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STABILITY OF SINGULAR JUMP-LINEAR SYSTEMS WITH A LARGE STATE SPACE: A TWO-TIME-SCALE APPROACH

DUNG TIEN NGUYEN[1], XUERONG MAO[2], G. YIN[✉3] and CHENGGUI YUAN[4]

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Abstract

This paper considers singular systems that involve both continuous dynamics and discrete events with the coefficients being modulated by a continuous-time Markov chain. The underlying systems have two distinct characteristics. First, the systems are singular, that is, characterized by a singular coefficient matrix. Second, the Markov chain of the modulating force has a large state space. We focus on stability of such hybrid singular systems. To carry out the analysis, we use a two-time-scale formulation, which is based on the rationale that, in a large-scale system, not all components or subsystems change at the same speed. To highlight the different rates of variation, we introduce a small parameter $\varepsilon > 0$. Under suitable conditions, the system has a limit. We then use a perturbed Lyapunov function argument to show that if the limit system is stable then so is the original system in a suitable sense for ε small enough. This result presents a perspective on reduction of complexity from a stability point of view.

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1. Introduction

Singular systems, which are also known by many other names such as descriptor systems, generalized systems and implicit systems, feature in differential-algebraic equations. They arise in various applications in the physical sciences, engineering and economics. Owing to their importance, such systems have been studied extensively and used widely in control and optimization [5–9, 13–16]. While the references

[1]Department of Electrical and Computer Engineering, University of British Columbia, Vancouver V6T 1Z4, Canada; e-mail: dungnt@ece.ubc.ca.
[2]Department of Mathematics and Statistics, University of Strathclyde, Glasgow G1 1XH, UK; e-mail: x.mao@strath.ac.uk.
[3]Department of Mathematics, Wayne State University, Detroit, Michigan 48202, USA; e-mail: gyin@math.wayne.edu.
[4]Department of Mathematics, University of Wales Swansea, Swansea SA2 8PP, UK; e-mail: C.Yuan@swansea.ac.uk.

mentioned are all concerned with deterministic systems, recent work also includes formulation, analysis and computation involving stochastic systems [1, 2, 13, 17].

In applications, dynamical systems are frequently not only time varying, but also associated with movements influenced by uncertain and exogenous discrete events driven by random disturbances. Many such systems involve noise of pure jump type, especially those arising in production planning, economics and stochastic networks. For example, in economics the coefficients of the classical Leontief model are fixed. In reality, more often than not they are changing with respect to time depending on the trend of the economy, and are subject to discrete switching regimes. The behaviour of the dynamical systems corresponding to different regimes is very different. As a result, a promising alternative model is to allow for the possibility of sudden, discrete changes in the values of the parameters, resulting in a "hybrid" or "switching" model governed by a Markov chain.

In this paper, we consider a hybrid model with randomly varying switching regime. The premise of our model is that, for example, many of the important movements in the economy arise from discrete events. A nation's economy sometimes appears quite calm and at other times rather volatile, and it is important to describe how this volatility changes over time. Monetary, fiscal or income policies often change in a way referred to in economics as shocks. Economists cannot observe these shifts directly, so these discrete events are governed by hidden random processes. Since the late 1980s, there has been increasing interest in using Markov-based models in economics. Although most of these efforts are devoted to time series analysis [3, 4, 11, 12], it is conceivable that the use of Markov-based models will play a more prominent role in the future.

A continuous-time Markov chain can be used to formulate the trend of the economy. Suppose that the economy has two possible "states": a fast growth phase and slow growth phase. At any given time, the economy will be in one of the two states, governed by the outcome of a Markov chain. Another example is the unemployment rate, which at any time is either rising or falling. The regimes or configurations of the system differ corresponding to the two states (for either example), resulting in different coefficients in the linear equations. This leads to a hybrid/switching model modulated by a Markov chain with finite state space.

One concern is on the reduction of complexity. Denote the Markov chain by $\alpha(\cdot)$. In a multi-sector economy, the state space of the Markov chain is likely to be large due to rapid growth in science and technology. The large number of states of the underlying chain gives a detailed representation of the position of the economy. However, the large-scale nature of the system makes design and control very difficult. To reduce the complexity of the system, we observe that not all states change at the same speed. Some vary rapidly and others slowly. The inherent fast and slow time scales give us the possibility of grouping the states in accordance with their transition rates. We introduce a small parameter $\varepsilon > 0$, and let $\alpha(t) = \alpha^{\varepsilon}(t)$. The parameter ε enters as a rate parameter. It highlights the fast and slow dynamics resulting in weak and strong interactions. The precise description of these effects is given in Section 2.

Suppose that we want to control a hybrid system in which the state space $\mathcal{M}$ consists of m elements, where m is large. Instead of examining the original complex system, by using appropriate asymptotic analysis we can consider a "reduced" system whose state space consists of a smaller number, l, of elements. If $l \ll m$, the complexity of the task is dramatically reduced. In the asymptotic analysis, to rigorously prove the desired result it is necessary to consider the limit as $\varepsilon \to 0$. In the actual applications, ε could be a constant; it need not go to 0. The asymptotic results provide us with guidance on the control, optimization and design of the actual system. We mention mainly applications in economic systems, but such modelling tools can also be used in manufacturing and production planning as well as many networked systems.

The main motivation for this paper is twofold. First, there are many systems in applications that are singular. The stability of such systems is of crucial importance. Motivated by the recent stability analysis of Huang and Mao [13] for stochastic systems with Markov regime switching, we also treat regime-switching singular systems. However, for us there is another difficulty, namely that the state space of the Markov chain is very large. Thus, it is essential to reduce the computational complexity. We use the idea of a two-time-scale formulation of Markov chains [17, 18] to carry out the analysis, and focus on stability analysis. By letting $\varepsilon \to 0$, we obtain a limit system with reduced state space for an aggregated switching process. Knowing the stability of the limit system, we aim to obtain stability of the original system under suitable conditions. While the stability of singular systems of switching diffusions has been analysed by Huang and Mao [13], this work deals with the case when the state space of the Markov chain becomes very large. Our new contributions provide a method for reduction of complexity through time-scale separations.

The rest of the paper is organized as follows. The precise problem formulation is given in Section 2, and a number of preliminary results are presented in Section 3. Stability of the underlying singular systems is discussed in Section 4. We use a two-time-scale approach. Under broad conditions, we show that by use of the limit system we can obtain stability of the original system. Examples are presented in Section 5, and final remarks in Section 6.

2. Problem formulation

Suppose that $\alpha^\varepsilon(t)$ is a continuous-time Markov chain taking values in a finite state space $\mathcal{M}$. We consider the switching process $\alpha^\varepsilon(t)$ as having quickly and slowly varying transitions in that the generator of the Markov chain is given by

$$Q^\varepsilon = \frac{\widetilde{Q}}{\varepsilon} + \widehat{Q},$$

with

$$\widetilde{Q} = \mathrm{diag}(\widetilde{Q}^1, \dots, \widetilde{Q}^l), \tag{2.1}$$

where $\mathrm{diag}(D^1, \dots, D^l)$ denotes a diagonal-block matrix with entries $D^1, \dots, D^l$ and $\widehat{Q}$ is another generator without specific structure. Because of the structure of the matrix $\widetilde{Q}$ in (2.1), we write the state space $\mathcal{M}$ as

$$\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \cdots \cup \mathcal{M}_l, \quad \text{where} \quad \mathcal{M}_i = \{s_{i1}, \ldots, s_{im_i}\} \text{ for } i = 1, \ldots, l.$$

Throughout the paper, we use the following condition for the fast-changing part of the generator $\widetilde{Q}$.

