does not depend on ν, therefore:

$$K_{1,0} = -\frac{\mathcal{G}M}{r\epsilon} \sum_{n=1}^{n_{max}} \sum_{j=-n}^{n} \left(\frac{\alpha}{r}\right)^n \big(\mathcal{G}^1_{n,0,j}(I)\cos(-j\theta) \\ +\mathcal{G}^2_{n,0,j}(I)\sin(-j\theta)\big) \tag{22}$$

so that $K_{1,0}$ is the averaging of $\tilde{H}_{1,0}$ over tha angle ν. Then, inverting (21):

$$\begin{array}{l} W_{1,0} = -\frac{1}{\omega}\int (U_{nmax} - K_{1,0}) d\nu \\ \quad = -\frac{1}{\omega}\int \big(-\frac{\mathcal{G}M}{r\epsilon} \sum_{n=1}^{n_{max}} \sum_{m=1}^{n} \sum_{j=-n}^{n} \left(\frac{\alpha}{r}\right)^n \cdot \\ \quad \cdot \big(\mathcal{G}^1_{n,m,j}(I)\cos(m\nu - j\theta) + \\ \quad +\mathcal{G}^2_{n,m,j}(I)\sin(m\nu - j\theta)\big)\big) d\nu \end{array} \tag{23}$$

which is periodic in ν.

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Second Relegation iteration:

By (12) we first evaluate

$$\begin{array}{l} [H_0 - G; W_{1,0}] = R\frac{\partial W_{1,0}}{\partial r} + \frac{\Theta}{r^2}\frac{\partial W_{1,0}}{\partial \theta} \\ \quad -(\frac{\Theta^2}{r^3} - \frac{M\mathcal{G}}{r^2})\frac{\partial W_{1,0}}{\partial R} \end{array} \tag{24}$$

with $W_{1,0}$ is as in (23).
Notice that $[H_0 - G; W_{1,0}]$ is still ν-periodic.

[1]Let A be an open subset of $C^n \times C^n$. If the mappings $f(X, x)$ and $g(X, x)$ from A to C are differentiable in A, the Poisson bracket of f and g ($[f; g]$), in that order, is the mapping from A to C ($[f; g] : A \to C$) that maps $(X, x) \to D_2 f(X, x) \cdot D_1 g(X, x) - D_1 f(X, x) \cdot D_2 g(X, x)$.

Now we have to find $K_{1,1}$ and $W_{1,1}$ such that:

$$\begin{cases} [H_0 - G; W_{1,0}] = K_{1,1} + [W_{1,1}; G] \\ K_{1,1} \in Ker(\mathcal{L}_G) \end{cases} \tag{25}$$

That is

$$\begin{cases} [H_0 - G; W_{1,0}] = K_{1,1} - \omega \frac{\partial W_{1,1}}{\partial \nu} \\ \frac{\partial K_{1,1}}{\partial \nu} = 0 \end{cases} \tag{26}$$

Thus $K_{1,1}$ is the collection of all the terms of $[H_0 - G; W_{1,0}]$ such that their derivative with respect to ν is zero, namely the collection of all the terms of $[H_0 - G; W_{1,0}]$ which does not depend on ν, but $[H_0 - G; W_{1,0}]$ is periodic in ν, therefore

$$K_{1,1} = 0 \tag{27}$$

And, inverting (26),

$$W_{1,1} = -\frac{1}{\omega} \int [H_0 - G; W_{1,0}] d\nu \tag{28}$$

$*****$

Other Relegation iterations:

In complete analogy with the second iteration of the relegation it is found that $\forall 2 \leq j \leq p$:

$$\begin{array}{l} K_{1,j} = 0 \\ W_{1,j} = -\frac{1}{\omega} \int [H_0 - G; W_{1,j-1}] d\nu \end{array} \tag{29}$$

$*****$

It must be noted that, at each step of relegation $1 \leq j \leq p$ the "remainder"

$$\begin{array}{l} [H_0 - G; W_{1,j}] = R \frac{\partial W_{1,j}}{\partial r} + \frac{\Theta}{r^2} \frac{\partial W_{1,j}}{\partial \theta} \\ \quad - (\frac{\Theta^2}{r^3} - \frac{M\mathcal{G}}{r^2}) \frac{\partial W_{1,j}}{\partial R} \end{array} \tag{30}$$

is of order $\frac{1}{r^{j+2}}$.

