

12 Teaching mathematics

Self-knowledge, pupil knowledge and content knowledge

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Introduction

Mathematical learning is significantly influenced by the quality of mathematics teaching (Hiebert and Grouws 2007). In spite of the evidence for teachers seeking to do what they believe to be in the best interests of their learners (Schuck 2009; Gholami and Husu 2010), research and policy reports (within the UK and beyond) draw attention to insufficient mathematical attainment (Williams 2008; Eurydice 2011). Why is there this discrepancy? On the one hand, teachers are open to improving their professional practices (Escudero and Sánchez 2007), and on the other, the findings of mathematical education research make little or no impact on teachers' practice (William 2003), even although teachers themselves think that they are enacting new or revised practices (Speer 2005).

This chapter seeks to address this discrepancy not through offering advice on what works in classrooms (because such advice rarely has the same meaning for all classrooms) but through causing teachers to reflect on their own expertise. Because teaching is understood not only as overt behaviour but also as the teacher's thinking – precipitated by the situational, developmental and contextual needs of particular learners (Shulman 1988) – this chapter explores three dimensions of the teacher's thinking: teachers' self-knowledge, pupil knowledge and content knowledge. Since dialogue, reflection and discussion are central to teacher learning, the chapter is structured to stimulate readers to think about their own practices as well as discussing practices with peers. The rationale for this approach is twofold. First, regardless of whether one is teaching at early or later stages of school, profound mathematics teaching involves learners constructing understandings of mathematics (Ginsburg and Amit 2008). Second, as teachers talk and work together on matters that are deeply situated in practice, they share and attempt to understand how each deals

214 IMPROVING PRIMARY MATHEMATICS TEACHING AND LEARNING

with specific concepts and procedures, they identify inconsistencies in their collective knowledge and they consult authoritative sources. Further, they set goals, monitor collaborative effort, and negotiate future courses of action. In so doing, they steer and organize the construction of their corporate knowledge, through taking account of different contributions in the context of their own teaching (Hurme et al. 2006; Damşa et al. 2010).

It is teachers' lived professional experience that on a daily basis they will be met with learners who vary in their prior mathematical experience and their motivation to learn. Because classes are constructed according to chronological age, this variation can include learners who are advanced, who underachieve, who grapple with the language of instruction, who come from diverse cultural and/or economic backgrounds, and who have learning problems. Indeed some learners may fit into more than one of the categories. The variation of some learners performing at, some performing below and some capable of performing well above, stage-level expectations is demanding in any curricular area, but teachers are particularly challenged by the mathematics curriculum because of learners' lack of expected progression and failure to achieve stage-level standards (Miller and Hudson 2007).

How teachers respond to this variation and capitalize on professional support to maximize their own learning can be understood in terms of teachers' expertise. Some will be overwhelmed by the vast amount of mathematics research information that is 'out there'. At the other end of the continuum will be teachers who will evaluate that information and select from it in ways that are fit-for-purpose. When people work in any domain they are constantly in the process of developing expertise. Through sensitizing ourselves to notice what was previously taken-for-granted or unnoticed, our observations can inform action in the future. By becoming aware of how our attention shifts, we can learn, intentionally, from experience. Dreyfus characterized this growth of expertise through five different stages (Dreyfus and Dreyfus 1980), which represent qualitatively different forms of learning behaviour (Berliner 1994; Flyvberg 2001).

Reflection

Consider the Novice-to-Expert Levels in Table 12.1 and also think of a mathematical topic that you teach.

- How expert do you consider yourself to be in teaching that topic?
- What implications for your continuing professional development arise from your earlier judgement?