(A1) For $i = 1, \ldots, l$, the generator $\widetilde{Q}^i$ is irreducible.

By irreducibility we mean that the system of equations

$$\nu^i \widetilde{Q}^i = 0, \quad \nu^i \mathbb{1}_{m_i} = 1$$

has a unique nonnegative solution. Here $\nu^i \in \mathbb{R}^{1\times m_i}$ and $\mathbb{1}_{m_i} = (1, \ldots, 1)' \in \mathbb{R}^{m_i \times 1}$; ν^i is the stationary distribution associated with the generator $\widetilde{Q}^i$. In what follows, we use $q_{s_{ij}s_{\kappa\iota}}$ to denote the $(m_1 + \cdots + m_i + j, m_1 + \cdots + m_\kappa + \iota)$ entry of a given matrix Q.

Suppose that for each $s_{ij} \in \mathcal{M}$, $A(s_{ij})$, $G(s_{ij})$ and $H(s_{ij})$ are $n \times n$ matrices and that $G(s_{ij})$ is singular. Our interest lies in the switching linear system

$$\begin{aligned} &G(\alpha^\varepsilon(t))\, dx^\varepsilon(t) = A(\alpha^\varepsilon(t))x^\varepsilon(t)\, dt + H(\alpha^\varepsilon(t))x^\varepsilon(t)\, dB(t), \\ &x^\varepsilon(0) = \xi, \quad \alpha^\varepsilon(0) = \iota = s_{ij}, \end{aligned} \tag{2.2}$$

for $i = 1, \ldots, l$ and $j = 1, \ldots, m_i$, where $x^\varepsilon(t) \in \mathbb{R}^{n\times 1}$ and $B(\cdot)$ is an n-dimensional standard Brownian motion. We study the stability of this system. The difficulty is that the system is singular, so standard stability analysis techniques do not carry over.

3. Preliminary results

Denote by $\deg(p)$ the degree of a polynomial p. We use the following assumptions.

(A2) For any $i = 1, \ldots, l$ and any $j = 1, \ldots, m_i$, the triplet (G, A, H) satisfies one of the following conditions:
   (a) $\det(sG(s_{ij}) - A(s_{ij})) \neq 0$ for some s, $\deg(\det(sG(s_{ij}) - A(s_{ij}))) = r_{ij}$ and $\mathrm{rank}([G(s_{ij})H(s_{ij})]) = r_{ij}$.
   (b) $\det(sG(s_{ij}) - H(s_{ij})) \neq 0$ for some s, $\deg(\det(sG(s_{ij}) - H(s_{ij}))) = r_{ij}$ and $\mathrm{rank}([G(s_{ij})A(s_{ij})]) = r_{ij}$.

Denote by $\{\tau_k^\varepsilon\}$ a sequence of jump times of the Markov chain $\alpha^\varepsilon(t)$, namely $\tau_0^\varepsilon = 0$ and $\tau_{k+1}^\varepsilon = \inf\{t > \tau_k^\varepsilon : \alpha^\varepsilon(t) \neq \alpha^\varepsilon(\tau_k^\varepsilon)\}$. Then $\alpha^\varepsilon(t) = \alpha^\varepsilon(\tau_k^\varepsilon)$ on $[\tau_k^\varepsilon, \tau_{k+1}^\varepsilon)$. Moreover, $\tau_k^\varepsilon \to \infty$ as $k \to \infty$.

Lemma 3.1. *If* (A2) *holds then* (2.2) *has a unique solution.*

Proof. Assume that (A2)(b) holds. The following argument is similar if (A2)(a) holds. For convenience, denote $s_{ij} = \alpha^\varepsilon_{\tau_0^\varepsilon}$. There exist nonsingular $n \times n$ matrices $L(s_{ij})$, $R(s_{ij})$

such that [13]

$$L(s_{ij})G(s_{ij})R(s_{ij}) = \begin{pmatrix} I_{r_{ij}} & 0 \\ 0 & 0 \end{pmatrix},$$
$$L(s_{ij})A(s_{ij})R(s_{ij}) = \begin{pmatrix} A_1(s_{ij}) & A_2(s_{ij}) \\ 0 & 0 \end{pmatrix},$$
$$L(s_{ij})H(s_{ij})R(s_{ij}) = \begin{pmatrix} H_1(s_{ij}) & 0 \\ 0 & I_{n-r_{ij}} \end{pmatrix},$$

where $A_1(s_{ij})$, $H_1(s_{ij})$ are $r_{ij} \times r_{ij}$ matrices and $A_2(s_{ij})$ is an $r_{ij} \times (n - r_{ij})$ matrix. Let

$$w^\varepsilon(t) = R^{-1}x^\varepsilon(t) = [[w_1^\varepsilon(t)]'[w_2^\varepsilon(t)]']'.$$

Then (2.2) is equivalent to

$$\begin{aligned} &dw_1^\varepsilon(t) = [A_1(\alpha^\varepsilon(t))w_1^\varepsilon(t) + A_2(\alpha^\varepsilon(t))y_2^\varepsilon(t)]\,dt + H_1(\alpha^\varepsilon(t))w_1^\varepsilon(t)\,dB(t),\\ &0 = w_2^\varepsilon(t)\,dB(t),\\ &w^\varepsilon(0) = R^{-1}(\iota)\xi, \quad \alpha^\varepsilon(0) = \iota \in \mathcal{M},\end{aligned}$$

or

$$\begin{aligned} &dw_1^\varepsilon(t) = A_1(\alpha^\varepsilon(t))w_1^\varepsilon(t)\,dt + H_1(\alpha^\varepsilon(t))w_1^\varepsilon(t)\,dB(t),\\ &w_2^\varepsilon(t) = 0,\\ &w^\varepsilon(0) = R^{-1}(\iota)\xi, \quad \alpha^\varepsilon(0) = \iota \in \mathcal{M},\end{aligned}$$

which has a unique solution on the interval $[\tau_0^\varepsilon, \tau_1^\varepsilon]$. Continuing this process, we can prove by induction that (2.2) has a unique solution for all $t \geq 0$. □

4. Stability

In order to find stability conditions on the limit process instead of the original process, we lump the states in each $\mathcal{M}_i$ into a single state and define

$$\overline{\alpha}^\varepsilon(t) = i \quad \text{if } \alpha^\varepsilon(t) \in \mathcal{M}_i.$$

Denote the state space of $\overline{\alpha}^\varepsilon(\cdot)$ by

$$\overline{\mathcal{M}} = \{1, \ldots, l\}, \quad \text{and let } \widetilde{\nu} = \mathrm{diag}(\nu^1, \ldots, \nu^l),$$

where ν^k is the stationary distribution corresponding to $\widetilde{Q}^k$. Define

$$\overline{Q} = \widetilde{\nu}\widehat{Q}\mathbb{1}, \quad \text{where } \mathbb{1} = \mathrm{diag}(\mathbb{1}_{m_1}, \ldots, \mathbb{1}_{m_l}),\ \mathbb{1}_k = (1, \ldots, 1)' \in \mathbb{R}^{k\times 1}.$$

For $i \in \overline{\mathcal{M}}$, denote

$$\widehat{G}(i) = \sum_{j=1}^{m_i} \nu_j^i G(s_{ij}), \quad \widehat{A}(i) = \sum_{j=1}^{m_i} \nu_j^i A(s_{ij}), \quad \widehat{H}(i) = \sum_{j=1}^{m_i} \nu_j^i H(s_{ij}).$$

We need the following assumption.