As we have considered only the potential up to the terms $\sim \frac{1}{r^{n_{max}+1}}$ we will stop the relegation algorithm once the reminder is of the same order of the first neglected terms, i.e. of order $\sim \frac{1}{r^{n_{max}+2}}$, thus we set the maximum order of relegation to be $p = n_{max} - 1$.

Thus, once we have relegated $n_{max} - 1$-times, for the first homological equation we obtain:

$$\begin{array}{l} W_1 = \sum_{j=0}^{p-1} W_{1,j} \\ K_1 = \sum_{j=0}^{p-1} K_{1,j} + R_1 = K_{1,0} \\ R_1 = [H_0 - G; W_{1,n_{max}-1}] \end{array} \tag{31}$$

With $K_{1,0}$ as in (22) and where in the second equation the reminder has been dropped at it made of orders higher than $\frac{1}{r^{n_{max}+1}}$ which must be truncated.

Second (and higher) Homological Equations:

By (12), considering that in (15) we have set $H_2 = 0$: $\tilde{H}_2 = 2[H_1, W_1] + [[H_0; W_1]; W_1]$.

By direct calculations it can be easily verified that $\tilde{H}_2 \sim \frac{1}{r^8}$. Therefore it must be noticed that we have to fix n_{max} to be greater than 8 to get any results from the second homological equations.

We set the second homological equation

$$[H_0; W_2] + \tilde{H}_2 = K_2 \tag{32}$$

and start the relegation by setting $K_{2,0}$ equal to the terms of $\tilde{H}_2$ which does not depend on ν and $W_{2,0} = -\omega \int (\tilde{H}_2 - K_{2,0}) d\nu$ and so on.

The transformed Hamiltonian $K(r, \theta, R, \Theta, N)$, obtained from this process, no longer depends on the argument of the node ν, the variable N becomes cyclic and therefore $-\omega N$ is a constant term, which can be dropped from the Hamiltonian. The relegation is iterated $p - 1$ times and, setting $W_{1,p}^* = 0$, yields:

$$\begin{array}{l} W_1 = \sum_{j=0}^{p-1} W_{1,j} \\ K_1 = \sum_{j=0}^{p-1} K_{1,j} + R_1 = K_{1,0} + R_1 \end{array} \tag{33}$$

with

$$R_1 = [H_0 - G; W_{1,p}] \tag{34}$$

with $K_{1,0}$ as in (22).

It must be noted that, at each step of relegation $1 \leq j \leq p$ the coefficient $W_{1,j}$ of the generator is $O(\frac{1}{r^{j+2}})$ which implies that when the algorithm is stopped at the p^{th} iteration the "remainder"

$$\begin{array}{l} [H_0 - G; W_{1,p}] = R \frac{\partial W_{1,p}}{\partial r} + \frac{\Theta}{r^2} \frac{\partial W_{1,p}}{\partial \theta} \\ \quad - (\frac{\Theta^2}{r^3} - \frac{M\mathcal{G}}{r^2}) \frac{\partial W_{1,p}}{\partial R} \end{array} \tag{35}$$

is of degree $O(\frac{1}{r^{p+3}})$.

When applying this algorithm to a real asteroid, only a finite number of spherical harmonic coefficients will be known, that is $\exists n_{max}$ such that we know all the $C_{n,m}$ and $S_{n,m}$ up to $C_{n_{max},n_{max}}$ and $S_{n_{max},n_{max}}$ included[2]. This implies that in the potential the terms of degree $O(\frac{1}{r^{(n_{max}+2)}})$ and higher have been truncated; thus, in the relegation algorithm the maximum number of iterations p is set to $n_{max} - 1$ such that the reminder of the algorithm $O(\frac{1}{r^{p+3}}) = O(\frac{1}{r^{n_{max}+2}})$ can be dropped and the Hamiltonian becomes

$$\begin{array}{l} K = H_0 + \epsilon K_{1,0} \\ \quad = \frac{1}{2}(R^2 + \frac{\Theta^2}{r^2}) - \frac{M\mathcal{G}}{r} - \omega N + \epsilon K_{1,0} \end{array} \tag{36}$$

with $K_{1,0}$ as in (22).

Moreover the resulting Hamiltonian is equivalent to the one in the main problem of the artificial satellite, in which the argument of node ν is cyclic, the coriolis term $-\omega N$ is constant and can be neglected from the Hamiltonian. Therefore a closed form Delaunay normalization can be performed, for a further reduction of the degrees of freedom, thus yielding an integrable Hamiltonian.