Table 12.1 Novice-to-expert levels

	Knowledge	Coping with complexity	Perception of context
Novice	Minimal or 'textbook' knowledge without connecting it to practice	Little or no conception of dealing with complexity	Tends to see actions in isolation
Beginner	Working knowledge of key aspects of practice	Appreciates complex situations but only able to achieve partial resolution	Sees actions as a series of steps
Competent	Good working and background knowledge of area of practice	Copes with complex situations through deliberate analysis and planning	Sees actions at least partly in terms of longer-term goals
Proficient	Depth of understanding of discipline and area of practice	Deals with complex situations holistically, decision-making more confident	Sees overall 'picture' and how individual actions fit within it
Expert	Authoritative knowledge of discipline and deep tacit understanding across area of practice	Holistic grasp of complex situations, moves between intuitive and analytical approaches with ease	Sees overall 'picture' and alternative approaches; vision of what may be possible

Source: Adapted from Lester (2005)

What has been considered as particularly significant in the Dreyfus model is that the development of proficiency and expertise is marked by an abandonment of the rule-based thinking which underpins the thinking of the first three levels (Dreyfus and Dreyfus 1980). Rather, expertise is characterized by an extensive, well-organized and flexibly accessible domain-specific knowledge base. Further, this knowledge base is one to which experts add all the time as they reflect on previous teaching episodes and strive to ensure greater refinement in subsequent episodes.

216 IMPROVING PRIMARY MATHEMATICS TEACHING AND LEARNING

This dynamic characteristic of expertise is consistent with the evidence that while teachers' mathematical knowledge is an important aspect of effective teaching (Hill et al. 2005), it is not of itself sufficient (Shechtman et al. 2010). In addition to, and in interaction with, teachers' mathematical knowledge, proficient and expert teachers are both highly motivated and self-regulating.

Teachers' self-knowledge

The professional status of teachers requires them to initiate, control and manage teaching episodes by themselves; in other words that they be self-regulating. Such autonomy is very necessary because the sheer complexity of the mathematics classroom means that there can be no algorithmic response available for every problem that presents itself. Not only do teachers need to be aware of/ monitor their own thinking, understanding and knowledge about teaching but also they need to be sensitive to the different kinds of knowledge which they can draw upon to help develop their practice (Parsons and Stephenson 2005). Such self-regulation is defined as:

the self-conscious monitoring of one's cognitive activities, the elements used in those activities, and the results deduced, particularly by applying skills in analysis and evaluation to one's own inferential judgments with a view toward questioning, confirming, validating, or correcting either one's reasoning or one's results.

(Facione 1990: 10)

Self-regulation is commonly understood as being an iteratively phased activity. Teacher thinking in relation to teaching episodes can readily map on to the different phases (see Table 12.2).

Reflection

Consider a mathematics teaching episode which has depended on your self-regulation. Map onto the framework above the Planning you did in advance of the lesson, the Monitoring and Control in which you engaged during the lesson and your Reflections after the lesson. This mapping is unlikely to be the same for different teachers and unlikely to be the same for different lessons. If you are working with others, compare your responses. If working on your own compare your self-regulation for two different mathematics teaching episodes. In either event, try to tease out the implications of your analyses for your further development.

Table 12.2 Teachers' cognitive self-regulation

Phase	Teacher cognitive self-regulation in relation to a teaching episode	Other considerations which arise from, and feed into, teachers' regulation
Planning	<ul style="list-style-type: none"> • Determines targets and standards; sets out objectives • Considers the prior knowledge learners need to have in order to benefit from teaching episode • Anticipates potential obstacles in the resources to be used by learners 	<ul style="list-style-type: none"> • Time and effort needed by teacher for planning • Judging how self-efficacious learners may be to episode • Judging how interested learners may be in episode
Monitoring	<ul style="list-style-type: none"> • Formal/informal judgements of learning, of task appropriateness and of contextual conditions • Self-assessment of one's own planning 	<ul style="list-style-type: none"> • Skill in diagnosing moment-to-moment interactions between and among learners and their context • Determining if and when to act in relation to indicators of learning
Control	<ul style="list-style-type: none"> • Adjusting episode to 'repair' what is going wrong • Adapting episode to accelerate learning when it is evident that planning underestimated learner competence 	<ul style="list-style-type: none"> • Options for changing/renegotiating task • Selection/adaptation of strategies to promote learning and manage motivation
Reflection	<ul style="list-style-type: none"> • Judges whether episode did or didn't work using predetermined criteria • Considers why/offers explanation of, the task 'worked' or didn't 'work' 	<ul style="list-style-type: none"> • Teachers' knowledge of attribution, motivation, self-efficacy • Autonomy to modify subsequent teaching episode