(A3) For $i \in \overline{\mathcal{M}}$, $\operatorname{rank}(G(s_{i1})) = \cdots = \operatorname{rank}(G(s_{im_i})) = \operatorname{rank}([G'(s_{i1})| \cdots |G'(s_{im_i})])$.

REMARK 4.1. The assumption (A3) is equivalent to the following statement.

(A3′) For $i = 1, \dots, l$, there exists a sequence of elementary row operations transforming $\{G'(s_{ij})\}_{j \in \mathcal{M}_i}$ into row echelon matrices.

We derive from (A3) that, for any $s_{ij} \in \mathcal{M}$, there exist nonsingular $n \times n$ matrices $L(s_{ij})$, $R(s_{ij})$ such that

$$L(s_{ij})G(s_{ij})R(s_{ij}) = \begin{pmatrix} I_{r_i} & 0 \\ 0 & 0 \end{pmatrix},$$

where $r_{ij} = r_i$ for all $j = 1, \dots, m_i$ and $R(s_{ij})$ could be chosen to be the same matrix, denoted by $\widehat{R}(i)$, for all $s_{ij} \in \mathcal{M}_i$. This is valid due to Remark 4.1.

For any $i \in \overline{\mathcal{M}}$, put

$$\widehat{L}(i) = \Big(\sum_{j=1}^{m_i} \nu_j^i L^{-1}(s_{ij})\Big)^{-1}.$$

Then

$$\widehat{L}(i)\widehat{G}(i)\widehat{R}(i) = \begin{pmatrix} I_{r_i} & 0 \\ 0 & 0 \end{pmatrix}.$$

For any $s_{ij} \in \mathcal{M}$, denote

$$\text{\c{G}}(s_{ij}) = \widehat{G}(i), \qquad \text{\c{L}}(s_{ij}) = \widehat{L}(i), \qquad \text{\c{R}}(s_{ij}) = \widehat{R}(i).$$

Then

$$\text{\c{L}}(s_{ij})\text{\c{G}}(s_{ij}) = L(s_{ij})G(s_{ij}).$$

Given any $U \in \mathbb{R}^{n\times n}$, $\text{\c{U}} = \frac{1}{2}(U + U')$ is a symmetric matrix. Let

$$V(x, s_{ij}) = x'\text{\c{G}}'(s_{ij})U\text{\c{G}}(s_{ij})x.$$

Then

$$V(x, s_{ij}) = x'\text{\c{G}}'(s_{ij})\text{\c{U}}\text{\c{G}}(s_{ij})x = x'G'(s_{ij})\widehat{U}(s_{ij})G(s_{ij})x,$$

where

$$\widehat{U}(s_{ij}) = L'(s_{ij})\text{\c{L}}'^{-1}(s_{ij})\text{\c{U}}\text{\c{L}}^{-1}(s_{ij})L(s_{ij}).$$

For a suitable function V, define

$$\mathcal{L}^\varepsilon V(x, \kappa) = \lim_{s\to 0^+} [E(V(x^\varepsilon(t + s), \alpha^\varepsilon(t + s)) \mid x^\varepsilon(t) = x, \alpha^\varepsilon(t) = \kappa) - V(x, \kappa)],$$

where E denotes the expectation. Then

$$\mathcal{L}^\varepsilon V(x, \kappa) = x'\Big\{A'(\kappa)\widehat{U}(\kappa)G(\kappa) + G'(\kappa)\widehat{U}(\kappa)A(\kappa) + H'(\kappa)\widehat{U}(\kappa)H(\kappa) + \sum_{\iota\in\mathcal{M}} \hat{q}_{\kappa\iota}G'(\iota)\widehat{U}(\iota)G(\iota)\Big\}x.$$

Denote by $\otimes$ the Kronecker product of matrices. We use operators $\llparenthesis \cdot \rrparenthesis$ and $\llparenthesis \cdot, \cdot \rrparenthesis$ defined by

$$\llparenthesis A \rrparenthesis = A \otimes A \quad \text{and} \quad \llparenthesis A, B \rrparenthesis = A \otimes B + B \otimes A.$$

Let $y^{\varepsilon}(t) = E\llparenthesis x^{\varepsilon}(t) \rrparenthesis$. Using an argument similar to that of Huang and Mao [13], we obtain

$$\begin{aligned} &\widetilde{G}(\alpha^{\varepsilon}(t))\dot{y}^{\varepsilon}(t) = \widetilde{A}(\alpha^{\varepsilon}(t))y^{\varepsilon}(t), \\ &y^{\varepsilon}(0) = \widetilde{\xi} = E\llparenthesis \xi \rrparenthesis, \quad \alpha^{\varepsilon}(0) = \iota, \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} &\widetilde{G}(s_{ij}) = \llparenthesis \underset{\sim}{G}(s_{ij}) \rrparenthesis = \llparenthesis \widehat{G}(i) \rrparenthesis \equiv \overline{G}(i), \\ &\underset{\sim}{A}(s_{ij}) = \underset{\sim}{L}^{-1}(s_{ij})L(s_{ij})A(s_{ij}), \\ &\underset{\sim}{H}(s_{ij}) = \underset{\sim}{L}^{-1}(s_{ij})L(s_{ij})H(s_{ij}), \\ &\widetilde{A}(s_{ij}) = \llparenthesis \underset{\sim}{G}(s_{ij}), \underset{\sim}{A}(s_{ij}) \rrparenthesis + \llparenthesis \underset{\sim}{H}(s_{ij}) \rrparenthesis + \sum_{\kappa=1}^{l} \sum_{\iota=1}^{m_\kappa} \hat{q}^{\varepsilon}_{s_{ij}s_{\kappa\iota}} \widetilde{G}(s_{\kappa\iota}). \end{aligned}$$

For $i \in \overline{\mathcal{M}}$, denote

$$\overline{A}(i) = \llparenthesis \widehat{G}(i), \widehat{A}(i) \rrparenthesis + \llparenthesis \widehat{H}(i) \rrparenthesis + \sum_{\kappa=1}^{l} \overline{q}_{i\kappa} \overline{G}(\kappa).$$

The following result can be proved in a similar fashion to the development of Yin and Zhang [17]; the details are omitted.

PROPOSITION 4.2. *Let $y^{\varepsilon}(t)$ be the solution of (4.1). Then as $\varepsilon \to 0$, $y^{\varepsilon}(\cdot)$ converges weakly to $y(\cdot)$, a solution of the singular system of differential equations*

$$\begin{aligned} &\overline{G}(\overline{\alpha}(t))\dot{y}(t) = \overline{A}(\overline{\alpha}(t))y(t), \\ &y(0) = \widetilde{\xi}, \end{aligned}$$

where $\overline{\alpha}(\cdot)$ is a Markov chain generated by $\overline{Q}$.

To carry out the stability analysis, we need more assumptions.

(A4) For each $i = 1, \dots, l$, there exists a nonsingular matrix $\overline{P}(i)$ such that
  (a) $\overline{G}'(i)\overline{P}(i) = \overline{P}'(i)\overline{G}'(i) \geq 0$,
  (b) $\overline{A}'(i)\overline{P}(i) + \overline{P}'(i)\overline{A}(i) + \sum_{i_1 \in \overline{\mathcal{M}}} \overline{q}_{ii_1} \overline{G}'(i_1)\overline{P}(i_1) < 0$.
  Here the inequalities are in the sense of ordering for positive definite matrices.