[2]Note that fixing the maximum degree n_{max} of terms for the potential means that all the terms up to $C_{nmax,nmax}$ and $S_{nmax,nmax}$ are taken into account, which is a total of $(nmax + 2)(nmax + 1)$ coefficients of the potential

IV.II DELAUNAY NORMALIZATION

The Delaunay coordinates are symplectic action-angle variables (L, G, H, ℓ, g, h), where the angles ℓ, g and h are conjugated to the actions L, G and H respectively, where

- ℓ is the *mean anomaly* measured from the pericenter;
- g is the argument of the pericenter;
- h is the argument of the node;
- L is related to the major semi-axis, a, by $L = \sqrt{\mathcal{G}Ma}$;
- G is the *total angular momentum* of the spacecraft with respect to the Asteroid (in the inertial frame), related to the eccentricity and the variable L by $e = \sqrt{1 - \frac{G^2}{L^2}}$;
- H is the z-component of the total angular momentum, i.e. $H = G \cos I$.

Moreover the relation between the *True anomaly* and the *Eccentric anomaly* u is defined as:

$$\tan\left(\frac{f}{2}\right) = \sqrt{\frac{1+e}{1-e}}\tan\left(\frac{u}{2}\right), \tag{37}$$

which, in particular, implies

$$r = a(1 - e\cos u) = a\frac{1-e^2}{1+e\cos f}. \tag{38}$$

The closed form normalization algorithm ([10]) is briefly illustrated here, which, instead of using the expansions of r and f in powers of the eccentricity, changes the independent variable from time to the true anomaly f.

Definition 1.
A formal series $K'(y, Y, \epsilon) = \sum_{s=0}^{\infty} \frac{\epsilon^s}{s!} K'_s(y, Y)$ *is said to be in Delaunay normal form if the Lie derivative* $L_{K'_0} J$ *is zero, that is* $[K'_s, K'_0] = 0 \ \forall s \geq 0$.

In our case, as $K'_0 = J_0 = -\frac{(\mathcal{G}M)^2}{2L^2}$, the Lie derivative

$$L_{K'_0}(\cdot) = \frac{(\mathcal{G}M)^2}{L^3}\frac{\partial(\cdot)}{\partial \ell}$$

therefore the new Hamiltonian will be in normal form if and only if

$$\frac{\partial K'_1}{\partial \ell} = 0$$

Note that, as for the relegation, the normalization degenerates into an average over the mean anomaly ℓ.

* * * * *

Recalling that $\tilde{K}'_1 = J_1$, we set the first homologic equation:

$$\begin{aligned} &[J_0; \bar{W}_1] + \tilde{K}'_1 = K'_1 \\ &\Leftrightarrow -\frac{(\mathcal{G}M)^2}{L^3}\frac{\partial W'_1}{\partial \ell} + J_1 = K'_1 \end{aligned} \tag{39}$$

Now, as we want K'_1 to be in Delaunay normal form ($\Leftrightarrow \frac{\partial K'_1}{\partial \ell} = 0$), we set

$$K'_1 = \frac{1}{2\pi}\int_0^{2\pi} J_1 d\ell \tag{40}$$

This integral is solved by changing the independent variable from ℓ to be the true anomaly f by the relation

$$\begin{aligned} \frac{df}{d\ell} &= \frac{df}{du}\frac{du}{d\ell} \\ &= \left(\frac{1+e\cos f}{\sqrt{1-e^2}}\right)\left(\frac{1}{1-e\cos u}\right) \\ &= \frac{a^2\sqrt{1-e^2}}{r^2} \end{aligned} \tag{41}$$

Finally, inverting (39), yields the first order generating function:

$$W'_1 = \int \frac{L^3}{(\mathcal{G}M)^2}\left(J_1 - \frac{1}{2\pi}\int_0^{2\pi} J_1 d\ell\right) d\ell \tag{42}$$

This leads to an integrable, two degree of freedom, Hamiltonian which approximates the first homologic equation of the system which includes arbitrary degree spherical harmonic coefficients. This approximated system can now be applied to every inhomogeneous body in order to determine possible orbits useful for scientific observation missions such as frozen orbits.
Again, in this paper, only the first Homologic equation is considered and explicitly evaluated and the explicit expressions of (40) and (42) used for the applications have been obtained using a script in the Mathematica software.
Notice that, restricting the result to the case where all the $S_{n,m}$ and $C_{n,m}$ coefficients are zero except for the ellipticity and oblateness terms, the resulting Hamiltonian is reduced to that obtained by [20] and [22].
However, with respect to the Hamiltonian obtained in these papers, it must be highlighted that, considering arbitrary degree of spherical harmonic coefficients, the resulting Hamiltonian will, in general, contain both the relegated variables G and g, thus the system is still integrable but the solution cannot be explicitly solved, i.e. it is no longer "trivially integrable" as in [20].