218 IMPROVING PRIMARY MATHEMATICS TEACHING AND LEARNING

In teaching, these phases of self-regulation need constant adaptation because the dynamics of the environment and the social relationships of the classroom. Thus it is through the many cycles of planning, monitoring, controlling and reflecting that practice develops, and it is through deliberate practice that expertise develops, although the practice may be neither enjoyable nor easy.

The importance of teacher self-regulation is evidenced in learners who are self-regulating since their initiative, motivation and personal responsibility are mirrored in academic success (Nota et al. 2004). Space limitations here prevent adequate consideration of how self-regulation develops, but the more teachers are aware of both cognitive factors (developmental differences in learners' working memory; the role of prior knowledge; self-awareness; awareness of task's constituent demands; feelings of familiarity, difficulty, confidence, satisfaction) and motivational factors (judgements of learning; competence; interest and value; goal orientation; self-efficacy and volition) (Pintrich and Zusho 2002) the more autonomous teachers can be in using their adaptive expertise (Berliner 2001) to enable learners' deeper understanding of the subject-matter. Expert teachers pay close attention not only to the factual accuracy of learners' responses but also to the logical persuasiveness of the learners' reasoning (Sato et al. 1993). If teachers are unaware of learners' thinking, they are less likely to engage learners in self-regulating activities (Parsons and Stephenson 2005) thereby inhibiting the acceleration of learning which is available even to low-achievers when self-regulation is deployed (Zohar and Ben David 2008; Zohar and Peled 2008). The reality of self-regulation is in the strong linguistic component which characterizes many classrooms and which provides the mechanism for learners to negotiate how they make rules, draw conclusions, approach/solve a task, justify choices, and evaluate the advantages/disadvantages of different strategic behaviour. It is through consciously embedding these opportunities in all teaching episodes that teachers can support learners' self-regulation (Carr et al. 2011).

Pupil knowledge

Concerns about mathematical achievement centre on learners' understanding (Hiebert and Lefevre 1986). Learning with understanding is increasingly coming into sharper focus through the realization that the memorization of facts or procedures without understanding often results in fragile learning. This is not to say that factual accuracy and procedural facilitation are unimportant: just that they are insufficient of themselves. Understanding is mental activity which makes use of knowledge: facts,

concepts, principles, procedures and phenomena which each individual stores in memory as a connection or network. This knowledge varies from person to person as do the unique ways in which each individual forms connections between pieces of knowledge. The different pieces of knowledge are representations which are (unique-to-the-individual) configurations of symbols, real objects or events and mental images (Janvier 1987). So, for example, one can 'picture in one's head' what a cat is through a picture, or a verbal description, or a recent visit to the zoo. Thus individual understanding of 3 might be linked to a number of representations: such as the counting sequence 1, 2, 3; to the numeral 3; to three fingers; to 3 being 1 less than 4; and/or to other experiences that individuals have had. Constructing representations can be complex. To solve word problems learners must create different mental representations: they need to understand the numerical values and the quantitative relations between them (mathematical understanding) and they need to understand what information is essential and what information is less important (contextual/situational understanding). They then mathematize the situation: lock the two representations together in terms of previous mathematical knowledge, to determine how to proceed. Appropriate operation(s) can then be enacted and results interpreted in terms of both the mathematics and the situation described in the problem. But if learners, instead, use a meaningless strategy (perhaps identifying salient words such as 'more' or 'less' with particular operations) without regard for the situation, they do not fully understand either what the problem asked or how good their solution is (Thevenot et al. 2007). Understanding is thus the process of constructing networks of meaning between existing bits of knowledge and integrating new representations. Although it is never complete, understanding is said to develop as representations are connected in progressively more elaborate networks.