(A5) For each $s_{ij} \in \mathcal{M}$, $(\widetilde{G}(s_{ij}), \widetilde{A}(s_{ij}))$ is impulse free, that is,

$$\deg[\det(s\widetilde{G}(s_{ij}) - \widetilde{A}(s_{ij}))] = \operatorname{rank} \widetilde{G}(s_{ij}).$$

REMARK 4.3. Note that $\widetilde{G}(s_{ij}) = \overline{G}(i)$. In addition, using the argument in the proof of Lemma 3.1, we can find nonsingular matrices $L(s_{ij})$ for any $s_{ij} \in \mathcal{M}_i$ in the following way. Use elementary row operations to transform

$$[G'(s_{i1})| \cdots |G'(s_{im_i})]$$

into a row echelon matrix

$$[\breve{G}'(s_{i1})|\cdots|\breve{G}'(s_{im_i})].$$

Again, use elementary row operations to transform $[\breve{G}(s_{ij})|I_n]$ into the reduced row echelon matrix

$$\left[\begin{pmatrix} I_{r_i} & 0 \\ 0 & 0 \end{pmatrix} \middle| L(s_{ij})\right],$$

where I_n is an $n\times n$ identity matrix. In addition, if the $L(s_{ij})$ are the same for any $s_{ij}\in\mathcal{M}_i$ then $\underline{L}(s_{ij}) = L(s_{ij})$, and

$$\widetilde{A}(s_{ij}) = \langle\!|\widehat{G}(i), A(s_{ij})|\!\rangle + \langle\!|H(s_{ij})|\!\rangle + \sum_{\kappa=1}^{l}\sum_{\iota=1}^{m_\kappa} \hat{q}^{\varepsilon}_{s_{ij}s_{\kappa\iota}}\widetilde{G}(s_{\kappa\iota}).$$

Lemma 4.4. *If* (A3)–(A5) *hold then there exist constants* $c_1>0$, $c_2>0$ *such that*

(i) $|\dot{y}^{\varepsilon}(t)| \le c_1|y^{\varepsilon}(t)|$,

(ii) $[y^{\varepsilon}(t)]'\overline{G}'(\overline{\alpha}^{\varepsilon}(t))\overline{P}(\overline{\alpha}^{\varepsilon}(t))y^{\varepsilon}(t) \ge c_2|y^{\varepsilon}(t)|^2$,

where the inequalities are in the sense of ordering for positive definite matrices.

Proof. The proof is divided into two steps. To prove (i), we derive from condition (A5) that

$$\underline{G}(s_{ij}) = \widetilde{L}(s_{ij})\widetilde{G}(s_{ij})\widetilde{R}(s_{ij}) = \begin{pmatrix} I_{r_{ij}^2} & 0 \\ 0 & 0 \end{pmatrix},$$

$$\underline{A}(s_{ij}) = \widetilde{L}(s_{ij})\widetilde{A}(s_{ij})\widetilde{R}(s_{ij}) = \begin{pmatrix} \breve{A}(s_{ij}) & 0 \\ 0 & I_{n^2-r_{ij}^2} \end{pmatrix},$$

where $\breve{A}(s_{ij})$ is a nonsingular $r_{ij}^2\times r_{ij}^2$ matrix. Let

$$z^{\varepsilon}(t) = \widetilde{R}^{-1}(\alpha^{\varepsilon}(t))y^{\varepsilon}(t) = [z_1'(t)z_2'(t)]'.$$

Then

$$\begin{aligned}&\underline{G}(\alpha^{\varepsilon}(t))\dot{z}^{\varepsilon}(t) = \underline{A}(\alpha^{\varepsilon}(t))z^{\varepsilon}(t),\\ &z^{\varepsilon}(0) = \widetilde{R}^{-1}(\iota)\widetilde{\xi},\\ &\alpha^{\varepsilon}(0) = \iota.\end{aligned}$$

So

$$\begin{aligned}\dot{z}_1^{\varepsilon}(t) &= \breve{A}(\alpha^{\varepsilon}(t))z_1^{\varepsilon}(t),\\ z_2^{\varepsilon}(t) &= 0,\\ z^{\varepsilon}(0) &= \widetilde{R}^{-1}(\iota)\widetilde{\xi}, \quad \alpha^{\varepsilon}(0) = \iota.\end{aligned}$$

Thus $\dot{z}_2(t) = 0$. Denote

$$\mathring{A}(i) = \begin{pmatrix} \breve{A}(i) & 0 \\ 0 & 0_{n^2-r_i^2} \end{pmatrix} \quad \text{and} \quad \breve{A}(i) = R(i)\mathring{A}(i)R(i)^{-1}.$$

Then

$$\dot{y}^\varepsilon(t) = \breve{A}(\alpha^\varepsilon(t))y^\varepsilon(t).$$

Furthermore, $\breve{A}(i)$ is bounded. Hence the proof of (i) is complete.

For (ii), put

$$\underset{\sim}{\mathrm{P}}(s_{ij}) = (\widetilde{L}^{-1})'(s_{ij})\overline{P}(i)\widetilde{R}(s_{ij}) = \begin{pmatrix} \underset{\sim}{\mathrm{P}}_{11}(s_{ij}) & \underset{\sim}{\mathrm{P}}_{12}(s_{ij}) \\ \underset{\sim}{\mathrm{P}}_{21}(s_{ij}) & \underset{\sim}{\mathrm{P}}_{22}(s_{ij}) \end{pmatrix} \quad \text{for all } s_{ij} \in \mathcal{M}.$$

Then

$$\underset{\sim}{\mathrm{G}}'(s_{ij})\underset{\sim}{\mathrm{P}}(s_{ij}) = \underset{\sim}{\mathrm{P}}'(s_{ij})\underset{\sim}{\mathrm{G}}(s_{ij}) \geq 0,$$

which implies that

$$\underset{\sim}{\mathrm{P}}_{12}(s_{ij}) = \underset{\sim}{\mathrm{P}}_{21}(s_{ij}) = 0 \quad \text{and} \quad \underset{\sim}{\mathrm{P}}_{11}(s_{ij}) > 0.$$

Thus

$$\begin{aligned} [y^\varepsilon(t)]'\overline{G}'(\overline{\alpha}^\varepsilon(t))\overline{P}(\overline{\alpha}^\varepsilon(t))y^\varepsilon(t) &= [y^\varepsilon(t)]'\widetilde{G}'(\alpha^\varepsilon(t))\overline{P}(\overline{\alpha}^\varepsilon(t))y^\varepsilon(t) \\ &= [z^\varepsilon(t)]'\underset{\sim}{\mathrm{G}}'(\alpha^\varepsilon(t))\underset{\sim}{\mathrm{P}}(\alpha^\varepsilon(t))z^\varepsilon(t) \\ &= [z_1^\varepsilon(t)]'\underset{\sim}{\mathrm{P}}_{11}(\alpha^\varepsilon(t))z_1^\varepsilon(t) \\ &\geq c_3|z_1^\varepsilon(t)|^2 = c_3|z^\varepsilon(t)|^2 \geq c_2|y^\varepsilon(t)|^2, \end{aligned}$$

and so (ii) is proved. □

Theorem 4.5. *If* (A3)–(A5) *hold then there exist constants* $\gamma > 0$, $c > 0$ *such that*

$$E|x^\varepsilon(t)|^2 \leq ce^{-\gamma t}\sqrt{\varepsilon}.$$

Proof. For any $\alpha \in \mathcal{M}$, define

$$V_0(y, \alpha) = y'\overline{G}'(\overline{\alpha})\overline{P}(\overline{\alpha})y = y'\widetilde{G}'(\alpha)\overline{P}(\overline{\alpha})y.$$