V. APPLICATIONS

The Hamiltonian obtained is of the form: $K'(L, G, H, \_, g, \_)$ thus the equations of motion are:

$$\begin{aligned} \ell'(t) &= \frac{\partial K'}{\partial L} \\ g'(t) &= \frac{\partial K'}{\partial G} \\ h'(t) &= \frac{\partial K'}{\partial H} \\ L'(t) &= 0 \\ G'(t) &= -\frac{\partial K'}{\partial g} \\ H'(t) &= 0, \end{aligned} \tag{43}$$

where L and H are constants and all the other motions will only depend on $G(t)$ and $g(t)$.

Definition 2. *(Frozen orbit)*
A frozen orbit is an orbit in which the Inclination, the Eccentricity and the Argument of perigee remains constant during the motion.

This in particular implies that such an orbit is then perfectly periodic except for the orbital plane precession.
A frozen orbit it thus described by the system:

$$\begin{aligned} \dot{e} &= \frac{d}{dt}\frac{\sqrt{L^2-G^2}}{L} = 0 \\ \dot{I} &= \frac{d}{dt}\arccos\frac{H}{G} = 0 \\ \dot{g} &= 0. \end{aligned} \tag{44}$$

For the properties of the Lie transformations, the "normalized" eccentricity, inclination and argument of perigee are related to their relative "real" equivalents by the generator of

the transformation (see [7]), and can thus be interpreted as a perturbed version of their real correspondents.
In the normalized variables (43), the system (44) is equivalent to:

$$\begin{array}{l}\dot{G} = 0\\ \dot{g} = 0.\end{array} \quad (45)$$

Thus fixing normalized eccentricity e and inclination I for the desired normalized frozen orbit, and solving the system gives:

$$\begin{array}{l}\dot{G} = 0\\ \dot{g} = 0\\ e = \frac{\sqrt{L^2-G^2}}{L}\\ I = \arccos\frac{H}{G},\end{array} \quad (46)$$

and the initial conditions (L_0, G_0, H_0, g_0) for normalized frozen orbits can be found.
Results show that these initial conditions can be used in the initial system describing the full dynamics (the one described by the Hamiltonian (4)) to generate a good initial guess for frozen orbits around any inhomogeneous body.

VI. CONCLUSIONS

Setting the desired eccentricity and inclination it is thus possible to determine initial conditions which lead to frozen orbits in the truncated system.
Such initial conditions are used to approximate the solutions for the secular motion of the satellite in the real system thus showing a good agreement between the approximated and the real dynamics.
Example applications of the method are shown for the asteroid 433-Eros, for different eccentricities, inclinations, and argument of pericenter. This highly irregular, elongated, Near Earth Asteroid, is the main example used in the literature, for which the spherical harmonic coefficients can be found up to the 15^{th} degree, i.e. 272 coefficients (see Appendix A).
The physical properties of Eros are summarized in the table (1):

Mass Kg	6.6904×10^{15}
Rotational velocity rad/s	3.31182×10^{-4}
Reference Radius Km	16

Table 1: Physical properties of 433-Eros

For illustration purposes, 3 different triples of initial parameters eccentricity E_0, inclination I_0 and argument of perigee g_0, are fixed. Each triple yield the initial conditions f_0, h_0, L_0, G_0, and H_0 for the corresponding frozen orbit found. The results obtained for 433-Eros are collected in Table (2). In the last row of the table, the initial semimajor axe a_0 of the resulting orbits has also been recorded.