Individual internal representations of any idea are extremely important, but being internal are not visible to others; they have to be brought into a form which others can see. To express any mathematical concept or problem, a representation must necessarily be used (Dreyfus and Eisenberg 1996). Symbols, pictures, language, counters, number lines, fraction bars, cubes, graphs, tables or formulae are common external representations. Being able to 'see' or identify the same mathematical concept or problem in different representations (and thereby move fluently among representations) indicates stronger understanding (Dufour-Janvier et al. 1987). This is analogous to travelling to a new destination. The traveller may have a set of directions listing street names and turns to follow, which is perfectly adequate albeit limiting. If, additionally, the traveller has a map of the area, he/she can determine the most efficient route among neighbouring streets, and accommodate road

220 IMPROVING PRIMARY MATHEMATICS TEACHING AND LEARNING

closures. In other words, understanding of navigation is strengthened by having more than one representation. Because any single external representation cannot describe fully a mathematical construct, the limitations of any particular representation can be overtaken by using multiple representations (just as a map is another resource to support the traveller), which help learners to construct a better picture of a mathematical concept.

Learners are presented regularly with representations since these are what teachers and learners use as carriers of knowledge and as thinking tools to describe or explain a concept, a relationship, or a problem. External representations are important because they interact with learners' internal representations and help the learner to make more sense (Greeno 1991). But learners tend to discount their common-sense knowledge of the real world and view mathematics as artificial and disconnected from reality (Greer et al. 2002). So the richness of available representations and the facility to 'translate' between them need to be an explicit focus of teaching if learners are to avoid confusing external representations of concepts with the concepts themselves (Janvier 1987; Greeno 1991). In other words external representations mustn't be privileged to the point of being manipulated without reference to the internal representation that the learner is using. Neither is it helpful to impose external representations or coerce learners to use particular external representations. Teachers therefore have to elicit learner's existing thinking and ideas so that new knowledge is not isolated from existing knowledge. The mechanism for this is discussion which allows:

- Teachers to identify what strategies learners are using (and thereby decide how to respond and/or whether to build on that particular learner's suggestion)
- Learners to share their thinking with the other members of the class
- Others to comment, question, elaborate on the suggestion.

A prototypical sequence for such discussion might be:

- Teacher poses a question or problem to the class (possibly making three or four answers available for consideration).
- Learners have time to think of answers or responses individually.
- Learners discuss possible responses (in groups of four or fewer).
- Individual learners 'vote' for an answer or a preferred solution.
- Teacher tallies and displays distribution of votes to the class.
- Class discussion ensues, with learners justifying their answers.
- Teacher moderates discussion to allow closure.

Reflection

The following three aspects of the significant concept, place value, are ones that you might, at times, want to emphasize in a teaching episode. Consider how much emphasis you put on the different representations which together constitute a robust understanding of the concept.

1. The relationship between the oral name and the numeral. Perhaps the easiest of the representations is the connection between what we call 'numbers' and their written form. Tasks such as 'Can you read these numbers: 17, 39, 50, 56, 71?' and 'Circle numbers on the chart as I read them: 14, 25, 42' focus on this relationship. Our 'twelve' in some other cultures is 'ten-two', and our 'forty-seven' is 'four-ten-seven'. We use 'ty' to mean 'ten', but 'twen', 'thir' and 'fif' don't sound like 'two', 'three' and 'five'. Finally, teen numbers are reversed. 'Seventeen' can be misunderstood as '71'.
2. The relationship between the oral name and the quantity. While counting in ones may be secure, learners need to appreciate that counting in groups – 2s, 5s, 10s – is more efficient. From a display of bundles of ten and a large collection of single counters, what are the learners doing when you pose tasks such as the following: 'Show me thirty', 'Show me thirty-five', 'Look at this collection of thirty-five. If we count them by ones, how many will we get?', 'If we count them by tens, what answer do we get?' and 'Show me seventy. How many tens is that?'
3. The relationship between the numeral and the quantity. Here the task is to emphasize the relationship of the written symbol with the numerosity, so the same type of tasks as above can be used.