By the irreducibility of $\widetilde{Q}^i$,

$$\widetilde{Q}V_0(y, \cdot)(\alpha) = 0.$$

Therefore,

$$\mathcal{L}^\varepsilon V_0(y^\varepsilon(t), \alpha^\varepsilon(t)) = [y^\varepsilon(t)]'[\widetilde{A}'(\alpha^\varepsilon(t))\overline{P}(\alpha^\varepsilon(t)) + \overline{P}'(\alpha^\varepsilon(t))\widetilde{A}(\alpha^\varepsilon(t)) + g(\alpha^\varepsilon(t))]y^\varepsilon(t),$$

where

$$g(s_{ij}) = \sum_{\kappa=1}^{l}\sum_{\iota=1}^{m_\kappa} \hat{q}_{s_{ij}s_{\kappa\iota}}\overline{G}'(s_{\kappa\iota})\overline{P}(s_{\kappa\iota}).$$

Denote

$$L = [y^\varepsilon(t)]'[\overline{A}'(\overline{\alpha}^\varepsilon(t))\overline{P}(\overline{\alpha}^\varepsilon(t)) + \overline{P}'(\overline{\alpha}^\varepsilon(t))\overline{A}(\overline{\alpha}^\varepsilon(t)) + \overline{g}(\overline{\alpha}^\varepsilon(t))]y^\varepsilon(t),$$

where

$$\overline{g}(i) = \sum_{\kappa=1}^{l} \overline{q}_{i\kappa}\overline{G}'(\kappa)\overline{P}(\kappa).$$

In order to obtain the desired stability result, we use the method of perturbed Lyapunov functions. Define the perturbations

$$\begin{aligned}
V_1^\varepsilon(y,t) &= E_t \int_t^\infty e^{t-u} y'[\widetilde{A}'(\alpha^\varepsilon(u)) - \overline{A}'(\overline{\alpha}^\varepsilon(u))]\overline{P}(\overline{\alpha}^\varepsilon(u)) y\, du,\\
V_2^\varepsilon(y,t) &= E_t \int_t^\infty e^{t-u} y'\overline{P}'(\overline{\alpha}^\varepsilon(u))[\widetilde{A}(\alpha^\varepsilon(u)) - \overline{A}(\overline{\alpha}^\varepsilon(u))] y\, du,\\
V_3^\varepsilon(y,t) &= E_t \int_t^\infty e^{t-u} y'[g(\alpha^\varepsilon(u)) - \overline{g}(\overline{\alpha}^\varepsilon(u))] y\, du.
\end{aligned}$$

In order to estimate $V_2^\varepsilon(y,t)$ and $\mathcal{L}^\varepsilon V_2^\varepsilon(y^\varepsilon(t),t)$, we consider

$$\begin{aligned}
V_{2A}^\varepsilon(y,t) &= E_t \int_t^\infty e^{t-u} y'\overline{P}'(\overline{\alpha}^\varepsilon(u))\\
&\qquad \times [\llparenthesis \underset{\sim}{\mathrm{G}}(\alpha^\varepsilon(u)), \underset{\sim}{\mathrm{A}}(\alpha^\varepsilon(u))\rrparenthesis - \llparenthesis \widehat{G}(\overline{\alpha}^\varepsilon(u)), \widehat{A}(\overline{\alpha}^\varepsilon(u))\rrparenthesis] y\, du,\\
V_{2H}^\varepsilon(y,t) &= E_t \int_t^\infty e^{t-u} y'\overline{P}'(\overline{\alpha}^\varepsilon(u))[\llparenthesis \underset{\sim}{\mathrm{H}}(\alpha^\varepsilon(u))\rrparenthesis - \llparenthesis \widehat{H}(\overline{\alpha}^\varepsilon(u))\rrparenthesis] y\, du,\\
V_{2Q}^\varepsilon(y,t) &= E_t \int_t^\infty e^{t-u} y'\overline{P}'(\overline{\alpha}^\varepsilon(u))[h(\alpha^\varepsilon(u)) - \overline{h}(\overline{\alpha}^\varepsilon(u))] y\, du,
\end{aligned}$$

where

$$h(s_{ij}) = \sum_{\kappa=1}^{l}\sum_{\iota=1}^{m_i} \hat{q}^\varepsilon_{s_{ij}s_{\kappa\iota}} \widetilde{G}(s_{\kappa\iota}) \quad \text{and} \quad \overline{h}(i) = \sum_{\kappa=1}^{l} \overline{q}_{i\kappa}\overline{G}(\kappa).$$

Then

$$V_2^\varepsilon(y,t) = V_{2A}^\varepsilon(y,t) + V_{2H}^\varepsilon(y,t) + V_{2Q}^\varepsilon(y,t).$$

On the one hand,

$$\begin{aligned}
&\llparenthesis \underset{\sim}{\mathrm{G}}(\alpha^\varepsilon(u)), \underset{\sim}{\mathrm{A}}(\alpha^\varepsilon(u))\rrparenthesis - \llparenthesis \widehat{G}(\overline{\alpha}^\varepsilon(u)), \widehat{A}(\overline{\alpha}^\varepsilon(u))\rrparenthesis\\
&\quad = \llparenthesis \widehat{G}(\overline{\alpha}^\varepsilon(u)), \underset{\sim}{\mathrm{A}}(\alpha^\varepsilon(u)) - \widehat{A}(\overline{\alpha}^\varepsilon(u))\rrparenthesis\\
&\quad = \llparenthesis \widehat{G}(\overline{\alpha}^\varepsilon(u)), \underset{\sim}{\mathrm{L}}^{-1}(\alpha^\varepsilon(u))L(\alpha^\varepsilon(u))[A(\alpha^\varepsilon(u)) - \widehat{A}(\overline{\alpha}^\varepsilon(u))]\rrparenthesis\\
&\qquad + \llparenthesis \widehat{G}(\overline{\alpha}^\varepsilon(u)), [\underset{\sim}{\mathrm{L}}^{-1}(\alpha^\varepsilon(u)) - L^{-1}(\alpha^\varepsilon(u))]L(\alpha^\varepsilon(u))\widehat{A}(\overline{\alpha}^\varepsilon(u))\rrparenthesis.
\end{aligned}$$

On the other hand,

$$A(\alpha^\varepsilon(u)) - \widehat{A}(\overline{\alpha}^\varepsilon(u)) = \sum_{i=1}^{l}\sum_{j=1}^{m_i} A(s_{ij})[\chi(\alpha^\varepsilon(u) = s_{ij}) - \nu^i_j \chi(\alpha^\varepsilon(u) \in \mathcal{M}_i)],$$

$$\underset{\sim}{\mathrm{L}}^{-1}(\alpha^\varepsilon(u)) - L^{-1}(\alpha^\varepsilon(u)) = -\sum_{i=1}^{l}\sum_{j=1}^{m_i} L^{-1}(s_{ij})[\chi(\alpha^\varepsilon(u) = s_{ij}) - \nu^i_j \chi(\alpha^\varepsilon(u) \in \mathcal{M}_i)],$$

and for $u \geq t$,

$$E_t^{\varepsilon}[\chi(\alpha^{\varepsilon}(u) = s_{ij}) - \nu_j^i \chi(\overline{\alpha}^{\varepsilon}(u) = i)] = O(\varepsilon + e^{-k_0(u-t)/\varepsilon}).$$