The resulting orbits for 433-Eros, in the cartesian inertial frame of reference centered in the center of mass of the inhomogeneous body:

	Fig.1	Fig.2	Fig.3
	$I_0 = 0.5$ $E_0 = 0.001$ $g_0 = -\pi/2$	$I_0 = 1.1$ $E_0 = 0.4$ $g_0 = -\pi/2$	$I_0 = 0.001$ $E_0 = 0.5$ $g_0 = \pi/2$
h_0	π	π	π
f_0	π	π	π
G_0	234.612	187.656	302.438
L_0	234.612	204.749	504.063
H_0	205.892	85.119	302.437
a_0	$\sim 100km$	$\sim 90km$	$500km$

Table 2: 433-Eros: initial conditions for frozen orbits

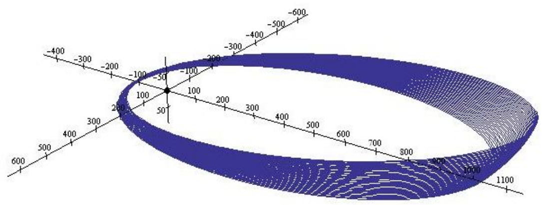

Figure 1: *433-Eros:* $I_0 = 0.5$, $E_0 = 0.001$, $g_0 = -\pi/2$*:* after 5 years

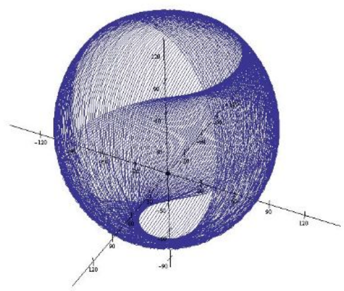

Figure 2: *433-Eros:* $I_0 = 1.1$, $E_0 = 0.4$, $g_0 = -\pi/2$*:* after 5 years

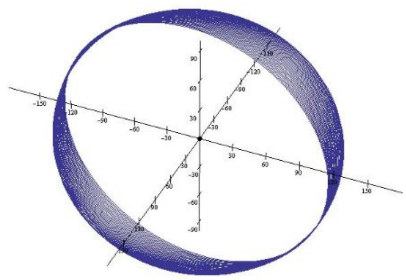

Figure 3: *433-Eros:* $I_0 = 0.001$, $E_0 = 0.5$, $g_0 = \pi/2$*:* after 5 years

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Appendix A

The un-normalized spherical harmonic coefficients of 433-Eros are here listed.
This coefficients are the harmonic coefficients gravity solution NEAR15A, a 15th degree and order model obtained from radiometric tracking (Doppler and range data) and landmark tracking of the NEAR spacecraft in orbit about Eros. The gravity model includes data from the entire mission beginning with orbit insertion on Feb. 14, 2000 and ending with the first descent maneuver for landing on Feb. 12, 2001

$C_{0,0}$	1
$C_{1,0}$	0
$C_{1,1}$	0
$C_{2,0}$	-1.65899$\times10^{-1}$
$C_{2,1}$	-2.11454$\times10^{-6}$
$C_{2,2}$	5.31886$\times10^{-2}$
$C_{3,0}$	-5.29244$\times10^{-3}$
$C_{3,1}$	4.38548$\times10^{-3}$
$C_{3,2}$	6.0659$\times10^{-4}$
$C_{3,3}$	-1.4525$\times10^{-3}$
$C_{4,0}$	5.48636$\times10^{-2}$
$C_{4,1}$	-9.52013$\times10^{-5}$
$C_{4,2}$	-3.90614$\times10^{-3}$
$C_{4,3}$	-1.79405$\times10^{-5}$
$C_{4,4}$	3.68808$\times10^{-4}$
$C_{5,0}$	3.09067$\times10^{-3}$
$C_{5,1}$	-2.36787$\times10^{-3}$
$C_{5,2}$	-1.26781$\times10^{-4}$
$C_{5,3}$	1.51169$\times10^{-4}$
$C_{5,4}$	3.86908$\times10^{-6}$
$C_{5,5}$	-2.51307$\times10^{-5}$
$C_{6,0}$	-2.53848$\times10^{-2}$
$C_{6,1}$	-1.91651$\times10^{-5}$
$C_{6,2}$	8.13891$\times10^{-4}$
$C_{6,3}$	5.9664$\times10^{-6}$
$C_{6,4}$	-2.13764$\times10^{-5}$
$C_{6,5}$	-3.93777$\times10^{-7}$
$C_{6,6}$	1.18484$\times10^{-6}$