Is it your practice to deal with these relationships explicitly and discretely? Do you give equal attention to each or put more emphasis on one? Although only an initial understanding of place value has been illustrated, analogous relationships have to be constructed for more advanced concepts to allow profound understanding of the relationship between a digit's place and its value.

Content knowledge

As has been argued throughout this chapter, knowledge is not a collection of static gobbets but the thinking and reasoning that people draw upon (and construct) when solving problems and engaging in non-routine

222 IMPROVING PRIMARY MATHEMATICS TEACHING AND LEARNING

activity. Furthermore the importance of understanding is irrefutable. Key mathematical understandings appear to coalesce round additive and multiplicative reasoning (Nunes et al. 2009) which can be most economically illustrated through the range of mathematical problems that are used in primary school (Carpenter et al. 1999).

Additive reasoning focuses on the sums of, and differences between, quantities and in part develops intuitively. Nevertheless, addition and subtraction as reflected in different types of problems are experienced variously by learners (see Table 12.3).

Table 12.3 Types of addition and subtraction problems

Problem type			
Join	<i>Result unknown</i>	<i>Change unknown</i>	<i>Start unknown</i>
	May has 5 apples. John gives her 8 more. How many apples does May have altogether?	John has 5 marbles. How many more does he need to have 13 marbles altogether?	May has some hoops. John gives her 5 more. Now she has 13 hoops. How many did she have to start with?
Separate	<i>Result unknown</i>	<i>Change unknown</i>	<i>Start unknown</i>
	John has 13 jigsaws. He gives 5 to the boy next door. How many does he have left?	May has 13 flowers. She planted some, now she is left with 5. How many did she plant?	John has some grapes. He gives 5 to May and now has 8 left. How many did he have to start with?
Part-part-whole	<i>Whole unknown</i>	<i>Part unknown</i>	
	May has 5 pink shirts and 8 blue ones. How many shirts does she have?	John has 13 pens. 5 are red and the rest, blue. How many blue ones has he got?	
Compare	<i>Difference unknown</i>	<i>Compare quantity unknown</i>	<i>Referent unknown</i>
	John has 13 arrows. May has 5. How many more arrows does John have?	May has 5 books. John has 8 more. How many books has John got?	May has 13 hats. She has 5 more than John. How many hats has May got?

Source: Adapted from Carpenter et al. (1999)

These problems contain the same key words but the structure of each is unique and influences how easy or how difficult learners may find them. A major task for many learners is to appreciate that these are alternative representations of additive reasoning.

Multiplicative reasoning, on the other hand, does not develop intuitively and requires formal instruction (Sowder et al. 1998). Multiplicative reasoning is what underpins understandings of common and decimal fractions, percentages, proportion and ratio, all critically important ideas for us to develop. While additive reasoning is a necessary first stage, it is important to enable learners to restructure their concept of number to understand the unit of quantification is not one but a set (such as a pair, a trio, or other composite unit) or a fractional quantity (such as one-quarter, one-third). There is now a new variable, the multiplier, which counts sets. Learners need to be able to understand a collection of items as a set (the multiplicand) which is operated on by the multiplier to result in product. Equally they need to understand the relationships of dividend, divisor and quotient (see Table 12.4).

Again, while these problems appear to be the same, their structural differences are mirrored, initially, in very different problem solutions by learners.