Therefore,

$$|V_{2A}^{\varepsilon}(y,t)| \leq \sum_{i=1}^{l} \sum_{j=1}^{m_i} |y|^2 \int_t^{\infty} e^{t-u} O(\varepsilon + e^{-k_0(u-t)/\varepsilon})\, du = O(\varepsilon)|y|^2.$$

Similarly, from the fact that

$$\begin{aligned}
\llparenthesis \underset{\sim}{H}(\alpha^{\varepsilon}(u)) \rrparenthesis - \llparenthesis \widehat{H}(\overline{\alpha}^{\varepsilon}(u)) \rrparenthesis &= \llparenthesis \underset{\sim}{H}(\alpha^{\varepsilon}(u)), \underset{\sim}{H}(\alpha^{\varepsilon}(u)) - \widehat{H}(\overline{\alpha}^{\varepsilon}(u)) \rrparenthesis \\
&\qquad + \llparenthesis \underset{\sim}{H}(\alpha^{\varepsilon}(u)) - \widehat{H}(\overline{\alpha}^{\varepsilon}(u)), \widehat{H}(\overline{\alpha}^{\varepsilon}(u)) \rrparenthesis, \\
h(\alpha^{\varepsilon}(u)) - \overline{h}(\overline{\alpha}^{\varepsilon}(t)) &= \sum_{i=1}^{l} \sum_{j=1}^{m_i} \widehat{Q}\widetilde{G}(\cdot)(s_{ij}) E_t^{\varepsilon}[\chi(\alpha^{\varepsilon}(u) = s_{ij}) - \nu_j^i \chi(\overline{\alpha}^{\varepsilon}(u) = i)],
\end{aligned}$$

we derive

$$|V_{2H}^{\varepsilon}(y,t)| \leq O(\varepsilon)|y|^2, \quad |V_{2Q}^{\varepsilon}(y,t)| \leq O(\varepsilon)|y|^2.$$

Thus

$$|V_2^{\varepsilon}(y,t)| \leq O(\varepsilon)|y|^2.$$

Furthermore,

$$\begin{aligned}
\mathcal{L}^{\varepsilon} V_2^{\varepsilon}(y^{\varepsilon}(t), t) &= \lim_{\delta \downarrow 0} E_t^{\varepsilon}[V_2^{\varepsilon}(y^{\varepsilon}(t+\delta), t+\delta) - V_2^{\varepsilon}(y^{\varepsilon}(t), t)] \\
&= \lim_{\delta \downarrow 0} E_t^{\varepsilon}[V_2^{\varepsilon}(y^{\varepsilon}(t+\delta), t+\delta) - V_2^{\varepsilon}(y^{\varepsilon}(t), t+\delta)] \\
&\qquad + \lim_{\delta \downarrow 0} E_t^{\varepsilon}[V_2^{\varepsilon}(y^{\varepsilon}(t), t+\delta) - V_2^{\varepsilon}(y^{\varepsilon}(t), t)] \\
&= -[y^{\varepsilon}(t)]' \overline{P}'(\overline{\alpha}^{\varepsilon}(t))[\widetilde{A}(\alpha^{\varepsilon}(t)) - \overline{A}(\overline{\alpha}^{\varepsilon}(t))] y^{\varepsilon}(t) + V_2^{\varepsilon}(y^{\varepsilon}(t), t) \\
&\qquad + E_t \int_t^{\infty} e^{t-u} [\dot{y}^{\varepsilon}(t)]' \overline{P}'(\overline{\alpha}^{\varepsilon}(t))[\widetilde{A}(\alpha^{\varepsilon}(u)) - \overline{A}(\overline{\alpha}^{\varepsilon}(u))] y^{\varepsilon}(t)\, du \\
&\qquad + E_t \int_t^{\infty} e^{t-u} [y^{\varepsilon}(t)]' \overline{P}'(\overline{\alpha}^{\varepsilon}(t))[\widetilde{A}(\alpha^{\varepsilon}(u)) - \overline{A}(\overline{\alpha}^{\varepsilon}(u))] \dot{y}^{\varepsilon}(t)\, du \\
&\leq -[y^{\varepsilon}(t)]' \overline{P}'(\overline{\alpha}^{\varepsilon}(t))[\widetilde{A}(\alpha^{\varepsilon}(t)) - \overline{A}(\alpha^{\varepsilon}(t))] y^{\varepsilon}(t) + O(1) V_2^{\varepsilon}(y^{\varepsilon}(t), t) \\
&\leq -[y^{\varepsilon}(t)]' \overline{P}'(\overline{\alpha}^{\varepsilon}(t))[\widetilde{A}(\alpha^{\varepsilon}(t)) - \overline{A}(\alpha^{\varepsilon}(t))] + O(\varepsilon)|y^{\varepsilon}(t)|^2.
\end{aligned}$$

Using a similar argument, we obtain

$$\begin{aligned}
&|V_1^{\varepsilon}(y,t)| = O(\varepsilon)|y|^2, \quad |V_3^{\varepsilon}(y,t)| = O(\varepsilon)|y|^2, \\
&\mathcal{L}^{\varepsilon} V_1^{\varepsilon}(y^{\varepsilon}(t), t) \leq -[y^{\varepsilon}(t)]'[\widetilde{A}'(\alpha^{\varepsilon}(t)) - \overline{A}'(\overline{\alpha}^{\varepsilon}(t))] \overline{P}(\overline{\alpha}^{\varepsilon}(t)) y^{\varepsilon}(t) + O(\varepsilon)|y^{\varepsilon}(t)|^2, \\
&\mathcal{L}^{\varepsilon} V_3^{\varepsilon}(y^{\varepsilon}(t), t) \leq -[y^{\varepsilon}(t)]'[g(\alpha^{\varepsilon}(u)) - \overline{g}(\overline{\alpha}^{\varepsilon}(u))] y^{\varepsilon}(t) + O(\varepsilon)|y^{\varepsilon}(t)|^2.
\end{aligned}$$

Define

$$V^\varepsilon(t) = V_0(y^\varepsilon(t), \alpha^\varepsilon(t)) + \sum_{i=1}^{3} V_i^\varepsilon(y^\varepsilon(t), t).$$

Then

$$V^\varepsilon(t) = V_0(y^\varepsilon(t), \alpha^\varepsilon(t)) + O(\varepsilon)|y^\varepsilon(t)|^2.$$

In addition,

$$\mathcal{L}^\varepsilon V^\varepsilon(t) \le L + O(\varepsilon)|y^\varepsilon(t)|^2.$$

By assumption, there exists a constant $\gamma > 0$ such that

$$L + \gamma V_0(y^\varepsilon(t), \alpha^\varepsilon(t)) \le L + \gamma c|y^\varepsilon(t)|^2 \le 0.$$