$C_{7,0}$	-2.50016$\times10^{-3}$
$C_{7,1}$	1.26047$\times10^{-3}$
$C_{7,2}$	3.82038$\times10^{-5}$
$C_{7,3}$	-3.48143$\times10^{-5}$
$C_{7,4}$	-5.15671$\times10^{-7}$
$C_{7,5}$	1.33563$\times10^{-6}$
$C_{7,6}$	2.25518$\times10^{-9}$
$C_{7,7}$	-1.25528$\times10^{-7}$
$C_{8,0}$	1.53478$\times10^{-2}$
$C_{8,1}$	-3.43765$\times10^{-5}$
$C_{8,2}$	-2.57667$\times10^{-4}$
$C_{8,3}$	-3.12096$\times10^{-6}$
$C_{8,4}$	3.61153$\times10^{-6}$
$C_{8,5}$	8.73471$\times10^{-8}$
$C_{8,6}$	-7.09764$\times10^{-8}$
$C_{8,7}$	-9.71194$\times10^{-10}$
$C_{8,8}$	2.89016$\times10^{-9}$
$C_{9,0}$	1.12427$\times10^{-3}$
$C_{9,1}$	-4.97634$\times10^{-4}$
$C_{9,2}$	-2.57824$\times10^{-5}$
$C_{9,3}$	1.07011$\times10^{-5}$
$C_{9,4}$	-4.14388$\times10^{-7}$
$C_{9,5}$	-1.7556$\times10^{-7}$
$C_{9,6}$	-3.11553$\times10^{-9}$
$C_{9,7}$	5.83725$\times10^{-9}$
$C_{9,8}$	1.43792$\times10^{-10}$
$C_{9,9}$	-2.52185$\times10^{-10}$
$C_{10,0}$	-2.23924$\times10^{-3}$

$C_{10,1}$	-3.65977$\times10^{-4}$
$C_{10,2}$	8.59725$\times10^{-5}$
$C_{10,3}$	2.44668$\times10^{-6}$
$C_{10,4}$	-2.12904$\times10^{-8}$
$C_{10,5}$	-3.91544$\times10^{-8}$
$C_{10,6}$	1.06018$\times10^{-8}$
$C_{10,7}$	6.6781$\times10^{-10}$
$C_{10,8}$	-1.03388$\times10^{-10}$
$C_{10,9}$	-2.93031$\times10^{-11}$
$C_{10,10}$	4.93363$\times10^{-12}$
$C_{11,0}$	1.04666$\times10^{-2}$
$C_{11,1}$	3.72982$\times10^{-4}$
$C_{11,2}$	3.37686$\times10^{-6}$
$C_{11,3}$	1.80367$\times10^{-6}$
$C_{11,4}$	-5.5386$\times10^{-7}$
$C_{11,5}$	7.57115$\times10^{-8}$
$C_{11,6}$	2.19576$\times10^{-9}$
$C_{11,7}$	5.19815$\times10^{-11}$
$C_{11,8}$	3.75133$\times10^{-11}$
$C_{11,9}$	1.74028$\times10^{-11}$
$C_{11,10}$	-2.76742$\times10^{-13}$
$C_{11,11}$	-2.57971$\times10^{-13}$
$C_{12,0}$	1.71922$\times10^{-3}$
$C_{12,1}$	3.7954$\times10^{-4}$
$C_{12,2}$	1.55553$\times10^{-4}$
$C_{12,3}$	6.86842$\times10^{-6}$
$C_{12,4}$	2.99064$\times10^{-7}$
$C_{12,5}$	-8.38626$\times10^{-8}$

$C_{12,6}$	4.07023×10^{-9}
$C_{12,7}$	-1.60746×10^{-10}
$C_{12,8}$	1.86662×10^{-11}
$C_{12,9}$	-4.62122×10^{-12}
$C_{12,10}$	-5.72445×10^{-13}
$C_{12,11}$	2.02689×10^{-14}
$C_{12,12}$	8.12551×10^{-15}
$C_{13,0}$	2.75545×10^{-2}
$C_{13,1}$	-2.9199×10^{-3}
$C_{13,2}$	-2.02593×10^{-6}
$C_{13,3}$	6.84023×10^{-6}
$C_{13,4}$	3.23691×10^{-7}
$C_{13,5}$	-3.54904×10^{-8}
$C_{13,6}$	2.59498×10^{-10}
$C_{13,7}$	4.03437×10^{-11}
$C_{13,8}$	-1.37277×10^{-11}
$C_{13,9}$	-7.31327×10^{-13}
$C_{13,10}$	-7.27471×10^{-14}
$C_{13,11}$	2.30772×10^{-14}
$C_{13,12}$	2.58196×10^{-16}
$C_{13,13}$	-2.18667×10^{-16}
$C_{14,0}$	-1.53377×10^{-2}
$C_{14,1}$	7.66068×10^{-4}
$C_{14,2}$	2.96292×10^{-4}
$C_{14,3}$	5.32869×10^{-6}
$C_{14,4}$	-5.87731×10^{-7}
$C_{14,5}$	-5.31799×10^{-8}
$C_{14,6}$	2.30096×10^{-9}