Reflection

Consider how much emphasis you put on these different representations of additive and multiplicative reasoning.

- Do you expect that having taught one representation of an operation that learners will automatically or easily generalize to others?
- Are there some problem types you shy away from?
- What changes to teaching the 'four rules' might you now consider, in the light of reading this chapter?

Conclusion

This chapter has been written for readers who strive to be experts in mathematics teaching and so has primarily been concerned with how teachers think about themselves, their practices and their learners' thinking. Teachers' own complex reflections on practice are important because it is through these that they implement change meaningfully in their

224 IMPROVING PRIMARY MATHEMATICS TEACHING AND LEARNING

Table 12.4 Types of multiplication and division problems

Problem type	Multiplication	Measurement division (quotitive division)	Sharing division (partitive division)
Grouping/ partitioning	Jack has 4 tomato plants. There are 6 tomatoes on each plant. How many tomatoes are there altogether?	Jack has some tomato plants. There are 6 tomatoes on each plant. In total there are 24 tomatoes. How many plants does Jack have?	Jack has 4 tomato plants, with the same number of tomatoes on each. There are 20 tomatoes altogether. How many are there on each plant?
Rate	Jenny walks 3 miles an hour. How many miles does she walk in 5 hours?	Jenny walks 3 miles an hour. How many hours will it take to walk 15 miles?	Jenny walked 15 miles. It took her 5 hours. What is her average hourly walking speed?
Price	Comics costs £2.00. How much do 12 comics cost?	Comics costs £2.00. How many comics can you buy for £24.00?	Mum bought 12 comics, spending £24.00. If each comic was the same price, what did it cost?
Multiplicative comparison	The tree is 3 times as tall as the house. The house is 9 feet high. How tall is the tree?	The tree is 27 feet tall. The house is 9 feet high. The tree is how many times taller than the house?	The tree is 27 feet high, 3 times as tall as the house. How high is the house?

Source: Adapted from Carpenter et al. (1999)

classrooms. The explicit opportunity to think about, question and elaborate on different perspectives of problem solving, both as learners and as teachers, allows teachers to focus more on deep understanding of task demands and on learner-centred teaching (Kramarski and Revach 2009). As well as questioning and reviewing one's practices, it can at times be helpful to resource one's reflections through further reading. The research

literature in mathematics education is vast but two relatively recent publications (noted below) stand out because they have been written with practising teachers in mind.

Further reading

Nunes, T., Bryant, P. & Watson, A. (eds) (2009) *Key Understandings in Mathematics Learning*. London: Nuffield Foundation.

This publication synthesizes the recent research literature of mathematics learning by children (aged 5–16 years). It explores issues such as the insights learners must have in order to understand basic mathematical concepts; the sources of these insights and how informal mathematics knowledge relates to learning mathematics in school; and the understandings learners must have in order to build new mathematical ideas using basic concepts. The publication is presented in a set of eight papers, all of which are available from the funder of the study, the Nuffield Foundation (www.nuffieldfoundation.org).

Carpenter, T., Fennema, E., Franke, M., Levi, L. & Emspon, S. (1999) *Children's Mathematics: Cognitively Guided Instruction*. Portsmouth, NH: Heinemann.

Cognitively Guided Instruction (CGI) is a professional development programme for teachers which started in the United States at the beginning of the twenty-first century and continues to develop. CGI is an approach to teaching mathematics rather than a curriculum programme and at its core is the practice of listening to learners' mathematical thinking and using this as a basis for teaching. Research into CGI shows that expert teachers use a variety of practices to extend learners' mathematical thinking; and that their professional judgment is central to making decisions about how to use information about children's thinking. See <http://ncisla.wceruw.org/publications/reports/NCISLARreport1.pdf>

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226 IMPROVING PRIMARY MATHEMATICS TEACHING AND LEARNING

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228 IMPROVING PRIMARY MATHEMATICS TEACHING AND LEARNING

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