Using

$$\mathcal{L}^\varepsilon(e^{\gamma t} V^\varepsilon(t)) = e^{\gamma t}(\gamma V^\varepsilon(t) + \mathcal{L}^\varepsilon V^\varepsilon(t)) \le e^{\gamma t} O(\varepsilon)|y^\varepsilon(t)|^2,$$

we obtain

$$\begin{aligned} E[e^{\gamma t} V^\varepsilon(t)] &\le EV^\varepsilon(0) + E\int_0^t e^{\gamma u} O(\varepsilon)|y^\varepsilon(u)|^2\, du \\ &\le O(\varepsilon)|\widetilde{\xi}|^2 + E\int_0^t e^{\gamma u} O(\varepsilon)|y^\varepsilon(u)|^2\, du. \end{aligned}$$

On the other hand,

$$V_0(y^\varepsilon(t), \alpha^\varepsilon(t)) > \frac{|y^\varepsilon(t)|^2}{c}.$$

Therefore,

$$E[e^{\gamma t}|y^\varepsilon(t)|^2] \le O(\varepsilon)|\widetilde{\xi}|^2 + E\int_0^t e^{\gamma u} O(\varepsilon)|y^\varepsilon(u)|^2\, du.$$

Gronwall's inequality [10, p. 36] yields

$$E[e^{\gamma t}|y^\varepsilon(t)|^2] \le O(\varepsilon)|\widetilde{\xi}|^2.$$

Hence

$$E|y^\varepsilon(t)|^2 \le e^{-\gamma t} O(\varepsilon)|\widetilde{\xi}|^2.$$

Therefore,

$$E|x^\varepsilon(t)|^2 \le CE|\widetilde{x}^\varepsilon(t)|^2 \le C\sqrt{E|y^\varepsilon(t)|^2} \le e^{-\gamma t/2} O(\sqrt{\varepsilon})|\xi|^2.$$

The proof is complete. □

5. Examples

In this section we provide examples to illustrate two-time-scale singular systems. In these examples, the matrix manipulations were carried out using Maple. Since the paper focuses on developing the two-time-scale models, but not on large-scale simulations, only simple examples are presented.

EXAMPLE 5.1. Let $\alpha^\varepsilon(t)$ be a switching process taking values in $\mathcal{M} = \{1, 2\}$ with generator

$$Q^\varepsilon = \frac{\widetilde{Q}}{\varepsilon}, \quad \text{where } \widetilde{Q} = \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \text{ and } \varepsilon = 0.1.$$

Consider the singular system

$$G(\alpha^\varepsilon(t))\dot{x}(t) = A(\alpha^\varepsilon(t))x(t)\,dt + H(\alpha^\varepsilon(t))x(t)\,dB(t), \tag{5.1}$$

where

$$G(1) = G(2) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A(1) = \begin{bmatrix} 0 & 0.5 \\ 0 & 0 \end{bmatrix}, \quad A(2) = \begin{bmatrix} -3.5 & 0 \\ 0 & 0 \end{bmatrix},$$

$$H(1) = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad H(2) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then

$$\widetilde{G}(1) = \widetilde{G}(2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\widetilde{A}(1) = \begin{bmatrix} 1 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.25 \end{bmatrix}, \quad \widetilde{A}(2) = \begin{bmatrix} -3.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now let us consider the corresponding limit singular system

$$\overline{G}\dot{\overline{y}}(t) = \overline{A}\overline{y}(t),$$

where

$$\overline{G} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \overline{A} = \begin{bmatrix} -2.3333 & 0.1667 & 0 & 0 \\ 0 & 0.1667 & 0 & 0 \\ 0 & 0 & 0.1667 & 0 \\ 0 & 0 & 0 & 0.75 \end{bmatrix}.$$

Then

$$\overline{G}'P = P'\overline{G} = \begin{bmatrix} P_{1,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P_{1,2} = P_{1,3} = P_{1,4} = 0.$$

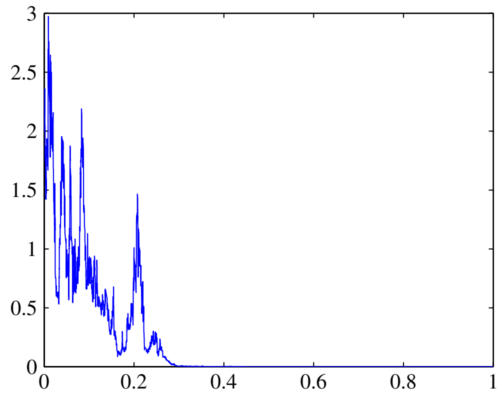

FIGURE 1. A trajectory of the singular dynamical system (5.1). Time between 0 and 1; $\varepsilon = 0.001$.

Thus

$$\overline{A}'P + P'\overline{A}$$
$$= \begin{bmatrix} -4.0\,P_{1,1} & 0.1\bar{6}\,P_{1,1} + 0.1\bar{6}\,P_{2,1} & 0.1\bar{6}\,P_{3,1} & 0.75\,P_{4,1} \\ 0.1\bar{6}\,P_{1,1} + 0.1\bar{6}\,P_{2,1} & 0.\bar{3}\,P_{2,2} & 0.1\bar{6}\,P_{2,3} + 0.1\bar{6}\,P_{3,2} & 0.1\bar{6}\,P_{2,4} + 0.75\,P_{4,2} \\ 0.1\bar{6}\,P_{3,1} & 0.1\bar{6}\,P_{2,3} + 0.1\bar{6}\,P_{3,2} & 0.\bar{3}\,P_{3,3} & 0.1\bar{6}\,P_{3,4} + 0.75\,P_{4,3} \\ 0.75\,P_{4,1} & 0.1\bar{6}\,P_{2,4} + 0.75\,P_{4,2} & 0.1\bar{6}\,P_{3,4} + 0.75\,P_{4,3} & 1.50\,P_{4,4} \end{bmatrix}.$$

Hence $\overline{G}$ and $\overline{A}$ satisfy (A3)–(A5) with

$$\overline{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Therefore, (5.1) is asymptotically mean-square stable.

Next we plot sample paths of the system, using MATLAB with step size $h = 0.0001$. We obtain Figures 1–3 for the first coordinate. The system quickly reaches its limit position. Figure 1 shows the result for $\varepsilon = 0.001$, and Figure 3 shows that result in four smaller intervals. Figure 2 displays the sample path and trajectory corresponding to $\varepsilon = 0.1$. It is seen that the smaller the value of ε, the faster the system decays.

EXAMPLE 5.2. In this example, we consider a slightly more complicated singular matrix. Let $\alpha^\varepsilon(t)$ be a switching process taking values in $\mathcal{M} = \{1, 2\}$ with generator $Q^\varepsilon = \widetilde{Q}/\varepsilon + \widehat{Q}$, where

$$\widetilde{Q} = \begin{bmatrix} -2 & 2 \\ 5 & -5 \end{bmatrix}, \quad \widehat{Q} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

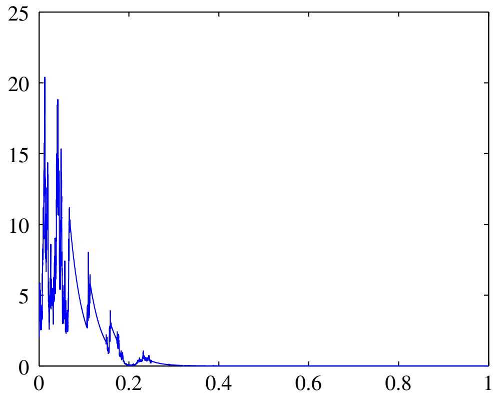

FIGURE 2. A trajectory of the singular dynamical system (5.1). Time between 0 and 1; $\varepsilon = 0.1$.