$C_{14,7}$	-7.86604×10^{-11}
$C_{14,8}$	-7.59718×10^{-12}
$C_{14,9}$	-1.0193×10^{-13}
$C_{14,10}$	5.29761×10^{-14}
$C_{14,11}$	7.4175×10^{-15}
$C_{14,12}$	-9.24318×10^{-16}
$C_{14,13}$	-2.2948×10^{-17}
$C_{14,14}$	1.81628×10^{-17}
$C_{15,0}$	2.06404×10^{-2}
$C_{15,1}$	-2.65164×10^{-3}
$C_{15,2}$	9.46812×10^{-6}
$C_{15,3}$	-3.69445×10^{-6}
$C_{15,4}$	3.18757×10^{-7}
$C_{15,5}$	-2.84101×10^{-8}
$C_{15,6}$	-1.9038×10^{-10}
$C_{15,7}$	-6.24463×10^{-11}
$C_{15,8}$	-1.06965×10^{-11}
$C_{15,9}$	-2.61478×10^{-13}
$C_{15,10}$	5.55852×10^{-16}
$C_{15,11}$	-1.64954×10^{-15}
$C_{15,12}$	4.81127×10^{-17}
$C_{15,13}$	2.5553×10^{-17}
$C_{15,14}$	5.60796×10^{-19}
$C_{15,15}$	-5.49434×10^{-19}

$S_{0,0}$	0
$S_{1,0}$	0
$S_{1,1}$	0
$S_{2,0}$	0
$S_{2,1}$	-1.80744×10^{-7}
$S_{2,2}$	-1.81446×10^{-2}
$S_{3,0}$	0
$S_{3,1}$	3.63836×10^{-3}
$S_{3,2}$	-2.40395×10^{-4}
$S_{3,3}$	-1.68328×10^{-3}
$S_{4,0}$	0
$S_{4,1}$	1.29913×10^{-4}
$S_{4,2}$	1.0351×10^{-3}
$S_{4,3}$	-7.12399×10^{-6}
$S_{4,4}$	-1.92384×10^{-4}
$S_{5,0}$	0
$S_{5,1}$	-1.04273×10^{-3}
$S_{5,2}$	6.17062×10^{-5}
$S_{5,3}$	1.16925×10^{-4}
$S_{5,4}$	-5.43531×10^{-6}
$S_{5,5}$	-1.43782×10^{-5}
$S_{6,0}$	0
$S_{6,1}$	-9.74106×10^{-5}
$S_{6,2}$	-1.48126×10^{-4}
$S_{6,3}$	1.56395×10^{-6}
$S_{6,4}$	1.56395×10^{-6}
$S_{6,5}$	3.86799×10^{-8}
$S_{6,6}$	-3.73278×10^{-7}

$S_{7,0}$	0
$S_{7,1}$	5.15445×10^{-4}
$S_{7,2}$	-1.97429×10^{-5}
$S_{7,3}$	-2.02322×10^{-5}
$S_{7,4}$	6.94006×10^{-7}
$S_{7,5}$	6.72634×10^{-7}
$S_{7,6}$	-3.44172×10^{-8}
$S_{7,7}$	-4.07766×10^{-8}
$S_{8,0}$	0
$S_{8,1}$	-1.24043×10^{-5}
$S_{8,2}$	2.30047×10^{-6}
$S_{8,3}$	-3.22691×10^{-7}
$S_{8,4}$	-6.27617×10^{-7}
$S_{8,5}$	-1.85513×10^{-8}
$S_{8,6}$	6.66802×10^{-10}
$S_{8,7}$	-3.57144×10^{-10}
$S_{8,8}$	1.74786×10^{-9}
$S_{9,0}$	0
$S_{9,1}$	-8.17618×10^{-5}
$S_{9,2}$	-1.31237×10^{-5}
$S_{9,3}$	7.54724×10^{-6}
$S_{9,4}$	-2.35188×10^{-7}
$S_{9,5}$	-1.00222×10^{-7}
$S_{9,6}$	1.12056×10^{-9}
$S_{9,7}$	1.6534×10^{-9}
$S_{9,8}$	-2.32921×10^{-11}
$S_{9,9}$	-5.56697×10^{-11}
$S_{10,0}$	0

$S_{10,1}$	6.94286×10^{-4}
$S_{10,2}$	-4.56443×10^{-5}
$S_{10,3}$	2.62557×10^{-6}
$S_{10,4}$	-4.14985×10^{-7}
$S_{10,5}$	-5.74199×10^{-8}
$S_{10,6}$	6.45742×10^{-9}
$S_{10,7}$	-7.47668×10^{-10}
$S_{10,8}$	-4.99191×10^{-12}
$S_{10,9}$	9.74982×10^{-13}
$S_{10,10}$	5.59573×10^{-12}
$S_{11,0}$	0
$S_{11,1}$	-8.17892×10^{-4}
$S_{11,2}$	-6.92074×10^{-5}
$S_{11,3}$	-1.13881×10^{-6}
$S_{11,4}$	-4.84678×10^{-7}
$S_{11,5}$	8.37324×10^{-8}
$S_{11,6}$	-1.09462×10^{-9}
$S_{11,7}$	-2.46115×10^{-10}
$S_{11,8}$	-2.79264×10^{-11}
$S_{11,9}$	9.02775×10^{-12}
$S_{11,10}$	1.31812×10^{-13}
$S_{11,11}$	-1.94565×10^{-13}
$S_{12,0}$	0
$S_{12,1}$	1.50676×10^{-3}
$S_{12,2}$	9.64141×10^{-5}
$S_{12,3}$	2.73675×10^{-6}
$S_{12,4}$	-2.39721×10^{-7}
$S_{12,5}$	-3.08972×10^{-8}

$S_{12,6}$	9.18684×10^{-9}
$S_{12,7}$	-5.56246×10^{-10}
$S_{12,8}$	5.98262×10^{-12}
$S_{12,9}$	1.23035×10^{-13}
$S_{12,10}$	-7.24925×10^{-13}
$S_{12,11}$	1.701×10^{-14}
$S_{12,12}$	1.63895×10^{-14}
$S_{13,0}$	0
$S_{13,1}$	-1.24564×10^{-3}
$S_{13,2}$	1.54632×10^{-5}
$S_{13,3}$	-6.74004×10^{-7}
$S_{13,4}$	-1.19607×10^{-6}
$S_{13,5}$	6.26074×10^{-9}
$S_{13,6}$	-1.26688×10^{-10}
$S_{13,7}$	-7.5178×10^{-13}
$S_{13,8}$	-1.60844×10^{-11}
$S_{13,9}$	-9.10394×10^{-14}
$S_{13,10}$	-7.19669×10^{-15}
$S_{13,11}$	-5.20369×10^{-15}
$S_{13,12}$	-1.01803×10^{-16}
$S_{13,13}$	-4.21829×10^{-16}
$S_{14,0}$	0
$S_{14,1}$	8.65044×10^{-4}
$S_{14,2}$	1.51562×10^{-4}
$S_{14,3}$	4.31479×10^{-7}
$S_{14,4}$	1.77234×10^{-7}
$S_{14,5}$	-1.76094×10^{-9}
$S_{14,6}$	4.30073×10^{-9}

$S_{14,7}$	-2.43475×10^{-10}
$S_{14,8}$	-1.42072×10^{-11}
$S_{14,9}$	4.1348×10^{-13}
$S_{14,10}$	8.33334×10^{-15}
$S_{14,11}$	6.89565×10^{-16}
$S_{14,12}$	-3.88959×10^{-16}
$S_{14,13}$	3.71979×10^{-18}
$S_{14,14}$	2.08219×10^{-17}
$S_{15,0}$	0
$S_{15,1}$	-6.5828×10^{-5}
$S_{15,2}$	9.63909×10^{-5}
$S_{15,3}$	9.90187×10^{-7}
$S_{15,4}$	-7.56365×10^{-7}
$S_{15,5}$	-3.05489×10^{-8}
$S_{15,6}$	-2.45565×10^{-10}
$S_{15,7}$	-1.12172×10^{-11}
$S_{15,8}$	2.66204×10^{-12}
$S_{15,9}$	-2.21231×10^{-14}
$S_{15,10}$	-7.67107×10^{-15}
$S_{15,11}$	-2.08224×10^{-15}
$S_{15,12}$	4.21957×10^{-17}
$S_{15,13}$	1.21087×10^{-17}
$S_{15,14}$	-3.91552×10^{-19}
$S_{15,15}$	-4.94421×10^{-19}