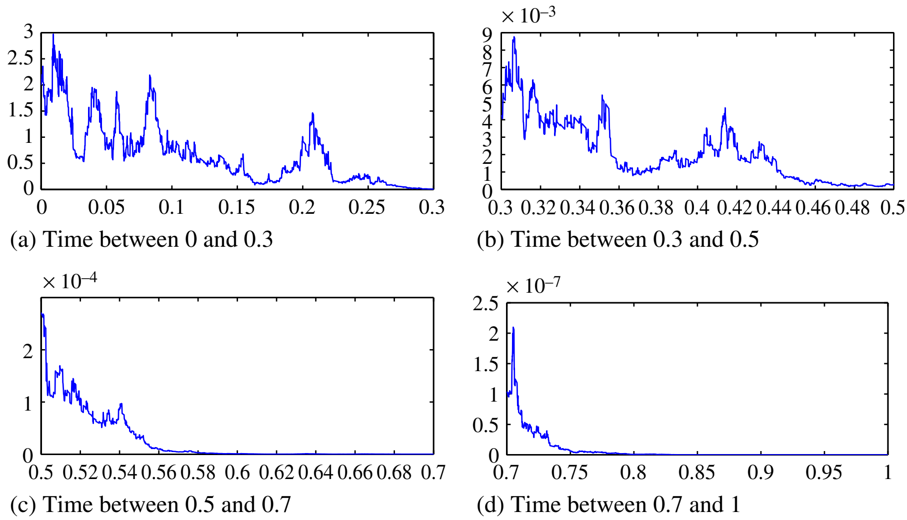

FIGURE 3. A trajectory of the singular dynamical system (5.1) with $\varepsilon = 0.001$.

and $\varepsilon = 0.1$. Consider the singular system

$$G(\alpha^{\varepsilon}(t))\dot{x}(t) = A(\alpha^{\varepsilon}(t))x(t)\,dt + H(\alpha^{\varepsilon}(t))x(t)\,dB(t), \qquad x(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \tag{5.2}$$

where

$$G(1) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad G(2) = \begin{bmatrix} -1 & -2 \\ 3 & 6 \end{bmatrix}, \quad A(1) = \begin{bmatrix} -4 & -8 \\ -8 & -16 \end{bmatrix},$$
$$A(2) = \begin{bmatrix} -4 & 2 \\ 12 & -6 \end{bmatrix}, \quad H(1) = \begin{bmatrix} 18 & -9 \\ 12 & -6 \end{bmatrix}, \quad H(2) = \begin{bmatrix} -9 & 2 \\ 7 & 4 \end{bmatrix}.$$

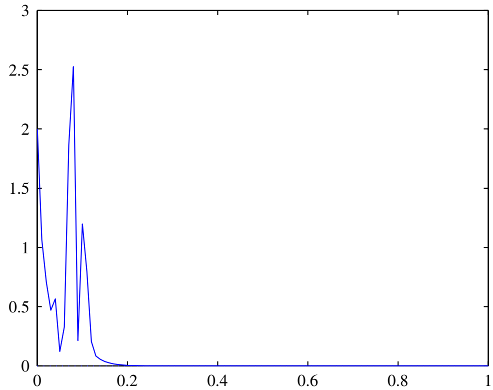

FIGURE 4. A trajectory of the singular dynamical system (5.2). Time between 0 and 1; $\varepsilon = 0.1$.

Analogous to Example 5.1, (5.2) is asymptotically mean-square stable as demonstrated in Figure 4.

EXAMPLE 5.3. Let $\alpha^{\varepsilon}(t)$ be a switching process taking values in $\mathcal{M} = \{1, 2, 3, 4\}$ with generator $Q^{\varepsilon} = \widetilde{Q}/\varepsilon$, where

$$\widetilde{Q} = \begin{bmatrix} -4 & 4 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 3 & -3 \end{bmatrix}, \quad \widehat{Q} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix},$$

and $\varepsilon = 0.1$. Consider the singular system

$$G(\alpha^{\varepsilon}(t))\dot{x}(t) = A(\alpha^{\varepsilon}(t))x(t)\,dt + H(\alpha^{\varepsilon}(t))x(t)\,dB(t), \tag{5.3}$$

where

$$G(1) = G(2) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad G(3) = G(4) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A(1) = \begin{bmatrix} 0 & 0.5 \\ 0 & 0 \end{bmatrix},$$
$$A(2) = \begin{bmatrix} -3.5 & 0 \\ 0 & 0 \end{bmatrix}, \quad A(3) = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, \quad A(4) = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix},$$
$$H(1) = \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad H(2) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad H(3) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad H(4) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

As $\varepsilon \to 0$, $E(x^{\varepsilon}(t)) \to z(t)$ such that

$$\overline{G}(\overline{\alpha}(t))\,dz(t) = \overline{A}(\overline{\alpha}(t))z(t)\,dt,$$

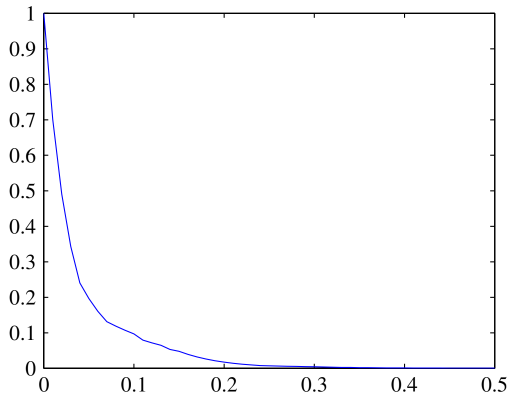

FIGURE 5. A trajectory of the singular dynamical system (5.3). Time between 0 and 0.5; $\varepsilon = 0.1$.

where $\overline{\alpha}(t)$ is the Markov chain generated by

$$\overline{Q} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \text{and} \quad \overline{G}(1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \overline{G}(2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\overline{A}(1) = \begin{bmatrix} -2.8 & 0.1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1.85 \end{bmatrix}, \quad \overline{A}(2) = \begin{bmatrix} -1.4 & 0 & 0 & 0.6 \\ 0 & 0.6 & 0 & 0 \\ 0 & 0 & -2.4 & 0 \\ 0 & 0 & 0 & 0.6 \end{bmatrix}.$$

Since $\overline{G}$ and $\overline{A}$ satisfy (A3)–(A5), (5.3) is asymptotically mean-square stable as demonstrated in Figure 5.

6. Further remarks

This work has been devoted to singular jump-linear systems whose switching processes have large state spaces. Thus, alternatively, such systems could be called singular systems with singularly perturbed Markov chains. The multi-scale structure and two-time-scale formulation are used to reflect that the discrete event process in the system has a large state space. We have established a reduction of complexity from the stability point of view, using perturbed Lyapunov function methods. The conclusion is that as the small parameter goes to 0, we can use the stability of the limit system to infer that of the original system. The original system is normally difficult to analyse because of its large dimension, whereas the limit system is relatively simpler. Thus the result provides a practical guideline for treating many such systems.

Some of the conditions used in our main results seem to be technical, such as some of the rank conditions. For example, (A3) and (A5) may not be easy to verify analytically, but could be verifiable using numerical methods in applications, and (A4) is a standard quadratic form which can be solved by linear matrix inequality techniques. In any case, relaxing these conditions will certainly be a worthwhile effort.

We have provided several examples for illustration only. The idea presented in the paper is applicable to large-scale systems. Working with large-scale simulation and numerical examples is another worthwhile direction for future research that will be important for various applications.

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